COMBINATORIAL ASPECTS OF ALGEBRAIC K-THEORY ARGENTINA K-THEORY SUMMER SCHOOL SUMMER 2018

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1. Day 1

Morally speaking, algebraic K-theory arises any time there is a group completion related to any kind of geometric or combinatorial data. Thus we have seen today that K_0-groups arise in situations ranging from addition of vector bundles over a space, to modules over a ring, to polygons in the Euclean plane. Higher K-groups arise because the K_0-group is *lossy*; for example, when looking at algebraic K-theory of any field, K_0 will always be $\mathbf{Z}$. One may hope that higher K-theory groups would retain more information. (In the case of fielts they do: for a field F, $K_1(F) \cong F^\times$; thus it is possible to distinguish (for example) finite fields by considering their K_1's.)

Before we go on to considering how to construct higher K-groups, however, I would like spend some more time on examples of K_0-groups that are pathological in various ways. There are many ways that monoids can have trivial group completions, of course. For example, if we take $\mathbf{Z}_{\geq 0} \cup \{\infty\}$ with standard addition, its group completion will be trivial: $n + \infty = \infty$ so $n = 0$ for all n. This is an example of the Eilenberg swindle, whose basic moral is that "any time infinity is involved, K-theory is trivial." If infinite sums are allowed, then for any modules A and B we define

$$C = (A + B) + (A + B) + (A + B) + \cdots$$

Then

$$B + C = B + (A + B) + (A + B) + (A + B) + \cdots = (B + A) + (B + A) + \cdots = C.$$

Thus inside the group completion, $[B] = 0$. K-theory is the study of *finiteness*, and the Eilenberg swindle demonstrates that, in general, if we try to add infinite objects and take their K-theories we get trivial results.

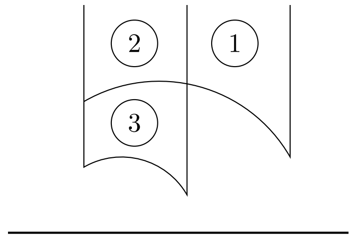
However, there are also other strange ways that K_0-groups can go wrong. We'll give some geometric examples here, although in general it is also possible to construct analogous algebraic examples.

First, consider the following partial monoid. The objects of the monoid are equivalence classes of rectangles with sides parallel to the axes; two rectangles are equivalent if they are related by compositions of translation and dilation. (In other words, any two rectangles with the same length/height ratio are equivalent.) In addition, if we have $R = \bigcup_{i=1}^n R_i$ with the measure of $R_i \cap R_j$ being 0 for $i \neq j$ then we declare that $[R] = \sum_{i=1}^n [R_i]$. This monoid is highly nontrivial, but its group competion is. To see this, note that for any rectangle R,

$$[R] = 4[R] = 9[R].$$

Thus $3[R] = 5[R] = 0$. From this we can conclude that $2[R] = 0$; subtracting this from $3[R] = 0$ we get $[R] = 0$. This example reflects the other side of the Eilenberg swindle: that if $X + X = X$ then $[X] = 0$ in the group completion.

Now let us consider an even stranger example. Consider the following partial monoid, whose group completion is the scissors congruence group of the hyperbolic plane with boundary. Let $X = \overline{H^2}$, the hyperbolic plane with boundary. One model for this is the complex half-plane of points with nonnegative imaginary part. The allowable lines are vertical lines and circles orthogonal to the real axis. A *triangle* in X is the convex hull of three points. A *polytope* in X is a finite union of triangles. Now consider the following three regions:



Since they are scalings of one another with center on the real axis, $[1 \cup 2] = [2 \cup 3]$. Thus $[1] + [2] = [2] + [3]$. In particular, in the group completion, $[1] = [3]$. However, it is definitely not the case that regions 1 and 3 are equal in the monoid: the relations do not allow any relation between a polytope with only finite vertices and a polytope with infinite vertices. Note that there is no Eilenberg swindle here: all of the regions have finite area, so there is no "infinity" which is eating up differences. The key point here is that there are different types of points, which are not interchangeable, and these points are the reason this phenomenon shows up.

Thus, examining this case, we see that even without an Eilenberg swindle group completion can be very destructive. In the exercises we develop a method for guaranteeing that group completions are not destructive in geometric cases.

Last example: consider varieties over a field k, with the relation that every closed embedding $Y \hookrightarrow X$ produces a relation $[X] = [Y] + [X \smallsetminus Y]$. We can define multiplication, as before, via the Cartesian product, setting $[X][Y] = [X \times Y]$. This gives a commutative semiring; it's group completion is called the *Grothendieck ring of varieties*. If X and Y are piecewise isomorphic (in the sense that there are stratifications

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X \qquad \text{and} \qquad \emptyset = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n = Y$$

such that $X_i \smallsetminus X_{i-1} \cong Y_i \smallsetminus Y_{i-1}$) then $[X] = [Y]$ in the Grothendieck ring; however, the converse is not true. In fact, a classification in the reverse direction is not known, even for $k = \mathbf{C}$.

However, we know a perfectly good case when group completion is not desctructive: the integers. The integers are constructed by taking isomorphism classes of finite sets. Addition is given by disjoint union; multiplication is given by Cartesian product. This gives the set of isomorphism classes of finite sets a semiring structure. When we group complete, this extends to the usual ring structure on integers.

Let us consider subtraction a little bit more carefully. Usually we define $5 - 3$ to be "the number we add to 3 to make 5." However, once we are working with finite sets this becomes problematic: there is no set we can disjoint union with $\{0, 1, 2\}$ to make $\{a, b, c, d, e\}$. Thus to make this work correctly, we must keep track of how we are associating sets to one another. Later on in the week you will see the $S_\bullet$ construction, which keeps track of sets and inclusions. However, for now, we'll just keep track of bijections between sets: for any two-element set T there exists a bijection $\{0, 1, 2\} \amalg A \longrightarrow \{a, b, c, d, e\}$. Although this is only well-defined up to permutation, as long as we keep track of all possible permutations and all possible choices, we can make this work correctly.[1] Thus we construct the following simplicial set, which we call $N_\bullet i\mathbf{FinSet}$:

$N_0 i\mathbf{FinSet}$**:** the set of finite sets.

$N_1 i\mathbf{FinSet}$**:** bijections of finite sets; the vertices are the source and target of the bijections.

$N_2 i\mathbf{FinSet}$**:** pairs of bijections of finite sets; the 0 and 2-faces are the two elements of the pair, and the 1-face is the composition.

$N_3 i\mathbf{FinSet}$**:** triples of bijections; the 0 and 3-faces are the beginning and ending pairs; the 1- and 2-faces are the compositions of the first two and second two, respectively.

And so on. This construction is called the *nerve*, and it can be generalized to any category by restricting tuples to composable morphisms. (Thus we get a way of assigning a space to the structure of a category.) The disjoint union and Cartesian product of finite sets give two operations on this simplicial set which are associative, commutative and distributive (up to homotopy).

