Approximation by group invariant subspaces

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Joint work with C. Cabrelli, E. Hernández and U. Molter

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Motivation I: dimensionality reduction

Approximation by linear subspaces of finite dimensional data in a vector space: Principal Component Analysis.

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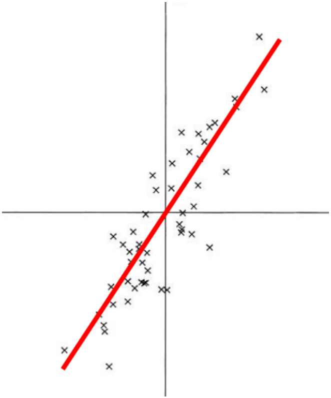

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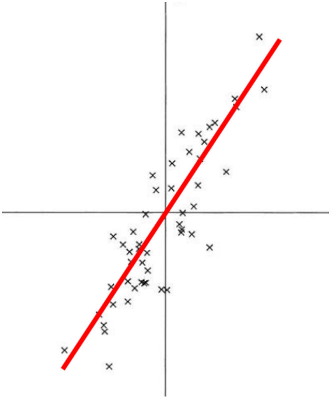

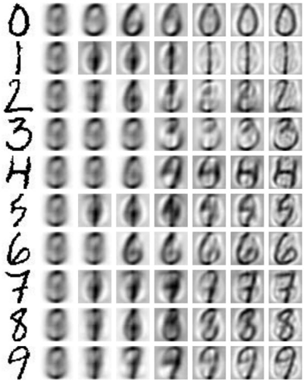

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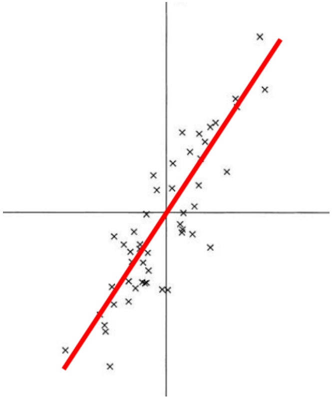
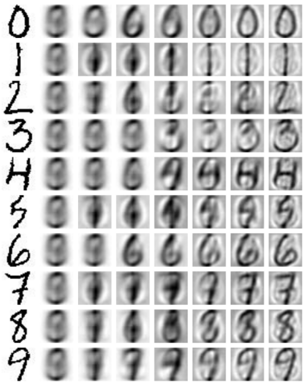
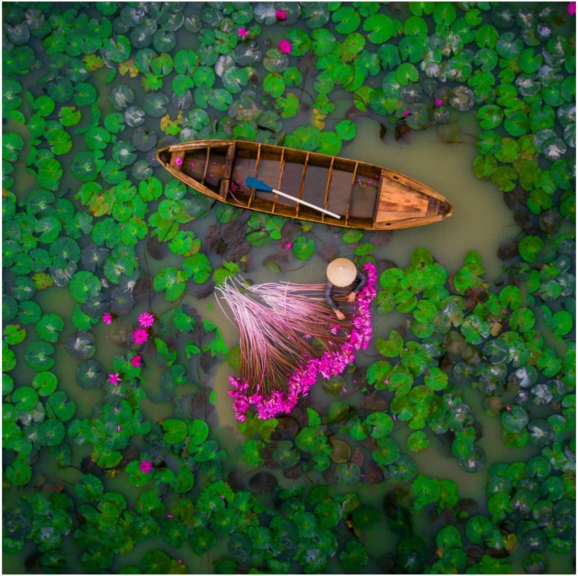




Approximation by shift-invariant subspaces of data in $L^2(\mathbb{R}^d)$: Aldroubi, Cabrelli, Hardin and Molter 2007.

Motivation II: symmetries in data - abelian



Motivation II: symmetries in data - abelian

Motivation II: symmetries in data - non abelian

Results

For non abelian symmetries on $L^2(\mathbb{R}^d)$, we will discuss:

1. characterizations of invariant spaces;
2. construction of group Parseval frames;
3. approximation by group invariant subspaces.

Definition of group invariance

Let $\Lambda \subset \mathbb{R}^d$ be a lattice subgroup[1], and let $G \subset O(d)$ be a finite group of isometries such that $g\Lambda = \Lambda$ for all $g \in G$.

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Let $\Gamma = \Lambda \rtimes G = \{(k, g) : k \in \Lambda, g \in G\}$, with composition law

$$(k, g) \cdot (k', g') = (gk' + k, gg').$$

Γ is a crystallographic group, which acts on $\mathbb{R}^d$ by

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The corresponding action on $L^2(\mathbb{R}^d)$ is given by the operators

$$T(k)f(x) = f(x-k)\,,\ R(g)f(x) = f(g^{-1}x)\,,\quad \text{for } f \in L^2(\mathbb{R}^d)$$

which indeed satisfy $T(k)R(g)T(k')R(g') = T(gk'+k)R(gg')$.

