Muckenhoupt weights with singularities on lower dimensional sets

Marilina Carena
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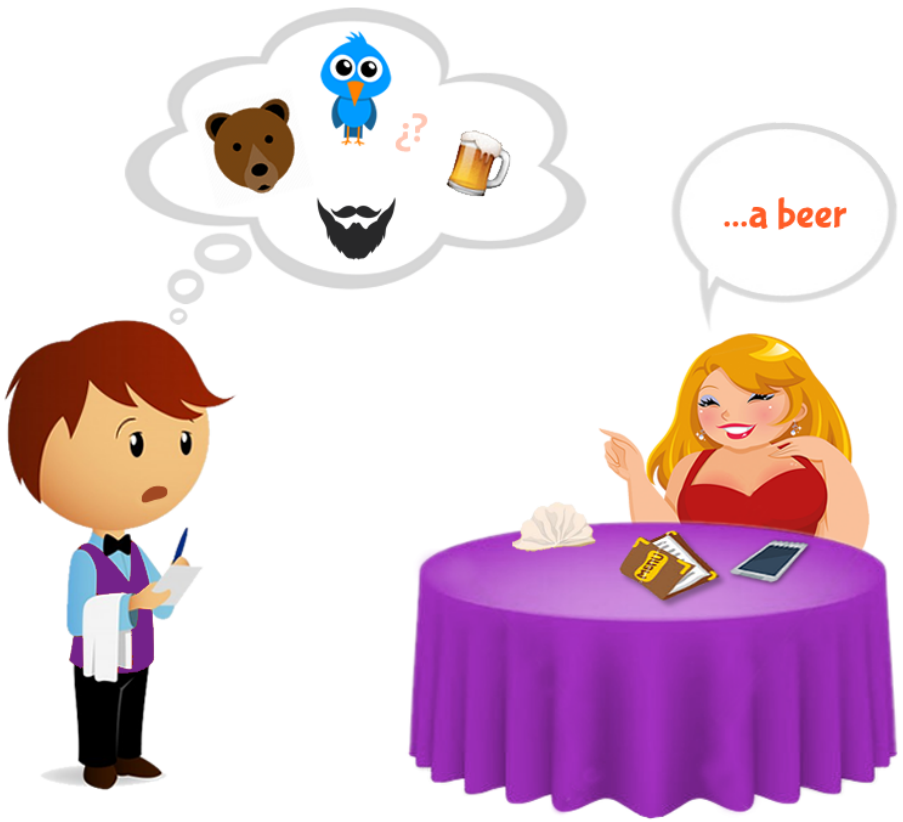
V CLAM, Barranquilla 2016

My Pronunciation



Mi pronunciación en inglés es tan mala, que cuando pido una cerveza no sé si me van a traer un oso, una barba, un pájaro o una birra

Muckenhoupt weights

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for every ball B in X.

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  for every ball B in X.
- $w \in A_1(X, d, \mu)$ if

$$\frac{1}{\mu(B)} \int_B w \, d\mu \leq C \, w(x)$$

  for every ball B in X and a.e. $x \in B$.

Weighted spaces

(X, μ) measure space, $1 \leq p < \infty$

- $L^p(X, \mu)$

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} < \infty$$

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Sobolev spaces

Motivation

Dirichlet boundary value problem

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Conditions on F and β such that $d(x, F)^\beta$ belongs to a Muckenhoupt class

Previous results

- [DST08]: if $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with $\partial\Omega$ in C^2, then $d(x, \partial\Omega)^\beta \in A_p$ for $-1 < \beta < p - 1$.

R. G. Durán, M. Sanmartino, and M. Toschi.
Weighted a priori estimates for the Poisson equation.
Indiana Univ. Math. J., 57(7):3463–3478, 2008.

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- [DLG10]: sufficient conditions on a compact set F of $(\mathbb{R}^n, |\cdot|, \lambda)$, and on β, such that $d(x, F)^\beta \in A_p$.

R. G. Durán and F. López García.
Solutions of the divergence and analysis of the Stokes equations in planar Hölder-α domains,
Math. Mod. Meth. Appl. Sci. **20** (2010) 95–120.

