Continuity of singular integral operators: Estimates on distribution type spaces

Lucas Oliveira - UFRGS
(in collaboration with Lucas Chaffee and Jarod Hart)

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Contents

Main Question

MAIN QUESTION:

Is it possible to obtain a generalization of the $T(1)$-theorem that holds for other function spaces and/or other classes of operators (fractional/singular/hypersingular)?

Motivation and known results

Generalized Calderón-Zygmund Operators - Meyer - Lemarie - David and Journé

Suppose that $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ has a kernel K and associated to it there are constants $C_1, C_2 > 0$ and $0 < \epsilon < 1$ verifying

1. $|K(x,y)| \leq C_1 |x-y|^{-n}$ and
2. $|K(x,y) - K(x',y)| \leq C_2 |x-y|^{-n-\epsilon} |x-x'|^{\epsilon}$

for all pairs $(x,y), (x',y) \in \mathbb{R}^{2n}$ verifying $x \neq y$, $x' \neq y$ and $2|x-x'| < |x-y|$. If T verifies the conditions above we will simply write $T \in CZO(\epsilon)$ and say that T is a generalized Calderón-Zygmund operator.

Motivation and known results

T1 type theorems

Suppose that $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a generalized Calderón-Zygmund operator (that is $T \in CZO(\epsilon)$). Then Definition of $\dot{B}_p^{s,q}$?

1. (**David-Journé - 1984**) (If also $T^* \in CZO(\epsilon)$)

$$T(1), T^*(1) \in BMO \text{ and } T \in WBP \iff T : L^2 \to L^2\,;$$

2. (**Lemarié - 1985**) For all $1 \le p, q \le \infty$ and $0 < s < \epsilon$

$$T(1) = 0 \text{ and } T \in WBP \Rightarrow T : \dot{B}_p^{s,q} \to \dot{B}_p^{s,q}\,;$$

3. (**M. Meyer - 1987**) For all $1 \le p, q \le \infty$ and $0 < s < \epsilon$ Definition of $BMO^S_{p,q}$

$$T(1) \in BMO^s_{p,q} \text{ and } T \in WBP \iff T : \dot{B}_p^{s,q} \to \dot{B}_p^{s,q}\,.$$

Motivation and known results

Because of the above results, it is natural to conjecture that $T(1) \in BMO^{s}_{p,q}$ is the right condition for a possible generalization for the $T(1)$ theorem for other settings...

BUT WE SHOULD BE CAREFUL HERE!

Motivation and known results

Generalized Calderón-Zygmund Operators II - Bourdad - Y. Meyer Youssfi - Torres

Suppose that $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ has a kernel K, and associated to it there are positive constants C_1, C_2, m such that

1. $|\partial_x^\alpha K(x,y)| \leq C_1|x-y|^{-n-|\alpha|}$ for all $|\alpha| \leq [m]$ and
2. $|\partial_x^\alpha K(x,y) - \partial_x^\alpha K(x',y)| \leq C_2|x-y|^{-n-m}|x-x'|^{m-[m]}$ for all $|\alpha| = [m]$,

for all pairs $(x,y), (x',y) \in \mathbb{R}^{2n}$ verifying $x \neq y$, $x' \neq y$ and $2|x-x'| < |x-y|$. If T verifies the conditions above we will simply write $T \in CZO(m)$ and say that T is a generalized Calderón-Zygmund operator.

Motivation and known results

T1 type theorems

Suppose that $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ and that $T \in CZO(m)$. Then

1. (**Y. Meyer - around 1983**) For all $0 < s < m$ and
$$T \in WBP \text{ and } T(x^\beta) = 0 \text{ for all } |\beta| \leq [m] \Rightarrow T : \Lambda^s \to \Lambda^s;$$
2. (**Frazier-Han-Jawerth-Weiss - 1987**) For $0 < s < m < 1$ and $1 \leq p, q \leq \infty$
$$T \in WBP \text{ and } T(1) = 0 \Rightarrow T : \dot{F}_p^{s,q} \to \dot{F}_p^{s,q};$$
3. (**Torres 1991**) For $s > 1$, $1 \leq p, q < \infty$ and $m = [s] + \delta$ (with $s_* < \delta < 1$),
$$T \in WBP \text{ and } T(x^\beta) = 0 \text{ for all } |\beta| \leq [m] \Rightarrow T : \dot{F}_p^{s,q} \to \dot{F}_p^{s,q};$$

