BMO, weights and the Schrödinger operator

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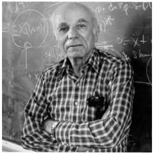
The space BMO



Fritz John   Louis Nirenberg

The BMO space is defined as the set of functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f - f_B| < \infty$$

where $f_B = \frac{1}{|B|} \int_B f$.

- $\| \cdot \|_{BMO}$ is a **semi-norm** in BMO.
- It is a **norm** if we consider the quotient $BMO/\mathfrak{C}$ where $\mathfrak{C}$ is the set of constant functions. $BMO/\mathfrak{C}$ is a Banach space.

Clasical versions of BMO

- When the domain is a ball or a cube: $BMO(Q)$. Given a ball $Q \subset \mathbb{R}^d$, the supremum is taken over balls contained in Q.

$$\sup_{B \subset Q} \frac{1}{|B|} \int_B |f - f_B| < \infty$$

- With weights: $BMO(w)$. Given a function $w \in L^1_{\text{loc}}(\mathbb{R}^d)$, we can ask f to satisfy

$$\sup_{B} \frac{1}{w(B)} \int_B |f - f_B| < \infty$$

Near L^∞

Let $f(x) = |\log |x||$, for $x \in \mathbb{R}$.

$$f \in BMO(\mathbb{R})$$

$$f \in L^p([-1,1]), \quad 1 \leq p < \infty$$

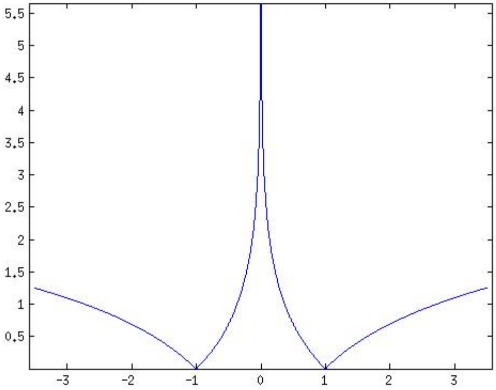
and

$$f \notin L^\infty([-1,1]).$$

On finite measure BMO belongs between L^∞ and L^p if $1 < p < \infty$:,

$$L^\infty([-1,1]) \subset BMO([-1,1]) \subset L^p([-1,1]) \subset L^1([-1,1])$$

“Sharp” singularities are allowed.



Graphic of f.

Power singularities are not allowed.
If $\alpha > 0$,

$$1/|x|^\alpha \notin BMO$$

It is easy to see, in particular, that

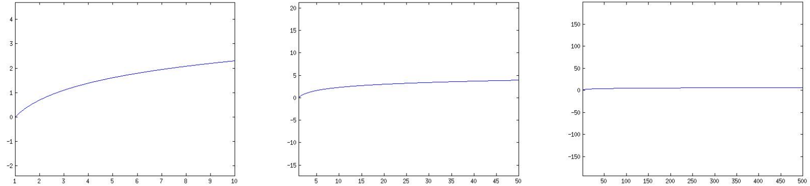
$$\frac{1}{\delta}\int_0^\delta |1/|x|^\alpha - (1/|x|^\alpha)_{(0,\delta)}|dx \ \to \ \infty$$

The function $1/|x|^\alpha$ "is not sharp enough" at the origin.

Globally, a function of *BMO* can be increasing at infinity, but not so fast.

$$\lim_{x\to\infty} \log|x| = \infty$$

For a computer $\log|x|$ turns a constant very fast:



(the graphic on the intervals $[1, 10]$, $[1, 50]$ and $[1, 500]$)

If $\alpha > 0$,

$$|x|^\alpha \notin BMO(\mathbb{R})$$

We can see this with

$$\frac{1}{r}\int_0^r \left||x|^\alpha - (|x|^\alpha)_{(0,r)}\right| dx \ \rightarrow \ \infty$$

This is because x^α "does not turns a constant fast enough".