In our previous definition of the integers, we defined

$$\mathbf{Z} \overset{\text{def}}{=} (\pi_0 N_\bullet i\mathbf{FinSet}, \amalg, \times)^{gp}.$$

[1] For those worried about set-theoretic issues: when we take "finite sets" we actually mean "finite subsets of U", where U is a chosen infinite universe. The choice of U does not matter; any set will give us a homotopy-equivalent space in the end. Because of this, we generally ignore the question of which U we are using.

Thus we see that we took π_0 first (thus destroying lots of interesting information) and then we took a group completion. It makes sense, therefore, that if we want to be less destructive we may want to "commute" the π_0 and the group completion past one another, so that the group completion takes place first. This will also show the natural way of constructing higher K-groups: they are simply the higher homotopy groups of this group completion.

Let us think a bit about what the group completion is. If M is a topological monoid and $a \in \pi_0 M$ is invertible (in the sense that there exists $b \in \pi_0 M$ with $a + b = 0$) then a is a unit in the Pontrjagin ring $H_*(M)$. Thus if a space N can be considered to be the "group completion" of M then we should have a map $M \longrightarrow N$ such that $\pi_0 N \cong (\pi_0 M)^{gp}$ and this map induces a map

$$H_*(M)[(\pi_0 M)^{-1}] \longrightarrow H_*(N).$$

For any topological monoid there is a canonical map $M \longrightarrow \Omega BM$; when $\pi_0 M$ is a group, this is a weak equivalence. The theorem of McDuff–Segal states that for (nice) general M, setting $N = \Omega BM$ gives an isomorphism on homology.

Thus we can rewrite the above formula as

$$K_0(\mathbf{FinSet}) = \pi_0 \Omega B(N_\bullet i\mathbf{FinSet}, \amalg, \times).$$

Substituting i for 0 gives a general formula for the K-theory of finite sets. We can then define more generally

$$K(\mathbf{FinSet}) = \Omega B(N_\bullet i\mathbf{FinSet}, \amalg, \times).$$

Note that the B on the right-hand side is incredibly difficult to work with. It is defined as the geometric realization of a simplicial space, and actually writing down anything about its homotopy groups is difficult. Thus it should not be surprising that these are difficult to compute. What may be more surprising is the following theorem:

Theorem 1.1 (Barratt–Priddy–Quillen)**.**

$$K(\mathbf{FinSet}) \simeq QS^0 \stackrel{\text{def}}{=} \operatorname*{colim}_{n\to\infty} \Omega^n \Sigma^n S^0.$$

Thus the K-theory of finite sets is the same as the stable homotopy of spheres.

Remark 1.2. This should actually not be that surprising, if we are thinking of K-theory as assigning spaces to rings in a coherent way. The integers are the initial commutative ring; moreover, they classify the combinatorics of addition and multiplication in rings. Thus, if we are constructing K-theory correctly, the space we produced should classify the combinatorics of adding and multiplying homotopy... which is exactly what the sphere spectrum does.

Remark 1.3. Barratt–Priddy–Quillen implies that K-theory is fundamentally a *spectrum*, not a space. Spectra are "abelianized spaces," in the following senses:

1. For any spectrum X, $\pi_1 X$ is always abelian.
2. For any two spectra X and Y, $X \vee Y \simeq X \times Y$, just like for abelian groups, $G \oplus H \cong G \times H$. In spaces this is not the case, just as in groups $G * H \ncong G \times H$.
3. Suspension is invertible in spectra, so anything that is true in the stable range in spaces is true everywhere in spectra.

A spectrum X is modeled as a sequence of spaces $X_0, X_1, \ldots$ together with some extra coherence data. The "0-space" of a spectrum, write $\Omega^\infty X$, is

$$\operatorname*{colim}_{n\to\infty} \Omega^n X_n.$$

Thus the definition of Q above is just the 0-space of the spectrum $S^0, S^1, \ldots$. This is generally called the *sphere spectrum* and denoted $\mathbb{S}$.

The upshot of this discussion is that defining K-theory (at least when there are no exact sequences to split) is easy. The hard part is working with it and computing it, because B is difficult to work with. This hard part is always there, but it is possible to move it around: if we move part of it into the *construction* of K-theory, it makes computations and proofs easier. Thus a lot of the machinery of K-theory is this kind of "shoving the lump under the rug around," trying to find the right balance between formality and simplicity to be able to prove things.

We finish today by discussing one construction of group completion which "moves the lump around" enough to make some things (somewhat) clearer. This construction has the benefit that it factors through Γ-spaces. Γ-spaces are a model of *connective* spectra. This means that they are often simpler to work with than standard spectra, but they have the downside that they cannot model negative spheres. When working with connective K-theory, however, this is occasionally "good enough."

Definition 1.4. Let $\mathbf{n}$ be the finite pointed set $\{0, \dots, n\}$ with 0 as the basepoint. We think of $\mathbf{FinSet}_*$ as the category of such sets.

A Γ-*space* X is a functor $\mathbf{FinSet}_* \longrightarrow \mathbf{Top}$. Note that for every n and for every $1 \le i \le n$ there is a map $\alpha_{ni}: X_n \longrightarrow X_1$ induced by the map $\mathbf{n} \longrightarrow \mathbf{1}$ taking i to 1 and everything else to 0. A Γ-space is called *special* if for all n, the map

$$X_n \xrightarrow{\alpha_{n1} \times \cdots \times \alpha_{nn}} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is a weak equivalence. This means that $X_0 \simeq *$ and that X_n is a model for X_1^n.

Consider the map $\mathbf{2} \longrightarrow \mathbf{1}$ taking 0 to 0 and everything else to 1. The induced map $X_2 \longrightarrow X_1$ is often called μ, because it behaves like a multiplication on X_2. The analogous maps are higher multiplications, and the structure of the Γ-space is an encoding of a space with commutative and associative multiplication.

Example 1.5. Consider the Γ-space taking $\mathbf{n}$ to the nerve of the full subcategory of $\mathbf{FinSet}^n$ whose objects are *disjoint* finite sets. For a map $\phi: \mathbf{n} \longrightarrow \mathbf{k}$ we define the map $\mathbf{FinSet}^n \longrightarrow \mathbf{FinSet}^k$ on an object $(U_1, \dots, U_n)$ (resp. morphism $(f_1, \dots, f_h)$) to have as its i-th coordinate $\bigcup_{j \in \phi^{-1}(i)} U_j$ (resp. $\bigcup_{j \in \phi^{-1}(i)} f_j$). Note that this is a special Γ-space, since for any finite tuple of finite sets we can choose an isomorphic model which where the finite sets are disjoint.

Example 1.6. Let M be a commutative monoid, and let $X: \mathbf{FinSet}_* \longrightarrow \mathbf{Top}$ be take $\mathbf{n}$ to M^n. This is a special Γ-space.

