UNIVERSIDAD DE BUENOS AIRES
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Distintos tipos de estructuras celulares en espacios topológicos

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Distintos Tipos de Estructuras Celulares en Espacios Topológicos

Resumen

Introducimos y desarrollamos la teoría de CW(A)-complejos, que son espacios que se construyen pegando celdas que se obtienen tomando conos de suspensiones iteradas de un *espacio base* A. Estos espacios generalizan a los CW-complejos y nuestras construcciones, aplicaciones y resultados mantienen la intuición geométrica y la estructura combinatoria de la teoría original de J.H.C. Whitehead. Investigamos a fondo las propiedades topológicas y homotópicas de CW(A)-complejos, su localización y los cambios de espacios base.

Como primeras aplicaciones, obtenemos generalizaciones de los teoremas homotópicos clásicos de CW-complejos y del teorema fundamental de Whitehead.

También desarrollamos la teoría de homología de los CW(A)-complejos, generalizando la teoría de homología celular clásica. En el caso de que la homología del espacio base A esté concentrada en cierto grado, definimos un complejo de cadenas A-celular que nos permite calcular los grupos de homología singular de un CW(A)-complejo X a partir de la homología de A y de la estructura A-celular de X. En el caso general, obtenemos una sucesión espectral construida a partir de los grupos de homología de A y de la estructura A-celular de X que converge a la homología de X. Además, utilizamos sucesiones espectrales y una pequeña modificación de las clases de Serre, para obtener información de los grupos de homotopía de los CW(A)-complejos a partir de los grupos de homología y homotopía de A y la estructura A-celular de dichos espacios.

Como una variante de la homología clásica, dado un CW-complejo A, definimos en esta tesis una teoría de homología llamada A-homología, que coincide con la homología singular en el caso $A = S^0$. Esta teoría de homología está inspirada en el teorema de Dold-Thom. Obtenemos de esta forma generalizaciones de resultados clásicos como el teorema de Hurewicz, que relaciona los grupos de A-homología con los grupos de A-homotopía.

Hacia el final de la tesis, damos dos teoremas de clasificación homotópica para CW(A)-complejos, estudiamos aproximación de espacios por CW(A)-complejos y comenzamos el desarrollo de la teoría de obstrucción para estos espacios.

Palabras clave: Estructuras celulares, CW-complejos, sucesiones espectrales, teorías de homología, grupos de homotopía, clases de Serre.

Different Types of Cellular Structures in Topological Spaces

Abstract

We introduce and develop the theory of CW(A)-complexes, which are spaces built up out of cells obtained by taking cones of iterated suspensions of a *base space* A. These spaces generalize CW-complexes and our constructions, applications and results keep the geometric intuition and the combinatorial structure of J.H.C. Whitehead's original theory. We delve deeply into the topological and homotopical properties of CW(A)-complexes, their localizations and changes of the base spaces.

As first applications, we obtain generalizations of classical homotopical theorems for CW-complexes and Whitehead's fundamental theorem.

We also develop the homology theory of CW(A)-complexes, generalizing classical cellular homology theory. In case the homology of the base space A is concentrated in certain degree, we define an A-cellular chain complex which allows us to compute singular homology groups of a CW(A)-complex X out of the homology of A and the A-cellular structure of X. In the general case, we obtain a spectral sequence constructed from the homology groups of A and the A-cellular structure of X which converges to the homology of X. Furthermore, we use spectral sequences and a slight modification of Serre classes to obtain information about the homotopy groups of CW(A)-complexes out of the homology and homotopy groups of A and the CW(A)-structure of those spaces.

As a variant of classical homology, given a CW-complex A, we define in this thesis a homology theory, called A-homology, which coincides with singular homology in the case $A = S^0$. This homology theory is inspired by the Dold-Thom theorem. We obtain generalizations of classical results such as Hurewicz's theorem, relating A-homology groups with A-homotopy groups.

Towards the end of the thesis, we give two homotopy classification theorems for CW(A)-complexes, investigate approximation of spaces by CW(A)-complexes and start developing the obstruction theory for these spaces.

Key words: Cell structures, CW-Complexes, spectral sequences, homology theories, homotopy groups, Serre classes.

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mi mamá, Aniela,
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Contents

Introducción

Los CW-complejos son espacios que se construyen a partir de bloques simples o *celdas*. Los discos son utilizados como modelos para las celdas y se adjuntan secuencialmente utilizando *funciones de adjunción*, que están definidas en esferas, que son los bordes de los discos. Desde su introducción a finales de la década de los '40 por J.H.C. Whitehead [22], los CW-complejos han jugado un rol esencial en geometría y topología. Una de las razones de esta importancia vital es el teorema de CW-aproximación 1.4.18, que implica que en cuanto a grupos de homotopía, homología y cohomología respecta, todo espacio es equivalente a un CW-complejo. Además, la estructura combinatoria de estos espacios permite el desarrollo de herramientas que simplifican considerablemente el cálculo de grupos de homología y cohomología (cf. p. 41) y también el cálculo de grupos de homotopía (1.4.21). La teoría de homotopía de CW-complejos es rica en resultados y su categoría homotópica sirve de modelo para otras categorías homotópicas.

Las propiedades principales de los CW-complejos surgen de los siguientes dos hechos básicos: El n-disco D^n es el cono topológico (reducido) de la $(n-1)$-esfera S^{n-1} y (2) la n-esfera es la n-ésima suspensión (reducida) de la 0-esfera S^0.

Por ejemplo, las propiedades de extensión de homotopías de CW-complejos se siguen de (1), porque la inclusión de la $(n-1)$-esfera en el n-disco es una cofibración cerrada. El item (2) está estrechamente relacionado con la definición de los grupos de homotopía clásicos y es usado para demostrar resultados como el teorema de Whitehead o el teorema de escisión homotópica y en la construcción de espacios de Eilenberg-MacLane. Estos dos hechos básicos sugieren que uno puede reemplazar el *núcleo* original S^0 por otro espacio cualquiera A y construir espacios a partir de celdas de diferentes *formas* o *tipos* utilizando suspensiones y conos del espacio base A.

El propósito principal de esta tesis es introducir y desarrollar la teoría de esos espacios. Definimos la noción de CW-complejos de tipo A (o CW(A)-complejos, para abreviar) generalizando la definición de CW-complejos (los cuales constituyen un caso particular y especial de CW(A)-complejos obtenido tomando $A = S^0$).

Debemos mencionar que existen muchas generalizaciones de CW-complejos en la literatura. Por ejemplo, la generalización de Baues de complejos en categorías de cofibraciones [2] y la aproximación categórica a complejos celulares de Minian [12]. La teoría de CW(A)-complejos que desarrollamos en esta tesis está también relacionada con trabajos de E. Dror Farjoun [5] y W. Chachólski [4]. Sin embargo, nuestro enfoque es muy diferente a ellos y mantiene la intuición geométrica y combinatoria de la teoría original de Whitehead. Además, nos da una visión más profunda de la teoría clásica de CW-complejos, como veremos.

Al igual que en el caso clásico, damos una definición constructiva y una descriptiva y las comparamos, obteniendo los siguientes resultados

Proposición 1. *Sea A un espacio T1. Si X es un CW(A)-complejo constructivo, entonces es un CW(A)-complejo descriptivo.*

Proposición 2. *Sea A un espacio compacto y sea X un CW(A)-complejo descriptivo. Si X es Hausdorff entonces es un CW(A)-complejo constructivo.*

Además, damos contraejemplos si las hipótesis no se satisfacen.

En este contexto, también analizamos construcciones clásicas, como conos, suspensiones, cilindros y productos smash y determinamos si estos funtores aplicados a CW(A)-complejos dan como resultado CW(A)-complejos. Sorpresivamente, algunos de estos resultados no son ciertos para todos los núcleos A y algunas hipótesis son necesarias. Por ejemplo, si el núcleo A es la suspensión de un espacio localmente compacto y Hausdorff, entonces el cilindro reducido de un CW(A)-complejo es también un CW(A)-complejo, pero esto no vale para núcleos arbitrarios A.

Mientras desarrollabámos esta teoría, nos encontramos naturalmente con espacios que se construyen de una manera similar que los CW-complejos, pero en los cuales las celdas no eran adjuntadas en orden de dimensión creciente. Es sabido que espacios de este tipo pueden no ser CW-complejos aunque tiene el tipo homotópico de un CW-complejo. Nosotros los llamamos *CW-complejos generalizados* e inmediatamente definimos la noción de *CW(A)-complejos generalizados*. Obtuvimos los siguientes resultados.

Proposición 3. *Si A es un CW-complejo y X es un CW(A)-complejo generalizado, entonces X tiene el tipo homotópico de un CW-complejo.*

Teorema 4. *Sea A un CW(B)-complejo generalizado con B compacto y sea X un CW(A)-complejo generalizado. Si A y B son T1 entonces X es un CW(B)-complejo generalizado.*

Además, damos un ejemplo de un CW(A)-complejo generalizado que no tiene el tipo homotópico de un CW(A)-complejo (ver 5.2.9).

Otra pregunta que estudiamos es la siguiente. Supongamos que X es un CW(A)-complejo, o en otras palabras, que X se puede construir con *bloques* de tipo A. Y supongamos, además, que A es un CW(B)-complejo. Es natural preguntar si X se puede construir con bloques de tipo B, es decir, si X es un CW(B)-complejo. En esta dirección obtuvimos el siguiente resultado.

Teorema 5. *Sean A y B espacios topológicos punteados. Sea X un CW(A)-complejo, y sean $\alpha : A \to B$ y $\beta : B \to A$ funciones continuas.*

i. Si $\beta\alpha = \mathrm{Id}_A$, entonces existen un CW(B)-complejo Y y funciones continuas $\varphi : X \to Y$ y $\psi : Y \to X$ tales que $\psi\varphi = \mathrm{Id}_X$.

ii. Supongamos que A y B tienen puntos base cerrados. Si β es una equivalencia homotópica, entonces existe un CW(B)-complejo Y y una equivalencia homotópica $\varphi : X \to Y$.

iii. *Supongamos que A y B tienen puntos base cerrados. Si $\beta\alpha = \mathrm{Id}_A$ y $\alpha\beta \simeq \mathrm{Id}_A$ entonces existe un CW(B)-complejo Y y funciones continuas $\varphi : X \to Y$ y $\psi : Y \to X$ tales que $\psi\varphi = \mathrm{Id}_X$ y $\varphi\psi \simeq \mathrm{Id}_Y$.*

Como corolario tenemos

Corolario 6. *Sea A un espacio contráctil (con punto base cerrado) y sea X un CW(A)-complejo. Entonces X es contráctil.*

Finalizando con las propiedades topológicas de los CW(A)-complejos, analizamos la localización en CW(A)-complejos. El resultado obtenido es el más bonito posible, ya que, en cierta forma, para localizar un CW(A)-complejo uno puede simplemente localizar cada celda.

Teorema 7. *Sea A un CW-complejo simplemente conexo y sea X un CW(A)-complejo abeliano. Sea $\mathcal{P}$ un conjunto de primos. Dada una $\mathcal{P}$-localización $A \to A_{\mathcal{P}}$ existe una $\mathcal{P}$-localización $X \to X_{\mathcal{P}}$ con $X_{\mathcal{P}}$ un CW($A_{\mathcal{P}}$)-complejo. Además, la estructura de CW($A_{\mathcal{P}}$)-complejo de $X_{\mathcal{P}}$ se obtiene localizando las funciones de adjunción de la estructura de CW(A)-complejo de X.*

Luego, comenzamos a desarrollar la teoría de homotopía de CW(A)-complejos, obteniendo muchas generalizaciones de teoremas clásicos (ver secciones 4.1 y 4.2). Uno de los resultados más notables es la generalización del teorema de Whitehead, que ya se sabía válida en el enfoque de Dror Farjoun.

Teorema 8. *Sean X, Y CW(A)-complejos y sea $f : X \to Y$ una función continua. Entonces f es una equivalencia homotópica si y sólo si es una A-equivalencia débil.*

Después estudiamos la teoría de homología de CW(A)-complejos buscando una suerte de *complejo de cadenas celular* que nos permitiera calcular los grupos de homología singular de estos espacios a partir de la homología del núcleo A y de la estructura de CW(A)-complejo del espacio, generalizando la homología celular clásica. Notamos que un hecho bastante significativo en el contexto clásico es que la homología (reducida) de S^0 (con coeficientes en $\mathbb{Z}$) está concentrada en un grado (grado cero) y es libre (como grupo abeliano). Teniendo esto en mente, estudiamos dos casos: cuando la homología reducida de A está concentrada en un cierto grado y cuando los grupos de homología de A son libres.

En el primer caso, dado un CW(A)-complejo X, pudimos construir un *complejo de cadenas A-celular*, muy similar al clásico, cuyos grupos de homología coinciden con los grupos de homología singular de X. Dos propiedades notables de este complejo de cadenas A-celular son que da una manera sencilla de calcular grupos de homología singular de X y que los diferenciales se describen explícitamente en términos de las funciones de adjunción de las celdas, en forma parecida a lo que ocurre en el caso clásico.

En el segundo caso, también construimos un complejo de cadenas que permite el cálculo de los grupos de homología singular de CW(A)-complejos finitos. Desafortunadamente, los diferenciales no están descriptos explícitamente.

Damos también un ejemplo (5.2.8) que muestra que si la homología del núcleo A no está concentrada en un grado ni es libre como grupo abeliano, entonces los grupos de homología de CW(A)-complejos no pueden calcularse mediante un complejo de cadenas A-celular como antes. En este ejemplo tomamos el núcleo a como un cierto espacio cuya homología singular (reducida) es $\mathbb{Z}_4$ en grados 1 y 2 y el grupo trivial en otros grados y construimos un CW(A)-complejo X tal que $H_3(X)$ tiene un elemento de orden 8. Entonces, sus grupos de homología no pueden calcularse mediante un complejo de cadenas A-celular, porque este complejo de cadenas consiste de una suma directa de grupos cíclicos de orden cuatro en cada grado.

Sin embargo, por medio de sucesiones espectrales pudimos estudiar también el caso general y obtuvimos en siguiente resultado.

Teorema 9. *Sea A un CW-complejo de dimensión finita y sea X un CW(A)-complejo. Entonces existe una sucesión espectral $\{E^a_{p,q}\}$ con $E^1_{p,q} = \bigoplus_{A-p-cells} H_q(A)$ que converge a $H_*(X)$.*

Además, damos una descripción explícita de los diferenciales de la cara 1 de esta sucesión espectral.

Aquí podemos pensar a las sucesiones espectrales como la generalización de los complejos de cadena adecuada para CW(A)-complejos. Es interesante remarcar que en el caso en que la homología de A está concentrada en un cierto grado, la sucesión espectral de arriba tiene sólo una fila no nula, dando lugar al complejo de cadenas A-celular que mencionamos antes.

Dentro de la teoría de homología de CW(A)-complejos, también definimos la A-característica de Euler χ_A de CW(A)-complejos, que resulta ser un invariante homotópico si A es un CW-complejo con $\chi(A) \neq 0$. Es fácil demostrar que, para un CW(A)-complejo finito X, $\chi(X) = \chi_A(X)\chi(A)$. También introducimos la *caracteristica de Euler multiplicativa* χ_m para CW(A)-complejos finitos con grupos de homología finitos, que es una versión multiplicativa de la característica de Euler, y demostramos que si A es un CW-complejo con homología finita y X es un CW(A)-complejo finito, entonces $\chi_m(X) = \chi_m(A)^{\chi_A(X)}$.

Pasando a un enfoque distinto para estudiar homología, definimos una teoría de homología 'con forma A' por $H^A_n(X) = \pi^A_n(SP(X))$ donde $SP(X)$ denota el producto simétrico infinito de X. Un resultado interesante es la siguiente generalización del teorema de Hurewicz

Teorema 10. *Sea A un CW-complejo arcoconexo de dimensión $k \geq 1$ y sea X un espacio topoógico n-conexo (con $n \geq k$). Entonces $H^A_r(X) = 0$ para $r \leq n-k$ y $\pi^A_{n-k+1}(X) \simeq H^A_{n-k+1}(X)$.*

Una de los capítulos más importantes de esta tesis trata del estudio de grupos de homología, homotopía y A-homotopía de CW(A)-complejos a la luz de las clases de Serre y de una generalización clásica del teorema de Hurewicz. Presentamos resultados variados que dan información de los grupos de homotopía de un CW(A)-complejo mostrando que depende fuertemente de los grupos de homotopía y homología de A, como es de esperar. Recordemos que una clase no vacía de grupos abelianos $\mathscr{C}$ se llama *clase de Serre* si para toda sucesión exacta de tres términos $A \to B \to C$, si $A, C \in \mathscr{C}$ entonces $B \in \mathscr{C}$. Una clase de Serre $\mathscr{C}$ se llama *anillo de grupos abelianos* si $A \otimes B$ y $\mathrm{Tor}(A, B)$ pertenecen a $\mathscr{C}$ para todos $A, B \in \mathscr{C}$.

Un espacio topológico X se llama *$\mathscr{C}$-acíclico* si $H_n(X) \in \mathscr{C}$ para todo $n \geq 1$. Si $\mathscr{C}$ es una clase de Serre, decimos que $\mathscr{C}$ es *acíclica* si para todo $G \in \mathscr{C}$, los espacios de Eilenberg - MacLane de tipo $(G, 1)$ son $\mathscr{C}$-acíclicos. Finalmente, un *anillo acíclico de grupos abelianos* es una clase de Serre acíclica que es también un anillo de grupos abelianos.

Ejemplos de anillos acíclicos de grupos abelianos son la clase de grupos abelianos finitos y la clase de grupos abelianos de torsión. Otro ejemplo es la clase $\mathcal{T}_{\mathcal{P}}$ de grupos abelianos de torsión cuyos elementos tienen órdenes divisibles sólo por primos en un conjunto fijo $\mathcal{P}$ de números primos.

Obtuvimos los siguientes resultados.