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- [DLG10]: sufficient conditions on a compact set F of $(\mathbb{R}^n, |\cdot|, \lambda)$, and on β, such that $d(x, F)^\beta \in A_p$.
- [KS08]: sufficient conditions on w such that $w(d(x, x_0))$ belongs $A_p(X, d, \mu)$, with $x_0 \in X$ and $w(t)$ generalizing t^β.

Vakhtang Kokilashvili and Stefan Samko.
The maximal operator in weighted variable exponent spaces on metric spaces.
Georgian Math. J., 15(4):683–712, 2008.

Aims

(1) Sufficient conditions on a compact set F of (X, d, μ) Ahlfors, and on β, such that $d(x, F)^\beta \in A_p(X, d, \mu)$.

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(2) Sufficient conditions on $w(t)$ and on a subset F of (X, d, μ) doubling, such that $w(d(x, F)) \in A_p(X, d, \mu)$

Doubling and Ahlfors spaces

- (X, d, μ) with μ a doubling measure:

$$\boldsymbol{0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty,}$$

for every $x \in X$, $r > 0$.

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(X, d, μ) α-Ahlfors $\Longrightarrow$ (X, d, μ) doubling

s-sets

Definition

A closed subset F of X is an s-set if there exists a Borel measure ν supported on F such that

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Remark

(X, d, μ) α-Ahlfors, F an s-set in X with $s < \alpha$, then $\mu(F) = 0$.

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Theorem

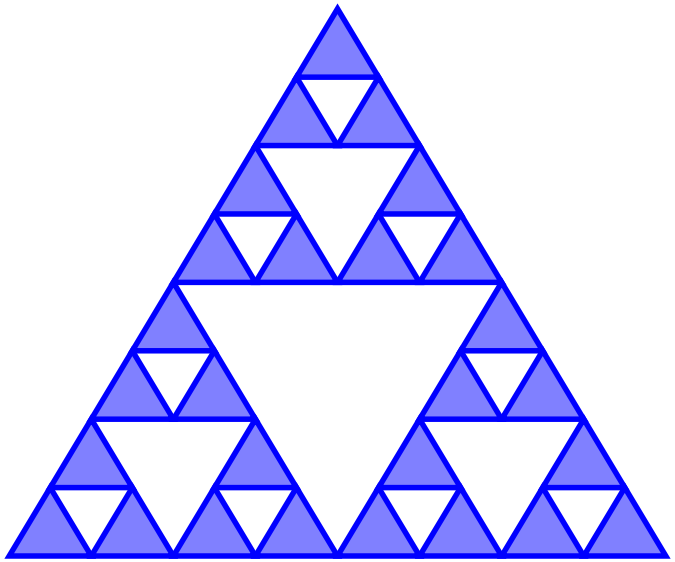
- (X, d, μ) α-*Ahlfors*
- F s-*set,* $0 \leq s < \alpha$

Then

$$d(x, F)^{\beta(s-\alpha)} \in A_p(X, d, \mu), \qquad 1 - p \leq \beta < 1$$

Hugo Aimar, Marilina Carena, Ricardo Durán, and Marisa Toschi.
Powers of distances to lower dimensional sets as Muckenhoupt weights.
Acta Math. Hungar., **143**(**1**):119–137, 2014.

Muckenhoupt weights on S



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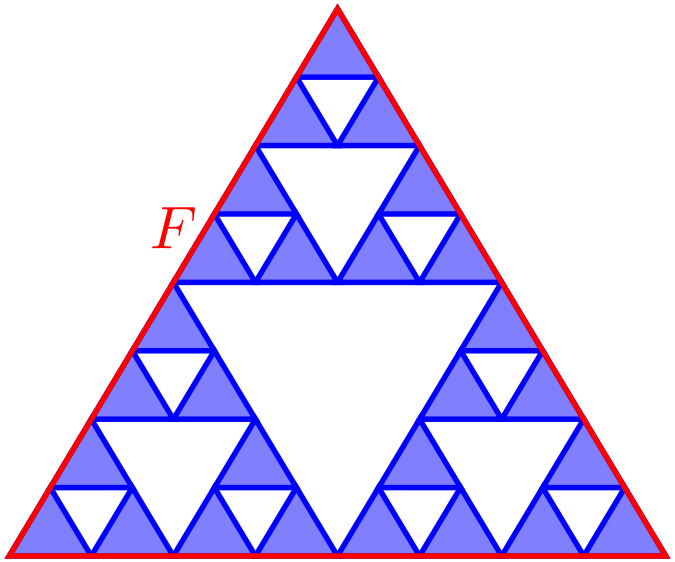