We can now do the following thing. Take any pointed simplicial set K, which is finite in each dimension. We can thus think of K as a functor $K: \Delta^{\mathrm{op}} \longrightarrow \mathbf{FinSet}_*$. For any Γ-space X we can then define a simplicial space $X \circ K$. For any Γ-space X, we define its *group completion* to be the spectrum

$$|X \circ (S^1)^{\wedge 0}|, |X \circ (S^1)^{\wedge 1}|, |X \circ (S^1)^{\wedge 2}|, \dots.$$

We don't have time to discuss why this is the correct model, but I want to end by pointing out a couple of important features of this construction:

(1) There are weak equivalences $|X \circ S^k| \xrightarrow{\sim} |X \circ S^{k+1}|$ for $k \ge 1$. Thus the 0-space of this spectrum is $\Omega|X \circ S^1|$.
(2) The simplicial maps in the 1-space encode ordered addition. The simplicial maps in the higher spaces encode "ordered addition" in a higher-dimensional setting, which requires addition to be commutative. (This is enforced by the general structure of the Γ-space.)
(3) As we go up the spectrum, the simplices representing our space are constructed out of higher and higher dimensional "grids" of copies of X_1. This is a common feature of constructions of K-theory, where the simplicies in the n-th space of the spectrum are represented by n-dimensional grids.

Theorem 1.7. $K(\mathbf{FinSet})$ *can be modeled by the* Γ*-space from Example 1.5.*

Exercises.

(1) Let A and B be any two abelian groups. Construct a ring R such that the group rings $R[A]$ and $R[B]$ are isomorphic.
(2) Both disjoint union and Cartesian product give symmetric monoidal structures on finite sets.
   (a) Prove that there exists a model of **FinSet** such that both of these are *strictly* monoidal, in the sense that all three natural transformations in the structure are identity transformations.
   (b) Write down the two natural transformations that define distributivity.
   (c) Show that it is not possible to make all eight of the above natural transformations identities simultaneously.

   This shows that $N_{\cdot} i\mathbf{FinSet}$ cannot be made into a strict ring object.
(3) Check that in Example 1.6, $X \circ S^1 \cong BM$.

(4) Look up the definition of a symmetric spectrum. Verify that the group completion of a Γ-space is a symmetric spectrum.

(5) In this exercise we will prove Zylev's theorem, which implies that the group completion of scissors congruence monoids is injective when the underlying space is Euclidean, spherical or hyperbolic space. More formally, we are trying to show the following. Let X be n-dimensional Euclidean, spherical or hyperbolic space. A *simplex* in X is the convex hull of $n+1$ points in general position; a *polytope* is a finite union of simplices. We define

$$\mathcal{P}(X) = \text{fr ab gp gen by polytopes} \Big/ \begin{array}{ll} [P \cup Q] = [P] + [Q] & \text{if } \operatorname{meas}(P \cap Q) = 0 \\ [P] = [Q] & \text{if } P \cong Q. \end{array}$$

Zylev's theorem states that if $[P] = [Q]$ in $\mathcal{P}(X)$ then P is scissors congruent to Q.

To prove this theorem it suffices to check the following. Suppose that P and Q are such that there exist R and S such that

- $\operatorname{meas}(P \cap R) = \operatorname{meas}(Q \cap S) = 0$
- R is scissors congruent to S
- $P \cup R$ is scissors congruent to $Q \cup S$.

Then P is scissors congruent to Q.

(a) Consider the following argument:
Since R is scissors congruent to S it is possible to write $R = \bigcup_{i=1}^n R_i$ and $S = \bigcup_{i=1}^n S_i$ with $R_i \cong S_i$ (and all unions appropriately disjoint). Since $P \cup R$ is scissors congruent to $Q \cup S$ it is possible to write $P = \bigcup_{i=1}^k X_i$, $R = \bigcup_{i=k+1}^m X_i$, $Q = \bigcup_{j=1}^\ell Y_j$, $S = \bigcup_{j=\ell+1}^m Y_j$ such that there is a permutation τ of $\{1, \ldots, m\}$ with $X_i \cong Y_{\tau(i)}$. This has two different subdivisions of R and S; take their common refinement and extend this scissors congruence to them.
Now draw a bipartite graph where each node is an X_i or a Y_j. Draw an edge connecting X_i to $Y_{\tau(i)}$, and an edge connecting X_i to Y_j which correspond to one another in the scissors congruence of R and S. Each piece of P and Q has degree 1, while each piece of R and S has degree 2; thus the graph splits into paths connecting pieces of P to pieces of Q. Since each edge connects congruent polytopes, this shows that P is scissors congruent to Q.
Explain why this argument is wrong. (Hint: what happens when you try it in the hyperbolic plane with infinite points example?)

(b) Suppose that in the scissors congruence between $P \cup R$ and $Q \cup S$, all pieces that hit S are inside R. Prove that P is scissors congruent to Q.

(c) Suppose that in the scissors congruence between $P \cup R$ and $Q \cup S$, all pieces that hit S are inside P. Prove that P is scissors congruent to Q.

(d) Suppose that $R \cong S$ and that P is "more than twice the size of R", in the sense that there is a subpolytope $P' \subseteq P$ which is scissors congruent to $R \amalg R$. Prove that P is scissors congruent to Q.

(e) Complete the proof.

2. Day 2

In this lecture we assume that the listener has seen Waldhausen's $S_\bullet$-construction. For more on this, see for example [Wal85] or [Wei13, IV.8].

Our goal for this lecture is to prove the Barratt–Priddy–Quillen theorem:

Theorem 2.1 (Barratt–Priddy–Quillen)**.**

$$K(\mathbf{FinSet}_*) \simeq \mathbb{S}.$$

More generally, if G is a discrete group we consider $G\mathbf{FinSet}_*$, then

$$K(G\mathbf{FinSet}_*) \simeq \Omega^\infty \bigvee_{\substack{H \leq G \\ \text{conj. class}}} \Sigma^\infty_+ BW_G(H).$$

We focus on proving the first case of the theorem. For this, we follow an approach of Rognes [Rog92]. First, a couple of observations:

(1) In the $S_\bullet$-construction, it is not necessary to keep track of the entire flag; instead, we can just keep track of the upper row. Similarly, in the iterated $S_\bullet^{(n)}$-construction it suffices to keep track of the k^n-cube of cofibrations, without worrying about chosen cofibers. This is explored further in the exercises.
(2) Suppose that we have a spectrum X with an increasing filtration $F_\bullet$. If the inclusion $F_1X \hookrightarrow F_nX$ is a weak equivalence for all n then $F_1X \simeq X$.

These two observations are our main tools for proving BPQ. We work in the category of symmetric spectra of simplicial sets (where the $S_\bullet$-construction naturally lands). Recall that the geometric realization of an n-simplicial set into simplicial sets is simply the restriction to the diagonal.

We will thus be analyzing a simplicial set whose elements are cubical diagrams of inclusions of finite sets. According to the definition of the $S_\bullet$-construction, these diagrams must satisfy an extra condition: given any square

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \hookrightarrow & D \end{array}$$

the induced map $B \cup_A C \longrightarrow D$ must also be a cofibration. Let us pause for a minute to discuss what this condition means. Let us suppose that A, B, C and D are subsets of some universe U, and that the inclusions of sets are just inclusions of subsets. (Up to isomorphism, this is always the case; we can just associate A, B and C with their images in D.) Then $B \cup C \subseteq D$ and $A \subseteq B \cap C$. What happens when $A \subsetneq B \cap C$, so that there is some $x \in B \cap C$ which is not in A? Note that $B \cup_A C$ has *two* copies of x in it, but D has only one; that means that the map $B \cup_A C \longrightarrow D$ is not injective (and thus not a cofibration). Thus in order for $B \cup_A C \longrightarrow D$ to be injective we must have $B \cap C = A$.