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A closed subspace $V \subset L^2(\mathbb{R}^d)$ is Γ-invariant if

$$T(k)R(g)V \subset V \quad \forall k \in \Lambda\,,\; g \in G.$$

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Shift-invariant spaces I

Shift-invariant spaces I

Let $\Lambda^{\perp} \subset \mathbb{R}^d$ be the annihilator[2] lattice of Λ, and let $\Omega \subset \mathbb{R}^d$ be $|\Omega \cap (\Omega + s)| = 0$ for $0 \neq s \in \Lambda^{\perp}$, and $|\mathbb{R}^d \setminus \bigcup_{s \in \Lambda^{\perp}} \Omega + s| = 0$.

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The map $\mathcal{T} : L^2(\mathbb{R}^d) \to L^2(\Omega, \ell_2(\Lambda^\perp))$ is the surjective isometry

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Since $\mathcal{T}[T(k)f](\omega) = e^{-2\pi i k\omega}\mathcal{T}[f](\omega)$, it is equivalent to have

- $V \subset L^2(\mathbb{R}^d)$ is Λ-invariant: $f \in V \Rightarrow T(k)f \in V$ for all $k \in \Lambda$
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Thus

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The range function $\mathcal{J}$ of V is the measurable map

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Theorem (B. Cabrelli Hernández Molter)

$V \subset L^2(\mathbb{R}^d)$ is Γ-invariant if and only if it is shift-invariant and its range function $\mathcal{J}$ satisfies, for all $g \in G$,

$$\mathcal{J}(g^t\omega) = r(g^{-1})\mathcal{J}(\omega)\,, \quad \text{a.e. } \omega \in \Omega.$$

where $r(g)\{c_s\}_{s\in\Lambda^\perp} = \{c_{g^t s}\}_{s\in\Lambda^\perp}$, for $c \in \ell_2(\Lambda^\perp)$.

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Proof.

This is based on the intertwining of the action R of G on $L^2(\mathbb{R}^d)$ with the isometry $\mathcal{T}$, which reads

$$\mathcal{T}[R(g)\psi](\omega) = r(g)\mathcal{T}[\psi](g^t\omega)\,, \quad \text{a.e. } \omega \in \Omega. \qquad \square$$

Shift-invariant spaces II

Let $\Phi = \{\phi_i\}_{i=1}^N \subset L^2(\mathbb{R}^d)$ be a finite family. The pre-Gramian $\mathscr{T}_\Phi$ is the (infinite) matrix-valued L^2 function of Ω

$$\mathscr{T}_\Phi(\omega) = \begin{pmatrix} \vdots & & \vdots \\ \mathcal{T}[\phi_1](\omega) & \dots & \mathcal{T}[\phi_N](\omega) \\ \vdots & & \vdots \end{pmatrix}.$$

The Gramian of Φ is the $N \times N$ matrix-valued L^1 function of Ω

$$\mathscr{G}_\Phi(\omega) = \mathscr{T}_\Phi^*(\omega)\mathscr{T}_\Phi(\omega) = \Big(\sum_{s \in \Lambda^\perp} \widehat{\phi}_j(\omega + s)\overline{\widehat{\phi}_i(\omega + s)} \Big)_{i,j}.$$

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The system of translates $\{T(k)\phi_i\}_{k,i}$ is a Parseval frame, i.e.

$$f = \sum_{i=1}^N \sum_{k\in\Lambda} \langle f, T(k)\phi_i\rangle_{L^2(\mathbb{R}^d)} T(k)\phi_i \quad \forall\ f \in \overline{\operatorname{span}\{T(k)\phi_i\}_{k,i}}$$

if and only if $\mathscr{G}_\Phi(\omega)$ is an orthogonal projection for a.e. $\omega \in \Omega$.

Γ-invariance and the Gramian

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a finitely generated SIS $V \subset \mathbb{R}^d$ is Γ-invariant if and only if there exist $N \times \#G$ vectors $\Psi = \{\psi_i^g\}_{i=1,\ g \in G}^N \subset L^2(\mathbb{R}^d)$ such that

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Lemma (B Cabrelli Hernández Molter)

Let $V \subset L^2(\mathbb{R}^d)$ be a SIS with $N \times \#G$ generators $\Psi = \{\psi_i^g\}_{i=1,\ g\in G}^N \subset L^2(\mathbb{R}^d)$. Then V is Γ-invariant if and only if

$$\mathscr{T}_\Psi(g^t\omega) = r(g^{-1})\mathscr{T}_\Psi(\omega)\lambda(g)$$

where $\lambda(g)c(j,g') = c(j,g^{-1}g')$ for $c \in \mathbb{C}^{(N\times\#G)}$.