The John-Nirenberg inequality

Threre exist constants C_1 and C_2 such that for every f in BMO, and every ball B and $t > 0$, we have

$$|\{x \in B : |f(x) - f_B| > t\}| \;\le\; C_1 |B|\, e^{-C_2 t/\|f\|_{BMO}}.$$

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K. O. Friedrichs anniversary issue

On Functions of Bounded Mean Oscillation*

F. JOHN and L. NIRENBERG

§ 1. We prove an inequality (Lemmas 1.1′) which has been applied by one of the authors and by J. Moser in their papers in this issue. The inequality expresses that a function, which in every subcube C of a cube C_0 can be approximated in the L^1 mean by a constant a_C with an error independent of C, differs then also in the L^p mean from a_C in C by an error of the same order of magnitude. More precisely, the measure of the set of points in C, where the function differs from a_C by more than an amount σ decreases exponentially as σ increases.

In Section 2 we apply Lemma 1′ to derive a result of Weiss and Zygmund [3], and in Section 3 we present an extension of Lemma 1′.

LEMMA 1. *Let $u(x)$ be an integrable function defined in a finite cube C_0 in n-dimensional space; $x = (x_1, \cdots, x_n)$. Assume that there is a constant K such that for every parallel subcube C, and some constant a_C, the inequality*

$$\frac{1}{m(C)}\int_C |u - a_C|\,dx \le K \tag{1}$$

holds. Here dx denotes element of volume and $m(C)$ is the Lebesgue measure of C. Then, if $\mu(\sigma)$ is the measure of the set of points where $|u - a_{C_0}| > \sigma$, we have

$$\mu(\sigma) \le B e^{-b\sigma/K} m(C_0) \quad \textit{for} \quad \sigma > 0, \tag{2}$$

where B, b are constants depending only on n.

Since for every continuously differentiable function $f(s)$, vanishing at the origin,

Equivalence of p-oscillations

A consequence of the J-N inequality:
The norm

$$\|f\|_{BMO} \;=\; \sup_B \frac{1}{|B|}\int_B |f - f_B|$$

is equivalent to

$$\sup_B \left(\frac{1}{|B|}\int_B |f - f_B|^p\right)^{1/p}$$

This is true for all $1 < p < \infty$.

The dual of *BMO*

The dual space of *BMO* is the Hardy space H^1.

$$H^1 = \left\{ f \in L^1 : \| \sup_{t>0} |f * \Phi_t| \|_{L^1} < \infty \right\},$$

where

$$\Phi_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{|x|^2/4t}.$$

G. H. Hardy

Frigyes Riesz

Charles Fefferman

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CHARACTERIZATIONS OF BOUNDED MEAN OSCILLATION

BY CHARLES FEFFERMAN

Communicated by M. H. Protter, December 14, 1970

BMO (bounded mean oscillation) is the Banach space of all functions $f \in L^1_{\mathrm{loc}}(R^n)$ for which

$$\|f\|_{\mathrm{BMO}} = \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - \mathrm{av}_Q f| \; dx \right) < \infty,$$

where the sup ranges over all cubes $Q \subseteq R^n$, and $\mathrm{av}_Q f$ is the mean of f over Q. See [5]. For convenience, we identify f and f' in BMO if $f - f'$ is constant.

THEOREM 1. *BMO is the dual of the Hardy space $H^1(R^n)$. The inner product is given by $\langle f, g \rangle = \int_{R^n} f(x)g(x)\,dx$ for $f \in BMO$ and g belonging to the dense subspace of C^∞ rapidly decreasing functions in H^1.*

Here, we regard H^1 as the space of $f \in L^1(R^n)$ whose Riesz transforms $R_j(f)$ are all in L^1. See [7].

