Poincaré, Sobolev and Rubio de Francia

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Outline

What?

Why?

How?

Outline

What?

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^q w\right)^{\frac{1}{q}} \leq C_w \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

Why?

$$\operatorname{div}(A(x)\nabla u)=0 \qquad , \qquad A(x)\xi.\xi \approx |\xi|^2 w(x)$$

How?

Unweighted L^1 inequalities involving "Self-improving functionals"

$$\fint_Q |f-f_Q|dx \leq a(Q)$$

Main problem

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^q w\right)^{\frac{1}{q}} \leq C_w \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

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- For a given $p \geq 1$.

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- There is a natural choice for a class A_p of weights.

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- For a given $p \geq 1$.
- There is a natural choice for a class A_p of weights.
- We try to reach the best possible $q = p_w^*$.
- Keeping track of the constant C_w!

Unweighted Poincaré in $(\mathbb{R}^n, dx)$

$(1,1)$ Poincaré inequality

$$\frac{1}{|Q|}\int_Q |f - f_Q|dx \lesssim \ell(Q)\frac{1}{|Q|}\int_Q |\nabla f|dx$$

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(p,p) Poincaré inequality, $\qquad 2 \leq n$, $1 \leq p < n$.

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Higher order Poincaré inequality with polynomials, $m \in \mathbb{N}$

$$\frac{1}{|Q|}\int_Q |f(y) - \pi_Q(y)| dy \lesssim \frac{\ell(Q)^m}{|Q|}\int_Q |\nabla^m f|\, dy$$

Poincaré - Sobolev

Poincaré-Sobolev inequality

$$\left(\frac{1}{|Q|}\int_Q |f - f_Q|^{p^*} dx\right)^{\frac{1}{p^*}} \lesssim \ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p\right)^{\frac{1}{p}}$$

Poincaré-Sobolev inequality

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$$p^* = \frac{np}{n-p}$$

Weights

$$[w]_{A_p} := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{1-p'} \right)^{p-1}$$

$$\frac{|E|}{|Q|} \leq [w]_{A_p}^{\frac{1}{p}} \left(\frac{w(E)}{w(Q)} \right)^{\frac{1}{p}}$$

$$[w]_{A_1} := \sup_Q \left(\fint_Q w \right) \|w^{-1}\|_{L^\infty(Q)}$$

Equivalently:

$$Mw(x) \leq Cw(x) \text{ a.e. } x \in \mathbb{R}^n$$

Hardy-Littlewood maximal function

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)| \, dy,$$

$$M : L^p(w\,dx) \to L^p(w\,dx) \iff w \in A_p \qquad 1 < p < \infty$$

$$M : L^1(w\,dx) \to L^{1,\infty}(w\,dx) \iff w \in A_1$$

$$\|M\|_{L^p(w)} \lesssim p'[w]_{A_p}^{\frac{1}{p-1}}, \qquad 1 < p < \infty$$

$$\|M\|_{L^{p,\infty}(w)} \approx [w]_{A_p}^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$

The following are equivalent

1) $$\fint_Q |f(x) - f_Q| dx \lesssim \ell(Q) \fint_Q |\nabla f(x)| dx$$

2) $$|f(x) - f_Q| \lesssim I_1(|\nabla f|\chi_Q)(x) = \int_{\mathbb{R}^n} \frac{(|\nabla f|\chi_Q)(y)}{|x-y|^{n-1}} dy$$

As a consequence of 2),

$$|f(x) - f_Q| \lesssim I_1(|\nabla f|\chi_Q)(x) \lesssim \ell(Q) M(|\nabla f|)(x)$$

$$\|f - f_Q\|_{L^{p,\infty}_{Q,w}} \lesssim \ell(Q) \|M(|\nabla f|)\|_{L^{p,\infty}_{Q,w}} \lesssim \ell(Q) [w]_{A_p}^{\frac{1}{p}} \|\nabla f\|_{L^p_Q(w)}$$

