Eigenvalues of the Hessian and concave/convex envelopes.

Julio D. Rossi

(with Pablo Blanc and Carlos Esteve)

U. Buenos Aires (Argentina)
jrossi@dm.uba.ar
www.dm.uba.ar/∼jrossi

XIV Encuentro Nacional de Analistas A. P. Calderón. 2018.

In honor to Prof. Agnes Benedek.

Eigenvalues of D^2u

For a function $u : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$ we denote its Hessian as

$$D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j}$$

and

$$\lambda_1(D^2u) \leq \lambda_2(D^2u) \leq .... \leq \lambda_j(D^2u) \leq ....\lambda_n(D^2u)$$

the ordered eigenvalues of the Hessian D^2u.

Notice that

$$\Delta u = \lambda_1(D^2u) + ... + \lambda_n(D^2u).$$

Main goals

For the problem

$$\begin{cases} \lambda_j(D^2u) = 0, & \text{in } \Omega, \\ u = F, & \text{on } \partial\Omega. \end{cases}$$

• Relate solutions to convex/concave envelopes of the boundary datum F.

• Find a necessary and sufficient condition on the domain Ω in such a way that this problem has a viscosity solution that is continuous in $\overline{\Omega}$ for every $F \in C(\partial\Omega)$.

• Show a connection with probability (game theory).

• Study a parabolic version of this problem.

Convex envelopes

A function $u : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$ is convex if

$$u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y).$$

Given $F : \partial\Omega \mapsto \mathbb{R}$ the convex envelope of F in Ω is

$$u^*(x) = \sup_{u \text{ convex},\, u|_{\partial\Omega} \leq F} u(x).$$

That is, u^* is the largest convex function that is below F on $\partial\Omega$.

Concave envelopes

u is concave if

$$u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y).$$

Given $F : \partial\Omega \mapsto \mathbb{R}$ the concave envelope of F in Ω is

$$u_*(x) = \inf_{u \text{ concave}, \, u|_{\partial\Omega} \geq F} u(x).$$

That is, u_* is the smallest concave function that is above F on $\partial\Omega$

Convex envelopes

If $u \in C^2$ is convex then $D^2u(x)$ must be positive semidefinite,

$$\langle D^2u(x)v, v\rangle \geq 0.$$

In terms of the eigenvalues of D^2u this can be written as

$$\lambda_1(D^2u(x)) = \inf_{|v|=1} \langle D^2u(x)v, v\rangle \geq 0.$$

Moreover, the convex envelope of F in Ω, u^*, is the largest viscosity solution to

$$\begin{cases} \lambda_1(D^2u) = 0, & \text{in } \Omega, \\ u \leq F, & \text{on } \partial\Omega. \end{cases}$$

A. Oberman – L. Silvestre (2011).

Convex envelopes

Notice that in an interval $(a, b) \subset \mathbb{R}$, it holds that

$$u^*(x) = u_*(x) = \frac{(u(b) - u(a))}{b - a}(x - a) + u(a).$$

Therefore, a convex function, u, has the following property: for every segment (a, b) inside Ω we have

$$u(s) \leq v(s) \qquad s \in (a, b)$$

being v the concave envelope of the boundary values $u(a)$, $u(b)$ in (a, b).

Concave / convex envelopes

Let H_j be the set of functions v such that

$$v \leq F \qquad \text{on } \partial\Omega,$$

and have the following property: for every S affine of dimension j and every $j-$dimensional domain $D \subset S \cap \Omega$ it holds that

$$v \leq z \qquad \text{in } D$$

where z is the concave envelope of $v|_{\partial D}$ in D.

Concave / convex envelopes

Theorem *The function*

$$u(x) = \sup_{v \in H_j} v(x)$$

is the largest viscosity solution to

$$\lambda_j(D^2 u) = 0 \qquad \textit{in} \;\; \Omega,$$

with $u \leq F$ *on* $\partial\Omega$*.*

The equation for the concave envelope of $F|_{\partial\Omega}$ in Ω is just $\lambda_n = 0$; while the equation for the convex envelope is $\lambda_1 = 0$.

Condition (H)

A comparison principle (hence uniqueness) for the equation

$$\lambda_j(D^2 u) = 0$$

was proved in

F.R. Harvey, H.B. Jr. Lawson, (2009).

Condition (G)

Our geometric condition on the domain reads as follows: Given $y \in \partial\Omega$ we assume that for every $r > 0$ there exists $\delta > 0$ such that for every $x \in B_\delta(y) \cap \Omega$ and $S \subset \mathbb{R}^n$ a subspace of dimension j, there exists $v \in S$ of norm 1 such that

$$(G_j) \qquad \{x + tv\}_{t\in\mathbb{R}} \cap B_r(y) \cap \partial\Omega \neq \emptyset.$$

We say that Ω satisfies condition (G) if it satisfies both (G_j) and (G_{N-j+1}).

Theorem *The problem*

$$\begin{cases} \lambda_j(D^2 u) = 0, & \textit{in } \Omega, \\ u = F, & \textit{on } \partial\Omega. \end{cases}$$

has a continuous solution (up to the boundary) for every continuous data F

if and only if

Ω *satisfies condition* (G).

The Laplacian and a random walk

Let us consider a final payoff function

$$F : \mathbb{R}^n \setminus \Omega \mapsto \mathbb{R}.$$

In a random walk with steps of size ϵ from x the position of the particle can move to

$$x \pm \epsilon e_j,$$

each movement being chosen at random with the same probability, $1/2n$.

We assumed that Ω is homogeneous and that every time the movement is independent of its past history.