Proposición 11. *Sea $\mathscr{C}$ una clase de Serre de grupos abelianos y sea A un CW-complejo finito. Sea $k \in \mathbb{N}$ y sea X un espacio topológico tal que $\pi_n(X) \in \mathscr{C}$ para todo $n \geq k$. Entonces $\pi_n^A(X) \in \mathscr{C}$ para todo $n \geq k$.*

Teorema 12. *Sea $\mathscr{C}$ una clase de Serre de grupos abelianos. Sea A un CW-complejo $\mathscr{C}$-acíclico y sea X un CW(A)-complejo generalizado finito. Entonces X es también $\mathscr{C}$-acíclico. Si, además, X es simplemente-conexo y $\mathscr{C}$ es un anillo acíclico de grupos abelianos, entonces $\pi_n(X) \in \mathscr{C}$ para todo $n \in \mathbb{N}$.*

Corolario 13. *Sea $\mathscr{C}$ un anillo acíclico de grupos abelianos. Sea A un CW-complejo finito y sea X un CW(A)-complejo generalizado finito. Supongamos que A es $\mathscr{C}$-acíclico y que X es simplemente conexo. Entonces $\pi_n^A(X) \in \mathscr{C}$ para todo $n \in \mathbb{N}$.*

Proponemos después una pequeña modificación de las clases de Serre y de los anillos de grupos abelianos para eliminar la hipótesis de finitud en los resultados previos e introducimos la noción de *clase de Serre especial* (6.2.5). Aunque este es un concepto más restrictivo, la clase de grupos abelianos de torsión y la clase $\mathcal{T}_{\mathcal{P}}$ son clases de Serre especiales. Éstas dan lugar a aplicaciones interesantes y concretas. Con este nuevo concepto pudimos generalizar los resultados anteriores obteniendo la siguiente proposición.

Proposición 14. *Sea $\mathscr{C}'$ una clase de Serre especial, sea A un CW-complejo $\mathscr{C}'$-acíclico y sea X un CW(A)-complejo generalizado. Entonces:*

(a) X es $\mathscr{C}'$-acíclico.

(b) Si, además, X es simplemente conexo y $\mathscr{C}'$ es un anillo acíclico de grupos abelianos, entonces $\pi_n(X) \in \mathscr{C}'$ para todo $n \in \mathbb{N}$.

(c) Si A es finito, X es simplemente conexo y $\mathscr{C}'$ es un anillo acíclico de grupos abelianos, entonces $\pi_n^A(X) \in \mathscr{C}'$ para todo $n \in \mathbb{N}$.

Otra parte clave de esta tesis está constituida por la clasificación homotópica de CW(A)-complejos y la CW(A)-aproximación, estrechamente relacionadas entre sí. El objetivo de esta última es aproximar un espacio dado X por un CW(A)-complejo Z, donde una 'aproximación' en teoría de homotopía significa una equivalencia débil $f : Z \to X$. Obtuvimos el siguiente resultado:

Proposición 15. *Sea A un espacio de Moore de tipo $(\mathbb{Z}_p, r)$ con p primo, y sea X un espacio topológico simplemente conexo. Entonces existen un CW(A)-complejo Z y una equivalencia débil $f : Z \to X$ si y sólo si $H_i(X) = 0$ para $1 \leq i \leq \max\{r-1, 1\}$ y $H_i(X) = \bigoplus_{J_i} \mathbb{Z}_p$ para todo $i \geq \max\{r, 2\}$.*

Y aplicando el teorema de Whitehead obtenemos un teorema de clasificación homotópica para CW(A)-complejos.

Teorema 16. *Sea A un espacio de Moore de tipo $(\mathbb{Z}_p, r)$ con p primo y sea X un espacio topológico simplemente conexo que tiene el tipo homotópico de un CW-complejo. Entonces X tiene el tipo homotópico de un CW(A)-complejo si y sólo si $H_i(X) = 0$ para $1 \leq i \leq \max\{r-1, 1\}$ y $H_i(X) = \bigoplus_{J_i} \mathbb{Z}_p$ para todo $i \geq \max\{r, 2\}$.*

También damos un teorema de clasificación homotópica para CW(A)-complejos generalizados.

Teorema 17. *Sea $m \in \mathbb{N}$ y sea A un espacio de Moore de tipo $(\mathbb{Z}_m, r)$, con $r \geq 2$. Sea X un CW-complejo $(r-1)$-conexo que satisface las siguientes condiciones*

(a) $H_r(X) = \bigoplus_{j \in J} \mathbb{Z}_{m_j}$ con $m_j | m$ para todo $j \in J$

(b) Para todo $n \geq r+1$, $H_n(X)$ es un grupo abeliano finitamente generado tal que los divisores primos de los órdenes de sus elementos también dividen a m.

Entonces X tiene el tipo homotópico de un CW(A)-complejo generalizado.

Vale la pena mencionar que, por la proposición 14 de antes, si un espacio topológico X tiene el tipo homotópico de un CW(A)-complejo generalizado, donde A es un espacio de Moore de tipo $(\mathbb{Z}_m, r)$, entonces X es $(r-1)$-conexo y para todo $n \geq r$, $H_n(X)$ es un grupo abeliano de torsión tal que los divisores primos de los órdenes de sus elementos también dividen a m. Así, el teorema previo es una recíproca débil de este hecho.

En el último capítulo de esta tesis, comenzamos a desarrollar la teoría de obstrucción para CW(A)-complejos. Observamos que el complejo de cadenas A-celular no era satisfactorio para este propósito. Entonces introdujimos un nuevo complejo de cadenas A-celular adecuado para teoría de obstrucción. Su definición se basa en los *grupos de A-homotopía estable* que se definen por $\pi_n^{A,\mathrm{st}}(X) = \operatorname*{colim}_j \pi_{n+j}^A(\Sigma^j X)$.

Imponemos en A la restricción de ser un CW-complejo l-conexo y compacto de dimensión k con $k \leq 2l$ y $l \geq 1$. Esto es para que la función $\Sigma : [\Sigma^n A, \Sigma^n A] = \pi_n^A(A) \to [\Sigma^{n+1} A, \Sigma^{n+1} A] = \pi_{n+1}^A(A)$ sea biyectiva para $n \geq 0$ y entonces un isomorfismo de grupos para $n \geq 1$. Notemos que la 0-esfera S^0 no cumple la hipótesis de ser por lo menos 1-conexa. Sin embargo, sabemos que en el caso $A = S^0$ también tenemos los isomorfismos anteriores. Entonces, esta teoría de obstrucción también funciona para $A = S^0$, dando lugar a la teoría de obstrucción clásica.

Tomamos $R = \pi_0^{A,\mathrm{st}}(X)$. Entonces R es isomorfo a $\pi_r^A(\Sigma^r A)$ para $r \geq 2$. Le damos a R una estructura de anillo como sigue. La suma $+$ está inducida por la operación usual de grupo abeliano en $\pi_r^A(\Sigma^r A)$ y el producto está inducido por $[f][g] = [g \circ f]$ en $\pi_r^A(\Sigma^r A)$.

Dado un CW(A)-complejo X, el nuevo complejo de cadenas A-celular se define como sigue. C_n es el R-módulo libre generado por las A-n-celdas de X y el morfismo de borde $d : C_n \to C_{n-1}$ se define de la siguiente manera. Sea e^n_α una A-n-celda de X, sea g_α su función de adjunción y sea J_{n-1} un conjunto que indexa las A-$(n-1)$-celdas. Para $\beta \in J_{n-1}$, sea $q_\beta : X^{n-1} \to X^{n-1}/(X^{n-1} - \overset{\circ}{e}{}^{n-1}_\beta) = \Sigma^{n-1}A$ la función cociente. Definimos $d(e^n_\alpha) = \sum\limits_{\beta \in J_{n-1}} [q_\beta g_\alpha] e^{n-1}_\beta$. Dada una función continua $f : X^{n-1} \to Y$, donde X es un CW(A)-complejo, definimos el cociclo de obstrucción $c(f) \in \mathrm{Hom}_R(C_n, \pi^A_{n-1}(Y))$ que satisface que $c(f) = 0$ si y sólo si f se puede extender a X^n. También, dado un CW(A)-complejo X y funciones continuas $f, g : X^n \to Y$ tales que $f|_{X^{n-1}} = g|_{X^{n-1}}$ definimos la *cocadena diferencia de f y g* $d(f,g) \in \mathrm{Hom}_R(C_n, \pi^A_n(Y))$.

Finalmente, demostramos las siguientes generalizaciones de teoremas clásicos de teoría de obstrucción.

Teorema 18. *Sean A, X y f como antes y sea $d \in \mathrm{Hom}_R(C_n, \pi^A_n(Y))$. Entonces existe una función continua $g : X^n \to Y$ tal que $g|_{X^{n-1}} = f|_{X^{n-1}}$ y $d(f,g) = d$.*

Teorema 19. *Sea X un CW(A)-complejo y sea $f : X^n \to Y$ una función continua. Entonces existe una función continua $g : X^{n+1} \to Y$ tal que $g|_{X^{n-1}} = f|_{X^{n-1}}$ si y sólo si $c(f)$ es un coborde.*

Teorema 20. *Sea A la suspensión de un CW-complejo y sea X un CW(A)-complejo. Sean $f, g : X^n \to Y$ funciones continuas. Entonces*

(a) $f \simeq g$ rel X^{n-1} si y sólo si $d(f,g) = 0$.

(b) $f \simeq g$ rel X^{n-2} si y sólo si $\overline{d(f,g)} = 0$ en $H^n(C^*, \delta)$.

Introduction

CW-complexes are spaces which are built up out of simple building blocks or *cells*. Balls are used as models for the cells and these are attached step by step using *attaching maps*, which are defined in the boundary spheres of the balls. Since their introduction in the late fourties by J.H.C. Whitehead [22], CW-complexes have played an essential role in geometry and topology. One of the reasons of this vital importance is the CW-approximation theorem 1.4.18, which implies that for the sake of homotopy, homology and cohomology groups, every space is equivalent to a CW-complex. Moreover, the combinatorial structure of these spaces allows the development of tools which considerably simplify the computation of homology and cohomology groups (cf. p. 41) and also the computation of homotopy groups (1.4.21). The homotopy theory of CW-complexes is pleasantly rich in results and its homotopy category serves as a model for other homotopy categories.

The main properties of CW-complexes arise from the following two basic facts: (1) The n-ball D^n is the topological (reduced) cone of the $(n-1)$-sphere S^{n-1} and (2) The n-sphere is the (reduced) n-th suspension of the 0-sphere S^0. For example, the homotopy extension properties of CW-complexes follow from (1), since the inclusion of the $(n-1)$-sphere in the n-disk is a closed cofibration. Item (2) is closely related to the definition of classical homotopy groups of spaces and it is used to prove results such as Whitehead's theorem or homotopy excision and in the construction of Eilenberg-MacLane spaces. These two basic facts suggest that one might replace the original *core* S^0 by any other space A and construct spaces from cells of different *shapes* or *types* using suspensions and cones of the base space A.

The main purpose of this dissertation is to introduce and develop the theory of such spaces. We define the notion of CW-complexes of type A (or CW(A)-spaces for short) generalizing the definition of CW-complexes (which constitute the particular and special case of CW(A)-complexes obtained by taking $A = S^0$).

We ought to mention that there exist many generalizations of CW-complexes in the literature. For instance, Baues' generalization of complexes in cofibration categories [2] and Minian's categorical approach to cell complexes [12]. The theory of CW(A)-complexes that we develop in this thesis is also related to works of E. Dror Farjoun [5] and W. Chachólski [4]. However, our approach is quite different from these and keeps the geometric and combinatorial intuition of Whitehead's original theory. Moreover, it gives us a deeper insight in the classical theory of CW-complexes, as we shall see.

As in the classical case, we give a constructive and a descriptive definition and compare them obtaining the following results

Proposition 1. *Let A be a T1 space. If X is a constructive CW(A)-complex, then it is a descriptive CW(A)-complex.*

Proposition 2. *Let A be a compact space and let X be a descriptive CW(A)-complex. If X is Hausdorff then it is a constructive CW(A)-complex.*

Furthermore, we give counterexamples if the hypotheses are not satisfied.

In this context, we also analyse classical constructions such as cones, suspensions, cylinders and smash products and determine whether those functors applied to CW(A)-complexes give CW(A)-complexes as result. Quite surprisingly, some of these results are not true for every core A and a couple of hypotheses are needed. For instance, if the core A is the suspension of a locally compact and Hausdorff space, then the reduced cylinder of a CW(A)-complex is also CW(A)-complex, but this does not hold for arbitrary cores A.

While developing this theory, we naturally encounter spaces which were constructed in a similar way as CW-complexes, but in which cells were not attached in a dimension-increasing order. It is known that spaces of this kind may not be CW-complexes altough they have the homotopy type of a CW-complex. We called them *generalized CW-complexes* and promptly define the notion of *generalized CW(A)-complexes*. The following results were obtained.

Proposition 3. *If A is a CW-complex and X is a generalized CW(A)-complex then X has the homotopy type of a CW-complex.*

Theorem 4. *Let A be a generalized CW(B)-complex with B compact, and let X be a generalized CW(A)-complex. If A and B are T1 then X is a generalized CW(B)-complex.*

Furthermore, we give an example of a generalized CW(A)-complex which does not have the homotopy type of a CW(A)-complex (see 5.2.9).

Another question that we studied is the following. Suppose X is a CW(A)-complex, or in other words, X can be built with *blocks* of type A. And suppose in addition that A is a CW(B)-complex. It seems natural to ask whether X can be built with blocks of type B, that is whether X is a CW(B)-complex. In this direction we obtained the following result.

Theorem 5. *Let A and B be pointed topological spaces. Let X be a CW(A)-complex, and let $\alpha : A \to B$ and $\beta : B \to A$ be continuous maps.*

i. If $\beta\alpha = \mathrm{Id}_A$, then there exists a CW(B)-complex Y and maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi\varphi = \mathrm{Id}_X$.

ii. Suppose A and B have closed base points. If β is a homotopy equivalence, then there exists a CW(B)-complex Y and a homotopy equivalence $\varphi : X \to Y$.

iii. Suppose A and B have closed base points. If $\beta\alpha = \mathrm{Id}_A$ and $\alpha\beta \simeq \mathrm{Id}_A$ then there exists a CW(B)-complex Y and maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi\varphi = \mathrm{Id}_X$ and $\varphi\psi \simeq \mathrm{Id}_Y$.

As a corollary we have

Corollary 6. *Let A be a contractible space (with closed base point) and let X be a CW(A)-complex. Then X is contractible.*

Finishing with the topological properties of CW(A)-complexes, we analysed localization in CW(A)-complexes. The result obtained is the nicest possible since, to a certain extent, to localize a CW(A)-complex one may simply localize each cell.

Theorem 7. *Let A be a simply-connected CW-complex and let X be an abelian CW(A)-complex. Let $\mathcal{P}$ be a set of prime numbers. Given a $\mathcal{P}$-localization $A \to A_{\mathcal{P}}$ there exists a $\mathcal{P}$-localization $X \to X_{\mathcal{P}}$ with $X_{\mathcal{P}}$ a CW($A_{\mathcal{P}}$)-complex. Moreover, the CW($A_{\mathcal{P}}$)-complex structure of $X_{\mathcal{P}}$ is obtained by localizing the adjunction maps of the CW(A)-complex structure of X.*

Afterwards, we started developing the homotopy theory of CW(A)-complexes, obtaining many generalizations of classical theorems (see sections 4.1 and 4.2). One of the most remarkable results is the generalization of Whitehead's theorem, which was already known to be valid in Dror Farjoun's approach.

Theorem 8. *Let X and Y be CW(A)-complexes and let $f : X \to Y$ be a continuous map. Then f is a homotopy equivalence if and only if it is an A-weak equivalence.*

Then, we studied homology theory of CW(A)-complexes looking for a kind of *cellular chain complex* which would allow us to compute the singular homology groups of these spaces out of the homology of the core A and the CW(A)-structure of the space, generalizing classical cellular homology. We noted that a quite significant fact in the classical setting was that the (reduced) homology of S^0 (with coefficients in $\mathbb{Z}$) is concentrated in one degree (degree zero) and is free (as an abelian group). Keeping this in mind, we studied two cases: when the reduced homology of A is concentrated in a certain degree and when the homology groups of A are free.

In the first case, given a CW(A)-complex X, we were able to construct an *A-cellular chain complex*, very similar to the classical one, whose homology groups coincide with the singular homology groups of X. Two remarkable properties of this A-cellular chain complex are that it gives an easy way to compute singular homology groups of X and that the differentials are described explicitly in terms of attaching map of cells, much as it occurs in the classical case.

In the second case, we also constructed a chain complex which permits computation of singular homology groups of finite CW(A)-complexes. Unfortunately, the differentials are not explicitly described.

We also give an example (5.2.8) which shows that if the homology of the core A is neither concentrated in one degree nor free as an abelian group, then the homology groups of CW(A)-complexes cannot be computed by an A-cellular chain complex as above. In this example, we take the core A to be a certain space whose (reduced) singular homology is $\mathbb{Z}_4$ in degrees 1 and 2 and the trivial group otherwise and we construct a CW(A)-complex X such that $H_3(X)$ has an element of order 8. Thus, its homology groups cannot be computed by an A-cellular chain complex, since this chain complex consists of a direct sum of cyclic groups of order four in each degree.

However, by means of spectral sequences, we could also study the general case and obtain the following result.

Theorem 9. *Let A be a finite dimensional CW-complex and let X be a CW(A)-complex. Then there exists a spectral sequence $\{E^a_{p,q}\}$ with $E^1_{p,q} = \bigoplus_{A-p-cells} H_q(A)$ which converges to $H_*(X)$.*

Moreover, we give a explicit description of the differentials of the first page of this spectral sequence.

Here, we may think of spectral sequences as the generalization of chain complexes suitable for CW(A)-complexes. It is interesting to remark that in case the homology of A is concentrated in a certain degree, the spectral sequence above has only one nontrivial row, giving rise to the A-cellular chain complex that we mentioned before.

Regarding homology theory of CW(A)-complexes, we also define the A-Euler characteristic χ_A of CW(A)-complexes, which turns out to be a homotopy invariant if A is a CW-complex with $\chi(A) \neq 0$. It is easy to prove that, for a finite CW(A)-complex X, $\chi(X) = \chi_A(X)\chi(A)$. We also introduce the *multiplicative Euler characteristic* χ_m for finite CW(A)-complexes with finite homology groups, which is a multiplicative version of the Euler characteristic, and we prove that if A is a CW-complex with finite homology and X is a finite CW(A)-complex, then $\chi_m(X) = \chi_m(A)^{\chi_A(X)}$.