Truncation or *weak implies strong* lemma:

Lemma

Let $g \geq 0$*, Lipschitz. Suppose a weak* $(1, p)$*-type estimate for the measures* μ, ν *and* $p > 1$*:*

$$\sup_{t>0} t\, \mu(\{x \in \mathbb{R}^n : g(x) > t\})^{1/p} \lesssim \int_{\mathbb{R}^n} |\nabla g(x)| d\nu$$

Then the strong estimate also holds, namely

$$\|g\|_{L^p_\mu} \lesssim \int_{\mathbb{R}^n} |\nabla g(x)| d\nu$$

Theorem

Let $w \in A_p$, then

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^p w\right)^{\frac{1}{p}} \leq [w]_{A_p}^{\frac{1}{p}} \ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

How to deal with higher order Poincaré inequality with polynomials? No truncation...

$$\frac{1}{|Q|}\int_Q |f(y)-\pi_Q(y)|dy \lesssim \frac{\ell(Q)^m}{|Q|}\int_Q |\nabla^m f|\, dy$$

Starting point

$$\fint_Q |f - f_Q| d\mu \leq a(Q), \qquad a : \mathcal{Q} \to (0, \infty)$$

Starting point

$$\fint_Q |f - f_Q| d\mu \leq a(Q), \qquad a : \mathcal{Q} \to (0, \infty)$$

Hypothesis on the functional a

$$\sum_{P \in \Lambda} a(P)^p w(P) \leq \|a\|^p a(Q)^p w(Q)$$

$$a \in D_p(w) \tag{1}$$

Self improving functionals

Theorem (Franchi-Perez-Wheeden - 1998)

Let $w \in A_\infty$ and $a \in D_p(w)$ for some $p > 0$. Let f such that

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq a(Q).$$

Then

$$\|f - f_Q\|_{L^{p,\infty}_{Q,\frac{w}{w(Q)}}} \leq C\|a\|a(Q).$$

- Only for the weak norm
- C depends exponentially on $[w]_\infty$.

New D_p-type condition

Small families

A family of pairwise disjoint subcubes $\{Q_i\} \subset \mathcal{D}(Q)$ is in $S(L), L > 1$ if

$$\sum_i |Q_i| \leq \frac{|Q|}{L}$$

Smallness preserving functionals

$a \in SD_p^s(w)$ for $0 \leq p < \infty$ and $s > 1$ if

$$\sum_i a(Q_i)^p w(Q_i) \leq \|a\|^p \left(\frac{1}{L}\right)^{\frac{p}{s}} a(Q)^p w(Q)$$

whenever $\{Q_i\} \in S(L)$

Main Theorem

Theorem (A)

Let w be **any** *weight, $p \geq 1, s > 1$ and $a \in SD_p^s(w)$. If*

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq a(Q),$$

then

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^p\, w\right)^{\frac{1}{p}} \leq C_n\, s\|a\|^s a(Q)$$

About the proof

Hypothesis: $$\fint_Q \frac{|f - f_Q|}{a(Q)} \leq 1, \qquad a \in SD_p^s(w)$$

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Calderon - Zygmund decomposition

$$\Omega_L := \left\{ x \in Q : M_Q^d \left(\frac{|f-f_Q|}{a(Q)} \chi_Q \right)(x) > L \right\} = \bigcup_j Q_j$$

$$L < \fint_{Q_j} \frac{|f-f_Q|}{a(Q)}\, dy \leq L\, 2^n$$

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Key step: Go from $(\cdot)_Q$ to $(\cdot)_{Q_j}$

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Key step: Go from $(\cdot)_Q$ to $(\cdot)_{Q_j}$

Triangular inequality is not a good idea

$$\int_{Q_j} |f-f_Q|^p w dx \leq 2^{p-1} \left(\int_{Q_j} |f-f_{Q_j}|^p w dx + \int_{Q_j} |f_{Q_j}-f_Q|^p w dx \right)$$