Definition 2.2. In an n-dimensional diagram $(0 < \cdots < q)^n \longrightarrow \mathcal{C}$, the *lower corner* is the object which is the image of $(q, \dots, q)$.

In general, this means that in a diagram in the $s_\bullet$-construction, an element of the lower corner "appears" at some minimal point and then is an element of every set with larger index, but not in any other set.

For conciseness, we write

$$X_n = \operatorname{diag} N_\bullet i S_\bullet^{(n)} \mathbf{FinSet}_*.$$

We define

$$F_k(X_n)_\ell = \{\ell\text{-simplices whose lower corner has cardinality at most } k\}.$$

Note that this is compatible both with the face/degeneracy maps, and with the structure maps of the K-theory spectrum. Thus this gives us a filtration on the spectrum $K(\mathbf{FinSet}_*)$. Our next goal is to compute how the filtration components fit together.

Definition 2.3. Let $D_{n,k}$ be the pointed simplicial set whose q-simplices are those diagrams in $s_q^{(n)}\mathbf{FinSet}_*$ which have the set $\{1,\ldots,k\}$ in the lower corner, plus the basepoint. All face and degeneracy maps are defined as in the $s_\bullet$-construction, except that any face map which would produce a diagram with a smaller set in the lower corner is instead takes the diagram to the basepoint.

Proposition 2.4.

$$F_kX_n/F_{k-1}X_n \simeq (D_{n,k})_{h\Sigma_k}.$$

Proof. Note that we can think of $F_kX_n/F_{k-1}X_n$ as the geometric realization of the nerve of the n-simplicial category whose objects are n-cubes whose lower corner is isomorphic to $\{1,\ldots,k\}$. Let $Y_\bullet$ be the full n-simplicial subcategory of those objects whose lower corner is *equal* to $\{1,\ldots,k\}$ and all of whose maps are inclusions of subsets. Since at each n-coordinate $\vec{i}$, $Y_{\vec{i}}$ is an equivalent category to $(F_kX_n/F_{k-1}X_n)_{\vec{i}}$, the inclusion of $Y_\bullet$ into $F_kX_n/F_{k-1}X_n$ is an equivalence on geometric realization. Thus $F_kX_n/F_{k-1}X_n \simeq N_\bullet Y_\bullet$.

The morphisms in $Y_\bullet$ are uniquely determined by their action on the lower corner. Since the lower corner is always equal to $\{1,\ldots,k\}$ every morphism is uniquely determined by its source and an element of Σ_k. Taking the objects of $Y_\bullet$ and restricting to the diagonal gives us $D_{n,k}$; thus $Y_\bullet \simeq (D_{n,k})_{h\Sigma_k}$, as desired. □

Therefore we see that to determine the homotopy type of $F_kX_n/F_{k-1}X_n$ it remains to calculate the homotopy type of $D_{n,k}$.

Proposition 2.5.

$$D_{n,k} \simeq S^{nk}.$$

Proof. We have a model for S^1 which is $\Delta^1/\partial\Delta^1$; the q-simplices of S^1 are therefore the set $\{*,1,\ldots,q\}$, where $*$ is the basepoint. The i-th face map takes j to j if j is at most i, and to $j+1$ otherwise; when $i=j=q$ it takes q to $*$. (The way to visualize this is to think of Δ^1 as the nerve of $0 \longrightarrow 1$ and to work accordingly.) The space $S^{nk} \cong (S^1)^{\wedge nk}$ has q-simplices $\{1,\ldots,q\}^{nk} \amalg \{*\}$. The face and degeneracy maps act coordinatewise.

We define an isomorphism $D_{nk} \longrightarrow S^{nk}$ in the following manner. Let σ be a non-basepoint q-simplex of D_{nk}. For each $i \in \{1,\ldots,k\}$ let $\vec{q_i} \in \{1,\ldots,q\}^n$ be the minimal coordinate of a finite set in σ which contains i. (Since the lower corner contains i, this exists.) We then take σ to the k-tuple $(\vec{q_1},\ldots,\vec{q_k}) \in \{1,\ldots,q\}^{nk}$. It is a simple exercise with the definitions to see that this is a map of simplicial sets.

To check that it is an isomorphism of simplicial sets it suffices to check that it is a levelwise bijection; this follows directly from the definitions. □

Corollary 2.6. *The inclusion $F_1K(\mathbf{FinSet}_*) \longrightarrow F_kK(\mathbf{FinSet}_*)$ is a weak equivalence for all k.*

Proof. When $k=1$ this is trivially true, so we focus on $k \geq 2$.

By the proposition above, $F_kX_n \simeq S^{nk}_{h\Sigma_k}$. By the homotopy fixed point spectral sequence

$$H_s(\Sigma_k, \pi_pS^{nk}) \Longrightarrow \pi_{p+s}S^{nk}_{h\Sigma_k}$$

$S^{nk}_{h\Sigma_k}$ is $(nk-1)$-connected. In particular, for $k \geq 2$ it is at least $2n-1$-connected. This means that as we go up the spectrum, the maps $F_1X_n \longrightarrow F_kX_n$ become more and more connected; thus, stably, the map $F_1K(\mathbf{FinSet}_*) \longrightarrow F_kK(\mathbf{FinSet}_*)$ is a weak equivalence. □

This completes the proof of BPQ, since $F_1X_n \simeq S^n$, as desired.

In this proof we did not use anything other than combinatorics and a basic understading of topology of spaces and spectra. This illustrates the fact that at the base of K-theory is some very clever combinatorics, which controls a lot of the underlying structure.

This structure tells us that there should be a simple combinatorial description of the stable homotopy groups of spheres. We already discussed last time the description of K_0: it is just the group completion of the cardinalities of finite sets. By tracing through the definitions it is also possible to directly extract a description of what K_1 is: it is the sign of a permutation. More concretely, given any permutation $\sigma \in \Sigma_n$ we have a map

$$S^1 \xrightarrow{\sigma} B\Sigma_n \longrightarrow N_\bullet\mathbf{FinSet}_* \longrightarrow \Omega B(N_\bullet\mathbf{FinSet}_*) \simeq K(\mathbf{FinSet}_*).$$

This gives an element in $\pi_1K(\mathbf{FinSet}_*) \cong \pi_1\mathbb{S} \cong \mathbf{Z}/2$; this is the sign of the permutation.

Unfortunately, such nice descriptions of the higher homotopy groups of spheres do not exist, although morally speaking they should; the main difficultly is in understanding the combinatorics of the $S_\bullet$ construction well enough to be able to extract such descriptions.