Turning to a different approach towards homology, we define an 'A-shaped' homology theory by $H^A_n(X) = \pi^A_n(SP(X))$ where $SP(X)$ denotes the infinite symmetric product of X. An interesting result is the following generalization of Hurewicz's theorem

Theorem 10. *Let A be a path-connected CW-complex of dimension $k \geq 1$ and let X be an n-connected topological space (with $n \geq k$). Then $H^A_r(X) = 0$ for $r \leq n - k$ and $\pi^A_{n-k+1}(X) \simeq H^A_{n-k+1}(X)$.*

One of the most important chapters of the thesis deals with the study of homology, homotopy and A-homotopy groups of CW(A)-complexes in the light of Serre classes and a classical generalization of Hurewicz's theorem. We present a variety of results which give information about the homotopy groups of a CW(A)-complex showing that it depends strongly on the homology and homotopy groups of A, as one would expect. Recall that a nonempty class of abelian groups $\mathscr{C}$ is called a *Serre class* if for any three term exact sequence $A \to B \to C$, if $A, C \in \mathscr{C}$ then $B \in \mathscr{C}$. A Serre class $\mathscr{C}$ is called an *ring of abelian groups* if $A \otimes B$ and $\mathrm{Tor}(A, B)$ belong to $\mathscr{C}$ whenever $A, B \in \mathscr{C}$.

A topological space X is called *$\mathscr{C}$-acyclic* if $H_n(X) \in \mathscr{C}$ for all $n \geq 1$. If $\mathscr{C}$ is a Serre class, we say that $\mathscr{C}$ is *acyclic* if for all $G \in \mathscr{C}$, Eilenberg - MacLane spaces of type $(G, 1)$ are $\mathscr{C}$-acyclic. Finally, an *acyclic ring of abelian groups* is an acyclic Serre class which is also a ring of abelian groups.

Examples of acyclic rings of abelian groups are the class of finite abelian groups and the class of torsion abelian groups. Another example is the class $\mathcal{T}_\mathcal{P}$ of torsion abelian groups whose elements have order divisible only by primes in a fixed set $\mathcal{P}$ of prime numbers.

We obtained the following results.

Proposition 11. *Let $\mathscr{C}$ be a Serre class of abelian groups and let A be a finite CW-complex. Let $k \in \mathbb{N}$ and let X be a topological space such that $\pi_n(X) \in \mathscr{C}$ for all $n \geq k$. Then $\pi^A_n(X) \in \mathscr{C}$ for all $n \geq k$.*

Theorem 12. *Let $\mathscr{C}$ be a Serre class of abelian groups. Let A be a $\mathscr{C}$-acyclic CW-complex and let X be a finite generalized CW(A)-complex. Then X is also $\mathscr{C}$-acyclic. If, in addition, X is simply-connected and $\mathscr{C}$ is an acyclic ring of abelian groups, then $\pi_n(X) \in \mathscr{C}$ for all $n \in \mathbb{N}$.*

Corollary 13. *Let $\mathscr{C}$ be an acyclic ring of abelian groups. Let A be a finite CW-complex and let X be a finite generalized CW(A)-complex. Suppose that A is $\mathscr{C}$-acyclic and that X is simply connected. Then $\pi_n^A(X) \in \mathscr{C}$ for all $n \in \mathbb{N}$.*

We then propose a slight modification of Serre classes and rings of abelian groups to get rid of the finiteness hypothesis in the previous results and introduce the notion of *special Serre class* (6.2.5). Although this is a more restrictive concept, the class of torsion abelian groups and the class $\mathcal{T}_{\mathcal{P}}$ are special Serre classes. These yield interesting and concrete applications. With this new concept we were able to generalize the above results obtaining the following proposition.

Proposition 14. *Let $\mathscr{C}'$ be a special Serre class, let A be a $\mathscr{C}'$-acyclic CW-complex and let X be a generalized CW(A)-complex. Then:*

(a) X is $\mathscr{C}'$-acyclic.

(b) If, in addition, X is simply connected and $\mathscr{C}'$ is an acyclic ring of abelian groups, then $\pi_n(X) \in \mathscr{C}'$ for all $n \in \mathbb{N}$.

(c) If A is finite, X is simply connected and $\mathscr{C}'$ is an acyclic ring of abelian groups, then $\pi_n^A(X) \in \mathscr{C}'$ for all $n \in \mathbb{N}$.

Another key part of this thesis is constituted by the homotopy classification of CW(A)-complexes and the CW(A)-approximation, closely related to each other. The aim of the last one is to approximate a given space X by a CW(A)-complex Z, where an 'approximation' in homotopy theory means a weak equivalence $f : Z \to X$. We obtained the following nice result:

Proposition 15. *Let A be a Moore space of type $(\mathbb{Z}_p, r)$ with p prime, and let X be a simply-connected topological space. Then there exists a CW(A)-complex Z and a weak equivalence $f : Z \to X$ if and only if $H_i(X) = 0$ for $1 \leq i \leq \max\{r-1, 1\}$ and $H_i(X) = \bigoplus_{J_i} \mathbb{Z}_p$ for all $i \geq \max\{r, 2\}$.*

And applying Whitehead's theorem we obtain a homotopy classification theorem for CW(A)-complexes.

Theorem 16. *Let A be a Moore space of type $(\mathbb{Z}_p, r)$ with p prime, and let X be a simply-connected topological space having the homotopy type of a CW-complex. Then X has the homotopy type of a CW(A)-complex if and only if $H_i(X) = 0$ for $1 \leq i \leq \max\{r-1, 1\}$ and $H_i(X) = \bigoplus_{J_i} \mathbb{Z}_p$ for all $i \geq \max\{r, 2\}$.*

We also give a homotopy classification theorem for generalized CW(A)-complexes.

Theorem 17. *Let $m \in \mathbb{N}$ and let A be a Moore space of type $(\mathbb{Z}_m, r)$, with $r \geq 2$. Let X be an $(r-1)$-connected CW-complex satisfying the following conditions*

(a) $H_r(X) = \bigoplus_{j \in J} \mathbb{Z}_{m_j}$ *with $m_j | m$ for all $j \in J$*

(b) For all $n \geq r+1$, $H_n(X)$ is a finite abelian group such that the prime divisors of the orders of its elements also divide m.

Then X has the homotopy type of a generalized CW(A)-complex.

It is worth mentioning that, by proposition 14 above, if a topological space X has the homotopy type of a generalized CW(A)-complex, where A is a Moore space of type $(\mathbb{Z}_m, r)$, then X is $(r-1)$-connected and for all $n \geq r$, $H_n(X)$ is a torsion abelian group such that the prime divisors of the orders of its elements also divide m. Thus, the previous theorem is a weak converse to this statement.

In the last chapter of this thesis, we started developing the obstruction theory for CW(A)-complexes. We found out that the A-cellular chain complex was not satisfactory for this purpose. Thus we introduced a new A-cellular chain complex suitable for obstruction theory. Its definition relies on the *stable A-homotopy groups* which are defined by $\pi_n^{A,\mathrm{st}}(X) = \operatorname{colim}_j \pi_{n+j}^A(\Sigma^j X)$.

We impose on A the restriction to be an l-connected and compact CW-complex of dimension k with $k \leq 2l$ and $l \geq 1$. This is for the map $\Sigma : [\Sigma^n A, \Sigma^n A] = \pi_n^A(A) \to [\Sigma^{n+1} A, \Sigma^{n+1} A] = \pi_{n+1}^A(A)$ to be a bijection for $n \geq 0$ and hence an isomorphism of groups for $n \geq 1$. Note that the 0-sphere S^0 does not satisfy the hypothesis of being at least 1-connected. However, we know that in case $A = S^0$ we also have the previous isomorphisms. Thus, this obstruction theory also works for $A = S^0$, yielding classical obstruction theory.

We take $R = \pi_0^{A,\mathrm{st}}(X)$. Then R is isomorphic to $\pi_r^A(\Sigma^r A)$ for $r \geq 2$. We give R a ring structure as follows. The sum $+$ is induced by the usual abelian group operation in $\pi_r^A(\Sigma^r A)$ and the product is induced by $[f][g] = [g \circ f]$ in $\pi_r^A(\Sigma^r A)$.

Given a CW(A)-complex X, the new A-cellular chain complex is defined as follows. C_n is the free R-module generated by the A-n-cells of X and the boundary map $d : C_n \to C_{n-1}$ is defined in the following way. Let e_α^n be an A-n-cell of X, let g_α be its attaching map and let J_{n-1} be an index set for the A-$(n-1)$-cells. For $\beta \in J_{n-1}$, let $q_\beta : X^{n-1} \to X^{n-1}/(X^{n-1} - \overset{\circ}{e}{}_\beta^{n-1}) = \Sigma^{n-1} A$ be the quotient map. We define $d(e_\alpha^n) = \sum_{\beta \in J_{n-1}} [q_\beta g_\alpha] e_\beta^{n-1}$. Given a continuous map $f : X^{n-1} \to Y$, where X is a CW(A)-complex, we define the obstruction cocycle $c(f) \in \mathrm{Hom}_R(C_n, \pi_{n-1}^A(Y))$ satisfying that $c(f) = 0$ if and only if f can be extended to X^n. Also, given a CW(A)-complex X and continuous maps $f, g : X^n \to Y$ such that $f|_{X^{n-1}} = g|_{X^{n-1}}$ we define *difference cochain of f and g* $d(f,g) \in \mathrm{Hom}_R(C_n, \pi_n^A(Y))$.

Finally, we prove the following generalizations of classical obstruction theory theorems

Theorem 18. *Let A, X and f be as above and let $d \in \mathrm{Hom}_R(C_n, \pi_n^A(Y))$. Then there exists a continuous map $g : X^n \to Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$ and $d(f,g) = d$.*

Theorem 19. *Let X be a CW(A)-complex and let $f : X^n \to Y$ be a continuous map. Then there exists a continuous map $g : X^{n+1} \to Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$ if and only if $c(f)$ is a coboundary.*

Theorem 20. *Let A be the suspension of a CW-complex and let X be a CW(A)-complex. Let $f, g : X^n \to Y$ be continuous maps. Then*

(a) $f \simeq g$ rel X^{n-1} if and only if $d(f,g) = 0$.

(b) $f \simeq g$ rel X^{n-2} if and only if $\overline{d(f,g)} = 0$ in $H^n(C^*, \delta)$.

Chapter 1

CW-complexes

CW-complexes are spaces which are built in sequential process of attaching *cells*. They were introduced by J.H.C. Whitehead [22] in the late fourties to meet the needs of homotopy theory. His idea was to work with a class of spaces which was broader than simplicial complexes, and in consequence, more flexible, but which still retained a combinatorial nature, so that computational considerations were not ignored.

In CW-complexes, cells are homeomorphic to disks, thus to simplices, and are attached by their boundaries, in much the same way as simplicial complexes. The key point is that in CW-complexes attaching maps are just continuous, which differs significantly from the much more rigid structure of simplicial complexes.

For example, smooth finite-dimensional manifolds are CW-complexes. Also, every topological space can be approximated in a homotopical sense by a CW-complex. Moreover, the homotopy category of CW-complexes is equivalent to the homotopy category of topological spaces. However, the combinatorial structure of these spaces allows the development of tools which simplify considerably computation of homology, cohomology and homotopy groups.

In this chapter we will give an introduction to CW-complexes and their homotopy theory. It is by no means exhaustive, though it includes a wide range of topics. Our aim is that it serves as a basis for the rest of this thesis. The interested reader might also want to consult [3, 7, 8, 20, 21]. Standard notation and terminology can be found in [20].

1.1 Adjunction spaces

In this section we recall some topological and homotopical properties of adjunction spaces for later application to CW-complexes and to our work. The main reference for this section is [3].

We begin with the definition of adjunction spaces.

Definition 1.1.1. Let X and B be topological spaces and let $A \subseteq B$ be a closed subspace. Let $f : A \to X$ be a continuous map. The *adjunction space* $X \underset{f}{\cup} B$ is defined by the pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "push"] & X \arrow[d, "\mathrm{in}_1"] \\ B \arrow[r, "\mathrm{in}_2"'] & X \underset{f}{\cup} B \end{tikzcd}$$

This is to say that $X \underset{f}{\cup} B$ is obtained from the disjoint union $X \sqcup B$ by identifying each point $a \in A$ with its image $f(a) \in X$.

Remark 1.1.2. Let $X \underset{f}{\cup} B$ be as above and let $q : X \sqcup B \to X \underset{f}{\cup} B$ be the quotient map. From the quotient topology, we know that a subset $U \subset X \underset{f}{\cup} B$ is open (resp. closed) in $X \underset{f}{\cup} B$ if and only if $q^{-1}(U)$ is open (resp. closed) in $X \sqcup B$. And this last statement holds if and only if $q^{-1}(U) \cap X$ is open (resp. closed) in X and $q^{-1}(U) \cap B$ is open (resp. closed) in B, or equivalently if and only if $(\mathrm{in}_1)^{-1}(U)$ is open (resp. closed) in X and $(\mathrm{in}_2)^{-1}(U)$ is open (resp. closed) in B.

Examples 1.1.3.

(a) Let A and X be topological spaces and let $f : A \to X$ be a continuous map. The cylinder of f, Z_f, is an adjunction space:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] \arrow[dr, phantom, "push"] & X \arrow[d] \\ A \times I \arrow[r] & Z_f \end{tikzcd}$$

(b) As in the previous example, if $f : A \to X$ is a continuous map then the cone of f, C_f, is an adjunction space:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "push"] & X \arrow[d] \\ \mathrm{C}A \arrow[r] & C_f \end{tikzcd}$$

(c) As a particular case of the previous example we have the following. If $A = S^{n-1}$ ($n \in \mathbb{N}$) and $g : S^{n-1} \to X$ is a continuous map then the space C_g is called X *with an n-cell attached* and denoted by $X \cup e^n$:

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "push"] & X \arrow[d] \\ D^n \arrow[r] & X \cup e^n \end{tikzcd}$$

Usually, the space X will be a Hausdorff space. This example will be of utter importance in next section.

Proposition 1.1.4. *Let $X \underset{f}{\cup} B$ be the adjunction space defined above. Then* $\mathrm{in}_1 : X \to X \underset{f}{\cup} B$ *is a closed subspace and* $\mathrm{in}_2|_{B-A} : B - A \to X \underset{f}{\cup} B$ *is an open subspace.*

Proof. For the first statement, we have to prove that in_1 is injective, initial and closed. Since in_1 is continuous and injective, it suffices to prove that in_1 is closed. Let $F \subseteq X$ be a closed subspace. We have that $(\mathrm{in}_1)^{-1}(\mathrm{in}_1(F)) = F$ which is closed in X and $(\mathrm{in}_2)^{-1}(\mathrm{in}_1(F)) = f^{-1}(F)$ which is closed in B. Hence, $\mathrm{in}_1(F)$ is closed in $X \underset{f}{\cup} B$.

In a similar way, for the second statement it suffices to prove that $\mathrm{in}_2|_{B-A}$ is an open map. Let $U \subseteq B - A$ be an open subspace. Since $B - A$ is open in B, U is also open in B. Then $(\mathrm{in}_1)^{-1}(\mathrm{in}_2|_{B-A}(U)) = \varnothing$ and $(\mathrm{in}_2)^{-1}(\mathrm{in}_2|_{B-A}(U)) = U$. Hence, $\mathrm{in}_2|_{B-A}(U)$ is open in $X \underset{f}{\cup} B$. □

The following proposition establishes conditions which assure that the adjunction space will be a Hausdorff space.

Proposition 1.1.5. *Let X and B be Hausdorff topological spaces and let $A \subseteq B$ be a closed subspace. Let $f : A \to X$ be a continuous map. Suppose that the following conditions hold:*

(a) For each $b \in B - A$ there exists a closed neighbourhood C_b of b in B such that $C_b \cap A = \varnothing$.

(b) There exists an open subset $U \subseteq B$ and a retraction $r : U \to A$.

Then the adjunction space $X \underset{f}{\cup} B$ is Hausdorff.

Proof. Let $\mathrm{in}_1 : X \to X \underset{f}{\cup} B$ and $\mathrm{in}_2 : B \to X \underset{f}{\cup} B$ be as in the definition of adjuntion spaces and let $x_1, x_2 \in X \underset{f}{\cup} B$. We must find disjoint open subsets $V_1, V_2 \subseteq X \underset{f}{\cup} B$ such that $x_1 \in V_1$ and $x_2 \in V_2$. We divide the proof in three cases.

(1) $x_1, x_2 \in B - A$. Since $B - A$ is Hausdorff there exist open and disjoint subsets $V_1, V_2 \subseteq B - A$ such that $x_1 \in V_1$ and $x_2 \in V_2$. But $B - A$ is open in $X \underset{f}{\cup} B$ by the previous proposition, hence V_1 and V_2 are also open in $X \underset{f}{\cup} B$.

(2) $x_1 \in X$ and $x_2 \in B - A$. We take $V_1 = X \underset{f}{\cup} B - C_{x_2}$ and $V_2 = (C_{x_2})^\circ$. Note that $(\mathrm{in}_1)^{-1}(V_1) = X$, $(\mathrm{in}_2)^{-1}(V_1) = B - C_{x_2}$, $(\mathrm{in}_1)^{-1}(V_2) = \varnothing$ and $(\mathrm{in}_2)^{-1}(V_2) = (C_{x_2})^\circ$. Hence V_1 and V_2 are open in $X \underset{f}{\cup} B$.

(3) $x_1, x_2 \in X$. Since X is Hausdorff there exist open and disjoint subsets $W_1, W_2 \subseteq X$ such that $x_1 \in W_1$ and $x_2 \in W_2$. But W_1 and W_2 might not be open in $X \underset{f}{\cup} B$. Using the retraction r we will enlarge the subsets W_1 and W_2 so that they are open in $X \underset{f}{\cup} B$ and remain disjoint. We take $V_1 = W_1 \cup r^{-1}f^{-1}(W_1)$ and $V_2 = W_2 \cup r^{-1}f^{-1}(W_2)$. Note that $V_1 \cap V_2 = \varnothing$ and that V_1 and V_2 are open in $X \underset{f}{\cup} B$ since $(\mathrm{in}_1)^{-1}(V_i) = W_i$, $(\mathrm{in}_2)^{-1}(V_i) = r^{-1}f^{-1}(W_i)$ for $i = 1, 2$. □

Important remark 1.1.6. If we take $A = S^{n-1}$ and $B = D^n$ then conditions (a) and (b) of the previous proposition hold. The same happens if we take $A = \bigsqcup_{i \in I} S^{n-1}$ and $B = \bigsqcup_{i \in I} D^n$.