The Laplacian, Δ

Let

$$u_\epsilon(x) = \mathbb{E}^x(F(x_N))$$

be the expected final payoff when we move with steps of size ϵ. Applying conditional expectations we get

$$u_\epsilon(x) = \sum_{j=1}^{n} \left(\frac{1}{2n} u_\epsilon(x + \epsilon e_j) + \frac{1}{2n} u_\epsilon(x - \epsilon e_j) \right).$$

That is,

$$0 = \sum_{j=1}^{n} \left\{ u_\epsilon(x + \epsilon e_j) - 2u_\epsilon(x) + u_\epsilon(x - \epsilon e_j) \right\}.$$

The Laplacian, Δ

Now, one shows that u_ϵ converge as $\epsilon \to 0$ to a continuous function u uniformly in $\overline{\Omega}$.

Then, we get that u is a **viscosity solution** to the Laplace equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = F & \text{on } \partial\Omega. \end{cases}$$

A game

Rules

- Two-person, zero-sum game: two players are in contest and the total earnings of one are the losses of the other. Player I, plays trying to minimize his expected outcome. Player II is trying to maximize.
- $\Omega \subset \mathbb{R}^n$, bounded domain and $F : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ a final payoff function.
- Starting point $x_0 \in \Omega$. At each turn, Player I chooses a subspace S of dimension j and then Player II chooses $v \in S$ with $|v| = 1$.
- The new position of the game is $x \pm \epsilon v$ with probability (1/2–1/2).
- Game ends when $x_N \notin \Omega$, Player I earns $F(x_N)$ (Player II earns $-F(x_N)$)

Remark

The sequence of positions $\{x_0, x_1, \cdots, x_N\}$ has some probability, which depends on

- The starting point x_0.
- The strategies of players, S_I and S_{II}.

Expected result Taking into account the probability defined by the initial value and the strategies:

$$\mathbb{E}^{x_0}_{S_I, S_{II}}(F(x_N))$$

"Smart" players

- Player I tries to choose at each step a strategy which minimizes the result.
- Player II tries to choose at each step a strategy which maximizes the result.

Extremal cases

- $$u_I(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^x_{S_I,S_{II}}(F(x_N))$$
- $$u_{II}(x) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^x_{S_I,S_{II}}(F(x_N))$$

The game has a value $\Leftrightarrow$ $u_I = u_{II}$.

Theorem *This game has a value*

$$u^\epsilon(x).$$

Dynamic Programming Principle

Main Property (Dynamic Programming Principle)

$$u^\epsilon(x) = \inf_{dim(S)=j} \sup_{v\in S,|v|=1} \left\{ \frac{1}{2} u^\epsilon(x+\epsilon v) + \frac{1}{2} u^\epsilon(x-\epsilon v) \right\}$$

$$0 = \inf_{dim(S)=j} \sup_{v\in S,|v|=1} \left\{ u^\epsilon(x+\epsilon v) - 2u^\epsilon(x) + u^\epsilon(x-\epsilon v) \right\}$$

Idea

If $\lambda_1 \leq ... \leq \lambda_N$ are the eigenvalues of $D^2 u(x)$, the $j-$st eigenvalue verifies

$$\min_{dim(S)=j} \max_{v\in S,|v|=1} \langle D^2 u(x) v, v\rangle = \lambda_j.$$

Theorem *It holds that*

$$u_\epsilon \to u, \qquad \text{as } \epsilon \to 0,$$

uniformly in $\overline{\Omega}$.

The limit u is the unique viscosity solution to

$$\begin{cases} \lambda_j(D^2 u) = 0, & \text{in } \Omega, \\ u = F, & \text{on } \partial\Omega. \end{cases}$$

A parabolic version

Consider

$$\begin{cases} u_t(x,t) - \lambda_j(D^2 u(x,t)) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = F(x), & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(0), & \text{in } \Omega, \end{cases}$$

This problem is the evolution version of our previous elliptic problem

$$\begin{cases} \lambda_j(D^2 z(x)) = 0, & \text{in } \Omega, \\ z(x) = F(x), & \text{on } \partial\Omega, \end{cases}$$

A parabolic version

Theorem *For the parabolic problem there is also an associated game.*

Theorem *For Ω strictly convex and u_0 compatible with F ($u_0|_{\partial\Omega} = F$), existence and uniqueness for the parabolic problem holds.*

Theorem (asymptotic behaviour) *There exist positive constants C (depending on the initial condition u_0) and $\mu > 0$ (depending only on Ω), such that*

$$\|u(\cdot, t) - z(\cdot)\|_\infty \leq Ce^{-\mu t}.$$

A parabolic version

In addition, we also describe an interesting behavior of the solution to

$$\begin{cases} u_t(x,t) - \lambda_j(D^2u(x,t)) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0, & \text{in } \Omega, \end{cases}$$

with u_0 any continuous function and $1 < j < N$.

Theorem *There exists $T > 0$ depending only on Ω, such that the viscous solution u satisfies $u(x,t) \equiv 0$, for any $t > T$. Moreover, for any affine function F the same phenomenon also holds (just apply the same argument to $\tilde{u} = u - z$).*

References

- I. Birindelli, G. Galise and I. Ishii, to appear.
- L. Caffarelli, L. Nirenberg and J. Spruck, (1985).
- M.G. Crandall, H. Ishii and P.L. Lions. (1992).
- F.R. Harvey, H.B. Jr. Lawson, (2009)
- J. J. Manfredi, M. Parviainen and J. D. Rossi, (2010).
- A. M. Oberman and L. Silvestre, (2011).

THANKS !!!.   GRACIAS !!!.