Group Parseval frames

Theorem (B Cabrelli Hernández Molter)

For any $\Phi = \{\varphi_i\}_{i=1}^N \subset L^2(\mathbb{R}^d)$ *there exists* $\widetilde{\Phi} = \{\widetilde{\varphi}_i\}_{i=1}^N \subset L^2(\mathbb{R}^d)$ *such that* $\overline{\text{span}}\{T(k)R(g)\varphi_i\}_{k,g,i} = \overline{\text{span}}\{T(k)R(g)\widetilde{\varphi}_i\}_{k,g,i}$*, and*

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Proof.

Let $\Psi = \{R(g)\varphi_i : g \in G, i = 1, \ldots, N\}$, and define

$$Q(\omega) = \mathscr{T}_\Psi(\omega)(\mathscr{G}_\Psi(\omega)^+)^{\frac{1}{2}}.$$

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Then $Q^*(\omega)Q(\omega) = \mathbb{P}_{\text{Range}(\mathscr{G}_\Psi(\omega))}$, so, denoting by $\{q_i^g\}_{i=1,\ g\in G}^N$ its columns and by $\widetilde{\varphi}_i^g = \mathcal{T}^{-1}[q_i^g]$, we have that

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Then $Q^*(\omega)Q(\omega) = \mathbb{P}_{\text{Range}(\mathscr{G}_\Psi(\omega))}$, so, denoting by $\{q_i^g\}_{i=1,\ g\in G}^N$ its columns and by $\widetilde{\varphi}_i^g = \mathcal{T}^{-1}[q_i^g]$, we have that

$$\{T(k)\widetilde{\varphi}_i^g\}_{k,g,i} \text{ is a Parseval frame.}$$

Moreover, $\widetilde{\varphi}_i^g = R(g)\widetilde{\varphi}_i^{\mathrm{e}}$, because $Q(g^t) = r(g^{-1})Q(\omega)\lambda(g)$. □

Best approximation problem

Let $\mathscr{F} = \{f_1, \ldots, f_m\} \subset L^2(\mathbb{R}^d)$, and let $\kappa \in \mathbb{N}$ be fixed. We want to minimize

$$\mathscr{E}[\Psi] = \sum_{i=1}^{m} \|f_i - \mathbb{P}_{S_\Gamma(\Psi)} f_i\|^2_{L^2(\mathbb{R}^d)}$$

over all $\Psi \subset L^2(\mathbb{R}^d)$ finite and such that $\#\Psi \leq \kappa$, where

$$S_\Gamma(\Psi) = \overline{\text{span}}\Big\{ T(k)R(g)\psi, k \in \Lambda, g \in G, \psi \in \Psi \Big\}.$$

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Note that, if $\{T(k)R(g)\psi\}$ is a Parseval frame, then

$$\mathbb{P}_{S_\Gamma(\Psi)} f = \sum_{k\in\Lambda} \sum_{g\in G} \sum_{\psi\in\Psi} \langle f, T(k)R(g)\psi \rangle_{L^2(\mathbb{R}^d)} T(k)R(g)\psi.$$

Fundamental domain

We know that the action by translations of $\Lambda^\perp$ on $\mathbb{R}^d$ has a fundamental domain $\Omega \subset \mathbb{R}^d$. But we will also need that the action of Γ has a fundamental domain P, that satisfies

$$|P \cap g^t P| = 0 \text{ for } g \neq \mathrm{e}\,, \text{ and } \left|\Omega - \bigcup_{g \in G} g^t P\right| = 0.$$

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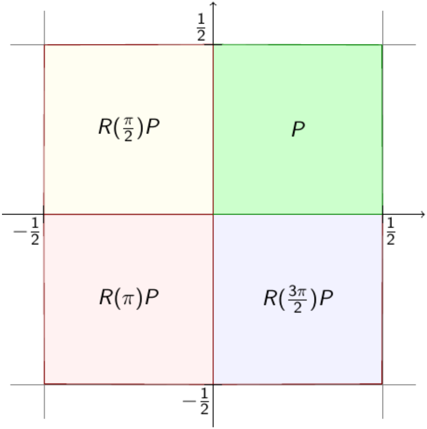

Approximation by Γ-invariant spaces

Theorem (B Cabrelli Hernández Molter)

The problem of finding the minimizer Ψ *of* $\mathscr{E}[\Psi]$ *over all* Ψ *with cardinality* $\leq \kappa$ *is equivalent to the problem of finding the range function* $\mathcal{J}$ *with* $\#\Psi \times \#G$ *generators, that minimizes*

$$\sum_{i=1}^{m} \sum_{g \in G} \|\mathcal{T}[R(g)f_i](\omega) - \mathbb{P}_{\mathcal{J}(\omega)}\mathcal{T}[R(g)f_i](\omega)\|^2_{\ell_2(\Lambda^\perp)}$$

for a.e. $\omega \in P$.

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This equivalent problem can be solved for each $\omega \in P$ by Eckhart-Young theorem (e.g. using SVD) over the data

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which allows us to obtain explicit expressions for the generators of the approximating Γ-invariant space in $L^2(\mathbb{R}^d)$ ...

Muchas gracias!