We want now to find conditions for two adjunction spaces to be homotopy equivalent. To this end, we will need to work with cofibrations.

Definition 1.1.7. Let $i : A \to X$ be a continuous map. We say that i is a *cofibration* if given a continuous map $f : X \to Z$ and a homotopy $H : IA \to Z$ such that $Hi_0 = fi$ there exists a homotopy $\overline{H} : IX \to Z$ such that $\overline{H}i_0 = f$ and $\overline{H}Ii = H$.

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & IA \arrow[d, "Ii"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "f"'] & IX \arrow[dr, dotted, "\overline{H}"] & \\
 & & Z
\end{tikzcd}$$

This property is called the *homotopy extension property.*

Examples 1.1.8. Let X be a topological space. Then:

(a) The inclusions $i_0, i_1 : X \to IX$ are cofibrations.

(b) The inclusion $i : X \times \{0, 1\} \to IX$ is a cofibration.

(c) The inclusion $i : X \to \mathrm{C}X$ is a cofibration.

(d) If $f : X \to Y$ is a continuous map, the inclusion $i : X \to Z_f$ is a cofibration.

Proposition 1.1.9. *Let $i : A \to X$ be a continuous map. Then i is a cofibration if and only if there exists a retraction $r : X \times I \to Z_i$.*

Proof. Suppose first that i is a cofibration. Then there exists a map r in the diagram

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times \mathrm{Id}_I"] \arrow[ddr, bend left, "i_1"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "i_2"'] & X \times I \arrow[dr, dotted, "r"] & \\
 & & Z_i
\end{tikzcd}$$

The map r is the desired retraction.

Conversely, suppose that there exists a retraction $r : X \times I \to Z_i$ and continuous maps $f : X \to Z$ and $H : IA \to Z$ such that $Hi_0 = fi$. Let F be the dotted arrow in the diagram

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & A \times I \arrow[d, "i\times \mathrm{Id}_I"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "f"'] & Z_i \arrow[dr, dotted, "F"] & \\
& & Z
\end{tikzcd}$$

and let $\overline{H} = Fr$. The map $\overline{H}$ is the desired homotopy extension. $\square$

The following proposition shows that it is not a coincidence that all the previous examples of cofibrations are inclusion maps.

Proposition 1.1.10. *Let $i : A \to X$ be a cofibration. Then i is a subspace map.*

Proof. Let $h : A \times I \to A \times I$ be defined by $h(a,t) = (a, 1-t)$ and let $\mathrm{inc} : A \times I \to Z_i$ and $j : X \to Z_i$ be the corresponding inclusion maps. We define $H : A \times I \to Z_i$ by $H = \mathrm{inc} \circ h$. Since i is a cofibration, there exists a continuous map $\overline{H} : X \times I \to Z_i$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times \mathrm{Id}_I"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "j"'] & X \times I \arrow[dr, dotted, "\overline{H}"] & \\
& & Z_i
\end{tikzcd}$$

Then $Hi_0 = \overline{H} \circ (i \times \mathrm{Id}_I) \circ i_0 = \overline{H} i_0 i$. Since H and i_0 are injective, it follows that i is injective. Also, Hi_0 is initial because it is a subspace map, it is initial. But $Hi_0 = \overline{H} i_0 i$ and since $\overline{H} i_0$ and i are continuous maps, it follows that i is initial. Therefore, i is a subspace map. $\square$

Proposition 1.1.11. *Let X be a topological space and let $A \subseteq X$ be a subspace such that the inclusion $i : A \to X$ is a cofibration. Then there exists a retraction $r : X \times I \to X \times \{0\} \cup A \times I$.*

Proof. Since i is a cofibration there exists a map r in the diagram

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i\times \mathrm{Id}_I"] \arrow[ddr, bend left, "\mathrm{inc}"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "\mathrm{inc}"'] & X \times I \arrow[dr, dotted, "r"] & \\
& & X \times \{0\} \cup A \times I
\end{tikzcd}$$

The map r is the desired retraction. $\square$

Proposition 1.1.12. *Let X be a topological space and let $A \subseteq X$ be a subspace such that the inclusion $i : A \to X$ is a cofibration. Then $i : X \times \{0\} \cup A \times I \to X \times I$ is a strong deformation retract.*

Proof. Let r be defined as in the proof of the previous proposition. We want to see that $ir \simeq \mathrm{Id}_{X\times I}$ rel $X \times \{0\} \cup A \times I$. We consider $\mathrm{inc} \circ r : X \times I \to X \times I$ and write it as $(\mathrm{inc} \circ r)(x, t) = (r_1(x, t), r_2(x, t))$.

We define $H : (X \times I) \times I \to X \times I$ by $H(x, s, t) = (r_1(x, st), s(1-t) + tr_2(x, s))$. Then H is continuous and satisfies

- $H(x, s, 0) = (x, s)$
- $H(x, s, 1) = r(x, s)$
- $H(x, 0, t) = (x, 0)$
- $H(a, s, t) = (a, s)$ for $a \in A$.

□

In a similar way we can prove that if $i : A \to X$ is a cofibration, then $i : X\times\{1\}\cup A\times I \to X \times I$ is a strong deformation retract.

It is quite interesting to note that the converse of propositions 1.1.11 and 1.1.12 hold if $i : A \to X$ is a closed cofibration. More precisely, we have the following result

Proposition 1.1.13. *Let $A \subseteq X$ be a closed subspace. Then the following are equivalent:*

(a) The inclusion $i : A \to X$ is a cofibration.

(b) $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

(c) $X \times \{0\} \cup A \times I \subseteq X \times I$ is a strong deformation retract.

Proof. The implication $(a) \Rightarrow (c)$ holds by 1.1.12 while the implication $(c) \Rightarrow (b)$ is trivial. So it only remains to prove $(b) \Rightarrow (a)$.

Suppose that $r : X \times I \to X \times \{0\} \cup A \times I$ is a retraction and that there are continuous maps $f : X \to Z$ and $H : IA \to Z$ such that $Hi_0 = fi$. Since $A \subseteq X$ is a closed subspace then IA, $X \times \{0\}$ and $A \times \{0\}$ are closed in IX. Hence, by the pasting lemma, there is a well-defined and continuous map $F : X \times \{0\} \cup A \times I \to Z$ such that $F(x, 0) = f(x)$ for all $x \in X$ and $F(a, t) = H(a, t)$ for all $a \in A$ and $t \in I$. Then the map $\overline{H} = Fr$ is the desired homotopy extension. □

Remark 1.1.14. Note that if $A \subseteq X$ is a closed subspace then there is a pushout diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{inc}"] \arrow[d, "i_0"'] \arrow[dr, phantom, "push"] & X \arrow[d] \\
IA \arrow[r] & X \times \{0\} \cup IA
\end{tikzcd}$$

since the space $X \times \{0\} \cup IA$ clearly satisfies the universal property of pushouts by the pasting lemma. However, this might not be true if A is not a closed subspace of X and it is easy to find counterexamples.

The following proposition follows from the exponential law

Proposition 1.1.15. *If $i : A \to X$ is a cofibration then $Ii : IA \to IX$ is also a cofibration.*

Now we will give a series of results which under certain conditions will tell us when two adjunction spaces are homotopy equivalent. We begin with the following proposition, which will be used many times throughout this thesis.

Proposition 1.1.16. *Let $i : A \to X$ be a cofibration and let $f, g : A \to Y$ be continuous maps such that f is homotopic to g. Then $X \underset{f}{\cup} Y$ and $X \underset{g}{\cup} Y$ are homotopy equivalent relative to Y.*

Proof. Let $H : A \times I \to Y$ be a homotopy between f and g. Consider the adjunction space

$$\begin{tikzcd} A \times I \arrow[r, "H"] \arrow[d, "i \times \mathrm{Id}_I"'] \arrow[dr, phantom, "push"] & Y \arrow[d] \\ X \times I \arrow[r] & X \times I \underset{H}{\cup} Y \end{tikzcd}$$

Note that $X \times \{0\} \underset{H|_{A \times \{0\}}}{\cup} Y = X \underset{f}{\cup} Y$ and $X \times \{0\} \underset{H|_{A \times \{0\}}}{\cup} Y \subseteq X \times I \underset{H}{\cup} Y$ is a strong deformation retract since $X \times \{0\} \cup A \times I \to X \times I$ is.

Hence, $X \underset{f}{\cup} Y \subseteq X \times I \underset{H}{\cup} Y$ is a strong deformation retract. In a similar way $X \underset{g}{\cup} Y \subseteq X \times I \underset{H}{\cup} Y$ is a strong deformation retract.

Thus, $X \underset{f}{\cup} Y$ and $X \underset{g}{\cup} Y$ are homotopy equivalent relative to Y. $\square$

The following proposition and its proof can be found in [3].

Proposition 1.1.17. *Let $i : A \to X$ be an inclusion. If i is a cofibration and a homotopy equivalence then $i : A \to X$ is a strong deformation retract.*

As a corollary we obtain the following.

Corollary 1.1.18. *Let $f : X \to Y$ be a continuous map. Then f is a homotopy equivalence if and only if X is a strong deformation retract of Z_f.*

Proof. Let $i : X \to Z_f$ be the inclusion and $r : Z_f \to Y$ be the standard strong deformation retraction. We have that $f = ir$. Hence, if f is a homotopy equivalence, then i is also a homotopy equivalence. Since i is also a cofibration, by the previous proposition we conclude that X is a strong deformation retract of Z_f.

Conversely, if X is a strong deformation retract of Z_f, then i is a homotopy equivalence. Hence $f = ri$ is also a homotopy equivalence. $\square$

The previous corollary will be useful because it allows us to replace a given homotopy equivalence by a strong deformation retract.

We give now some results of [3] regarding cofibrations and homotopy equivalences that are needed for our work.

Proposition 1.1.19. *Let X and B be topological spaces and let $A \subseteq B$ be a closed subspace such that the inclusion $i : A \to B$ is a cofibration. Let $f, g : A \to X$ be continuous maps such that $f \simeq g$. Then $X \underset{f}{\cup} B \simeq X \underset{g}{\cup} B$ rel X.*

Theorem 1.1.20. *Consider the commutative diagram*

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dr, "\phi_A"] \arrow[dd, "i"'] & & X \arrow[dr, "\phi_X"] \arrow[dd, "j" near start] & \\
& A' \arrow[rr, "g'" near start] \arrow[dd, "i'" near start] & & X' \arrow[dd, "j'"] \\
B \arrow[rr, "f"' near end] \arrow[dr, "\phi_B"'] & & Y \arrow[dr, dotted, "\phi_Y"] & \\
& B' \arrow[rr, "f'"'] & & Y'
\end{tikzcd}$$

where the front and back faces are pushouts. If i and i' are closed cofibrations and ϕ_A, ϕ_X and ϕ_B are homotopy equivalences, then ϕ_Y is a homotopy equivalence.

As a corollary of the previous theorem we obtain another useful result for our work.

Corollary 1.1.21. *Let*

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & X \arrow[d] \\
B \arrow[r, "f"'] & Y
\end{tikzcd}$$

be a pushout diagram. If i is a closed cofibration and g is a homotopy equivalence, then f is a homotopy equivalence.

We end this section with another result about cofibrations and homotopy equivalences that will be needed later.

Proposition 1.1.22. *Let*

$$\begin{tikzcd}
X_0 \arrow[r, "i_0"] \arrow[d, "f_0"] & X_1 \arrow[r, "i_1"] \arrow[d, "f_1"] & X_2 \arrow[r, "i_2"] \arrow[d, "f_2"] & \cdots \\
Y_0 \arrow[r, "j_0"'] & Y_1 \arrow[r, "j_1"'] & Y_2 \arrow[r, "j_2"'] & \cdots
\end{tikzcd}$$

be a commutative diagram such that for all $n \in \mathbb{N}_0$, the maps i_n and j_n are closed inclusions and cofibrations. Let $X = \operatorname{colim} X_n$ and $Y = \operatorname{colim} Y_n$ and let $f : X \to Y$ be the induced map. If f_n is a homotopy equivalence for all $n \in \mathbb{N}_0$ then f is a homotopy equivalence.

A proof can be found in [7] (proposition A.5.11).

1.2 Definition of CW-complexes

In this section we recall the definition of CW-complexes and some standard examples and basic properties. We analyse both the constructive and descriptive approachs and we prove that they are equivalent. Finally, we give the definition of subcomplexes and relative CW-complexes and we study product cellular structures.

1.2.1 Constructive definition

Definition 1.2.1. We say that a topological space X *is obtained from a topological space B by attaching an n-cell* if X is the adjunction space $B \underset{g}{\cup} D^n$ for some continuous map $g : S^{n-1} \to X$, i.e. if there exists a pushout diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & B \arrow[d] \\ D^n \arrow[r, "f"'] & X \end{tikzcd}$$

The *cell* is the image of f. The *interior of the cell* is $f(D^n - S^{n-1})$ and the *boundary of the cell* is $f(S^{n-1})$. The map g is the *attaching map* of the cell, and f is its *characteristic map.*

For example, S^n can be obtained from the singleton $*$ attaching and n-cell. Also, the disk D^n can be obtained from S^{n-1} attaching an n-cell by the identity map.

Remark 1.2.2.

(a) Attaching a 0-cell means adding a disjoint point.

(b) The interior of an n-cell is homeomorphic to $(D^n)^\circ = D^n - S^{n-1}$.

(c) The space X of the definition above is the adjunction space $X = B \underset{g}{\cup} D^n$. It can also be seen as the mapping cone of the map g.

We can attach many n-cells at the same time by taking various copies of S^{n-1} and D^n.

$$\begin{tikzcd} \coprod\limits_{\alpha\in J} S^{n-1} \arrow[r, "\coprod\limits_{\alpha\in J} g_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & B \arrow[d] \\ \coprod\limits_{\alpha\in J} D^n \arrow[r, "\coprod\limits_{\alpha\in J} f_\alpha"'] & X \end{tikzcd}$$

Definition 1.2.3. Let X be a topological space. A *CW-complex structure on X* is a sequence $\varnothing = X^{-1}, X^0, X^1, \dots, X^n, \dots$ of subspaces of X such that the following three conditions are satisfied.

(a) For all $n \in \mathbb{N}_0$, X^n is obtained from X^{n-1} by attaching n-cells (*b*) $X = \bigcup_{n\in\mathbb{N}} X^n$.

(*c*) The space X has the final topology with respect to the inclusions $X^n \hookrightarrow X$, $n \in \mathbb{N}$.

The space X^n is called the *n-skeleton* of X.
We say that the space X is a *CW-complex* if it admits some CW-complex structure.

Clearly, if X is a CW-complex it will generally admit many different CW-complex structures.

Important remark 1.2.4. Condition (c) says that a map $f : X \to Z$ is continuous if and only if $f|_{X^n} : X^n \to Z$ is continuous for all $n \in \mathbb{N}_0$. Equivalently, $U \subseteq X$ is open in X if and only if $U \cap X^n$ is open in X^n for all $n \in \mathbb{N}_0$.

Examples 1.2.5.

(*a*) The n-sphere S^n is a CW-complex. We will consider two different structures:

1) The m-skeleton of S^n is $*$ for all $0 \leq m < n$ and S^n for $m \geq n$. In this structure we have 1 0-cell and 1 n-cell and the n-skeleton is obtained from the $(n-1)$-skeleton by attaching one n-cell:

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & * \arrow[d] \\ D^n \arrow[r, "f"'] & S^n \end{tikzcd}$$

2) $(S^n)^m = S^m$ for all $m \leq n$. The $(m-1)$-skeleton S^{m-1} is the equator of the m-skeleton S^m for all $m \leq n$ and the last one is obtained from the first one by attaching 2 m-cells which correspond to the northern and southern hemispheres of S^m.

(*b*) The n-disk D^n is a CW-complex. We will consider two different CW-complex structures on D^n, both of which satisfy that $(D^n)^{n-1} = S^{n-1}$ and that the n-cell is attached by the identity map. These two different structures are obtained giving each of the structures of the previous example to the $(n-1)$-skeleton S^{n-1}. Hence one of them has 1 0-cell, 1 $n-1$-cell and 1 n-cell and the other has 2 k-cells for each $0 \leq k \leq n-1$ and one n-cell.

(*c*) Polyhedra are CW-complexes with CW-complex structure induced by the simplicial structure.

(*d*) The torus is a CW-complex with 1 0-cell, 2 1-cells and one 2 cell. The 1-skeleton is a wedge of 2 copies of S^1.

(*e*) The infinite dimensional sphere S^∞ is a CW-complex. Recall that S^∞ is defined as follows. Let $\mathbb{R}^{(\mathbb{N})}$ be the set of sequences of real numbers of finite support. We give $\mathbb{R}^{(\mathbb{N})}$ the final topology with respect to the inclusions

$$\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \subseteq \dots$$

The infinite dimensional sphere is defined as $S^\infty = \{x \in \mathbb{R}^{(\mathbb{N})} : \ \|x\|_2 = 1\}$. We give S^∞ the following CW-complex structure. Its n-skeleton is S^n for all $n \in \mathbb{N}_0$ and it is the equator of the $(n+1)$-skeleton, as before. Hence $S^\infty = \bigcup_{n\in\mathbb{N}} S^n$. The n-skeleton S^n is obtained from the $(n-1)$-skeleton S^{n-1} by attaching two n-cells as the second structure of example (a).

(*f*) The real proyective plane $\mathbb{P}^2$ is a CW-complex with 1 0-cell, 1 1-cell and 1 2-cell. The 1-skeleton of this structure is S^1 and the 2-cell is attached by the map $g : S^1 \subseteq \mathbb{C} \to S^1 \subseteq \mathbb{C}$ defined by $g(z) = z^2$.

(*g*) More generally, the n-dimensional real projective space $\mathbb{P}^n$ is a CW-complex with one m-cell for each $m \leq n$. Moreover, the m-skeleton of this CW-complex structure is $\mathbb{P}^m$ for all $2 \leq m \leq n$.