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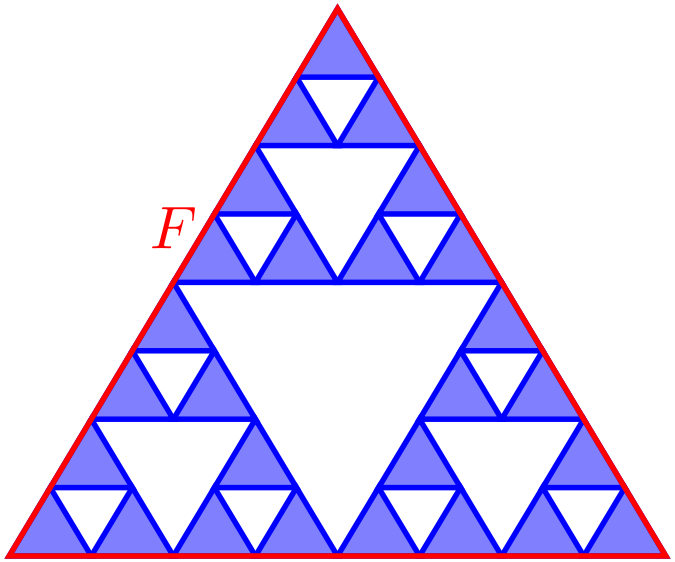

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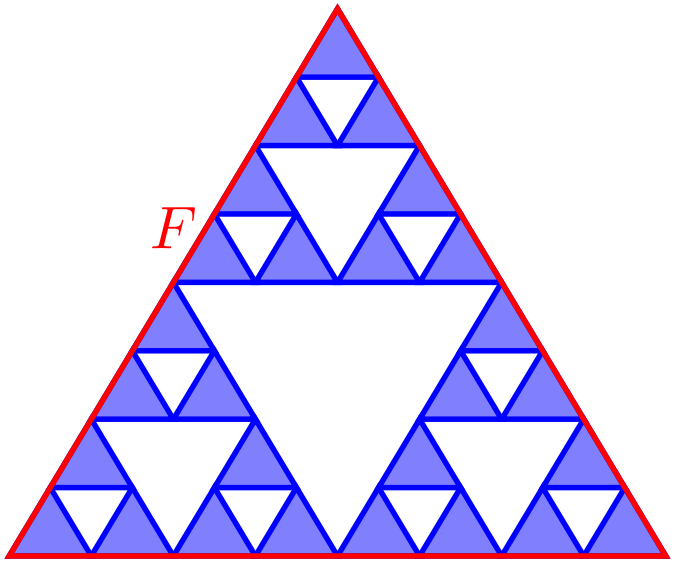

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Case $F = \{x_0\}$ in [KS08]

Definition ($w \in \mathcal{W}^0$)

$$\int_0^r \frac{\mu(B(x_0,t))}{tw(t)}\,dt \lesssim \frac{\mu(B(x_0,r))}{w(r)},$$

for every $0 < r < \operatorname{diam}(X)$.

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$$w^{-1} \in \mathcal{W}^0 \text{ and } w^{\frac{1}{p-1}} \in \mathcal{W}^0 \Longrightarrow w(d(x,x_0)) \in A_p(X,d,\mu)$$

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$$\int_0^r \frac{\mu([x_0]_t\cap B(x_0,r))}{tw(t)}\,dt$$

Definition (t-enlargement)

$$[E]_t = \bigcup_{x\in E} B(x,t)$$

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$$w^{-1} \in \mathcal{W}^F \text{ and } w^{\frac{1}{p-1}} \in \mathcal{W}^F \Longrightarrow w(d(x,F)) \in A_p(X,d,\mu)$$