Now we make a digression into an examination of the $S_\bullet$-construction and the Q-construction. One of the more unsatisfying aspects of the definition of a Waldhausen category is that it requires taking $\mathbf{FinSet}_*$, not just **FinSet**. In the discussion above we never used the basepoint in our finite sets (in fact, I omitted it from the notation and likely nobody noticed). The only reason that it is necessary is because the definition of a Waldhausen category required a zero object.

This is one of the advantages of Γ-spaces: there is no similar restriction, and we can just take finite sets and union as described in Example 1.5. However, we then run into the problem that Γ-spaces are somewhat too general: we lose all of the nice structure of the category of finite sets when we just look at the space model.

There are two solutions to this. The first is just to bite the bullet and keep track of the combinatorics of Γ-spaces directly. With this, it is actually possible to do a similar Rognes-style proof of Barratt–Priddy–Quillen, so we don't lose too much. But it's more annoying and harder to generalize to other situations.

The second is to look very hard at the $S_\bullet$-construction and notice that the only maps that it requires are injections and pushouts of maps $A \longrightarrow *$. Such a pushout is a map which is injective everywhere away from the preimage of the basepoint. What we realize when we think about it this way is that this is *the inverse of an injection*. The reason we were having so much trouble with it is that we were looking at it backwards, just like Segal did in his definition of Γ-spaces.

Looking back at the Q-construction, we see that a morphism in the Q-construction is a formal composition of a backwards epic and a forwards monic. We can do a similar construction for finite sets, where we simply declare that the map is a formal composition of two injections, one "red" and one "blue," with a way of commuting them past one another. This definition produces a similar Q-construction for finite sets, which does not use formal inverses.

Exercises.

(1) Let $\mathcal{C}$ be a Waldhausen category, $\mathcal{D}$ be the category $* \longleftarrow * \longrightarrow *$, and let $F\colon \mathcal{C}^{\mathcal{D}} \longrightarrow \mathcal{C}$ be the colimit functor. We define $\widetilde{S}_n\mathcal{C}$ to be the category with objects diagrams

$$0 = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n$$

and with morphisms natural transformations. We define s_i to repeat A_i (and insert the identity morphism), d_i for $i > 0$ to skip A_i. Let $G_i\colon \mathcal{D} \longrightarrow \mathcal{C}$ be the diagram $0 \longleftarrow A_i \hookrightarrow A_i$; we let d_0 take the diagram above to

$$F(G_1) \hookrightarrow \cdots \hookrightarrow F(G_n).$$

  (a) What condition is required on F to make F well-defined?
  (b) What condition is required on F to make $\widetilde{S}_\bullet\mathcal{C}$ a simplicial object?
  (c) When F is well-defined, check that $|wS_\bullet\mathcal{C}| \simeq |w\widetilde{S}_\bullet\mathcal{C}|$.

(2) Let $\mathcal{C}$ be a Waldhausen category. Explicitly describe the structure maps in the spectrum $K(\mathcal{C})$.

(3) Use a similar technique to the lecture to show that the Γ-space associated to addition of finite pointed sets is weakly equivalent to the sphere spectrum.

3. Day 3

Recall the following definition:

Definition 3.1. The *Grothendieck ring of varieties* $K_0(\mathrm{Var}_k)$ is the free abelian group generated by varieties over k, modulo the relation that for every closed immersion $Y \hookrightarrow X$, $[X] = [Y] + [X \smallsetminus Y]$. Multiplication is defined by Cartesian product, $[X][Y] = [X \times Y]$.

The Grothendieck ring is a very complicated ring. For example, it's not an integral domain [Poo02]; in fact, the class of the affine line is itself a zero divisor [Bor18]. (It is not nilpotent, luckily, because if two varieties are equal in the Grothendieck ring then they have the same dimension.) In order to study this ring, people often consider various additive measures on it (morphisms out of it to other groups/rings) but we will take a different tack.

The set of varieties comes with an increasing filtration by dimension, and the relation is well-behaved relative to this filtration. It is therefore tempting to use this filtration to analyze the Grothendieck ring. We make the following definition:

Definition 3.2. Let $K_0(\mathrm{Var}_k^{(n)})$ be generated by varieties up to dimension n, and the same relation as $K_0(\mathrm{Var}_k)$ for varieties up to dimension n.

Definition 3.3. Let B_n be the set of birational isomorphism classes of varieties of dimension n. Write $\mathrm{BiratAut}(X)$ for the birational automorphism group of X.

We begin with a bit of wishful thinking. **WARNING: This is not correct mathematics. However, the motivation for it is important enough, and it is closely enough connected with correct mathematics that it will still be a useful thought-experiment.**

Incorrect discussion 1. We have an injective map $K_0(\mathrm{Var}_k^{(n)}) \longrightarrow K_0(\mathrm{Var}_k)$, which gives the filtration on $K_0(\mathrm{Var}_k)$. The ratio

$$K_0(\mathrm{Var}_k^{(n)})/K_0(\mathrm{Var}_k^{(n-1)}) \cong \mathbf{Z}\{B_n\}.$$

To see this, note that quotienting out by $K_0(\mathrm{Var}_k^{(n-1)})$ kills the generators associated to all varieties of dimension $n-1$; thus the only thing left is the n-dimensional generators, which are related by the relation $[X] = [X \smallsetminus Y]$, when Y has lower dimension than X. In this picture, a variety becomes equal to the sum of its irreducible components, so we don't need to worry about the cases when X is not irreducible.

But this means that $K_0(\mathrm{Var}_k)$ is filtered by free groups, which tells us that it is also free.

This is a very strong statement about $K_0(\mathrm{Var}_k)$. However, it is unfortunately not true: the incorrect statement is in the very beginning of the discussion, when we assumed that the map $K_0(\mathrm{Var}_k^{(n)}) \longrightarrow K_0(\mathrm{Var}_k)$ is injective. In fact, this inclusion does induce a filtration on $K_0(\mathrm{Var}_k)$; however, as the map is not injective we do not have a nice expression for the filtration quotient and thus we cannot compute the associated graded to the filtration. The injectivity of the map above is equivalent to the statement that higher-dimensional varieties do not induce relations between lower-dimensional varieties.

Let us see how this might happen. Suppose that we have a birational automorphism φ of a variety W. This means that $\varphi\colon U \longrightarrow V$ is an isomorphism between dense open subsets of W. In the Grothendieck ring this means that

$$[W \smallsetminus U] = [W] - [U] = [W] - [V] = [W \smallsetminus V].$$

However, from this analysis we do not even necessarily see that $W \smallsetminus U$ and $W \smallsetminus V$ have the same dimension, much less that they are piecewise isomorphic! Thus the question of injectivity above is a question about birational automorphisms, and how their exceptional loci can be located. The problem with the above analysis is that it had no way of keeping track of automorphisms, instead of isomorphism classes.

This is exactly where K-theory comes in. This tells us that we need to keep track of higher information in order to make this work, and that means that we need a K-theory of varieties.