Definition 1.2.6. Let X be a non-empty CW-complex. The *dimension* of X is defined as $\dim X = \sup\{n \in \mathbb{N}_0 / \ X^{n-1} \neq X^n\}$. The dimension may be $+\infty$.

We ought to mention that the dimension of a CW-complex is well defined, i.e. it does not depend on the CW-complex structure given to it. This can be proved using the invariance of domain theorem.

If X is a CW-complex then, by 1.1.4, we obtain that X^n is a closed subspace of X for all n, and if $\dim X = m$, the interior of m-cells are open in X.

Proposition 1.2.7. *If X is a CW-complex then X is a Hausdorff space.*

Proof. By 1.1.5 and induction we get that the n-skeleton, X^n is a Hausdorff space for all $n \in \mathbb{N}$. So, if X is finite-dimensional we are done.

For the general case, let x and y be distinct points in X. There exists $n \in \mathbb{N}$ such that $x, y \in X^n$. Since X^n is Hausdorff there exist open and disjoint subsets $U_n, V_n \subseteq X^n$ such that $x \in U_n$, $y \in V_n$. However, U_n and V_n might not be open in X. Since we are under the hypotheses of 1.1.5, we may proceed as in its proof to enlarge U_n and V_n to open subsets U_{n+1} and V_{n+1} of X^{n+1} such that $U_{n+1} \cap X^n = U_n$, $V_{n+1} \cap X^n = V_n$ and $U_{n+1} \cap V_{n+1} = \varnothing$. Repeating this process inductively we obtain sequences $(U_j)_{j\geq n}$ and $(V_j)_{j\geq n}$ satisfying

- U_j and V_j are open in X^j
- $U_{j+1} \cap X^j = U_j$ and $V_{j+1} \cap X^j = V_j$
- $U_j \cap V_j = \varnothing$

for all $j \geq n$.

Let $U = \bigcup_{j\geq n} U_j$ and $V = \bigcup_{j\geq n} V_j$. Then $x \in U$, $y \in V$ and $U \cap V = \varnothing$. Since for all $m \geq n$, $U \cap X^m = U_m$ is open in X^m then U is open in X. In the same way V is open in X. $\square$

1.2.2 Descriptive definition

We will give now the descriptive definition of CW-complexes and study some of its properties. In the next subsection we will prove that it is equivalent to the constructive definition given above. This equivalence is useful not only because it gives more insight into the definition and theory of CW-complexes, but also because it provides one with two different ways to work with CW-complexes. The constructive definition is needed to build CW-complexes step by step, while the descriptive one is more suitable for proving that a given space is a CW-complex by just decomposing it into cells and then checking that the conditions are satisfied.

Definition 1.2.8. Let X be a Hausdorff space. A *cell complex* on a space X is a collection $K = \{e^n_\alpha : n \in \mathbb{N}_0, \alpha \in J_n\}$ of subsets of X, called *cells*, which satisfy the properties below. The cell e^n_α is called a *cell of dimension* n or n-*cell* and the set J_n, $n \in \mathbb{N}$ is an index set for the n-cells.

For $n \geq 0$, we define the n-*skeleton of* K as $K^n = \{e^r_\alpha : r \leq n, \alpha \in J_r\}$. We also define $K^{-1} = \varnothing$. Let $|K^n| = \bigcup\limits_{\substack{r\leq n\\ \alpha\in J_r}} e^r_\alpha \subseteq X$.

For each cell e^n_α we define the *boundary of* e^n_α as $\overset{\bullet}{e}{}^n_\alpha = e^n_\alpha \cap |K^{n-1}|$ and the *interior of* e^n_α as $\overset{\circ}{e}{}^n_\alpha = e^n_\alpha - \overset{\bullet}{e}{}^n_\alpha$.

The collection K must satisfy

(*a*) $X = \bigcup\limits_{n,\alpha} e^n_\alpha$

(*b*) $\overset{\circ}{e}{}^n_\alpha \cap \overset{\circ}{e}{}^m_\beta = \varnothing$ if $e^n_\alpha \neq e^m_\beta$

(*c*) For each $n \in \mathbb{N}_0$ and $\alpha \in J_n$ there exists a continuous and surjective map $f^n_\alpha : (D^n, S^{n-1}) \to (e^n_\alpha, \overset{\bullet}{e}{}^n_\alpha)$ such that $f^n_\alpha(\overset{\circ}{D}{}^n) \subseteq \overset{\circ}{e}{}^n_\alpha$ and $f^n_\alpha|_{\overset{\circ}{D}{}^n} : \overset{\circ}{D}{}^n \to \overset{\circ}{e}{}^n_\alpha$ is a homeomorphism.

The map f^n_α is called the *characteristic map* of e^n_α.

Note that, by condition (c), cells are compact subspaces of X and hence closed, since X is Hausdorff.

Now, fix a cell e^n_α and consider the equivalence relation in D^n defined by $x \sim y$ if and only if $f^n_\alpha(x) = f^n_\alpha(y)$. Then f^n_α induces a well defined map $\overline{f^n_\alpha} : D^n/\sim \to e^n_\alpha$ which is continuous and bijective. Since $D^n/\sim$ is compact and e^n_α is Hausdorff it follows that $\overline{f^n_\alpha}$ is a homeomorphism. Thus, the cell e^n_α is homeomorphic to $D^n/\sim$ and $A \subseteq e^n_\alpha$ is closed (resp. open) in e^n_α if and only if $(f^n_\alpha)^{-1}(A)$ is closed (resp. open) in D^n.

Note also that $X = \bigcup\limits_{n,\alpha} \overset{\circ}{e}{}^n_\alpha$ and $\overset{\circ}{e}{}^0_\alpha = e^0_\alpha$.

Definition 1.2.9. Let X be a Hausdorff space and let K be a cell complex on X. We define the *dimension* of K as $\dim K = \sup\{n \in \mathbb{N}_0 /\ J_n \neq \varnothing\}$. The dimension may be $+\infty$.

Definition 1.2.10. Let K be a cell complex on X and let e^n_α, e^m_β be cells of K. We say that e^n_α is an *immediate face* of e^m_β if $\overset{\circ}{e}{}^n_\alpha \cap e^m_\beta \neq \varnothing$. If $e^n_\alpha \neq e^m_\beta$ this implies that $\overset{\circ}{e}{}^n_\alpha \cap \overset{\bullet}{e}{}^m_\beta \neq \varnothing$.

Note that if e^n_α is an immediate face of e^m_β and $e^n_\alpha \neq e^m_\beta$ then $n < m$.

Definition 1.2.11. We will say that the cell e^n_α is a *face* of e^m_β if there exists a finite sequence of cells $e_0 = e^n_\alpha, e_1, \dots, e_r = e^m_\beta$ such that e_j is an immediate face of e_{j+1} for $1 \leq j \leq r-1$.

A cell e^n_α is called *principal* if it is not a face of any other cell.

Note that the faces of a cell e^n_α are exactly those cells which we must attach first in order to be able to attach the cell e^n_α. This intuitive statement will become clearer after introducing the notion of subcomplexes in subsection 1.2.4.

Remark 1.2.12. A cell complex K on X does not give much information on the topology of X. For example, we may take $K = X$, that is every point of X is a 0-cell. This is a cell complex which does not give any data on the topology of X.

This is certainly not among the sort of things one would like to accept. So we will impose two extra conditions on a cell complex to call it a CW-complex.

Definition 1.2.13. Let X be a Hausdorff space. A *CW-complex structure on* X is a cell complex K such that the following conditions are satisfied:

(C) Each cell of K has only a finite number of faces.

(W) The space X has the weak topology induced by the cells of K, that is, $A \subseteq X$ is closed if and only if $A \cap e^n_\alpha$ is closed in e^n_α for all $n \in \mathbb{N}$, $\alpha \in J_n$.

A space X is called a *CW-complex* if it admits some CW-complex structure.

The following propositions follow easily from the definition of cell complex and condition (W).

Proposition 1.2.14. *If X is a CW-complex and e^n_α is a principal cell, then $\overset{\circ}{e}{}^n_\alpha$ is open in X.*

Condition (W) can also be stated in a couple of other ways, which allow us to understand topology of CW-complexes better.

Proposition 1.2.15. *Let X be a CW-complex with CW-structure K. The following are equivalent:*

(a) $A \subseteq X$ *is closed (resp. open).*

(b) $A \cap e^n_\alpha$ *is closed (resp. open) in* e^n_α *for all* n, α.

(c) $(f^n_\alpha)^{-1}(A) \subseteq D^n$ *is closed (resp. open) for all* n, α.

(d) $A \cap |K^n|$ *is closed (resp. open) in* $|K^n|$ *for all* n.

These equivalent statements can be reformulated in terms of continuous maps as next proposition shows.

Proposition 1.2.16. *Let X be a CW-complex with CW-structure K, let Y be a topological space and let $f : X \to Y$ be a map. Then the following are equivalent:*

(a) $f : X \to Y$ *is continuous.*

(b) $f|_{e^n_\alpha} : e^n_\alpha \to Y$ *is continuous for all* n, α.

(c) $f \circ f^n_\alpha : D^n \to Y$ *is continuous for all* n, α.

(d) $f|_{|K^n|}$ *is continuous for all* n.

This proposition will be useful when defining maps with domain a CW-complex. Usually, we will define maps skeleton by skeleton, continuous at each stage, and by the equivalences above we will conclude that they are continuous.

The same argument will be used when defining homotopies from CW-complexes. The following is the analogous of the previous proposition and follows from it applying the exponential law.

Proposition 1.2.17. *Let X be a CW-complex with CW-structure K, let Y be a topological space and let $H : X \times I \to Y$ be a map. Then the following are equivalent:*

(a) $H : X \times I \to Y$ *is continuous.*

(b) $H|_{e^n_\alpha \times I} : e^n_\alpha \times I \to Y$ *is continuous for all* n, α.

(c) $H \circ (f^n_\alpha \times \mathrm{Id}_I) : D^n \times I \to Y$ *is continuous for all* n, α.

(d) $H|_{|K^n| \times I}$ *is continuous for all* n.

The next proposition shows a key point in the theory of CW-complexes, as will be evident later on.

Proposition 1.2.18. *Let X be a CW-complex and let $K \subseteq X$ be a compact subset. Then K intersects only a finite number of interiors of cells.*

In particular, X is compact if and only if it is finite (i.e. has a finite number of cells).

Proof. For each n and α such that $K \cap \overset{\circ}{e}{}^n_\alpha \neq \varnothing$ we choose $x^n_\alpha \in C \cap \overset{\circ}{e}{}^n_\alpha$. Let $T = \{x^n_\alpha : n \in \mathbb{N}_0, \alpha \in J_n\}$. Then $T \subseteq K$. We shall prove that T is finite.

We will show that $T \subseteq K$ is closed (hence compact) and discrete. It suffices to prove that every $T' \subseteq T$ is closed in X.

By (W), $T' \subseteq X$ is closed if and only if $T' \cap e^n_\alpha$ is closed in e^n_α for all n, α. But by (C), each cell has a finite number of faces, hence $T' \cap e^n_\alpha$ is finite. Since X is Hausdorff it follows that $T' \cap e^n_\alpha$ is closed in X. Thus, T' is closed in X for all $T' \subseteq T$. $\square$

The following remark will be very important for our descriptive definition of CW(A)-complexes (cf. 3.2.2) since it gives us the right way to generalize the descriptive definition of CW-complexes.

Important remark 1.2.19. Conditions (C) and (W) are equivalent to (C') and (W) where

(C') Every compact subspace intersects only a finite number of interiors of cells.

Indeed, (C) and (W) imply (C') and (W) by the previous proposition. Conversely, (C') implies (C) because cells are compact subspaces.

1.2.3 Equivalence of the two definitions

We will show now that the two definitions are equivalent. The first implication is stated in the following proposition, which is easy to prove.

Proposition 1.2.20. *Let X be a Hausdorff topological space and let K be a CW-complex structure on X. Then X is a constructive CW-complex (i.e. according to definition 1.2.3) where the skeletons X^n coincide with $|K^n|$ and where the characteristic maps of the cells are the same in the two structures.*

For the converse we need the following lemma.

Lemma 1.2.21. *Let X be a descriptive CW-complex of dimension $n-1$ with cellular structure $K = \{e^r_\alpha\}_{r,\alpha}$. Suppose that Y is obtained from X attaching n-cells $\{e^n_\alpha\}_{\alpha\in J_n}$. Then $K' = K \cup \{e^n_\alpha\}_{\alpha\in J_n}$ is a CW-complex structure for Y.*

Proof. It is clear that $Y = \bigcup\limits_{r\leq n} e^r_\alpha$. From the pushout

$$\begin{tikzcd}
\bigsqcup\limits_{\alpha\in J_n} S^{n-1} \arrow[r, "\bigsqcup_{\alpha\in J_n} g_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & X \arrow[d] \\
\bigsqcup\limits_{\alpha\in J_n} D^n \arrow[r, "\bigsqcup_{\alpha\in J_n} f_\alpha"'] & Y
\end{tikzcd}$$

we deduce that $\mathring{e}^n_\alpha \cap \mathring{e}^m_\beta = \varnothing$ if $e^n_\alpha \neq e^m_\beta$.

Note that $\overset{\bullet}{e}{}^n_\alpha = g^n_\alpha(S^{n-1})$. For each $\alpha \in J_n$ we define the characteristic map of the cell e^n_α as $f^n_\alpha : (D^n, S^{n-1}) \to (e^n_\alpha, \overset{\bullet}{e}{}^n_\alpha)$. From the previous pushout it is easily deduced that f^n_α is surjective, $f^n_\alpha(\mathring{D}^n) \subseteq \mathring{e}^n_\alpha$ and $f^n_\alpha|_{\mathring{D}^n} : \mathring{D}^n \to \mathring{e}^n_\alpha$ is a homeomorphism.

From proposition 1.1.5 it follows that Y is Hausdorff. Thus K' is a cell complex on Y. It remains to prove that it satisfies (C) and (W).

(C) Let $\alpha \in J_n$. Since $\overset{\bullet}{e}{}^n_\alpha = g_\alpha(S^{n-1}) \subseteq X$ is compact and X is a CW-complex, then $\overset{\bullet}{e}{}^n_\alpha$ intersects a finite number of interiors of cells. Thus e^n_α has a finite number of faces.

(W) The space Y has the final topology with respect to $\{e^r_\alpha\}_{r\leq n}$ because it has the final topology (the pushout topology) with respect to X and $\{e^n_\alpha\}_{\alpha\in J_n}$. □

As a corollary we obtain that the constructive definition implies the descriptive one. More precisely,

Proposition 1.2.22. *Let X be a constructive CW-complex. Then there exists a CW-complex structure K on X (i.e. X is a descriptive CW-complex) such that $X^n = |K^n|$ for all $n \in \mathbb{N}_0$ and where the characteristic maps of the cells in the two structures coincide.*

Proof. By the previous lemma and induction, each X^n is a descriptive CW-complex of dimension n. Hence items (a), (b) and (c) of definition 1.2.8 and conditions (C) and (W) of 1.2.13 hold. By 1.2.7, X is a Hausdorff space. Hence, X is a descriptive CW-complex. □

1.2.4 Subcomplexes and relative CW-complexes

We give first a quick glance at subcomplexes. As mentioned before, given a cell e^n_α, its faces can be interpreted as the cells that need to be pasted first so that the cell e^n_α can be attached. For if any of the immediate faces of e^n_α is not attached first then the adjunction map of e^n_α will not be well defined. A similar argument applies for the immediate faces of e^n_α, and repeating this process we get the statement.

Definition 1.2.23. Let K be a cell complex on X. Let $L \subseteq K$. We say that L is a *subcomplex* of K if for each cell $e^n_\alpha \in L$ all its faces also belong to L.

The following proposition enlights and justifies the definition above.

Proposition 1.2.24. *Let X be a topological space, K a cell complex on X and $L \subseteq K$ a subcomplex. Let $|L| = \bigcup_{e^n_\alpha \in L} e^n_\alpha \subseteq X$ with the subspace topology. Then*

(a) L is a cell complex on $|L|$ with structure inherited from K (i.e. the characteristic maps are the same).

(b) If K is a CW-complex structure on X then L is a CW-complex structure on $|L|$.

(c) $|L|$ is closed in X.

Definition 1.2.25. A *CW-pair* is a topological pair (X, A) where X is a CW-complex and $A \subseteq X$ is a subcomplex.

In the constructive definition of CW-complexes we begin with the empty set and start attaching cells of different dimensions. If instead we begin with a Hausdorff topological space A, the space obtained is called relative CW-complex.

Definition 1.2.26. A *relative CW-complex* is a pair (X, A), where A and X are topological spaces such that $A \subseteq X$, A is Hausdorff and there exists a sequence of subspaces of X

$$A = X_A^{-1} \subseteq X_A^0 \subseteq X_A^1 \subseteq \ldots \subseteq X_A^n \subseteq \ldots$$

satisfying that, for all $n \in \mathbb{N}_0$, X_A^n is obtained from X_A^{n-1} by attaching n-cells, $X = \bigcup_{n \in \mathbb{N}} X_A^n$ and X has the final topology with respect to $\{X_A^n\}_{n \geq -1}$.

As in the absolute case, the subspace X_A^n is called the *n-skeleton* of (X, A).

Remark 1.2.27.

(a) Let (X, A) be a relative CW-complex. Then X is Hausdorff and $A \subseteq X$ is a closed subspace.

(b) If (X, A) is a CW-pair, then it is a relative CW-complex.

Definition 1.2.28. Let X and Y be CW-complexes. A continuous map $f : X \to Y$ is called *cellular* if $f(X^n) \subseteq Y^n$ for all $n \geq 0$.

Proposition 1.2.29. *Let X and Y be CW-complexes and let $f : X \to Y$ be a cellular map. Then the cylinder of f, Z_f, is a CW-complex and $X \subseteq Z_f$ is subcomplex.*

1.2.5 Product of cellular spaces

We want to give a CW-complex structure to the cartesian product of CW-complexes. Note that $D^n \times D^m$ is homeomorphic to D^{n+m}. Hence, if X is a CW-complex with structure K and Y is a CW-complex with structure K', it is reasonable that cells of the cartesian product $X \times Y$ would be products of one cell of K with one cell of K'. However, the product topology in $X \times Y$ is not always the right one, as we shall see.