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Corollary (KS08: $F = \{x_0\}$)

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(X, d, μ) g-Ahlfors and F an h-set

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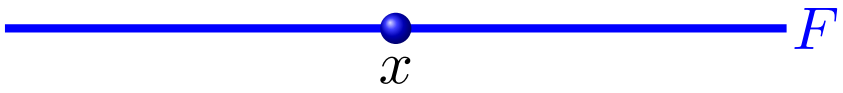

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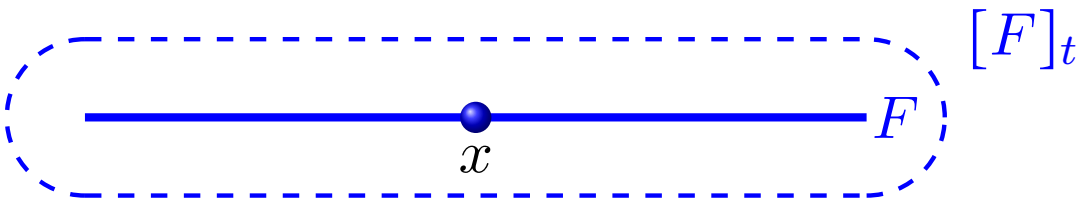

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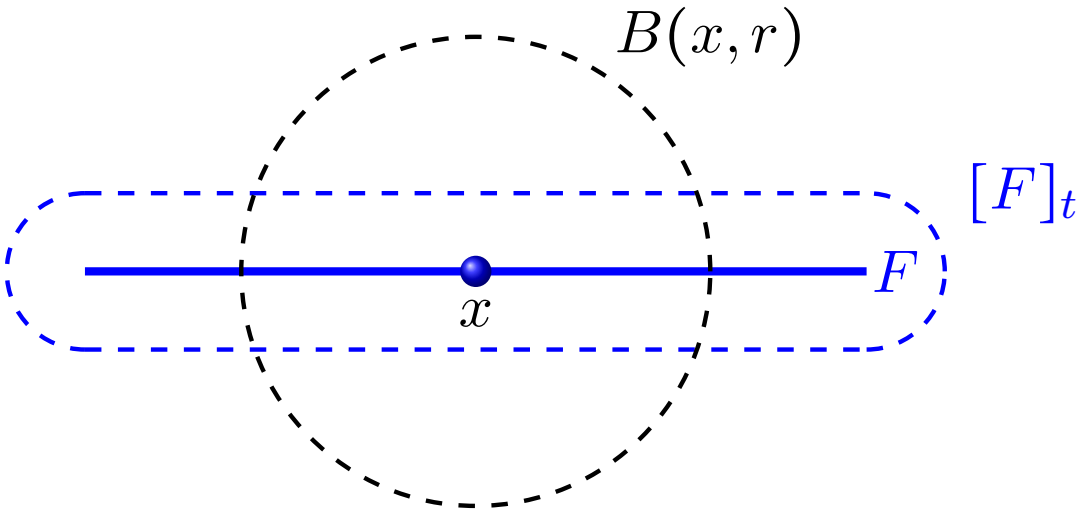

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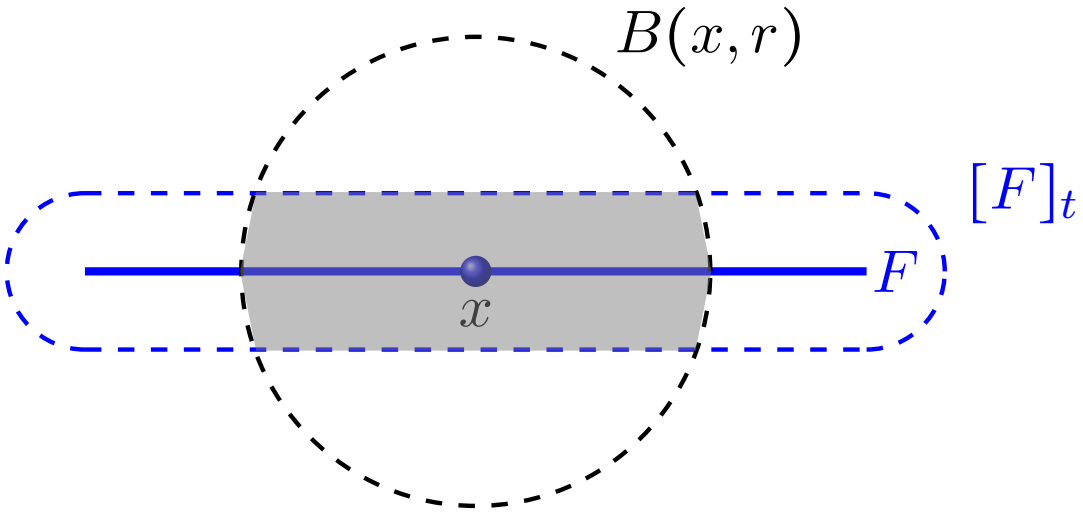

