Problems (Weibel)

1. Let G be a finite group. The Burnside ring $A(G)$ is K_0 of the category of finite G-sets X. As an abelian group, it is free on the classes G/H as H runs over conjugacy classes of subgroups.

   (a) Show that $A(G)$ is a ring with product $[X] \cdot [Y] = [X \times Y]$.

   (b) For every G-module M, show that $Maps(X, M)$ is a G-module.

   (c) Show that there is a pairing $A(G) \times K_0(\mathbb{Z}G) \to K_0(\mathbb{Z}G)$ satisfying $[X] \cdot [M] = Maps(X, M)$, and that it makes $K_0(\mathbb{Z}G)$ into an $A(G)$ module. In fact, all of the $K_n(\mathbb{Z}G)$ are $A(G)$-modules.

2. Let R be a ring. A chain complex M_* of R-modules is called *perfect* if there is a quasi-isomorphism $P_* \simeq M_*$, where P_* is a bounded complex of finitely generated projective R-modules, i.e., P_* is a complex in $\mathcal{C}h^b(\mathbb{P}(R))$. The perfect complexes form a Waldhausen subcategory $Perf(R)$ of $\mathcal{C}h(R)$. Show that $K_0(Perf(R))$ is isomorphic to $K_0(\mathcal{C}h^b(\mathbb{P}(R)))$.

3. Let $\mathcal{A}^+$ denote the idempotent completion of an exact category $\mathcal{A}$. Its objects are pairs (A, e) with e an idempotent endomorphism of an object A of $\mathcal{A}$. For example, the idempotent completion of free R-modules is projective R-modules.

   (a) Show that there is a natural way to make the idempotent completion of $\mathcal{A}$ into an exact category, with $\mathcal{A}$ an exact subcategory.

   (b) Show that $K_0\mathcal{A}$ is a subgroup of $K_0\mathcal{A}^+$, and that $K_n\mathcal{A} \cong K_n\mathcal{A}^+$ for all $n \geq 1$.

4. Let $\mathcal{A}$ be a Waldhausen category. Show that there is canonical map from $Bw(\mathcal{A})$ to $\Omega B(wS_\bullet\mathcal{A}) = K(\mathcal{A})$. If $\mathcal{A} = \mathbb{P}(R)$, this maps $BGL_n(R)$ to $K(R) = K(\mathbb{P}(R))$, and is part of the canonical map $BGL(R)^+ \to K(R)$.