If $K = \{e^n_\alpha\}_{n,\alpha}$ is a cellular structure in a Hausdorff space X and $K' = \{e^m_\beta\}_{m,\beta}$ is a cellular structure in another Hausdorff space Y, we define the *product cellular structure* in $X \times Y$ by $K \times K' = \{e^n_\alpha \times e^m_\beta\}_{n,m,\alpha,\beta}$. It is easy to prove that $K \times K'$ is a cellular structure in $X \times Y$.

It is clear that if the cellular structures K and K' satisfy condition (C) then $K \times K'$ also satisfies (C). However, this is not always true for condition (W). Hence we define the following.

Definition 1.2.30. Let X and Y be CW-complexes with cellular structures K and K' respectively. We define the CW-complex $X \underset{w}{\times} Y$ as the space $X \times Y$ with cellular structure $K \times K'$ and with the final topology with respect to the cells of $K \times K'$, i.e. $F \subseteq X \times Y$ is closed if and only if $F \cap (e^n_\alpha \times e^m_\beta)$ is closed in $e^n_\alpha \times e^m_\beta$ for all cells $e^n_\alpha \times e^m_\beta \in K \times K'$.

Note that the weak topology in $X \times Y$ (that is, the final topology with respect to the cells) has fewer open sets than the product topology. Nevertheless, the subspace topology in $e^n_\alpha \times e^m_\beta$ is the same for the two topologies.

The next proposition is not difficult to prove and stablishes a relation between the two topologies.

Proposition 1.2.31. *Let X and Y be CW-complexes. If X or Y is locally compact then $X \underset{w}{\times} Y = X \times Y$.*

In particular, if we take $Y = I$ with cellular structure $\{e^0_0 = \{0\}, e^0_1 = \{1\}, e^1 = I\}$ and X is a CW-complex with cellular structure K, then $X \times I$ is also a CW-complex and its cellular structure is $\{e^n_\alpha \times e^0_0, e^n_\alpha \times e^0_1, e^n_\alpha \times e^1 : e^n_\alpha \in K\}$. Note that $e^n_\alpha \times e^0_0$ and $e^n_\alpha \times e^0_1$ are n-cells and $e^n_\alpha \times e^1$ is a $(n+1)$-cell. Hence, if X is finite dimensional then $\dim(X \times I) = \dim(X) + 1$.

The following proposition sums up several properties of CW-complexes.

Proposition 1.2.32.

(a) *Let (X, A) be a relative CW-complex. Then X/A is a CW-complex with cellular structure inherited from (X, A).*

(b) *If X is obtained from A by attaching n-cells then $X/A = \bigvee_{\alpha \in J_n} S^n$ (where J_n is the set that indexes the n-cells).*

Moreover, there is a homeomorphism between X/A and $\bigvee_{\alpha \in J_n} S^n$ such that the following diagram commutes for all $\alpha \in J_n$.

$$\begin{tikzcd}
(D^n, S^{n-1}) \arrow[r, "q"] \arrow[dr, "f_\alpha"'] & (S^n, *) \arrow[r, "i_\alpha"] & (\bigvee_{\beta \in J_n} S^n, *) \arrow[d, "\simeq"] \\
& (X, A) \arrow[r, "q'"'] & (X/A, *)
\end{tikzcd}$$

where f_α is the characteristic map of the cell, i_α is the inclusion and q and q' are the respective quotient maps.

(c) If (X, A) is a relative CW-complex (X, A) then $X^n_A/X^{n-1}_A = \bigvee_{\alpha \in J_n} S^n$ for all $n \in \mathbb{N}_0$, where J_n is the set that indexes the n-cells of (X, A).
In particular, if $A = \varnothing$ we obtain that $X^n/X^{n-1} = \bigvee_{\alpha \in J_n} S^n$.

Another basic construction in homotopy theory is the smash product. Recall that if (X, x_0) and (Y, y_0) are pointed topological spaces, then the *smash product of X and Y* is defined as $X \wedge Y = (X \times Y)/(X \times \{y_0\} \cup \{x_0\} \times Y)$.

Since $X \times \{y_0\} \cup \{x_0\} \times Y$ is homeomorphic to $X \vee Y$, the definition above is usually written as $X \wedge Y = (X \times Y)/(X \vee Y)$.

In case (X, x_0) and (Y, y_0) are pointed CW-complexes such that either X or Y is locally compact, we know that $X \times Y$ is a CW-complex. It is easy to verify that $X \times \{y_0\} \cup \{x_0\} \times Y$ is a subcomplex of $X \times Y$. Hence, $X \wedge Y$ is a CW-complex.

1.3 Homology theory of CW-complexes

In this section we recall the homology theory of CW-complexes. The combinatorial structure of these spaces allows one to compute homology in a quite simple way.

1.3.1 Cellular homology

Definition 1.3.1. Let (X, A) be a topological pair. We say that (X, A) is a *good pair* if A is a closed subspace of X and there exists an open subset $U \subseteq X$ such that $U \supseteq A$ and the inclusion $i : A \to U$ is a strong deformation retract.

Important example 1.3.2. If X is a CW-complex and $A \subseteq X$ is a subcomplex then (X, A) is a good pair.

Using excision for homology groups we can prove the following proposition which is essential for computing homology of CW-complexes and developing cellular homology theory.

Proposition 1.3.3. *Let (X, A) be a topological pair and let $q : (X, A) \to (X/A, *)$ be the quotient map. If (X, A) is a good pair then q induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, *) \simeq \widetilde{H}_n(X/A)$.*

The proof is not difficult and will be omitted.

The definition and properties of cellular homology are based on the following lemma.

Lemma 1.3.4. *Let X be a CW-complex. Then:*

(a) $H_k(X^n, X^{n-1}) = 0$ if $k \neq n$ and $H_n(X^n, X^{n-1}) \simeq \bigoplus_{i \in J_n} \mathbb{Z}$, where J_n is an index set for the n-cells of X.

(b) $H_k(X^n) = 0$ if $k > n$.

(c) The inclusion $i : X^n \to X$ induces isomorphisms $i_* : H_k(X^n) \to H_k(X)$ for $k < n$.

Proof.
(a) Since (X^n, X^{n-1}) is a good pair, $H_k(X^n, X^{n-1}) \simeq H_k(X^n/X^{n-1}) \simeq H_k(\bigvee_{i \in J_n} S^n)$ and the result follows.

(b) Consider the long exact sequence in homology of the pair (X^n, X^{n-1}):

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow H_k(X^n, X^{n-1}) \longrightarrow \cdots$$

If $k > n$, $H_{k+1}(X^n, X^{n-1}) = 0$ and $H_k(X^n, X^{n-1}) = 0$ by (a). Hence $H_k(X^n) \simeq H_k(X^{n-1})$ for $k > n$. Repeating this argument we obtain $H_k(X^n) \simeq H_k(X^{n-1}) \simeq \ldots \simeq H_k(X^0) = 0$.

(c) Proceeding as in (b), we get $H_k(X^n) \simeq H_k(X^m)$ if $k < n \leq m$. Hence (c) is proved if X is finite dimensional. For the general case, let $[a] \in H_k(X)$. Hence a is a singular k-chain such that $d_k(a) = 0$. Since the image of each singular simplex is compact, a is also a singular k-chain in some skeleton X^m for a sufficiently large value of m. It follows that the map $i_* : H_k(X^n) \to H_k(X)$ is surjective.

Now suppose that $[a] \in H_k(X^n)$ is such that $[a] = 0$ in $H_k(X)$. Hence the singular k-chain a is the image of a singular $(k+1)$-chain b by d_{k+1}. As above, b is also a singular k-chain in some skeleton X^m for sufficiently large m. It follows that $[a] = 0$ in $H_k(X^m)$. $\square$

To define the cellular chain complex of a CW-complex X we consider the long exact sequences in homology of the pairs (X^n, X^{n-1}) for $n \in \mathbb{N}$ and arrange them in the following diagram

$$\begin{tikzcd}
 & H_n(X^n) \arrow[dr, "j_n"] & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[u, "\partial_{n+1}"] \arrow[r, "d'_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[rr, "d'_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] & &
\end{tikzcd}$$

where d'_n is defined as $d'_n = j_{n-1}\partial_n$ for all $n \in \mathbb{N}$. Since $\partial_n j_n = 0$, we obtain that $d'_n d'_{n+1} = 0$. The horizontal row is the *cellular chain complex of* X and the group $H_n(X^n, X^{n-1})$ corresponds to degree n. Note that, by the previous lemma, this group is $\bigoplus_{i \in J_n} \mathbb{Z}$, where J_n is an index set for the n-cells of X.

The *cellular homology groups of* X are defined as the homology groups of the cellular chain complex of X. As we shall see shortly, cellular homology coincides with singular homology and the differentials d'_n can be calculated in terms of the attaching maps of the cells. Hence, singular homology of CW-complexes can be computed directly from the combinatorial structure of CW-complexes by means of cellular homology. Moreover, it is clear that the cellular chain complex is far more simple that the singular one.

Before going on, we will prove that cellular homology groups coincide with singular homology ones. Consider the long exact sequence

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n) \longrightarrow \cdots$$

By the previous lemma, $H_n(X^{n+1}) \simeq H_n(X)$ and $H_n(X^{n+1}, X^n) = 0$. In consequence, $H_n(X) \simeq H_n(X^n)/\mathrm{Im}\, \partial_{n+1}$.

On the other hand, given $n \in \mathbb{N}$ we consider the exact sequence

$$\cdots \longrightarrow H_n(X^{n-1}) \longrightarrow H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \longrightarrow \cdots$$

By the previous lemma, $H_n(X^{n-1}) = 0$. Hence, j_n is injective. Thus, the maps $j_n|_{\mathrm{Im}\, \partial_{n+1}} : \mathrm{Im}\, \partial_{n+1} \to \mathrm{Im}\, (j_n \partial_{n+1}) = \mathrm{Im}\, (d'_{n+1})$ and $j_n : H_n(X^n) \to \mathrm{Im}\, j_n = \ker \partial_n$ are isomorphisms. Now, $\ker \partial_n = \ker d'_n$ since j_{n-1} is injective. Then j_n induces an isomorphism $H_n(X^n)/\mathrm{Im}\, \partial_{n+1} \simeq \ker d'_n/\mathrm{Im}\, (d'_{n+1})$. Hence, cellular homology groups coincide with singular homology ones.

As mentioned before, the differentials d'_n can be computed in terms of the attaching maps of the cells as stated by the following proposition.

Proposition 1.3.5. *Let X be a CW-complex. For each $n \in \mathbb{N}$, let J_n be an index set for the n-cells. We consider the cells e^n_α, $\alpha \in J_n$, as generators of the free abelian group $H_n(X^n, X^{n-1}) \simeq \bigoplus_{\alpha \in J_n} \mathbb{Z}$. Then the differential d'_n is defined by*

$$d'_n(e^n_\alpha) = \sum_{\beta \in J_{n-1}} \deg(q^{n-1}_\beta g^n_\alpha) e^{n-1}_\beta$$

where g^α_n is the attaching map of e^n_α and $q^{n-1}_\beta : X^{n-1} \to S^{n-1}$ is the quotient map which collapses $X^{n-1} - \overset{\circ}{e}{}^{n-1}_\beta$ to a point.

Note that the sum above has finite support since the image of g^n_α is compact and hence it intersects only a finite number of cells.

The proof of this proposition will not be given here. However, a generalization of this result will be proved in chapter 5.

We will give now some examples of application of the above results which will show the usefulness of cellular homology.

Examples 1.3.6.

(a) The real projective plane $\mathbb{P}^2$ has a CW-complex structure consisting of one 0-cell, one 1-cell and one 2-cell attached by a map of degree 2. Hence its cellular chain complex is

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{.2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

It follows that $H_0(\mathbb{P}^2) = \mathbb{Z}$, $H_1(\mathbb{P}^2) = \mathbb{Z}_2$ and $H_n(\mathbb{P}^2) = 0$ for $n \geq 2$.

(b) In a similar way, the real n-dimensional projective space has a CW-complex structure consisting of one i-cell for all $0 \leq i \leq n$. It is not hard to see that its cellular chain complex is

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{d'_n} \mathbb{Z} \xrightarrow{d'_{n-1}} \cdots \longrightarrow \mathbb{Z} \xrightarrow{.2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

where, for $1 \leq i \leq n$, the map d'_i is multiplication by 2 if i is even and trivial if i is odd. Hence, if n is even,

$$H_i(\mathbb{P}^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_2 & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ 0 & \text{if } i \text{ is even and } 1 \leq i \leq n \\ 0 & \text{if } i > n \end{cases}$$

and, if n is odd,

$$H_i(\mathbb{P}^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_2 & \text{if } i \text{ is odd and } 1 \leq i < n \\ 0 & \text{if } i \text{ is even and } 1 \leq i < n \\ \mathbb{Z} & \text{if } i = n \\ 0 & \text{if } i > n \end{cases}$$

(c) The torus $S^1 \times S^1$ has a CW-complex structure consisting of one 0-cell, two 1-cells and one 2-cell attached by the map induced by $aba^{-1}b^{-1}$ where a and b are the generators of $\pi_1(S^1 \vee S^1)$ given by the inclusions $i_1, i_2 : S^1 \to S^1 \vee S^1$. Hence its cellular chain complex is

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

It follows that $H_0(S^1 \times S^1) = \mathbb{Z}$, $H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$, $H_2(S^1 \times S^1) = \mathbb{Z}$ and $H_n(S^1 \times S^1) = 0$ for $n \geq 3$.

1.3.2 Moore spaces

Definition 1.3.7. Let G be an abelian group and let $n \in \mathbb{N}$. A *Moore space of type* (G, n) is a CW-complex X such that $H_n(X) \simeq G$, $\widetilde{H}_i(X) = 0$ if $i \neq n$ and such that X is simply-connected if $n > 1$.

For example, the projective plane $\mathbb{P}^2$ is a Moore space of type $(\mathbb{Z}_2, 1)$ and its $(n-1)$th suspension is a Moore space of type $(\mathbb{Z}_2, n)$. Also, if we attach an $(n+1)$-cell to S^n by a map $S^n \to S^n$ of degree m we obtain a Moore space of type $(\mathbb{Z}_m, n)$.

Note that if X is a Moore space of type (G, n) and Y is a Moore space of type (H, n) then $X \vee Y$ is a Moore space of type $(G \oplus H, n)$. Thus, if G is a finitely generated abelian group, it is easy to construct a Moore space of type (G, n) by taking wedge sums of Moore spaces of type (H_i, n), with H_i a cyclic group.

The following proposition states that this can be done for every abelian group G.

Proposition 1.3.8. *Let G be an abelian group and let $n \in \mathbb{N}$. Then there exists a Moore space of type (G, n).*

Proof. Let $A \subseteq G$ be a set of generators of G and let F be a free abelian group with basis A. Let $\phi : F \to G$ be the induced group homomorphism. Then $\ker \phi$ is also a free abelian group. Let $\{h_\alpha : \alpha \in J\}$ be a basis of $\ker \phi$. We will construct a Moore space of type (G, n) attaching $(n+1)$-cells to $\bigvee_{i \in A} S^n$.

For $i \in A$ let $q_i : \bigvee_{i \in A} S^n \to S^n$ be the quotient map which collapses everything except the i-th copy of S^n to a point. We write $h_\alpha = \sum_{i \in A} d_{i,\alpha} i$ (with $d_{i\alpha} \in \mathbb{Z}$) for $\alpha \in J$. Note that for each α, $d_{i,\alpha} = 0$ except for a finite number of indexes i. For each $\alpha \in J$ let $g_\alpha : S^n \to \bigvee_{i \in A} S^n$ be a continuous map such that $q_i g_\alpha$ is a map of degree $d_{i,\alpha}$. This map can be constructed in the following way.

Let $m = \#\{i \,; d_{i,\alpha} \neq 0\}$ and let $D_1, D_2, \ldots, D_m$ be m disjoint subsets of S^n, all of which are homeomorphic to D^n. Let ∂D_i denote the border of D_i, i.e. ∂D_i is the subset of D_i which corresponds to $S^{n-1} \subseteq D^n$ by the homeomorphism $\varphi_i : D_i \to D^n$. Let $q : D^n \to S^n \simeq D^n / S^{n-1}$ be the quotient map and let $\mathrm{in}_i : S^n \to \bigvee_{i \in A} S^n$ be the inclusion in the i-th copy. We define $g_\alpha : S^n \to \bigvee_{i \in A} S^n$ by

$$g_i(x) = \begin{cases} \gamma_{d_{i,\alpha}} q \varphi_i(x) & \text{if } x \in D_i \\ * & \text{if } x \in S^n - \bigcup_{i=1}^m D_i \end{cases}$$

where $\gamma_{d_{i,\alpha}} : S^n \to S^n$ is a map of degree $d_{i,\alpha}$. It is easy to prove that the map g_α satisfies the required conditions.

Let X be defined by the pushout

$$\begin{tikzcd} \coprod_{\alpha \in J} S^n \arrow[r, "\coprod_{\alpha \in J} g_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & \bigvee_{i \in A} S^n \arrow[d] \\ \coprod_{\alpha \in J} D^{n+1} \arrow[r] & X \end{tikzcd}$$

It follows that the cellular chain complex of X is

$$\cdots \longrightarrow 0 \longrightarrow \ker\phi \xrightarrow{d_{n+1}} F \longrightarrow 0 \longrightarrow \cdots$$

where the map d_{n+1} is the inclusion. Hence, X is a Moore space of type (G,n). □

As a corollary we obtain that we can build CW-complexes with arbitrary homology groups.

Corollary 1.3.9. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of abelian groups. Then there exists a connected CW-complex X such that $H_n(X) = G_n$ for all $n \in \mathbb{N}$.*

Proof. Take $X = \bigvee_{n\in\mathbb{N}} X_n$, where X_n is a Moore space of type (G_n, n). □

1.4 Homotopy theory of CW-complexes

In this section we will study the homotopy theory of CW-complexes, which is rich in results. Among them we ought to mention Whitehead's Theorem, which is one of the most important theorems of homotopy theory and asserts that a map between CW-complexes which induces isomorphisms in all homotopy groups is a homotopy equivalence (1.4.14). Another key result which will be given in this section is that every topological space can be homotopically approximated by a CW-complex (1.4.18).