THEOREM 2. *A function belongs to BMO if and only if it can be written in the form $g_0 + \sum_{j=1}^n R_j(g_j)$ with $g_0, g_1, \cdots, g_n \in L^\infty(R^n)$.*

Note that the usual definition

$$R_j(g)(x) = \lim_{\epsilon \to 0; M \to \infty} \int_{\epsilon < |x-y| < M} K_j(x-y) f(y)\, dy$$

with $K_j(y) = c y_j / |y|^{n+1}$ need not make sense for all $g \in L^\infty$. (Consider $g(x) = \mathrm{sgn}(x)$ on the line.) Therefore, we define

$$R_j(g)(x) = \lim_{\epsilon \to 0} \int_{\epsilon < |x-y|} [K_j(x-y) - K_j^0(-y)] g(y)\, dy,$$

where $K_j^0(y) = K_j(y)$ for $|y| > 1$ and $K_j^0(y) = 0$ for $|y| \leq 1$. This makes sense for all $g \in L^\infty$, and agrees with the usual definition up to an additive constant if g has compact support. See [3, p. 105].

The main idea in proving Theorems 1 and 2 is to study the Poisson integral of a function in BMO. Recall that any function f satisfying

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$$(*) \qquad \int_{R^n} \frac{|f(x)|}{(|x| + 1)^{n+1}}\, dx < \infty$$

has a Poisson integral $u(x, t) = \mathrm{P.I.}(f)$ defined on $R^{n+1}_+ = R^n \times (0, \infty)$.

THEOREM 3. *A function f belongs to BMO if and only if $(*)$ holds and $\iint_{|x-x_0|<\delta;\ 0<t<\delta} t |\nabla u(x,t)|^2\, dx\, dt \leq C\delta^n$ for all $x_0 \in R^n$ and $\delta > 0$.*

Theorems 1–3 and their proofs can be used to study H^1. For example,

THEOREM 4. *Let $F = (u_0(x,t); u_1(x,t), \cdots, u_n(x,t))$ be an $(n+1)$-tuple of harmonic functions on R^{n+1}_+, satisfying the Cauchy-Riemann equations of* [7]. *If the nontangential maximal function $u_0^*(x) \equiv \sup_{|x'|<t;\ t>0} |u_0(x-x', t)|$ belongs to L^1, then F is in H^1.*

Different techniques enable us to replace L^1 and H^1 by L^p and H^p, $0 < p < \infty$. This generalizes a one-dimensional result of D. Burkholder, R. Gundy, and M. Silverstein (see [1] and [2]).

Further applications of Theorems 1–3 appear in [4] and [6]. [4] contains detailed proofs of the results stated here.

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A substitue of L^∞

In Harmonic Analysis the BMO space appears in many situations playing the role of the space L^∞.

- Some operators fail to be bounded on L^∞ but they are on BMO.

The Schrödinger operator

We consider the Schrödinger operator in $\mathbb{R}^d$, with $d \geq 3$

$$\mathcal{L} = -\Delta + V$$

where $V \geq 0$ is a function satisfying for $q > \frac{d}{2}$, the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) \, dy \tag{1}$$

for all ball $B \subset \mathbb{R}^d$.

The space $BMO_{\mathcal{L}}$

In [DZ-1999] it is defined the space $H^1_{\mathcal{L}}$ associated to $\mathcal{L}$ and the authors find an atomic decomposition.
Later in [DGMTZ-2005] they find that the dual of $H^1_{\mathcal{L}}$ is a space that they call $BMO_{\mathcal{L}}$ similar to the clasical BMO, defined as the space of functions $f \in L^1_{loc}$ such that

$$\frac{1}{|B|}\int_B |f - f_B| \leq C \qquad (f_B = \frac{1}{|B|}\int_B f)$$

and

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f| \leq C, \quad r \geq \rho(x).$$

The critical radius function

A very important quantity to develop this theory is

$$\rho(x) = \text{ínf}\left\{ r > 0 : \ \frac{1}{r^{d-2}} \int_{B(x,r)} V \leq 1 \right\}, \quad x \in \mathbb{R}^d.$$

This function plays a crucial rol in the description of the spaces and the estimates of the operators associated to $\mathcal{L}$ [Shen-1995], [DGMTZ-2005], [DZ-2002], [DZ-2003].