Definition 3.4 ([CZ])**.** A *CGW-category* $(\mathcal{C}, \phi, c, k)$ is a double category $\mathcal{C} = (\mathcal{E}, \mathcal{M})$, with vertical arrows $\mathcal{E}$ and horizontal arrows $\mathcal{M}$, denoted $\looparrowright$ and $\rightarrowtail$, respectively. $\phi\colon i\mathcal{E} \longrightarrow i\mathcal{M}$ is an *isomorphism* of categories. The functors c and k are equivalences of categories

$$k\colon \mathrm{Ar}_\square\, \mathcal{E} \longrightarrow \mathrm{Ar}_\triangle\, \mathcal{M} \qquad \text{and} \qquad c\colon \mathrm{Ar}_\square\, \mathcal{M} \longrightarrow \mathrm{Ar}_\triangle\, \mathcal{E}$$

which satisfy:

(Z) $\mathcal{C}$ contains an object $\varnothing$ which is initial in both $\mathcal{E}$ and $\mathcal{M}$.
(M) Every morphism in the categories $\mathcal{E}$ and $\mathcal{M}$ is monic.
(K) For every $g\colon A \mapsto B$ in $\mathcal{E}$, $k(g\colon A \mapsto B) = A^{k/g} \xrightarrow{g^k} B$ and there exists a distinguished square

$$\begin{array}{ccc} \varnothing & \rightarrowtail & A \\ \downarrow & & \downarrow g \\ A^{k/g} & \overset{g^k}{\rightarrowtail} & B \end{array} .$$

Dually, for every $f\colon A \rightarrowtail B$ in $\mathcal{M}$, $c(A \overset{f}{\rightarrowtail} B) = A^{c/f} \xmapsto{f^c} B$ and there exists a distinguished square

$$\begin{array}{ccc} \varnothing & \rightarrowtail & A^{c/f} \\ \downarrow & & \downarrow f^c \\ A & \overset{f}{\rightarrowtail} & B \end{array} .$$

Example 3.5. Any exact category produces a CGW-category, by taking the horizontal morphisms to be admissible monics and the vertical morphisms to be the *opposite* of admissible epics. The map c takes the cokernel and k takes the kernel. The distinguished squares are the stable squares: those that are both pushouts and pullbacks.

Example 3.6. The category **FinSet** of finite sets is a CGW-category, where both $\mathcal{E}$ and $\mathcal{M}$ are injective maps. Both c and k take the complement of a finite set. The distinguished squares are the stable squares.

Example 3.7. The category Var_k is a CGW-category, where $\mathcal{M}$ is closed embeddings and $\mathcal{E}$ is open embeddings. c and k both take the complement. The distinguished squares are the stable squares.

Lemma 3.8. *For any diagram $A \overset{f}{\rightarrowtail} B \xmapsto{g} C$ there is a unique (up to unique isomorphism) distinguished square*

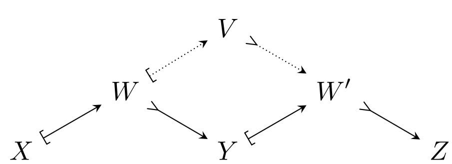
$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ \downarrow & & \downarrow g \\ D & \rightarrowtail & C \end{array} .$$

The analogous statement holds for any diagram $A \xmapsto{f} B \rightarrowtail C$.

This tells us that we can do a Q-construction for a CGW-category in the exact "same" way that we did it for an exact category:

Definition 3.9. For a CGW-category $\mathcal{C}$, we define a category $Q\mathcal{C}$ by

objects: $\mathrm{ob}\,\mathcal{C}$
morphisms: $X \longrightarrow Y$ are equivalence classes of diagrams $X \mapsto Z \rightarrowtail Y$. Composition works by applying Lemma 3.8 to the following diagram:



Interestingly enough, Quillen's proof of dévissage can be made to work almost identically in the context of CGW-categories as it does in exact categories.

Thus let us consider Var_k as a CGW-category. Then dimension gives us a filtration $\mathrm{Var}_k^{(n)}$ on Var_k, and this produces a filtration on the space $K(\mathrm{Var}_k^{(n)})$.[2]

So now the only question is: can we compute the associated graded of this filtration?

To do this, we need to compute the cofiber on K-theory of the inclusion $\mathrm{Var}_k^{(n-1)} \longrightarrow \mathrm{Var}_k^{(n)}$. Quillen does this for abelian categories using Serre categories: he shows that for a Serre subcategory $\mathcal{A}$ of an abelian categeory $\mathcal{B}$, there is a cofiber sequence in K-theory

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}/\mathcal{A}).$$

However, his proof heavily exploits the additive structure on an abelian category, and in our geometric context we have no such structure. It is our hope that we can prove an analog of this in [CZ], but we currently do not have a complete proof of this. However, using some more advanced machinery it is possible to compute this cofiber anyway:

Theorem 3.10 (Z)**.**

$$K(\mathrm{Var}^{(n)})/K(\mathrm{Var}^{(n-1)}) \simeq \bigvee_{[X]\in B_n} \Sigma^\infty_+ B\,\mathrm{BiratAut}(X).$$

Note that this has the expected form: on π_0 this is exactly $\mathbf{Z}\{B_n\}$, as before, and the higher homotopy groups contain the information of birational automorphism groups. Unfortunately, the higher homotopy groups are not easy to describe, since the stable homotopy groups of BG are not just G in degree 1 and 0 elsewhere. However, we do know that $\pi_1^s BG \cong G^{ab}$, so there is some analysis that can be done.

This filtration produces a spectral sequence, which on the E^1-page has $\mathbf{Z}\{B_n\}$ in the diagonal converging to $K_0(\mathrm{Var}_k)$, and $\bigoplus_{[X]\in B_n} \mathrm{BiratAut}(X)^{ab}$ in the diagonal converging to $K_1(\mathrm{Var}_k)$. What this means is that the difference between the (incorrect) filtration we computed before and the correct filtration comes from differentials arising from birational automorphisms, which is exactly what we expected from our informal analysis. In fact, the differentials in this spectral sequence are obtained in the following manner: for a birational automorphism $\varphi \in \mathrm{BiratAut}(X)$, defined as an isomorphism $U \longrightarrow V$, $d[\varphi] = [X \smallsetminus V] - [X \smallsetminus U]$. Thus these differentials exactly measure the failure of the map $K_0(\mathrm{Var}_k^{(n)}) \longrightarrow K_0(\mathrm{Var}_k)$ to be injective.

Remark 3.11. Borisov [Bor18] proved that when $k = \mathbf{C}$ the maps $K_0(\mathrm{Var}_{\mathbf{C}}^{(n)}) \longrightarrow K_0(\mathrm{Var}_{\mathbf{C}})$ are not injective, and most other such results require at least that k have characteristic 0. In the above analysis we did not use the characteristic of k at all; the question of what differentials in the spectral sequence look like and whether the maps $K_0(\mathrm{Var}_k^{(n)}) \longrightarrow K_0(\mathrm{Var}_k)$ are injective is open in finite characteristic.

Exercises.