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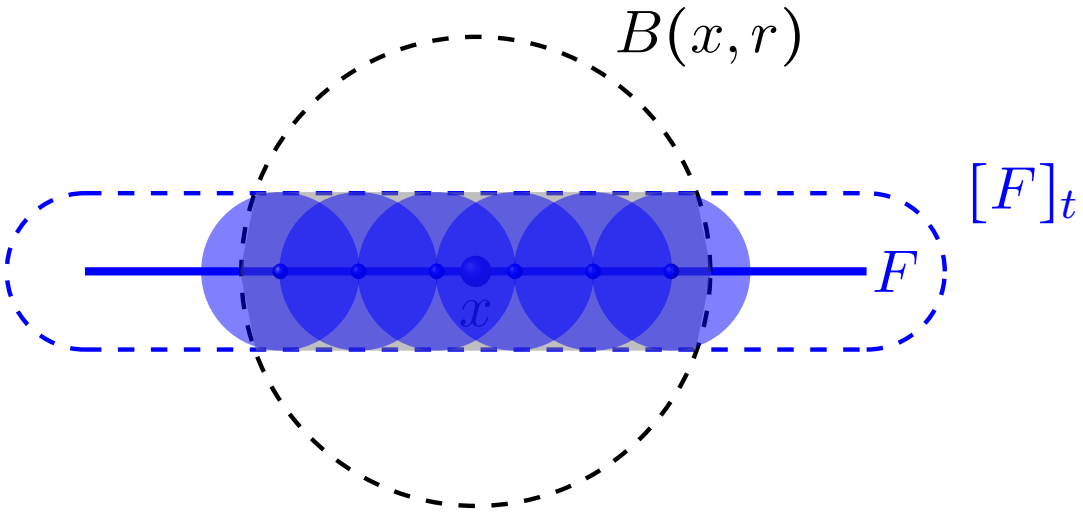

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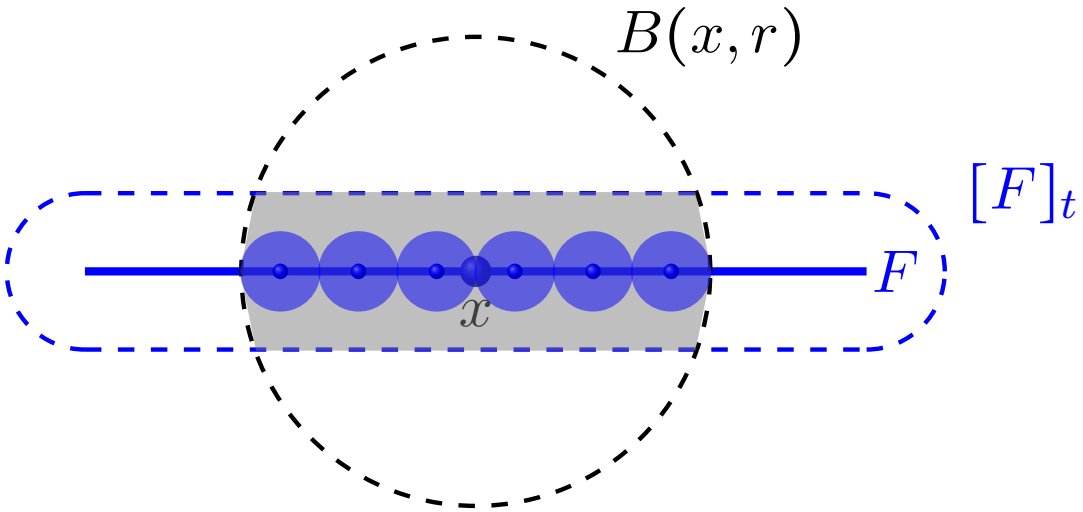