We will also recall the excision theorem for computing homotopy groups of CW-complexes and some corollaries of it. We also mention the cellular approximation theorem which says that every continuous map between CW-complexes is homotopic to a cellular one. This is of much importance in the homotopy theory of CW-complexes as we shall see.

We also study Eilenberg - MacLane spaces, which are the homotopical counterpart of Moore spaces.

We then recall Hurewicz's theorem, which is another key theorem in homotopy theory and relates homotopy and homology groups of topological spaces. In chapter 2 we will give an important and useful generalization of it, due to Serre [18].

In the last subsection we recall homology decomposition of spaces.

1.4.1 Basic properties

The following remark is one of the key properties of CW-complexes and is essential when developing their homotopy theory.

Important remark 1.4.1. If X is obtained from A by attaching n-cells then the inclusion $A \hookrightarrow X$ is a cofibration. In particular, if (X,A) is a relative CW-complex, then the inclusion $X_A^{n-1} \hookrightarrow X_A^n$ is a cofibration.

The following proposition generalizes the previous remark.

Proposition 1.4.2. *If (X,A) is a relative CW-complex then the inclusion $i : A \to X$ is a cofibration.*

Proof. Given a topological space Y and continuous maps $f : X \to Y$ and $H : A \times I \to Y$ such that $Hi_0 = fi$ we need to find a homotopy $G : X \times I \to Y$ extending H and such that $Gi_0 = f$.

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times \mathrm{Id}_I"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dotted, "G"] & \\
 & & Y
\end{tikzcd}$$

Since $X_A^{n-1} \hookrightarrow X_A^n$ is a cofibration for all n, we may construct inductively a sequence of continuous maps $\{G_n\}_{n \geq -1}$ with $G_n : X_A^n \times I \to Y$ such that

(a) $G_{-1} = H$

(b) $G_n|_{X_A^{n-1} \times I} = G_{n-1}$

(c) $G_n i_0 = f|_{X_A^n}$

We define $G : X \times I \to Y$ by $G(x,t) = G_n(x,t)$ if $x \in X_A^n$. The map G is well defined by (b) and continuous because X has the final topology with respect to the cells. By construction it is clear that the map G is the required extension. □

From the above proposition we get the following corollary.

Corollary 1.4.3. *If (X, A) is a relative CW-complex and A is contractible then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

Its proof is not difficult and can be found in [20] (proposition 6.6 p.75). It can also be deduced from the previous proposition and 1.1.21.

As an example of an application consider the following. Let G be a graph, i.e. a CW-complex of dimension 1. Let $T \subseteq G$ be a maximal tree. Then (G, T) is a relative CW-complex and thus the inclusion $T \hookrightarrow G$ is a cofibration. If J indexes the edges of G that do not belong to T we have that $q : G \to G/T \simeq \bigvee\limits_{\alpha \in J} S^1$ is a homotopy equivalence.

1.4.2 Cellular approximation

The cellular approximation theorem says that every continuous map between CW-complexes can be approximated by a cellular one.

Theorem 1.4.4 (Cellular approximation theorem)**.** *Let X and Y be CW-complexes and let $f : X \to Y$ be a continuous map. Then there exists a cellular map $f' : X \to Y$ such that $f' \simeq f$.*

Moreover, if $A \subseteq X$ is a subcomplex such that $f|_A$ is cellular, then we may take a cellular approximation map f' satisfying $f'|_A = f|_A$ and $f' \simeq f$ rel A.

Its proof can be found in [20].

We will see now some important applications of this theorem. For the first of them we need the following classical lemma about relative homotopy groups.

Lemma 1.4.5. *Let (X, x_0) be a pointed topological space and let $A \subseteq X$ be a subspace such that $x_0 \in A$. Let $f : (D^n, S^{n-1}) \to (X, A)$ be a continuous map. Then the following are equivalent.*

i) There exists a base point preserving homotopy $H : (D^n \times I, S^{n-1} \times I) \to (X, A)$ such that $Hi_0 = f$, $Hi_1(x) = x_0$ $\forall x \in D^n$.

ii) There exists a (base point preserving) homotopy $G : D^n \times I \to X$, relative to S^{n-1}, such that $Gi_0 = f$, $Gi_1(D^n) \subseteq A$.

iii) There exists a (base point preserving) homotopy $G : D^n \times I \to X$, such that $Gi_0 = f$, $Gi_1(D^n) \subseteq A$.

We omit the proof which is not difficult. Besides, we will give later a generalization of this result (4.1.5). As an immediate application we obtain the following corollary.

Corollary 1.4.6. *Let (X, A, x_0) be a pointed topological pair (i.e. (X, x_0) is a pointed topological space and $A \subseteq X$ is a subspace such that $x_0 \in A$) and let $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ be a continuous map. Then $[f] = 0$ in $\pi_n(X, A)$ if and only if f is homotopic relative to S^{n-1} to a map g such that $g(D^n) \subseteq A$.*

With this results at hand, we are able to give the first application of the cellular approximation theorem.

Corollary 1.4.7. *If X is a CW-complex, then the topological pair (X, X^n) is n-connected, i.e, $\pi_r(X, X^n) = 0$ for all $r \leq n$, or equivalently, the morphism $i_* : \pi_r(X^n) \to \pi_r(X)$, induced by the inclusion, is an isomorphism for $r < n$ and an epimorphism for $r = n$.*

Proof. Let $r \leq n$ and let $[f] \in \pi_r(X, X^n)$. We can take any 0-cell as base point for $\pi_r(X, X^n)$ since every point can be joined to a 0-cell by a continuous path. Hence, we may suppose that f sends the base point of D^r to a 0-cell.

We consider in S^{r-1} the cellular structure which consists of one 0-cell and one $(r-1)$-cell. Note that $f|_{S^{r-1}}$ is cellular. Thus, by the cellular approximation theorem there exists a cellular map $f' : D^r \to X$ such that $f'|_{S^{r-1}} = f|_{S^{r-1}}$ and $f' \simeq f$ rel S^{r-1}.

Since f' is celular and $r < n$ then $\operatorname{Im} f' \subseteq X^n$. Hence f is homotopic (relative to S^{r-1}) to a map f' with $\operatorname{Im} f' \subseteq X^n$. Thus, $[f] = 0$ in $\pi_r(X, X^n)$ by the corollary above. □

Corollary 1.4.8. *$\pi_r(S^n) = 0$ for $r < n$.*

Proof. Let $r < n$. We consider in S^r the cellular structure which consists of one 0-cell and one r-cell, and in S^n the analogous structure. Let $[f] \in \pi_r(S^n)$. By the cellular approximation theorem, there exists a cellular map $f' : S^r \to S^n$, such that $f' \simeq f$ rel $*$ (where $*$ is the 0-cell). But then $f'(S^r) \subseteq (S^n)^r = *$, i. e. f' is constant.

Thus, $[f] = 0$. □

We give now two variations of the cellular approximation theorem: for CW-pairs and for relative CW-complexes.

Proposition 1.4.9 (Cellular approximation theorem for CW-pairs)**.** *Let $f : (X, A) \to (Y, B)$ be a continuous map between CW-pairs. Then there exists a cellular map $g : (X, A) \to (Y, B)$ such that $f \simeq g$ as maps between topological pairs.*

Proposition 1.4.10 (Cellular approximation theorem for relative CW-complexes)**.** *Let $f : (X, A) \to (Y, B)$ be a continuous map between relative CW-complexes. Then there exists a cellular map $g : (X, A) \to (Y, B)$ (i.e. $g(X^n_A) \subseteq Y^n_B$ for all n) such that $f \simeq g$ relative to A.*

As a corollary we get the following theorem, which is analogous to 1.4.7.

Corollary 1.4.11. *Let (X, A) be a relative CW-complex and let $n \in \mathbb{N}_0$. If (X, A) has no cells of dimension less than or equal to n, then the topological pair (X, A) is n-connected.*

1.4.3 Whitehead's theorem

Whitehead's theorem is one of the most important theorems of the homotopy theory of classical CW-complexes. The proof we will give here will use the cellular approximation theorem. A bit more elementary proof, without using the cellular approximation theorem can be found in [20] and a generalization of this proof is given in chapter 4 of this thesis, in our generalization of Whitehead's theorem (4.2.4).

For the proof of Whitehead's theorem we will need the following lemma, interesting for its own sake.

Lemma 1.4.12. *Let (X, A) be a relative CW-complex and let (Y, B) be a topological pair with $B \neq \varnothing$. Suppose that for all $n \in \mathbb{N}_0$ such that there exists at least one n-cell in $X - A$ we have that $\pi_n(Y, B, b_0) = 0$ for all $b_0 \in B$. Then every continuous map $f : (X, A) \to (Y, B)$ is homotopic relative to A to a map g such that $g(X) \subseteq B$.*

Note that this lemma generalizes 1.4.6.

In particular if $\dim(X, A) = n$ (resp. $\dim(X, A) = \infty$) and (Y, B) is n-connected (resp. $\pi_n(Y, B) = 0$ for all $n \in \mathbb{N}$) then the hypothesis of the lemma are satisfied.

Proof. By induction, suppose $h : X \to Y$ is a continuous map such that $h(X^{n-1}_A) \subseteq B$. Let $e^n \in X^n_A$ and let $\varphi : (D^n, S^{n-1}) \to (X^n_A, X^{n-1}_A)$ be the characteristic map of e^n. Then $h\varphi : (D^n, S^{n-1}) \to (Y, B)$. Since $\pi_n(Y, B) = 0$ then there exists a map ψ such that $h\varphi \simeq \psi$ rel S^{n-1} and $\psi(D^n) \subseteq B$.

Hence, there exists a continuous map $h' : X^{n-1}_A \cup e^n \to Y$ such that $h'(X^{n-1}_A \cup e^n) \subseteq B$ and $h|_{X^{n-1}_A \cup e^n} \simeq h'$ rel X^{n-1}_A.

Doing this for all cells of dimension n we obtain a continuous map $\hat{h} : X^n_A \to Y$ with $\hat{h}(X^n_A) \subseteq B$ and a homotopy $H : h|_{X^n_A} \simeq \hat{h}$ rel X^{n-1}_A. Since the inclusion $i : X^n_A \to X$ is a cofibration, we extend H to $X \times I$:

$$\begin{tikzcd}
X_A^n \arrow[r, "i_0"] \arrow[d, "i"'] & X_A^n \times I \arrow[d, "i\times \mathrm{Id}_I"] \arrow[ddr, bend left, "H"] & \\
X \arrow[r, "i_0"] \arrow[drr, bend right, "h"'] & X \times I \arrow[dr, dotted, "G"] & \\
& & Y
\end{tikzcd}$$

Note that the homotopy G is relative to X_A^{n-1} since H is.

Hence, we may construct a sequence of continuous maps and homotopies

$$f = f_{-1} \underset{H_0}{\simeq} f_0 \underset{H_1}{\simeq} f_1 \underset{H_2}{\simeq} \cdots$$

with $f_n(X_A^n) \subseteq B$ and H_n relative to X_A^{n-1}.

Finally, we define $H : X \times I \to Y$ by

$$H(x,t) = \begin{cases} H_n(x, 2^{n+1}(t - (1 - \frac{1}{2^n}))) & \text{if } t \in [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}] \\ f_n(x) & \text{if } t = 1 \text{ and } x \in X_A^n \end{cases}$$

It is easy to verify that H is continuous and $Hi_0 = f$. We take $g = Hi_1$. □

From this lemma, we obtain the following result.

Corollary 1.4.13. *Let (X, A) be a relative CW-complex of dimension n (resp. of dimension ∞) such that (X, A) is n-connected (resp. $\pi_k(X, A) = 0$ for all $k \in \mathbb{N}$). Then $A \subseteq X$ is a strong deformation retract.*

Proof. By the previous lemma we obtain the retraction r in the following diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{Id}_A"] \arrow[d, "i"'] & A \arrow[d, "i"] \\
X \arrow[r, "\mathrm{Id}_X"'] \arrow[ur, dotted, "\exists r"] & X
\end{tikzcd}$$

where the upper left triangle commutes, while the lower right one commutes homotopically relative to A (that is, there exists a homotopy $ir \simeq \mathrm{Id}_X$ relative to A). □

Now we state Whitehead's theorem. Recall that a continuous map $f : X \to Y$ is called a *weak equivalence* if it induces isomorphisms $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all $x_0 \in X$ and for all $n \in \mathbb{N}_0$.

Theorem 1.4.14 (Whitehead's theorem)**.** *A continuous map $f : X \to Y$ between CW-complexes is a homotopy equivalence if and only if it is a weak equivalence.*

Proof. The first implication holds for every topological space (see [19]).

For the converse note that if f is the inclusion of a subcomplex in a CW-complex, the result follows from the previous corollary.

For the general case, let $f : X \to Y$ be any continuous map. By the cellular approximation theorem there exists a cellular map $f' : X \to Y$ such that $f' \simeq f$. Note that f' is a weak equivalence since f is.

By 1.2.29, $Z_{f'}$ is a CW-complex and $X \subseteq Z_{f'}$ is subcomplex. We have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "j"'] & Y \arrow[dl, shift left, "i"] \\ Z_{f'} \arrow[ur, shift left, "r"] & \end{tikzcd}$$

where j and i are inclusion maps and where $r : Z_{f'} \to Y$ is the standard strong deformation retraction.

Since $f' : X \to Y$ is a weak equivalence, then $j : X \to Z_{f'}$ is a weak equivalence. By the previous case, j is a homotopy equivalence. Then $f' = rj$ is a homotopy equivalence and since $f \simeq f'$, then f is a homotopy equivalence. $\square$

1.4.4 CW-approximations

We will now take study CW-approximations, that is, given a topological space X we want to find a CW-complex which 'homotopically approximates' the space X. More precisely,

Definition 1.4.15. Let X be a topological space. A *CW-approximation of* X is a CW-complex Z together with a weak equivalence $f : Z \to X$.

We will prove that every topological space X admits a CW-approximation as a corollary of a stronger result, for which we need the following definition.

Definition 1.4.16. Let (X, A) be a topological pair where $A \subseteq X$ is a non-empty CW-complex and let $n \in \mathbb{N}_0$. An *n-connected CW-model of* (X, A) (or simply an *n-model of* (X, A)) is a CW-pair (Z, A) together with a continuous map $f : Z \to X$ such that $f|_A = \mathrm{Id}$ and such that

(a) The pair (Z, A) is n-connected.

(b) For every $z_0 \in Z$ the morphism $f_* : \pi_r(Z, z_0) \to \pi_r(X, f(z_0))$ is an isomorphism for all $r > n$ and a monomorphism for $r = n$.

Note that an n-model (Z, A) of (X, A) is a kind of 'homotopic mixture' between A and X. If $i : A \to Z$ and $j : A \to X$ are the inclusion maps and $f : Z \to X$ is as in the previous definition, then $i_* : \pi_r(A) \to \pi_r(Z)$ is a isomorphism for all $r < n$ and $f_* : \pi_r(Z) \to \pi_r(X)$ is a isomorphism for all $r > n$.

Moreover, $i_* : \pi_n(A) \to \pi_n(Z)$ is an epimorphism and $f_* : \pi_n(Z) \to \pi_n(X)$ is a monomorphism and since $fi = j$, then $f_* i_* = j_*$. Hence, in some way $\pi_n(Z)$ can be thought as the image of $\pi_n(A)$ in $\pi_n(X)$.

Note also that if A consists of a point in each path-connected component of X, then a 0-model of (X, A) is a CW-approximation of X.

We give now the theorem of n-models from which we will deduce that every topological space admits a CW-approximation.

Theorem 1.4.17. *Let $n \in \mathbb{N}_0$ and let (X, A) be a topological pair where A is a non-empty CW-complex. Then there exists an n-model $f : (Z, A) \to (X, A)$.*

Moreover, the n-model (Z, A) can be taken in such a way that Z is built from A by attaching cells of dimension greater than n.

Proof. We build inductively CW-complexes Z_m for $m \geq n$ together with maps $f_m : Z_m \to X$ such that

$$A = Z_n \subseteq Z_{n+1} \subseteq Z_{n+2} \subseteq \ldots$$

Z_m is obtained from Z_{m-1} attaching m-cells, $f_m|_{Z_{m-1}} = f_{m-1}$, $f_m|_A$ is the inclusion of A in X and $(f_m)_* : \pi_r(Z_m) \to \pi_r(X)$ is a monomorphism for $n \leq r < m$ and an epimorphism for $n < r \leq m$.

Suppose we have constructed Z_k and $f_k : Z_k \to X$ such that $(f_k)_* : \pi_r(Z_k) \to \pi_r(X)$ is monomorphism for $n \leq r < k$ and epimorphism for $n < r \leq k$. We will build Z_{k+1} and extend f_k to $f_{k+1} : Z_{k+1} \to X$.

Henceforward, we will work in each path-connected component of A.

For each element $\alpha \in \ker(f_k)_* \subseteq \pi_k(Z_k)$ we choose a continuous map $\varphi_\alpha : S^k \to Z_k$, which may be supposed cellular, such that $[\varphi_\alpha] = \alpha$. For each α we attach a $(k+1)$-cell e_α^{k+1} to Z_k with adjunction map φ_α. Let $Y_{k+1} = Z_k \cup \bigcup_{\alpha \in \ker(f_k)_*} e_\alpha^{k+1}$.

Since $[\varphi_\alpha] \in \ker(f_k)_*$ then $f \circ \varphi_\alpha : S^k \to X$ is nullhomotopic. Thus, it can be extended to D^{k+1}. There is a commutative diagram

$$\begin{tikzcd}
S^k \arrow[r, "\varphi_\alpha"] \arrow[d] \arrow[dr, phantom, "push"] & Z_k \arrow[d] \arrow[ddr, bend left, "f_k"] & \\
D^{k+1} \arrow[r] \arrow[drr, bend right] & Z_k \cup e_\alpha^{k+1} \arrow[dr, dotted, "f_k^{(\alpha)}"] & \\
& & X
\end{tikzcd}$$

Pasting all the extensions $f_k^{(\alpha)}$ we extend the map f_k to Y_{k+1} as $\overline{f_k} : Y_{k+1} \to X$. Note that $(Y_{k+1})^k = Z_k$.