- A critical ball: $B(x, \rho(x))$.

Properties of ρ

- Threre exist C and $k_0 \geq 1$ such that,

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}$$

  for all $x, y \in \mathbb{R}^d$.
- If x and y belong to $B(x, \rho(x))$, then $\rho(x) \approx \rho(y)$.
- We have a useful covering of $\mathbb{R}^d$.

Proposition

There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^d$, such that the family of balls $B_k = B(x_k, \rho(x_k))$, $k \geq 1$, satisfy

1. $\cup_k B_k = \mathbb{R}^d$.
2. *There exists N such that, for all $k \in \mathbb{N}$,* $card\{j : \; 4B_j \cap 4B_k \neq \emptyset\} \leq N$.

The function ρ says how to make calculations.

The associated fractional integral

We have study boundedness of some operators associated to $\mathcal{L}$. One of them is the fractional integral associated to $\mathcal{L}$, defined for $\alpha > 0$, as

$$\mathcal{L}^{-\alpha/2} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x)\, t^{\alpha/2} \,\frac{dt}{t}\,,$$

where $\{e^{-t\mathcal{L}}\}_{t>0}$, is the heat semigroup associated to $\mathcal{L}$.

Boundedness of $\mathcal{L}^{-\alpha/2}$

Theorem (DGMTZ-2005)

If $0 < \alpha < d$ the operator $\mathcal{L}^{-\alpha/2}$ is bounded form $L^{d/\alpha}$ into $BMO_{\mathcal{L}}$.

- We use weights.
  For $\eta \geq 1$ we say that the weight $w \in D_\eta$ if there exists a constant C such that

$$w(tB) \leq C\, t^{d\eta}\, w(B)$$

  for all ball $B \subset \mathbb{R}^d$.
- Is it possible to go beyond $L^{d/\alpha}$?

Smoother spaces: the BMO_β

In (HSV-1997) the authors define spaces BMO_β with weights w where a function f has to satisfy

$$\int_B |f - f_B| \leq C\, w(B)\, |B|^{\beta/d}, \;\text{ con } f_B = \frac{1}{|B|}\int_B f,$$

for every ball B.

If we join both definitions of (DGMTZ-2005) and (HSV-1997), we have:

For $\beta \geq 0$ we define the space $BMO_{\mathcal{L}}^{\beta}(w)$ as the set of functions $f \in L^1_{loc}$ such that,

$$\int_B |f - f_B| \leq C\, w(B)\, |B|^{\beta/d} \tag{2}$$

and

$$\int_{B(x,R)} |f| \leq C\, w(B(x,R))\, |B(x,R)|^{\beta/d} \qquad R \geq \rho(x). \tag{3}$$

A Lipschitz version with weights

Following (HSV-1997), for $\beta > 0$, $w \in L^1_{\text{loc}}$, we define the quantity

$$W_\beta(x,r) = \int_{B(x,r)} \frac{w(z)}{|z-x|^{d-\beta}}\, dz \qquad x \in \mathbb{R}^d \quad r > 0.$$

The Lipschitz space associated to $\mathcal{L}$ denoted by $\Lambda^\beta_{\mathcal{L}}(w)$ is defined as the set of functions f such that

$$|f(x) - f(y)| \;\le\; C\,[W_\beta(x,|x-y|) + \; W_\beta(y,|x-y|)] \tag{4}$$

and

$$|f(x)| \;\le\; C\, W_\beta(x,\rho(x)) \tag{5}$$

for almost all x and y in $\mathbb{R}^d$.

The norm is the maximum of the two infimum of the constants in (4) and (5).