Motivation and known results

- In a different direction, Youssfi (1989) obtained a reduction in the spirit of the original $T(1)$ theorem of David and Journé: for all $1 \leq p, q \leq \infty$ and $m-1 \leq s < m$ if $m > 1$ and $0 < s < 1$ if $m = 1$, we have that

$$T : \dot{B}_p^{s,q} \to \dot{B}_p^{s,q} \iff T \in WBP \text{ and } \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} \Pi_{b_\alpha}^{\alpha} : \dot{B}_p^{s,q} \to \dot{B}_p^{s,q},$$

(where $b_\alpha = \Gamma^\alpha(T)(1)$ (a kind of iterated commutator applied to the function $T(1)$) and where

$$\Pi_b^\alpha f = \sum_{k \in \mathbb{Z}} \Lambda_k(b) S_{k-3}(\partial^\alpha f)$$

is a generalized Bony's paraproduct)

Motivation and known results

What is the right condition? Something in terms of cancellation conditions like $T(x^\alpha) = 0$ or $T^*(x^\alpha) = 0$ or a condition of the type $BMO^s_{p,q}$?

Motivation and known results

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I don't have an answer in general but when $s < 0$, it seems that the cancellation needs to enter in the game...

Contents

General classes of singular integral operators

Generalized Singular Integral Operators

Suppose that there are $\nu \in \mathbb{R}$, $\gamma \in (0,1]$, $M \in \mathbb{N}$ and $C > 0$ such that

1. $|\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C\,|x-y|^{-n-\nu-|\alpha|-|\beta|}$ for all $|\alpha|+|\beta| \leq M$
2. $|\partial_x^\alpha \partial_y^\beta K(x+h,y) - \partial_x^\alpha \partial_y^\beta K(x,y)| \leq C\,|x-y|^{-(n+\nu+M+\gamma)}|h|^\gamma$ for all $|\alpha|+|\beta| = M$,
3. $|\partial_x^\alpha \partial_y^\beta K(x,y+h) - \partial_x^\alpha \partial_y^\beta K(x,y)| \leq C\,|x-y|^{-(n+\nu+M+\gamma)}|h|^\gamma$ for all $|\alpha|+|\beta| = M$,

for all pairs $x, y, h \in \mathbb{R}^n$ verifying $x \neq y$ and $2|h| < |x-y|$. If the kernel of the operator $T : \mathcal{S}_P(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ verifies the conditions above we will simply write $T \in SIO_\nu(M+\gamma)$ (here $\mathcal{S}_P$ = *functions in* $\mathcal{S}$ *with vanishing moments up to order* $P \in \mathbb{Z}_+$).

General classes of singular integral operators

- Observe that when $-n < \nu < 0$ the above definition is related to fractional integral operators;
- When $\nu = 0$ and $M = 0$ we recover the class of generalized *CZO* of David-Journé;
- When $\nu > 0$ we have hyper-singular integral operators. Derivatives of *CZO* are good examples, but more operators belong to this class: the generalized paraproducts of Youssfi and members of the exotic/fordidden symbol class $OpS^0_{\delta,\rho}$ are in $SIO_\nu(M+\gamma)$ too (for correct choice of parameters).

General classes of singular integral operators

Generalized Calderón-Zygmund Operators III

For $\nu \in \mathbb{R}$, $\gamma \in (0,1]$ and $M \in \mathbb{N}$, an operator $T \in SIO_\nu(M+\gamma)$ is a ν-order Calderón-Zygmund operator, denoted by $T \in CZO_\nu(M+\gamma)$, if T can be extended continuously to a bounded operator from $\dot{W}^{\nu,p} \equiv \dot{F}_p^{\nu,2}$ into L^p for all $1 < p < \infty$.

We will also use two notations:

$$CZO_\nu = \cup_{M\in\mathbb{N}} \cup_{0<\gamma\leq 1} CZO_\nu(M+\gamma)$$

$$SIO_\nu(\infty) = \cap_{M\in\mathbb{N}} \cap_{0<\gamma\leq 1} SIO_\nu(M+\gamma)$$

General classes of singular integral operators

Weak Boundedness Property of order ν

An operator $T \in SIO_\nu(M+\gamma)$ verifies the ν-order weak bounded property, denoted by $T \in WBP_\nu$, if there are integers $L, N \geq 0$ and a constant $C > 0$ such that

$$|\langle T\psi, \varphi\rangle + \langle T^*\psi, \varphi\rangle| \leq C|B|^{1-\nu/n}$$

for any ball $B \subset \mathbb{R}^n$, $\psi \in \mathcal{D}_L$, $\varphi \in C_0^\infty$, $\operatorname{supp}(\psi) \cup \operatorname{supp}(\varphi) \subset B$ and

$$\|\partial^\alpha \psi\|_\infty \,,\, \|\partial^\alpha \varphi\|_\infty \leq |B|^{-|\alpha|/n}$$

for all $|\alpha| \leq N$ (where D_L = all functions in C_0^∞ with vanishing moments up to order L).