- $\{x_1, x_2, \ldots, x_{I_{t,r}}\} \subseteq F$, $B_i = B(x_i, t)$
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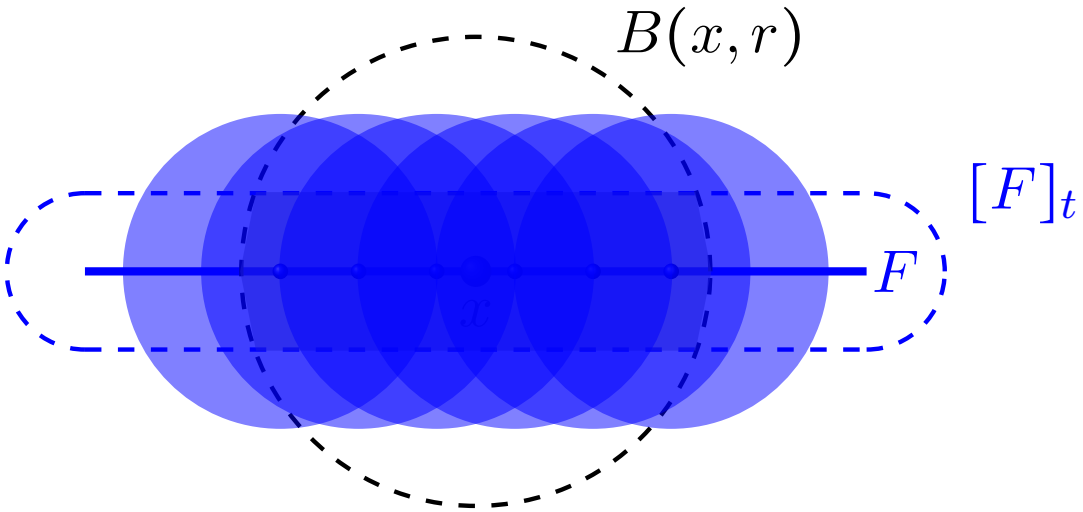

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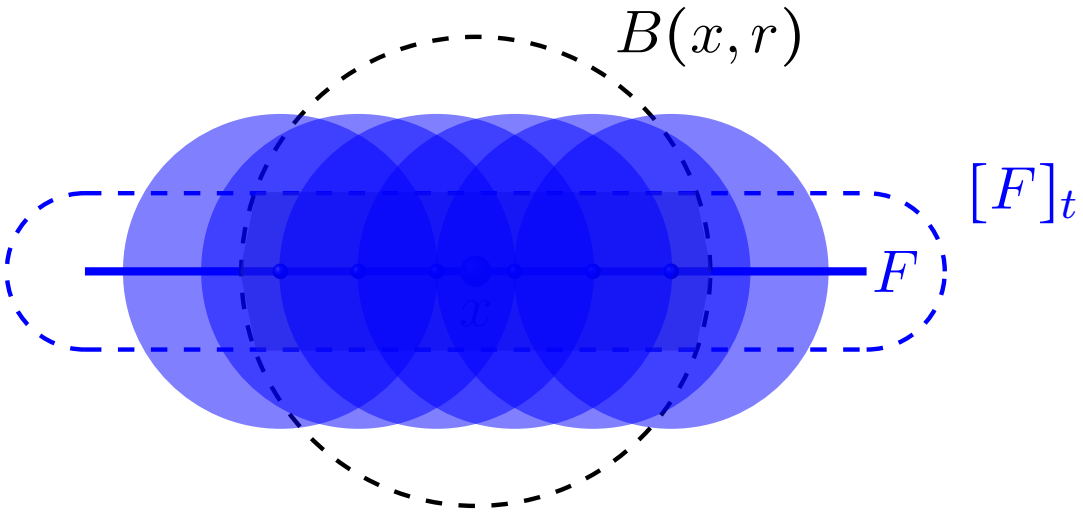