Observation

For almost all $x \in \mathbb{R}^d$, $W_\beta(x,r)$ is finite for all $r > 0$, and increases with r.

Coincidence with the Lipschitz version

As in the classical case, we have a Lipschitz description of $BMO^{\beta}_{\mathcal{L}}(w)$.

Proposition

If for $\beta > 0$ the weight w satisfies the doubling condition, then

$$\Lambda^{\beta}_{\mathcal{L}}(w) = BMO^{\beta}_{\mathcal{L}}(w),$$

with equivalent norms.

Boundedness of $\mathcal{L}^{-\alpha/2}$ in $BMO^{\beta}(w)$

A theorem with weights

Theorem

If $0 < \alpha < d$ and $w \in RH_{p'} \cap D_\eta$, where $1 \leq \eta < 1 + \frac{\delta_0}{d}$ with $\delta_0 = \text{mín}(1, 2 - \frac{d}{q})$, then the operator $\mathcal{L}^{-\alpha/2}$ is bounded form $L^{d/\alpha}(w)$ into $BMO_{\mathcal{L}}(w)$.

Beyond $L^{d/\alpha}$

Theorem

If $0 < \alpha < d$ and $\frac{d}{\alpha} \leq p < \frac{d}{(\alpha - \delta_0)^+}$ with $\delta_0 = \text{mín}(1, 2 - \frac{d}{q})$; $w \in RH_{p'} \cap D_\eta$, where $1 \leq \eta < 1 - \frac{\alpha}{d} + \frac{\delta_0}{d} + \frac{1}{p}$, then the operator $\mathcal{L}^{-\alpha/2}$ is bounded from $L^p(w)$ into $BMO_{\mathcal{L}}^{\alpha - d/p}(w)$.

Riesz Transforms

Classical Riesz Transforms

$$\mathbf{R}_i = \frac{\partial}{\partial x_i}(-\Delta)^{-1/2}, \quad i = 1, 2, \ldots, d;$$

- They are bounded in $L^p(w)$, for $1 < p < \infty$, whenever

$$\left(\int_B w\right)\left(\int_B w^{-\frac{1}{p-1}}\right)^{p-1} \leq C|B|^p, \quad (\text{Concidión } A_p)$$

  for all ball B of $\mathbb{R}^d$,
- They are NOT bounded on L^∞.
- If the weight $w = 1$, this extreme is replaced by BMO: the space of functios $f \in L^1_{loc}$ such that

$$\sup_B \frac{1}{|B|}\int_B |f(x) - f_B|\, dx < \infty.$$

Results in classical $BMO^{\beta}(w)$

- (Muckenhoupt-Wheeden, 1975) Boundedness of Riesz in BMO spaces with weights.
- (Morvidone, 2003) Boundedness of the Hilbert transform in $BMO^{\psi}(w)$.

Remind: $0 \leq \beta < 1$, $f \in BMO^{\beta}(w)$ if and only if $f \in L^1_{loc}$ such that

$$\sup_B \frac{1}{|B|^{\beta/d} w(B)} \int_B |f(x) - f_B| \, dx < \infty.$$

Theorem

$\mathbf{R}_i$ *are bounded on* $BMO^{\beta}(w)$, *whenever* $w \in A_\infty = \cup_{p=1}^{\infty} A_p$ *and*

$$|B|^{\frac{1-\beta}{d}} \int_{B^c} \frac{w(y)}{|x_B - y|^{d+1-\beta}} \leq C \frac{w(B)}{|B|}.$$

Riesz transforms associated to the Schrödinger operator

New Riesz transforms:

$$\mathcal{R}_i = \frac{\partial}{\partial x_i}(-\Delta + V)^{-1/2}, \quad i = 1, 2, \ldots, d.$$

They where studied by Shen in 1995.