(1) Prove Lemma 3.8.
(2) Check that if $\mathcal{C}$ is an exact category, then $K^Q(\mathcal{C}) \cong K^C(\mathcal{C})$.
(3) Prove that $K^c(\mathbf{FinSet}) \simeq K^W(\mathbf{FinSet}_*)$. (Hint: Waldhausen [Wal85] gives a relation between the $S_\bullet$-construction and the Q-construction.)
(4) Show that there is an $S_\bullet$-construction on CGW-categories.

[2] We are going to be a bit lax on whether we are working in a spectrum or a space. There is a way, analogous to the $S_\bullet$-construction, that we can deloop this, so it does work correctly. See [Cam, Zak17].

4. Day 4

In this lecture we will only be working over $\mathbf{C}$.

We have the following theorem of Larsen and Lunts:

Theorem 4.1 ([LL03])**.**

$$K_0(\mathrm{Var}_{\mathbf{C}})/(\mathbb{L}) \cong \mathbf{Z}[SB],$$

where SB is the multiplicative monoid of stable birational isomorphism types.

The proof of this theorem relies on the Weak Factorization Theorem:

Theorem 4.2 (Weak Factorization Theorem)**.** *Let $\phi\colon X \longrightarrow Y$ be a birational map of smooth complete varieties, and let $U \subseteq X$ be an open subset on which ϕ is an isomorphism. Then ϕ cna be factored as a zigzag of blowups/blowdowns with smooth centers disjoint from U.*

Outline of Proof of Larsen–Lunts. Let $\mathcal{M}$ be the multiplicative monoid of smooth complete irreducible varieties. Let $\Phi\colon \mathcal{M} \longrightarrow G$ be a homomorphism of monoids such that $\Phi([X]) = \Phi([Y])$ if X and Y are birational and such that $\Phi(\mathbb{P}^n) = 1$ for all n. The main step of the theorem is proving that Φ extends to a map $\widetilde{\Phi}\colon K_0(\mathrm{Var}) \longrightarrow \mathbf{Z}[\mathcal{M}]$. Once this is proved, we simply take $\Phi([X]) = [X] \in SB$, and we are done.

The construction of $\widetilde{\Phi}$ is given as follows:

(1) If X is smooth and complete, then $\widetilde{\Phi}(X) \stackrel{\text{def}}{=} [\Phi(X)]$.
(2) If X is smooth with connected components $X_1, \ldots, X_m$, pick a smooth irreducible completion of X_i, $\overline{X}_i$. Then define

$$\widetilde{\Phi}(X) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \widetilde{\Phi}(\overline{X}_i) - \widetilde{\Phi}(\overline{X}_i - X_i).$$

(3) If X is an arbitrary variety, let X^{sing} be its singular locus. Then we define $\widetilde{\Phi}(X) = \widetilde{\Phi}(X - X^{sing}) - \widetilde{\Phi}(X^{sing})$.

To prove the theorem, we proceed by induction on these three definitions simultaneously. Note that in (2), for example, the right-hand side of the definition relies only on definition (1) for a variety of dimension n and the definition of $\widetilde{\Phi}$ for a variety of dimension less than n. This suggests that we can do joined induction on these definitions to check that this is well-defined. ☐

Larsen and Lunts use this to prove that Kapranov's motivic zeta function,

$$\zeta(X,t) = \sum_{n \geq 0} [X^{(n)}] t^n$$

is not rational.

Our goal for this lecture is to give a different proof of this theorem using the higher homotopical structure of $K(\mathrm{Var}_{\mathbf{C}})$. These techniques will also allow us to extend results away from only smooth projective varieties.

Recall from last time that we have a filtration on $K(\mathrm{Var})$, whose n-filtered part is $\bigvee_{[X]\in B_n} \Sigma^\infty_+ B\,\mathrm{BiratAut}(X)$. There is thus a spectral sequence

$$E^1_{p,q} = \pi_p K(\mathrm{Var}^{(q)})/K(\mathrm{Var}^{(q-1)}) \Longrightarrow K_p(\mathrm{Var}).$$

(Note that this is highly nonstandard grading; the goal of this grading is to have the spectral sequence nicely fit into the first quadrant. As $\pi_0 \Sigma^\infty_+ BG \cong \mathbf{Z}$ and $\pi_1 \Sigma^\infty_+ BG \cong G^{ab}$, the first two columns of this spectral sequence are pictured in Figure 1.

Lemma 4.3. *The differentials in the above spectral sequence can be described in the following manner. For every birational automorphism $\varphi\colon X \longrightarrow X$, defined as an isomorphism $U \longrightarrow V$, map $[\varphi]$ to $[X \smallsetminus V] - [X \smallsetminus U]$, mapping each $n-1$-dimensional irreducible component to its birational isomorpihsm class.*

If $d_i[\varphi] = 0$ this means that $X \smallsetminus U$ and $X \smallsetminus V$ are birationally isomorphic in dimension $n-i$. To compute $d_{i+1}[\varphi]$ we pick a birational isomorphism between their components and subtract those out; what is left is some $n-i-1$-dimensional varieties, and we can test to see if those are birationally isomorphic. Any indeterminacy about the choice of birational isomorphism is killed on a previous page (as it comes from birational isomorphisms between lower-dimensional varieties), and thus d_{i+1} will be well-defined.

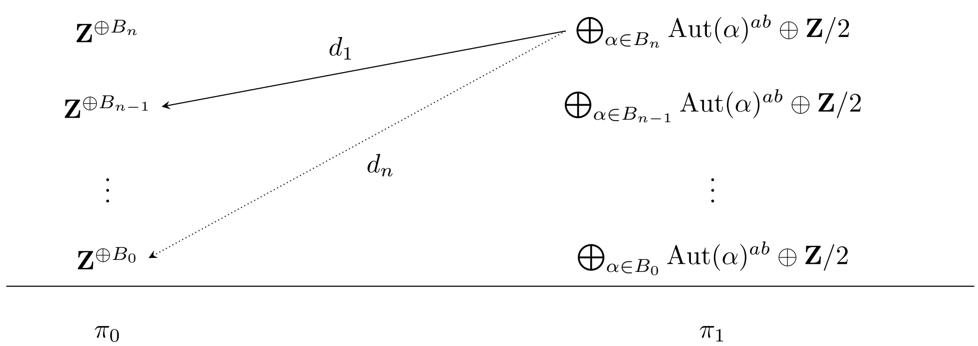


FIGURE 1. Spectral sequence for $K(\mathrm{Var})$

Note that the previous discussion ignored the $\mathbf{Z}/2$-component. This component comes from the basepoint in Σ^∞_+, which produces an $\mathbb{S}$-summand for each birational isomorphism class. This summand keeps track of the combinatorics of how birational components of varieties are being rearranged. By tracing through the maps defining the differentials, we can see that all differentials on these components are zero; thus we can safely ignore them.

Remark 4.4. This is because differentials measure how far off from an isomorphism a birational automorphism is. The morphism representing the $\mathbf{Z}/2$ simply swaps two copies of X. Thus this is defined everywhere, and from our formula we see that it must therefore be zero.