Our main results

Main results / Chaffee - Hart - Oliveira (2018)

In what follows $\nu \in \mathbb{R}$, $L \in \mathbb{N}$ with $L \geq |\nu|$, $M \geq \max(L, L-\nu)$ and $\gamma > 0$ verifies $(L-\nu)_* < \gamma \leq 1$. Suppose that $T \in SIO_\nu(M+\gamma)$, $T \in WBP_\nu$ and $T^*(x^\alpha) = 0$ for all $|\alpha| \leq L$. Then

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1. $T : \dot{B}^{\nu-t,q}_{p,w} \to \dot{B}^{-t,q}_{p,w}$ for all $\nu < t < \nu + [L-\nu] + \gamma$, $1 < p < \infty$, $0 < q < \infty$ and $w \in A_p$;

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1. $T : \dot{B}^{\nu-t,q}_{p,w} \to \dot{B}^{-t,q}_{p,w}$ for all $\nu < t < \nu + [L-\nu] + \gamma$, $1 < p < \infty$, $0 < q < \infty$ and $w \in A_p$;
2. $T : \dot{F}^{\nu-t,q}_{p,w} \to \dot{F}^{-t,q}_{p,w}$ for all $\nu < t < \nu + [L-\nu] + \gamma$, $1/\lambda < p, q < \infty$, and $w \in A_{\lambda p}$, where $\lambda = \frac{n+\nu+[L-\nu]+\gamma-t}{n}$;

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1. $T : \dot{B}^{\nu-t,q}_{p,w} \to \dot{B}^{-t,q}_{p,w}$ for all $\nu < t < \nu + [L-\nu] + \gamma$, $1 < p < \infty$, $0 < q < \infty$ and $w \in A_p$;
2. $T : \dot{F}^{\nu-t,q}_{p,w} \to \dot{F}^{-t,q}_{p,w}$ for all $\nu < t < \nu + [L-\nu] + \gamma$, $1/\lambda < p, q < \infty$, and $w \in A_{\lambda p}$, where $\lambda = \frac{n+\nu+[L-\nu]+\gamma-t}{n}$;
3. If $A = F$ or B, then
   - $T^* : \dot{A}^{t,q}_p \to \dot{A}^{t-\nu,q}_p$ for all $1 < p, q < \infty$ and $\nu < t < \nu + [L-\nu] + \gamma$;
   - $T^* : \dot{F}^{t,q}_\infty \to \dot{F}^{t-\nu,q}_\infty$ for all $1 < q < \infty$ and $\nu < t < \nu + [L-\nu] + \gamma$;
   - $T^* : \dot{B}^{t,\infty}_\infty \to \dot{B}^{t-\nu,\infty}_\infty$ for all $\nu < t < \nu + [L-\nu] + \gamma$.

Our main results

Main results / Chaffee - Hart - Oliveira (2018) - cont.

4. If $s, t \in \mathbb{R}$ verifies the two conditions $\nu < s < \nu + [L - \nu] + \gamma$ and $0 < t < [L - \nu] + \gamma$, then there exists an operator $T_{s,t} \in CZO_{\nu+t-s}$ such that
$$|\nabla|^{-s} T |\nabla|^{t} f - T_{s,t} f$$
is a polynomial for all $f \in \mathcal{S}_\infty$(infinity vanishing moments = $\cap_P \mathcal{S}_P$) ;

Our main results

Main results / Chaffee - Hart - Oliveira (2018) - cont.

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$$|\nabla|^{-s} T |\nabla|^{t} f - T_{s,t} f$$
is a polynomial for all $f \in \mathcal{S}_\infty$(infinity vanishing moments = $\cap_P \mathcal{S}_P$) ;
5. $T^*(x^\alpha) = 0$ for all $|\alpha| \leq L \iff T : \dot{F}_p^{\nu-t,q} \to \dot{F}_p^{-t,q}$ under the above conditions (in fact, the same holds if we use the conclusions in 1, 2, 3 or 4 above).