- $\mathcal{R}_i$ are Calderón-Zygmund if $V \in RH_q$.
  - They are bounded on L^2.
  - They have kernels satisfying for certain constants C and δ, the condition
    - $|K(x,y)| \leq \dfrac{C}{|x-y|^d}$
    - $|K(x+h,y) - K(x,y) \leq \dfrac{C\,h^\delta}{|x-y|^{d+\delta}}$, whenever $|h| < |x-y|/2$. (the same for the other variable)
- If $V \in RH_q$ for some $\frac{d}{2} < q < d$, then $\mathcal{R}_i$ becomes bounded in L^p, for $1 < p < \frac{1}{d} - \frac{1}{q}$.

Some results for the new Riesz Transforms

The reverse Hölder index of V: $\quad q_0 = \sup\{q : V \in RH_q\}$

Theorem

Let $V \in RH_d$ and $w \in A_\infty \cap D_\eta$.

(a) *For all $0 \leq \beta < 1 - d/q_0$ y $1 \leq \eta < 1 + \frac{1-d/q_0-\beta}{d}$, the operators $\mathcal{R}_j$, $1 \leq j \leq d$, are bounded on $BMO^\beta_{\mathcal{L}}(w)$.*

(b) *For all $0 \leq \beta < 1$ and $1 \leq \eta < 1 + \frac{1-\beta}{d}$, the operators $\mathcal{R}^*_j$, $1 \leq j \leq d$, are bounded on $BMO^\beta_{\mathcal{L}}(w)$.*

Theorem

Let $V \in RH_{d/2}$ such that $q_0 \leq d$, $0 \leq \beta < 2 - \frac{d}{q_0}$, and $w \in D_\eta \cap \cup_{s>p_0'}(A_{p_0/s'} \cap RH_s)$ where $\frac{1}{p_0} = \frac{1}{q_0} - \frac{1}{d}$ y $1 \leq \eta < 1 + \frac{2-d/q_0-\beta}{d}$. The operators $\mathcal{R}^*_j$, $1 \leq j \leq d$, are bounden on $BMO^\beta_{\mathcal{L}}(w)$.

Inequalities with weights in L^p

We deal with the following operators associated to $\mathcal{L}$:

- Maximal of the semi-group

$$\mathcal{T}^* f(x) = \sup_{t>0} e^{-t\mathcal{L}} f(x).$$

- $\mathcal{L}$-Ries potentials ($\mathcal{L}$-Fractional Integral)

$$\mathcal{I}_\alpha f(x) = \mathcal{L}^{-\alpha/2} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x)\, t^{\alpha/2} \frac{dt}{t}, \qquad 0 < \alpha < d.$$

- $\mathcal{L}$-Riesz transforms

$$\mathcal{R} = \nabla \mathcal{L}^{-1/2},$$

  adjoints

$$\mathcal{R}^* = \mathcal{L}^{-1/2} \nabla.$$

- $\mathcal{L}$-Square Function

$$\mathfrak{g}(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} e^{-t\mathcal{L}}(f)(x) \right|^2 t\, dt \right)^{1/2}.$$

Weights related to ρ

We define new classes of weights in terms of critical radii, more suitable for this context.
For $p \geq 1$ we define $A_p^{\rho,\infty} = \cup_{\theta \geq 0} A_p^{\rho,\theta}$, where $A_p^{\rho,\theta}$ is defined as the weights w such that

$$\left(\int_B w\right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \leq C|B|\left(1+\frac{r}{\rho(x)}\right)^{\theta},$$

for every ball $B = B(x,r)$.

- $A_p^{\rho,\theta}$ increasing with θ
- For $\theta = 0$ they become the classical Muckenhoupt classes A_p.
- $A_p \subsetneq A_p^{\rho,\infty}$. An example: $\rho \equiv 1$ and $w(x) = 1 + |x|^{\gamma}$. For $\gamma > d(p-1)$, the weight w belongs to $A_p^{\rho,\infty}$, but it is not in A_p.