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(X, d, μ) g-Ahlfors and F an h-set

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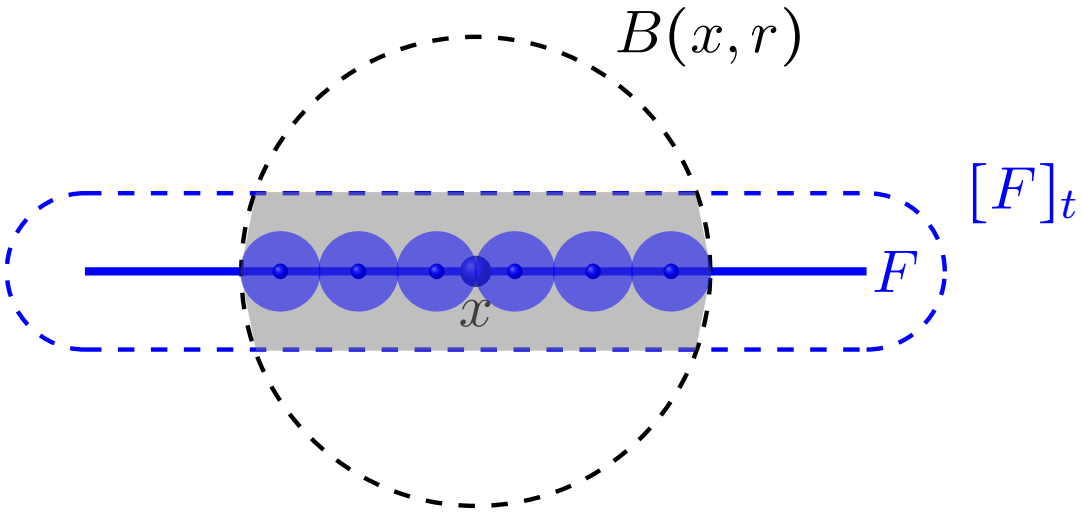

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- $I_{t,r} \lesssim \dfrac{h(r)}{h(t)}$ for every $x \in F$, $r > 0$ and $0 < t \leq r$

(X, d, μ) g-Ahlfors and F an h-set

Sufficient condition for $w \in \mathcal{W}^F$

$$\int_0^r \frac{\mu([F]_t \cap B(x,r))}{tw(t)}\, dt$$

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$$\int_0^r \frac{f(t)}{t}\, dt \lesssim f(r), \qquad \forall 0 < r < \operatorname{diam}(X)$$

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If $I(h) < i(g)$, *then* $\left(\frac{g}{h}\right)^{\delta} \in \mathcal{Z}^0$, *for every* $\delta > 0$

- $I(\phi) = \inf\{\alpha : \phi$ is of upper type $\alpha\}$,

$$\phi(\lambda t) \lesssim \lambda^{\alpha}\phi(t), \quad \forall \lambda \geq 1.$$

- $i(\phi) = \sup\{\alpha : \phi$ is of lower type $\alpha\}$,

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$w^{-1}, w^{\frac{1}{p-1}} \in \mathcal{W}^F$ → $w(d(x,F)) \in A_p(X, d, \mu)$

Proposition (Particular case)

- (X, d, μ) g-*Ahlfors* ($\mu(B(x,r)) \approx g(r)$)
- $F \subseteq X$ h-*set* ($\nu(B(x,r)) \approx h(r)$)

If $I(h) < i(g)$, *then* $\left(\dfrac{h(d(x,F))}{g(d(x,F))}\right)^{\beta} \in A_p(X, d, \mu)$, *for every* $(1-p) < \beta < 1$

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Corollary (ACDT14: $g(t) = t^{\alpha}$, $h(t) = t^{s}$)

If $s < \alpha$, *then* $d(x,F)^{\beta(s-\alpha)} \in A_p(X, d, \mu)$ *for every* $(1-p) < \beta < 1$.

$\mathbb{R}^n$ with the usual distance and Lebesgue measure

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Thanks!