About the proof

Calderón - Zygmund decomposition into good and bad parts

$$\frac{f-f_Q}{a(Q)} = g_Q + b_Q, \qquad \begin{cases} |g(x)| \leq 2^n L \\ b_Q(x) = \sum_i \frac{f(x)-f_{Q_i}}{a(Q)} \chi_{Q_i}(x) \end{cases}$$

$$\left(\frac{1}{w(Q)} \int_Q \frac{|f-f_Q|^p}{a(Q)^p} \, wdx \right)^{\frac{1}{p}} \leq 2^n L + \left(\frac{1}{w(Q)} \int_{\Omega_L} \sum_j |b_{Q_j}|^p \, wdx \right)^{\frac{1}{p}}$$

About the proof

$$\begin{aligned}
\int_{\Omega_L} |\sum_j b_{Q_j}|^p \, w dx &\leq \sum_i \int_{Q_i} \left|b_{Q_j}\right|^p \, w dx \\
&= \frac{1}{a(Q)^p} \sum_i \frac{a(Q_i)^p w(Q_i)}{w(Q_i)} \int_{Q_i} \left| \frac{f - f_{Q_i}}{a(Q_i)} \right|^p \, w dx \\
&\leq \frac{X^p}{a(Q)^p} \sum_i a(Q_i)^p w(Q_i),
\end{aligned}$$

where X is the quantity defined by

$$X = \sup_Q \left(\frac{1}{w(Q)} \int_Q \left| \frac{f - f_Q}{a(Q)} \right|^p \, w dx \right)^{1/p}.$$

About the proof

$$\left(\frac{1}{w(Q)}\int_Q \frac{|f-f_Q|^p}{a(Q)^p}\,wdx\right)^{\frac{1}{p}} \leq 2^n L + \left(\frac{1}{w(Q)}\int_{\Omega_L}\sum_j |b_{Q_j}|^p\,wdx\right)^{\frac{1}{p}}$$

$$X \leq 2^n L + X\frac{\|a\|}{L^{1/s}}.$$

Choose $L = 2e\,\text{máx}\{\|a\|^s, 1\}$ to conclude

$$X \leq 2^n 2e\|a\|^s\left((2e)^{1/s}\right)' \leq e2^{n+1}s\|a\|^s$$

Consequences I: weighted (p, p) Poincaré

Model example: $\alpha, p > 0$

$$a(Q) = \ell(Q)^\alpha \left(\frac{1}{w(Q)}\mu(Q)\right)^{1/p} \quad \rightsquigarrow \quad \begin{cases} a \in SD_p^{n/\alpha}(w) \\ \|a\| = 1 \end{cases}$$

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From unweighted $(1,1)$ to weighted (p, p) Poincaré inequalities

$$\fint_Q |f - f_Q| dx \;\; \lesssim \;\; \ell(Q) \fint_Q |\nabla f| dx$$

Consequences I: weighted (p, p) Poincaré

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From unweighted $(1, 1)$ to weighted (p, p) Poincaré inequalities

$$\begin{aligned} \fint_Q |f - f_Q| dx &\lesssim \ell(Q) \fint_Q |\nabla f| dx \\ &\lesssim [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx\right)^{\frac{1}{p}} \end{aligned}$$

We have then the starting point:

$$\fint_Q |f - f_Q| dx \lesssim a(Q) \in SD_p^n(w)$$

Consequences I: weighted (p,p) Poincaré

Corollary (Theorem A - A_p case)

Let $w \in A_p$, $p \geq 1, n > 1$. Since

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w \; dx\right)^{\frac{1}{p}} =: a(Q)$$

and $a \in SD_p^s(w)$ with $s = n$, $\|a\| = 1$, then

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^p \, w\right)^{\frac{1}{p}} \leq C_n [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

Consequences II - A weighted variant of the Keith-Zhong phenomenom

Theorem (Keith and Zhong, Ann. of Math., 2008)

Let $p > 1$ and let (X, d, μ) be a complete metric measure space with μ Borel and doubling, that admits a (1, p)-Poincaré inequality. Then there exists $\varepsilon > 0$ such that (X, d, μ) admits a $(1, q)$-Poincaré inequality for every $q > p - \varepsilon$, quantitatively.