Boundedness of the maximal of the semigroup

Theorem

If $1 < p < \infty$, *the operator* $\mathcal{T}^* f(x) = \sup_{t>0} e^{-t\mathcal{L}} f(x)$ *is bounded on* $L^p(w)$ *for* $w \in A_p^{\rho,\infty}$, *and of weak type* $(1,1)$ *for* $w \in A_1^{\rho,\infty}$.

Boundedness of the new Riesz transforms

Theorem

Let $\mathcal{R} = \nabla(-\Delta + V)^{-1/2}$ and $V \in RH_q$.

I) *If $q \geq d$, the operators $\mathcal{R}$ and $\mathcal{R}^*$ are bounded on $L^p(w)$, $1 < p < \infty$, for $w \in A_p^{\rho,\infty}$, and of weak type $(1,1)$ for $w \in A_1^{\rho,\infty}$.*

II) *If $d/2 < q < d$, and s is such that $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$, the operator $\mathcal{R}^*$ is bounded on $L^p(w)$, for $s' < p < \infty$ and $w \in A_{p/s'}^{\rho,\infty}$, and by duality $\mathcal{R}$ is bounded on $L^p(w)$, for $1 < p < s$, with w such that $w^{-\frac{1}{p-1}} \in A_{p'/s'}^{\rho,\infty}$. Moreover, $\mathcal{R}$ is of weak type $(1,1)$ for $w^{s'} \in A_1^{\rho,\infty}$.*

Localized weights for localized operators

Given an operator T we define T_{loc}, the ρ-localization of T, as

$$T_{\text{loc}}(f)(x) = T(f\chi_{B(x,\rho(x))})(x). \tag{6}$$

In order to study the ρ-localizations version of some classical operators we define the a ρ-localized class of weights $A_p^{\rho,\text{loc}}$ as follows:
The weights w such that

$$\left(\int_B w\right)^{1/p}\left(\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \;\leq\; C|B|, \tag{7}$$

for every ball $B(x,r)$ with $r \leq \rho(x)$.

Boundedness of some localized operators

Theorem

Given a critical radius function ρ we have

a) *The operators M_{loc}, T^*_{loc}, R_{loc} and $\mathbf{g}_{loc}$ are bounded on $L^p(w)$, whenever $1 < p < \infty$ and $w \in A_p^{\rho,loc}$, and of weak type $(1,1)$ when $w \in A_1^{\rho,loc}$.*

b) *If $0 < \alpha < d$, the operator $(I_\alpha)_{loc}$ is bounded form $L^p(w)$ into $L^s(w^{s/p})$, whenever $1 < p < d/\alpha$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{d}$, and $w^{s/p} \in A^{\rho,loc}_{1+\frac{s}{p'}}$. Moreover, it is of weak type $(1, \frac{d}{d-\alpha})$ whenever $w^{\frac{d}{d-\alpha}} \in A_1^{\rho,loc}$.*

Commutators with the multiplication operator

Given an operator T and a function b,

we deal with the commutator

$$[b, T]f(x) = T(bf)(x) - b(x)Tf(x), \quad x \in \mathbb{R}^d.$$

We study inequalities on $L^p(\mathbb{R}^d)$, $1 < p \leq \infty$, for the commutators

$$[b, \mathcal{R}_i] \quad \text{and} \quad [b, \mathcal{R}_i^*],$$

for certain functions b.

Previous results

- [R.R. Coifman, R. Rochberg, and G. Weiss], 1976.

  Commutators of classical Riesz transforms are bounded on L^p $\iff b \in BMO$.

- [Z. Guo, P. Li and L. Peng.], 2008.

  If $q > d/2$, $1 < p < \infty$ and $b \in BMO \implies [b, \mathcal{R}_i]$ and $[b, \mathcal{R}_i^*]$ are bounded on $L^p(\mathbb{R}^d)$.