Consequences of the main results

- Provide an easy way to understand fractional or hyper-singular integral operators by using CZO;
- We can prove new estimates for operators $T \in OpS^0_{1,1}$;
- Re-obtain some results (and provide new ones) for generalized paraproducts Π^α_β;
- Decompositions in the spirit of ”sparse domination”;
- Algebras of singular integral operators: we can prove that if $G \subset \mathbb{R}$ is closed under addition, then

$$\mathbb{U}_G = \{T \in SIO_\nu(\infty) : T(x^\alpha) = T^*(x^\alpha) = 0, T \in WBP_\nu, \nu \in G\}$$

is an operator algebra (closed under composition and transposition).

Contents

Final remarks and open questions

1. Can we change the conditions $T^*(x^\alpha) = 0$ by something like $T(x^\alpha) \in X$ for some space X?

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1. Can we change the conditions $T^*(x^\alpha) = 0$ by something like $T(x^\alpha) \in X$ for some space X?
    - When $\alpha = 0 = \nu$ the answer is yes, since by the results of M. Meyer this is the right condition to guarantee the boundedness on $\dot{F}_p^{s,q}$ spaces.
    - The situation may change if we increase/decrease the order of decay in the kernel;

Final remarks and open questions

2. Is it possible to characterize the spaces $BMO^s_{p,q}$ in a better way?

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   - $BMO^s_{1,1}$ is the space of functions formed by the elements $\beta \in \dot{B}^{0,\infty}_{\infty}$ for which the measure $|Q_t\beta|dxdt/t$ is a Carleson measure (M. Meyer);

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   - $BMO^{0,2}_2 = BMO$ and $BMO^{s,2}_2$ admits a characterization in terms of capacities (M. Meyer) and Carleson measures (Y. Meyer and Youssfi);
   - Can we do something similar in the general case?

Final remarks and open questions

3. What about extensions and limitations of our ”calculus”?

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   - Right now we have some results for the case were the operators involved in the calculus our of non-convolution type, but just for the ”fractional integral part” in the decomposition in 4; for the other part we still have some difficulties to overcome;

Final remarks and open questions

3. What about extensions and limitations of our ”calculus”?
   - Right now we have some results for the case were the operators involved in the calculus our of non-convolution type, but just for the ”fractional integral part” in the decomposition in 4; for the other part we still have some difficulties to overcome;
   - Our main objective right now is break the barrier of the smoothness in both variables. We are trying to combine ideas from Torres, M. Meyer and Youssfi with our current results in order to overcome this problems.

!!! Gracias !!!

Definition of WBP

Weak Boundedness Property

For $f \in \mathcal{S}$, $z \in \mathbb{R}^n$ and $u > 0$ define

$$f_{z,u}(x) = f\left(\frac{x-z}{u}\right) \text{ for all } x \in \mathbb{R}^n$$

We say that $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ verifies the weak boundedness property if there is a constant $C > 0$ such that for all $\varphi, \psi \in \mathcal{S}$, for all $x_0 \in \mathbb{R}^n$ and all $t > 0$ we have

$$|\langle T(\varphi_{x_0,t}, \psi_{x_0,t})\rangle| \leq C\, t^n$$

If T verifies this condition we will simply write $T \in WBP$.

Comeback

Definition of $BMO^s_{p,q}$

$BMO^s_{p,q}$ spaces

Given $\beta \in \dot{B}^{0,\infty}_{\infty}$, we will say that $\beta \in BMO^s_{p,q}$ if the paraproduct Π_β

$$\Pi_b f = \sum_{k\in\mathbb{Z}} \Lambda_k(b) S_{k-3}(f)$$

is bounded on $\dot{B}^{s,q}_p$.

Comeback

Definition of $\dot{B}^{s,q}_p$ and $\dot{F}^{s,q}_p$

$\dot{B}^{s,q}_p$ and $\dot{F}^{s,q}_p$ spaces

Given $s \in \mathbb{R}$, $0 < p, q \leq \infty$, the spaces $\dot{B}^{s,q}_p$ and $\dot{F}^{s,q}_p$ are the collection of all $f \in \mathcal{S}'/\mathcal{P}$ (tempered distribuitions modulo polynomials) for which the corresponding norms

$$\|f\|_{\dot{B}^{s,q}_p} = \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j f\|^q_{L^p} \right)^{1/q}$$

and

$$\|f\|_{\dot{F}^{s,q}_p} = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{jsq} |\Delta_j f|^q \right)^{1/q} \right\|_{L^p}$$

are finite (and where $\Delta_j f = \varphi_j \star f$, φ bump function). Comeback