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If

$$\fint_{B(x,r)} |f - f_Q| d\mu \lesssim r \left(\fint_{B(x,\lambda r)} |\nabla f|^p d\mu \right)^{1/p},$$

then for some $\varepsilon > 0$ and any $q \in (p - \varepsilon, p]$,

$$\fint_{B(x,r)} |f - f_Q| d\mu \lesssim r \left(\fint_{B(x,\lambda r)} |\nabla f|^q d\mu \right)^{1/q},$$

A weighted variant of the Keith-Zhong phenomenom

Theorem

$w \in A_{p_0}$, $1 < p_0 < \infty$, $\varphi : [1,\infty) \to (0,\infty)$ non-decreasing. If the pair (f,g) satisfies the weighted Poincaré $(1,p_0)$ for any $w \in A_{p_0}$,

$$\frac{1}{|Q|}\int_Q |f - f_Q|dx \leq \varphi([w]_{A_{p_0}})\ell(Q)\left(\frac{1}{w(Q)}\int_Q g^{p_0}\, wdx\right)^{\frac{1}{p_0}},$$

Then, for any p such that $1 < p < p_0$ the following estimate holds for any $w \in A_p$:

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^p\, wdx\right)^{\frac{1}{p}} \lesssim \varphi([w]_{A_p}^{\frac{p_0-1}{p-1}})\ell(Q)\left(\frac{1}{w(Q)}\int_Q g^p\, w\right)^{\frac{1}{p}}$$

A weighted variant of the Keith-Zhong phenomenom

$$a(Q) = \varphi([w]_{A_{p_0}})\ell(Q)\left(\frac{1}{w(Q)}\int_Q g^{p_0}\, wdx\right)^{1/p_0}.$$

$$a \in SD^n_{p_0}(w)$$

Then

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^{p_0} w\right)^{\frac{1}{p_0}} \lesssim \varphi([w]_{A_{p_0}})\ell(Q)\left(\frac{1}{w(Q)}\int_Q g^{p_0} w\right)^{\frac{1}{p_0}}$$

By Rubio de Francia's extrapolation technique,

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^{p} w\right)^{\frac{1}{p}} \lesssim \varphi(c_{p,p_0}[w]_{A_p}^{\frac{p_0-1}{p-1}})\ell(Q)\left(\frac{1}{w(Q)}\int_Q g^{p} w\right)^{\frac{1}{p}}$$

Consequences III: weighted (p_w^*, p) Poincaré - Sobolev

Unweighted Poincaré-Sobolev

$$\left(\fint_Q |f - f_Q|^{p^*} dx\right)^{\frac{1}{p^*}} \lesssim \ell(Q) \left(\fint_Q |\nabla f|^p\right)^{\frac{1}{p}}, \qquad \frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$$

Again, we start from:

$$\fint_Q |f - f_Q| dx \lesssim [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx\right)^{\frac{1}{p}} = a(Q)$$

Goal

Obtain that $a \in SD^s_{p^*_w}$ with some control on p^*_w, s and $\|a\|$.