¿Is there more suitable functions b?

The $BMO_\theta(\rho)$ space of symbols

Definition Let $\theta > 0$. The function b belongs to $BMO_\theta(\rho)$, when

$$\frac{1}{|B|}\int_B |b(y) - b_B|\, dy \ \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta,$$

with $B = B(x, r)$, and $b_B = \frac{1}{|B|}\int_B b$. We denote

$$BMO_\infty(\rho) = \cup_{\theta>0} BMO_\theta(\rho).$$

1. When $\theta = 0$, then $BMO_\theta(\rho) = BMO$.
2. If $0 < \theta < \theta'$, then $BMO \subset BMO_\theta(\rho) \subset BMO_{\theta'}(\rho)$.
3. $BMO_\theta(\rho) \neq BMO$.

Example: Let $V(x) = |x|^2$, then $\rho(x) \simeq \frac{1}{1+|x|}$. The function $b(x) = |x_j|^2$, belongs to $BMO_\infty(\rho)$, but not in BMO.

A result with b in $BMO_\infty(\rho)$

Theorem

Let $V \in RH_{d/2}$, $q_0 = \sup\{q : V \in RH_q\}$ be the Reverse Hölder index of V. For $b \in BMO_\infty(\rho)$ and p_0 such that $\frac{1}{p_0} = \left(\frac{1}{q_0} - \frac{1}{d}\right)^+$, we have

(I) *If $1 < p < p_0$, then $\|[b, \mathcal{R}_i]f\|_p \leq C_b \|f\|_p$.*

(II) *If $p_0' < p < \infty$, then $\|[b, \mathcal{R}_i^*]f\|_p \leq C_b \|f\|_p$.*

A problem: ¿what happens in the extreme L^∞?

- [E. Harboure, C. Segovia, and J. L. Torrea], 1997.

There is no functions $b \in BMO$ (up to constants) such that $[b, H]$ is bounded from $L^\infty(\mathbb{R})$ into BMO when H is the Hilbert trasform.

In the context of the $\mathcal{L}$-Riesz transforms, we have a positive answer.

The substitute of L^∞

We have see that

1. $\mathcal{R}_i$ y $\mathcal{R}_i^*$ are bounded on $BMO_{\mathcal{L}}$, when $q_0 > d$.
2. and also $\mathcal{R}_i^*$, when $q_0 > d/2$.

¿What kind of symbols b produce a bounded commutator from L^∞ into $BMO_{\mathcal{L}}$?

An other class of sybols b

Definición Let $\theta > 0$, denote by $BMO_\theta^{\log}(\rho)$ to the class of functions b such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b - b_B| \;\leq\; C\,\frac{(1 + r/\rho(x))^\theta}{1 + \log^+(\rho(x)/r)},$$

for all $x \in \mathbb{R}^d$ y $r > 0$. We denote by

$$BMO_\infty^{\log}(\rho) = \cup_{\theta>0} BMO_\theta^{\log}(\rho).$$

Theorem

Let $V \in RH_{d/2}$ and $b \in BMO_\infty(\rho)$, then

I) $[b, \mathcal{R}_i^*] : \; L^\infty \mapsto BMO_{\mathcal{L}} \quad \Longleftrightarrow \quad b \in BMO_\infty^{\log}(\rho)$.

II) *If $V \in RH_d$, the previous result is true for $[b, \mathcal{R}_i]$.*

Further works

- Commutators with weights.
- Boundedness on L^p of singular integrals associated to $-\Delta + V$.
- Extrapolation with a family of maximal functions associated to ρ.
- Resuls on Hardy type spaces.
- Boundedness of singular integrals on $BMO_\beta(w)$.

Now we are dealing with:

- Boundedness of the maximal of a family of operators that seems like the semi-group.
- Two weighted inequalities of the form

$$\int Tf\, w \leq C \int f\, Mw$$

Thanks!!