Larsen and Lunts computed the cofiber of $K_0(\mathrm{Var}) \xrightarrow{\cdot\mathbb{L}} K_0(\mathrm{Var})$ and showed that it is $\mathbf{Z}[SB]$. Our goal is to do the same, using the higher homotopical structure: we wish to compute $\pi_0(K_0(\mathrm{Var})/(\cdot\mathbb{L}))$. The key step in this computation is the following observation:

The map $\cdot\mathbb{L}$ is a map $K(\mathrm{Var}^{(n-1)}) \longrightarrow K(\mathrm{Var}^{(n)})$. Thus it is a map of *filtered* spectra; this induces a filtration on the cofiber via the following diagram:

$$\begin{array}{ccccccccc}
K(\mathrm{Var}^{(n-1)}) & \hookrightarrow & K(\mathrm{Var}^{(n)}) & \hookrightarrow & K(\mathrm{Var}^{(n+1)}) & \hookrightarrow & \cdots & \hookrightarrow & K(\mathrm{Var}) \\
\downarrow{\scriptstyle\cdot\mathbb{L}} & & \downarrow{\scriptstyle\cdot\mathbb{L}} & & \downarrow{\scriptstyle\cdot\mathbb{L}} & & & & \downarrow{\scriptstyle\cdot\mathbb{L}} \\
K(\mathrm{Var}^{(n)}) & \hookrightarrow & K(\mathrm{Var}^{(n+1)}) & \hookrightarrow & K(\mathrm{Var}^{(n+2)}) & \hookrightarrow & \cdots & \hookrightarrow & K(\mathrm{Var}) \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
K(\mathrm{Var}^{(n)})/\mathbb{L} & \hookrightarrow & K(\mathrm{Var}^{(n+1)})/\mathbb{L} & \hookrightarrow & K(\mathrm{Var}^{(n+2)})/\mathbb{L} & \hookrightarrow & \cdots & \hookrightarrow & K(\mathrm{Var})/\mathbb{L}
\end{array}$$

Thus the spectral sequence for $K(\mathrm{Var})/\mathbb{L}$ requires a computation of the cofiber of $K(\mathrm{Var}^{(n)})/\mathbb{L} \longrightarrow K(\mathrm{Var}^{(n+1)})/\mathbb{L}$. The key observation here is this: it is weakly equivalent to the cofiber of

$$K(\mathrm{Var}^{(n)})/K(\mathrm{Var}^{(n-1)}) \xrightarrow{\cdot\mathbb{L}} K(\mathrm{Var}^{(n+1)})/K(\mathrm{Var}^{(n)}).$$

Given that we have an expression for the components of these, this computation is much simpler. We get the spectral sequence shown in Figure 2. There, $\ell\colon B_{n-1} \longrightarrow B_n$ is the map induced by multiplication by $\mathbb{L}$.

$$\pi_1\widetilde{C}_\beta \cong \mathrm{Aut}(\beta)^{ab} \Big/ \iota\Big(\bigoplus_{\alpha\in\ell^{-1}(\beta)} \mathrm{Aut}(\alpha)\Big)$$

In addition, $\widetilde{\mathbf{Z}}^{\oplus S}$ is $\mathbf{Z}/2$ if S is empty, and the kernel of adding up all of the coordinates if S is nonempty.

The key to analyzing this spectral sequence is the following observations:

(1) d_r is 0 on $\pi_1\widetilde{C}_\beta$ and on $\widetilde{\mathbf{Z}}^{\oplus\ell^{-1}(\beta)}$ if $\ell^{-1}(\beta)$ is empty.
(2) Let $\deg\alpha \in B_n$ be the largest integer r such that $\alpha \in \operatorname{im}\ell^r$.

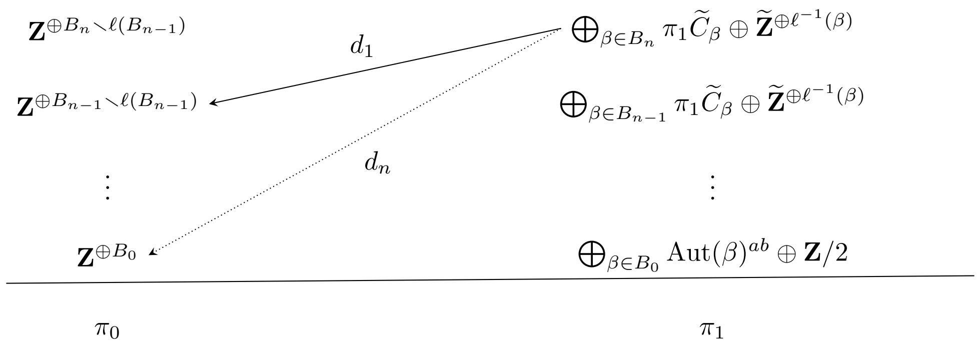


FIGURE 2. Spectral sequence for $K((\mathrm{Var}/\mathbb{L}).)$

If $\ell^{-1}(\beta) \neq \emptyset$, then

$$d_r([\alpha]-[\alpha']) = \begin{cases} 0 & 1 \leq r \leq \min(\deg\alpha, \deg\alpha') \\ [(\ell^{r-1})^{-1}(\alpha)] - [(\ell^{r-1})^{-1}(\alpha')] & r = \min(\deg\alpha,\deg\alpha')+1. \end{cases}$$

Here, $(\ell^{r-1})^{-1}(\alpha)$ is any preimage of α under ℓ^{r-1}.

So the upshot of this is this. To see what remains of the spectral sequence at E^∞, we ignore the $\pi_1\widetilde{C}_\beta$-component and any β with $\ell^{-1}(\beta) = 0$. For a formal difference of birational isomorphism classes, the nonzero differential depends on the degree of the birational isomorhism class; at that degree, the differential takes it to the difference between the classes. This means that on the r-th page of the spectral sequence, the n-th place is the free abelian group generated by birational isomorphism classes $B_n \smallsetminus \ell(B_{n-1})$, modulo the relation that if $\ell^{r+1}(\alpha) = \ell^{r+1}(\alpha')$ then $[\alpha] = [\alpha']$. When we take r to infinity, it thus produces the *stable* birational isomorphism classes, where $[\alpha] = [\alpha']$ if there is some r such that $\ell^r(\alpha) = \ell^r(\alpha')$. This proves the theorem of Larsen and Lunts.

Exercises.

(1) Suppose that $\widetilde{Phi}$ has been constructed for varieties of dimension up to n.
  (a) Show that $\widetilde{\Phi}(X\times\mathbb{P}^m) = \widetilde{\Phi}(X)$ for all X with $\dim X \leq n-m$.
  (b) Show that $\widetilde{\Phi}(X) = \widetilde{\Phi}(Y) + \widetilde{\Phi}(X \smallsetminus Y)$ for all closed embeddings $Y \hookrightarrow X$ if $\dim X \leq n$.
  (c) Show that $\widetilde{\Phi}(X\times Y) = \widetilde{\Phi}(X)\widetilde{\Phi}(Y)$ for $\dim X + \dim Y \leq n$.
  (d) Use this to construct $\widetilde{\Phi}$ for varieties of dimension up to $n+1$, thus completing the proof of Theorem 4.1.

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