Consequences II: weighted (p_w^*, p) Poincaré - Sobolev

Back to the model example:

$$a(Q) = \ell(Q) \left(\frac{\mu(Q)}{w(Q)} \right)^{1/p}$$

Modified Poincaré-Sobolev index: $p^* = p^*(q, M), \quad (q \geq 1, M > 1)$

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{nqM}$$

Lemma

$$w \in A_q, 1 \leq q \leq p \Longrightarrow a \in SD^s_{p^*}(w), \; s = nM', \; \|a\| = [w]_{A_q}^{\frac{1}{nqM}}$$

Key property:

$$\left(\frac{|E|}{|Q|} \right)^q \leq [w]_{A_q} \frac{w(E)}{w(Q)}$$

Consequences II: weighted (p_w^*, p) Poincaré - Sobolev

Choosing $M = 1 + \frac{1}{q}\log[w]_{A_q}$, we obtain

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n(q + \log[w]_{A_q})}$$

Theorem (B)

Let $1 \leq p < n$ and let $w \in A_q$ with $1 \leq q \leq p$. If

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq a(Q) = \ell(Q)\left(\frac{1}{w(Q)}\mu(Q)\right)^{1/p},$$

then

$$\left(\frac{1}{w(Q)}\int_Q |f - f_Q|^{p_w^*}\, w\right)^{\frac{1}{p_w^*}} \leq Ca(Q).$$

Consequences II: weighted (p_w^*, p) Poincaré

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{n(q + \log[w]_{A_q})}$$

Corollary

Let $1 \leq p < n$ *and let* $w \in A_q$ *with* $1 \leq q \leq p$.

$$\left(\frac{1}{w(Q)} \int_Q |f - f_Q|^{p_w^*} w\right)^{\frac{1}{p_w^*}} \leq [w]_{A_p}^{\frac{1}{p}} \left(\frac{\ell(Q)^p}{w(Q)} \int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

Further improvements on p_w^* - Template

Theorem (C)

Let a be a functional satisfying:

1. $a \in SD_p^n(w), p \geq 1, \|a\|_{SD_p^n(w)} = 1$
2. *For some* $r > p$, $a \in D_r(w)$

If

$$\frac{1}{|Q|}\int_Q |f - f_Q| \leq a(Q),$$

and $w \in A_p$, *then*

$$\|f - f_Q\|_{L^{r,\infty}(Q,\frac{w}{w(Q)})} \lesssim \|a\|_{D_r(w)}[w]_{A_p}^{\frac{1}{p}}\, a(Q).$$

Only for the weak norm...

$$\fint_Q |f - f_Q| dx \lesssim [w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \, dx \right)^{\frac{1}{p}} = a(Q)$$

- $a \in SD_p^n$ with $\|a\| = 1$
- $w \in A_q, \frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{nq} \Longrightarrow a \in D_{p_w^*}(w)$ with $\|a\| = [w]_{A_q}^{\frac{1}{nq}}$

Corollary

Let $1 \leq p < n$, $w \in A_q$ with $1 \leq q \leq p$. Then

$$\left(\frac{1}{w(Q)} \int_Q |f - f_Q|^{p_w^*} w \right)^{\frac{1}{p_w^*}} \lesssim [w]_{A_q}^{\frac{1}{nq}} [w]_{A_p}^{\frac{2}{p}} \left(\frac{\ell(Q)^p}{w(Q)} \int_Q |\nabla f|^p w \right)^{\frac{1}{p}}$$

The case of A_1 weights

From Theorem (C): $\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$, (Sobolev!)

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^{p^*} w\right)^{\frac{1}{p^*}} \lesssim [w]_{A_1}^{\frac{1}{n}}[w]_{A_p}^{\frac{2}{p}}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

The case of A_1 weights, again but different

By using a completely different method (details not included)

Theorem (D)

Let w be any weight in $\mathbb{R}^n$, $n \geq 2$. Then if $1 \leq p < n$ we have that

$$\left(\int_Q |f - f_Q|^{p^*} w dx \right)^{\frac{1}{p*}} \lesssim \left(\int_Q |\nabla f|^p \frac{(M(w\chi_Q))^{\frac{p}{n'}}}{w^{p-1}} dx \right)^{\frac{1}{p}},$$

for $\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$.

Corollary (D)

Let $w \in A_1$, $n \geq 2$. Then we have that

$$\left(\int_Q \frac{|f - f_Q|^{p*} w}{w(Q)} \right)^{\frac{1}{p*}} \lesssim [w]_{A_1} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w \right)^{\frac{1}{p}}$$

The case of A_1 weights

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}$$

From Theorem (C):

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^{p^*} w\right)^{\frac{1}{p^*}} \lesssim [w]_{A_1}^{\frac{1}{n}}[w]_{A_p}^{\frac{2}{p}}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

From Theorem (D):

$$\left(\int_Q |f-f_Q|^{p^*} wdx\right)^{\frac{1}{p*}} \lesssim [w]_{A_1}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}.$$

Lemma

If

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^{p*}w\right)^{\frac{1}{p*}} \leq C[w]_{A_1}^{\beta}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

then $\beta \geq \frac{1}{p}$.

Lemma

If

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^{p*}w\right)^{\frac{1}{p*}} \leq C[w]_{A_1}^{\beta}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}$$

then $\beta \geq \frac{1}{p}$.

Conjecture

Let w *be an* A_1 *weight in* $\mathbb{R}^n$, $n \geq 2$. *Then if* $1 \leq p < n$

$$\left(\frac{1}{w(Q)}\int_Q |f-f_{Q,w}|^{p*}w\right)^{\frac{1}{p*}} \leq C[w]_{A_1}^{\frac{1}{p}}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f|^p w\right)^{\frac{1}{p}}.$$

Thank you!

Thank you!

Some extra stuff

Higher order Poincaré

Theorem

Let w be any weight, $p \geq 1$, $a \in SD_p^s(w)$
If

$$\frac{1}{|Q|}\int_Q |f - P_Q f| \leq a(Q),$$

then

$$\left(\frac{1}{w(Q)}\int_Q |f - P_Q f|^p \, w dx\right)^{\frac{1}{p}} \leq C_{n,m} 2^{\frac{s+1}{p'}} s \|a\|^s a(Q)$$

Corollary

Let $1 \leq p < \frac{n}{m}$ *and let* $w \in A_p$.

$$\left(\frac{1}{w(Q)}\int_Q |f - P_Q f|^p w\right)^{\frac{1}{p}} \lesssim [w]_{A_p}^{\frac{1}{p}} \ell(Q)^m \left(\frac{1}{w(Q)}\int_Q |\nabla^m f|^p w\right)^{\frac{1}{p}}$$

The model example and $SD_p^{n/\alpha}$

Model example: $\alpha, p > 0$

$$a(Q) = \ell(Q)^{\alpha}\left(\frac{1}{w(Q)}\mu(Q)\right)^{1/p} \quad \rightsquigarrow \quad \begin{cases} a \in SD_p^{n/\alpha}(w) \\ \|a\| = 1 \end{cases}$$

$$\begin{aligned}
\sum_i a(Q_i)^p w(Q_i) &\leq \sum_i \ell(Q_i)^{p\alpha}\mu(Q_i) = \sum_i |Q_i|^{\frac{p\alpha}{n}}\mu(Q_i) \\
\color{red}{\left(\frac{p\alpha}{n} < 1\right)} \qquad &\leq \left(\sum_i |Q_i|\right)^{\frac{p\alpha}{n}} \left(\sum_i \mu(Q_i)^{(\frac{n}{p\alpha})'}\right)^{\frac{1}{(\frac{n}{p\alpha})'}} \\
&\leq \left(\frac{|Q|}{L}\right)^{\frac{p\alpha}{n}} \sum_i \mu(Q_i) \\
&= \left(\frac{1}{L}\right)^{\frac{p\alpha}{n}} \ell(Q)^{p\alpha}\mu(Q) = \left(\frac{1}{L}\right)^{\frac{p\alpha}{n}} a(Q)^p w(Q).
\end{aligned}$$

Consequences II: weighted (p_w^*, p) Poincaré - Sobolev

$$\begin{aligned}
\sum_i a(Q_i)^{p^*} w(Q_i) &= \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left(\frac{\ell(Q_i)}{w(Q_i)^{\frac{1}{p}-\frac{1}{p^*}}} \right)^{p^*} \\
&= \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left(\frac{|Q_i|}{w(Q_i)^{\frac{1}{qM}}} \right)^{\frac{p^*}{n}} \\
&\leq [w]_{A_q}^{\frac{p^*}{nqM}} \left(\frac{|Q|^q}{w(Q)} \right)^{\frac{p^*}{nqM}} \sum_i \mu(Q_i)^{\frac{p^*}{p}} |Q_i|^{\frac{p^*}{nM'}}
\end{aligned}$$

Consequences II: weighted (p_w^*, p) Poincaré - Sobolev

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$$= \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left(\frac{|Q_i|}{w(Q_i)^{\frac{1}{qM}}} \right)^{\frac{p^*}{n}}$$

$$\leq [w]_{A_q}^{\frac{p^*}{nqM}} \left(\frac{|Q|^q}{w(Q)} \right)^{\frac{p^*}{nqM}} \sum_i \mu(Q_i)^{\frac{p^*}{p}} |Q_i|^{\frac{p^*}{nM'}}$$

Hölder's inequality plus some magic...

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$$\sum_i a(Q_i)^{p^*} w(Q_i) = \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left(\frac{\ell(Q_i)}{w(Q_i)^{\frac{1}{p}-\frac{1}{p^*}}} \right)^{p^*}$$

$$= \sum_i \mu(Q_i)^{\frac{p^*}{p}} \left(\frac{|Q_i|}{w(Q_i)^{\frac{1}{qM}}} \right)^{\frac{p^*}{n}}$$

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Hölder's inequality plus some magic...

$$\leq [w]_{A_q}^{\frac{p^*}{nqM}} a(Q)^{p^*} w(Q) \left(\frac{1}{L} \right)^{\frac{p^*}{nM'}}$$

A_1 weights, which is better?

For A_1 weights, we can define p_w^* as

$$\frac{1}{p} - \frac{1}{p_p^*} = \frac{1}{n(p + \log[w]_{A_p})} \quad \text{or} \quad \frac{1}{p} - \frac{1}{p_1^*} = \frac{1}{n(1 + \log[w]_{A_1})}$$

Compare

$$\frac{1}{n(p + \log[w]_{A_p})} \leq \frac{1}{n(1 + \log[w]_{A_1})}$$

Equivalently,

$$[w]_{A_1} \leq e^{p-1}[w]_{A_p}$$

Is this true? Always? Never? Sometimes?

The case of A_1 weights

Let $w \in A_1$, namely $Mw \leq [w]_{A_1} w$ a.e.

$$\left(\int_Q |f - f_Q|^{p^*} w dx \right)^{\frac{1}{p*}} \lesssim \left(\int_Q |\nabla f|^p \frac{(M(w\chi_Q))^{\frac{p}{n'}}}{w^{p-1}} dx \right)^{\frac{1}{p}}$$

We write $v = \left(\frac{(Mw)^{1/n'}}{w} \right)^p w$

$$\begin{aligned} \left(\int_Q \frac{|f - f_Q|^{p*} w}{w(Q)} \right)^{\frac{1}{p*}} & \lesssim \frac{w(Q)^{1/p}}{w(Q)^{1/p*}} \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p v \right)^{\frac{1}{p}} \\ & \lesssim \left(\frac{w(Q)}{|Q|} \right)^{1/n} |Q|^{1/n} \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p v \right)^{\frac{1}{p}} \\ & \lesssim \inf_{x \in Q} (Mw(x))^{\frac{1}{n}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p v \right)^{\frac{1}{p}} \end{aligned}$$

The case of A_1 weights

Do not forget that $v = \left(\frac{(Mw)^{1/n'}}{w}\right)^p w$

$$
\begin{aligned}
\left(\int_Q \frac{|f - f_Q|^{p*} w}{w(Q)}\right)^{\frac{1}{p*}} &\lesssim \inf_{x\in Q} (Mw(x))^{\frac{1}{n}}\, \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p v\right)^{\frac{1}{p}} \\
&\lesssim \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p \left(\frac{Mw}{w}\right)^p w\right)^{\frac{1}{p}} \\
&\lesssim [w]_{A_1} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f|^p w\right)^{\frac{1}{p}}.
\end{aligned}
$$

A weighted variant of the Keith-Zhong phenomenom

$$a(Q) = \varphi([w]_{A_{p_0}})\ell(Q)\left(\frac{1}{w(Q)}\int_Q g^{p_0}\, wdx\right)^{1/p_0}.$$

$$a \in SD^n_{p_0}(w) \Longrightarrow \left(\frac{1}{w(Q)}\int_Q |f - f_Q|^{p_0}\, wdx\right)^{\frac{1}{p_0}} \leq C_n\, a(Q).$$

For any $h \in L^p$, the Rubio de Francia's operator is

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{M^k(h)}{\|M\|^k_{L^p(w)}}$$

(A) $h \leq R(h)$

(B) $\|R(h)\|_{L^p(w)} \leq 2\,\|h\|_{L^p(w)}$

(C) $[R(h)]_{A_1} \leq 2\,\|M\|_{L^p(w)}$

A weighted variant of the Keith-Zhong phenomenom

$$
\begin{aligned}
\left(\int_Q |f-f_Q|^p\, w dx\right)^{\frac{1}{p}} &= \left(\int_Q |f-f_Q|^p R(\chi_Q g)^{-\alpha p} R(\chi_Q g)^{\alpha p}\, w dx\right)^{\frac{1}{p}} \\
&\leq I.II
\end{aligned}
$$

$$
I = \left(\int_Q |f-f_Q|^{p_0} R(\chi_Q g)^{-\alpha p_0}\, w dx\right)^{1/p_0}
$$

$$
II = \left(\int_Q R(\chi_Q g)^{\alpha p \left(\frac{p_0}{p}\right)'}\, w dx\right)^{\frac{1}{p\left(\frac{p_0}{p}\right)'}}
$$

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$$\leq I.II$$

$$I = \left(\int_Q |f-f_Q|^{p_0} R(\chi_Q g)^{-(p_0-p)}\, w dx\right)^{1/p_0}$$

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$$\alpha = \frac{p_0 - p}{p_0}$$

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A weighted variant of the Keith-Zhong phenomenom

$$[R(\chi_Q g)^{-(p_0-p)}\, w]_{A_{p_0}} \leq c_{p,p_0,n}\, [w]_{A_p}^{\frac{p_0-1}{p-1}}.$$

$$\begin{aligned} I &= \left(\int_Q |f - f_Q|^{p_0} R(\chi_Q g)^{-(p_0-p)}\, w dx \right)^{1/p_0} \\ &\lesssim \varphi(c_{p,p_0,n}\, [w]_{A_p}^{\frac{p_0-1}{p-1}})\, \ell(Q) \left(\int_Q g^{p_0}\, R(\chi_Q g)^{-(p_0-p)}\, wdx \right)^{1/p_0} \\ &\lesssim \varphi(c_{p,p_0,n}\, [w]_{A_p}^{\frac{p_0-1}{p-1}})\, \ell(Q) \left(\int_Q g^p\, wdx \right)^{1/p_0} \end{aligned}$$

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$$II = \left(\int_Q R(\chi_Q g)^p\, w dx\right)^{\frac{1}{p\left(\frac{p_0}{p}\right)'}} \leq 2 \left(\int_Q g^p\, w dx\right)^{\frac{p_0-p}{p_0 p}}$$