We will prove that $(\overline{f_k})_* : \pi_r(Y_{k+1}) \to \pi_r(X)$ is a monomorphism for $n \leq r < k+1$. Since $(Y_{k+1})^k = Z_k$, then $(\overline{f_k})_*$ is a monomorphism for $n \leq r < k$. Let $\phi : S^k \to Y_{k+1}$ be a continuous map such that $(\overline{f_k})_*[\phi] = 0$. We may suppose that ϕ is cellular. Then $\phi(S^k) \subseteq (Y_{k+1})^k = Z_k$. Hence $[\phi] \in \ker((f_k)_*)$. Thus, $[\phi] = [\varphi_\alpha]$ for some $\alpha \in \ker(f_k)_*$. Let $j : Z_k \to Y_{k+1}$ be the inclusion map. Since $j\varphi_\alpha : S^k \to Y_{k+1}$ can be extended to the disk D^{k+1} it follows that $[\phi] = [\varphi_\alpha] = 0$ in $\pi_k(Y_{k+1})$.

Thus $(\overline{f_k})_* : \pi_r(Y_{k+1}) \to \pi_r(X)$ is a monomorphism for $n \leq r < k+1$. Note also that $(\overline{f_k})_* : \pi_r(Y_{k+1}) \to \pi_r(X)$ is an epimorphism for $n < r \leq k$ since $\overline{f_k} \circ j = f_k$. However, $(\overline{f_k})_* : \pi_{k+1}(Y_{k+1}) \to \pi_{k+1}(X)$ may not be an epimorphism.

For each $\beta \in \pi_{k+1}(X)$ we take a continuous map $\varphi_\beta : S^{k+1} \to X$ such that $[\varphi_\beta] = \beta$. Let $Z_{k+1} = Y_{k+1} \vee \bigvee_{\beta \in \pi_{k+1}(X)} S^{k+1}$ and $f_{k+1} : Z_{k+1} \to X$ be defined by $f_{k+1}|_{Y_{k+1}} = \overline{f_k}$ and $f_{k+1}|_{S^{k+1}_\beta} = \varphi_\beta$.

Note that $(Z_{k+1})^k = (Y_{k+1})^k = Z_k$. Hence $\pi_r(Y_{k+1}) = \pi_r(Z_{k+1}) = \pi_r(Z_k)$ for $n \leq r < k$. Thus $(f_{k+1})_* : \pi_r(Z_m) \to \pi_r(X)$ is a monomorphism for $n \leq r < k$ and an epimorphism for $n < r < k$.

By construction, $(f_{k+1})_* : \pi_{k+1}(Z_{k+1}) \to \pi_{k+1}(X)$ is an epimorphism. Indeed, if $i_\beta : S^{k+1} \to Z_{k+1}$ denotes the inclusion in the β-th copy of S^{k+1} then $(f_{k+1})_*([i_\beta]) = [f_{k+1} i_\beta] = [\varphi_\beta]$.

We will see now that $(f_{k+1})_* : \pi_k(Z_{k+1}) \to \pi_k(X)$ is an isomorphism. Since $Y_{k+1} \subseteq Z_{k+1}$ is a retract (with retraction $r : Z_{k+1} \to Y_{k+1}$ sending $\bigvee_{\beta \in \pi_{k+1}(X)} S^{k+1}$ to a point) then $ri = \mathrm{Id}_{Y_{k+1}}$, where $i : Y_{k+1} \to Z_{k+1}$ is the inclusion. Then, $r_* i_* = \mathrm{Id}$, and it follows that $i_* : \pi_k(Y_{k+1}) \to \pi_k(Z_{k+1})$ is a monomorphism. Moreover, by cellular approximation, $i_* : \pi_k(Y_{k+1}) \to \pi_k(Z_{k+1})$ is an epimorphism. Then, it is an isomorphism. Since $f_{k+1} i = \overline{f_k}$ it follows that $(f_{k+1})_* : \pi_k(Z_{k+1}) \to \pi_k(X)$ is an isomorphism.

We take $Z = \bigcup_{k \geq n} Z_k$ and $f : Z \to X$ defined by $f|_{Z_k} = f_k$. It follows that $f : (Z, A) \to (X, A)$ is an n-model of (X, A). □

As said before, we obtain as a corollary the following theorem.

Theorem 1.4.18. *Let X be a topological space. Then there exists a CW-approximation for X.*

Proof. We take A consisting of a point in each path-connected component of X. Let $f : (Z, A) \to (X, A)$ be a 0-connected model of (X, A). The map $f : Z \to X$ is a CW-approximation for X. □

There is also a CW-approximation theorem for topological pairs.

Theorem 1.4.19. *Let (X, A) be a topological pair. Then there exists a CW-pair (Z, B) and a continuous map $f : (Z, B) \to (X, A)$ such that $f_* : \pi_n(B) \to \pi_n(A)$, $f_* : \pi_n(Z) \to \pi_n(X)$ and $f_* : \pi_n(Z, B) \to \pi_n(X, A)$ are isomorphisms for all n.*

Proof. Let $g : B \to A$ be a CW-approximation for A. Let $\mathrm{in} : A \to X$ be the inclusion map and let $\alpha : (Z, B) \to (Z_{\mathrm{in} \circ g}, B)$ be a 0-connected CW-model for $(Z_{\mathrm{in} \circ g}, B)$. Let $r : Z_{\mathrm{in} \circ g} \to B$ be the standard strong deformation retraction and let $f = r\alpha$. It follows that $f|_B = g$ and that $f_* : \pi_n(B) \to \pi_n(A)$ and $f_* : \pi_n(Z) \to \pi_n(X)$ are isomorphisms for all $n \in \mathbb{N}_0$. Thus, from the five-lemma, $f_* : \pi_n(Z, B) \to \pi_n(X, A)$ is also an isomorphism for all n. □

Another important consequence of 1.4.17 is the following proposition.

Proposition 1.4.20. *If (X, A) is an n-connected relative CW-complex, then there exists a relative CW-complex (Z, A) such that (X, A) is homotopy equivalent to (Z, A) relative to A and (Z, A) has no cells of dimension less than $n + 1$.*

1.4.5 More homotopical properties

In a similar way as in homology, for homotopy groups of CW-complexes there is an excision theorem. However, it has a dimensional restriction.

Theorem 1.4.21 (Homotopy excision theorem)**.** *Let X be a topological space and let A and B be subspaces of X such that $X = A \cup B$, $(A, A \cap B)$ is an n-connected relative CW-complex with $n \geq 1$ and $(B, A \cap B)$ is an m-connected relative CW-complex. Let $j : (A, A \cap B) \to (X, B)$ be the inclusion. Then $j_* : \pi_r(A, A \cap B) \to \pi_r(X, B)$ is an isomorphism for $1 \leq r < n + m$ and an epimorphism for $r = m + n$.*

We will not give the proof here as it is quite technical and the ideas do not shed any light on our work. However, it is an important result and shows the difference in difficulty between dealing with homology groups and with homotopy groups.

As a corollary, we obtain the following proposition which is a refined version of 1.4.3.

Proposition 1.4.22. *Let (X, A) be a CW-pair such that A is m-connected and (X, A) is n-connected with $n \geq 1$. Let $q : (X, A) \to (X/A, *)$ be the quotient map. Then the induced map $q_* : \pi_r(X, A) \to \pi_r(X/A)$ is an isomorphism for $2 \leq r \leq m + n$ and an epimorphism for $r = m + n + 1$.*

Another important theorem concerning the homotopy theory of CW-complexes is the Freudhental suspension theorem. Recall that for homology groups one always has an isomorphism $H_n(X) \simeq H_{n+1}(\Sigma X)$. This is not the case for homotopy groups, where some conditions must be imposed.

Theorem 1.4.23 (Freudhental suspension theorem)**.** *Let X be an n-connected CW-complex, with $n \geq 0$. Then $\Sigma : \pi_r(X) \to \pi_{r+1}(\Sigma X)$ is an isomorphism for $1 \leq r \leq 2n$ and an epimorphism for $r = 2n + 1$.*

Using that $\pi_2(S^2) = \mathbb{Z}$ and Freudhental suspension theorem we obtain the following.

Corollary 1.4.24. *$\pi_n(S^n) = \mathbb{Z}$ for all $n \in \mathbb{N}$.*

Proof. Since S^n is $(n-1)$-connected, Freudhental suspension theorem gives an isomorphism $\Sigma : \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ if $1 \leq n \leq 2n - 2$ (i.e. if $n \geq 2$) and an epimorphism if $n = 2n - 1$ (i.e. if $n = 1$). □

The fact that $\pi_2(S^2) = \mathbb{Z}$ will be proved in next chapter (p. 65) as an application of the exact sequence of homotopy groups associated to a fibration.

We define now the stable homotopy groups of a space X.

Definition 1.4.25. Let X be a topological space and let $n \in \mathbb{N}_0$. We define the *n-th stable homotopy group of X* as $\pi_n^{\mathrm{st}}(X) = \operatorname{colim}_k \pi_{n+k}(\Sigma^k X)$.

If X is a CW-complex then $\Sigma^k X$ is $(k-1)$-connected. From the Freudhental suspension theorem it follows that $\pi_{n+k}(\Sigma^k X) \simeq \pi_{n+k+1}(\Sigma^{k+1} X)$ if $n + k \leq 2(k-1)$ or equivalently if $k \geq n + 2$. Then the groups $\pi_{n+k}(\Sigma^k X)$, $k \in \mathbb{N}$, stabilize for k sufficiently large and $\pi_n^{\mathrm{st}}(X) \simeq \pi_{n+k}(\Sigma^k X)$ for all $k \geq n + 2$.

Continuing with our comparison between homology and homotopy properties, we will analyse now the analogue for the homotopy groups of the wedge axiom for homology. As before, some hypothesis on the degrees of connectedness of the spaces are needed so that the wedge axiom holds.

Proposition 1.4.26. *Let X be an n-connected CW-complex and let Y be an m-connected CW-complex. Let $i_X : X \to X \vee Y$ and $i_Y : Y \to X \vee Y$ be the inclusion maps. Suppose that X or Y is locally compact. Then the induced map $(i_X)_* \oplus (i_Y)_* : \pi_r(X) \oplus \pi_r(Y) \to \pi_r(X \vee Y)$ is an isomorphism for $2 \leq r \leq n+m$.*

Proof. Since X or Y is locally compact the space $X \times Y$ with the product topology is a CW-complex with the product CW-structure. Let x_0 and y_0 be the base points of X and Y respectively. We know that $X \vee Y$ is homeomorphic to $X \times \{y_0\} \cup \{x_0\} \times Y \subseteq X \times Y$. Hence we will consider $X \vee Y$ as a subspace of $X \times Y$.

Since X is an n-connected CW-complex we may suppose that $X^n = \{x_0\}$. In a similar way, we may suppose that $Y^m = \{y_0\}$. Then $(X \times Y, X \vee Y)^{n+m+1} = X \vee Y$. Let $j : X \vee Y \to X \times Y$ be the inclusion map. From the long exact sequence of homotopy groups associated to the pair $(X \times Y, X \vee Y)$ we obtain that $j_* : \pi_r(X \vee Y) \to \pi_r(X \times Y)$ is an isomorphism for $1 \leq r \leq n+m$.

Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projections. Clearly, the induced map $((p_X)_*, (p_Y)_*) : \pi_r(X \times Y) \to \pi_r(X) \times \pi_r(Y)$ is an isomorphism for all r. But $\pi_r(X) \times \pi_r(Y)$ is isomorphic to $\pi_r(X) \oplus \pi_r(Y)$ for $r \geq 2$, so we consider $((p_X)_*, (p_Y)_*) : \pi_r(X \times Y) \to \pi_r(X) \oplus \pi_r(Y)$ for $r \geq 2$.

Since $((p_X)_*, (p_Y)_*) \circ j_* \circ ((i_X)_* \oplus (i_Y)_*)$ is the identity map of $\pi_r(X) \oplus \pi_r(Y)$, the result follows. $\square$

As a corollary we obtain the following

Corollary 1.4.27. *Let I be an index set. For $\alpha \in I$, let $i_\alpha : S^n \to \bigvee_{i \in I} S^n$ denote the inclusion in the α-th copy of S^n. Then the induced morphism $\bigoplus_{\alpha \in I} (i_\alpha)_* : \bigoplus_{\alpha \in I} \pi_n(S^n) \to \pi_n(\bigvee_{i \in I} S^n)$ is an isomorphism for $n \geq 2$.*

The proof follows easily from the previous proposition if I is a finite set. For the general case the result is proved by a standard compactness argument.

Recall that $\pi_1(\bigvee_{\alpha \in I} S^n)$ is the free group generated by $\{[i_\alpha] : \alpha \in I\}$.

1.4.6 Eilenberg - MacLane spaces

Definition 1.4.28. Let $n \in \mathbb{N}$ and let G be a group, which we require to be abelian if $n \geq 2$. An *Eilenberg - MacLane space of type* (G, n) is a path-connected CW-complex X such that $\pi_n(X) \simeq G$ and $\pi_i(X) = 0$ if $i \neq n$.

For example, the circle S^1 is an Eilenberg - MacLane space of type $(\mathbb{Z}, 1)$. Despite this simple example, Eilenberg - MacLane spaces are in general much more complicated than

Moore spaces. We will show now how to construct arbitrary Eilenberg - MacLane spaces. In contrast to the construction of Moore spaces, we will need to attach an infinite number of cells of different dimensions.

In order to make the key idea more explicit, we prove first the following lemma.

Lemma 1.4.29. *Let X be a path-connected CW-complex and let $n \in \mathbb{N}$, $n \geq 2$. Then there exists a CW-complex X' such that $\pi_i(X) \simeq \pi_i(X')$ for $i < n$, $\pi_n(X') = 0$ and X' is obtained from X by attaching $(n+1)$-cells.*

Proof. Let $\{\phi_\alpha : \alpha \in J\}$ be a set of generators of $\pi_n(X)$. For each $\alpha \in J$ let $g_\alpha : S^n \to X$ be a continuous map such that $[g_\alpha] = \phi_\alpha$. By cellular approximation, we may suppose that g_α is a cellular map. Let X' be the CW-complex obtained by attaching $(n+1)$-cells to X by the maps g_α, $\alpha \in J$.

Now we will see that X' satisfies the required conditions. Consider the long exact sequence in homotopy groups associated to the pair (X', X):

$$\cdots \longrightarrow \pi_r(X) \longrightarrow \pi_r(X') \longrightarrow \pi_r(X', X) \longrightarrow \pi_{r-1}(X) \longrightarrow \cdots$$

Note that $\pi_r(X', X) = 0$ for $r \leq n$. Indeed, let $[\alpha] \in \pi_r(X', X)$. Then $\alpha : (D^r, S^{r-1}) \to (X', X)$. By the cellular approximation theorem for CW-pairs α is homotopic (as maps between pairs) to a cellular map β. But since $r \leq n$, the image of β must be contained in X. Thus, by 1.4.6, $[\alpha] = 0$.

Hence, from the long exact sequence above it follows that the inclusion i_* induces isomorphisms $i_* : \pi_r(X) \to \pi_r(X')$ for $r < n$ and an epimorphism $i_* : \pi_n(X) \to \pi_n(X')$. Now, note that $i_*([g_\alpha]) = 0$ in $\pi_n(X')$ since g_α can be extended to D^n (the extension is the characteristic map of the $(n+1)$-cell attached). Since generators of $\pi_n(X)$ are mapped to 0 by the epimorphism $i_* : \pi_n(X) \to \pi_n(X')$, it follows that $\pi_n(X') = 0$. □

Proposition 1.4.30. *Let $n \in \mathbb{N}$ and let G be a group if $n = 1$ and an abelian group if $n \geq 2$. Then there exists an Eilenberg - MacLane space of type (G, n).*

Proof. We prove the case $n \geq 2$. The case $n = 1$ is similar and can be found in [8] (corollary 1.28).

Let $\{g_i : i \in I\} \subseteq G$ be a set of generators of G and let F be a free abelian group with basis $\{g_i : i \in I\}$. Let $\phi : F \to G$ be the induced group homomorphism. Then $\ker \phi$ is also a free abelian group. Let $\{h_\alpha : \alpha \in J\}$ be a basis of $\ker \phi$. We will construct an Eilenberg - MacLane space of type (G, n) attaching cells to $\bigvee_{i \in I} S^n$. Note that $\pi_n(\bigvee_{i \in I} S^n) = \bigoplus_{i \in I} \mathbb{Z} = F$.

For each $\alpha \in J$ let $g_\alpha : S^n \to \bigvee_{i \in I} S^n$ be a continuous map such that $[g_\alpha] \in \pi_n(\bigvee_{i \in I} S^n)$ corresponds to $h_\alpha \in F$.

Let $X^n = \bigvee_{i\in I} S^n$ and let X^{n+1} be defined by the pushout

$$\begin{tikzcd} \coprod_{\alpha\in J} S^n \arrow[r, "\coprod_{\alpha\in J} g_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & \bigvee_{i\in I} S^n \arrow[d] \\ \coprod_{\alpha\in J} D^{n+1} \arrow[r] & X^{n+1} \end{tikzcd}$$

Consider the following diagram

$$\begin{tikzcd} \pi_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d, "\varphi"', "\simeq"] & \pi_n(X^n) \arrow[r, "i_*"] \arrow[d, "\beta"', "\simeq"] & \pi_n(X^{n+1}) \arrow[r] & \pi_n(X^{n+1}, X^n) = 0 \\ \ker\phi \arrow[r, "\text{inc}"'] & F \arrow[r, "\phi"] & G \arrow[r] & 0 \end{tikzcd}$$

where the isomorphism φ is given by the composition

$$\pi_{n+1}(X^{n+1}, X^n) \simeq \pi_{n+1}(X^{n+1}/X^n) \simeq \pi_{n+1}(\bigvee_{\alpha\in J} S^n) \simeq \ker\phi.$$

Hence, the characteristic maps $\{f_\alpha : \alpha \in J\}$ form a basis of $\pi_{n+1}(X^{n+1}, X^n)$ since they correspond to the inclusions $i_\alpha : S^n \to \bigvee_{\alpha\in J} S^n$. Note that the above square commutes since $\partial([f_\alpha]) = [g_\alpha] = \beta^{-1}(h_\alpha)$.

Thus, $\pi_n(X^{n+1}) \simeq G$. And $\pi_r(X^{n+1}) \simeq \pi_r(X^n) = 0$ for $r < n$, since X^{n+1} is obtained from X^n by attaching $(n+1)$-cells.

Applying the previous lemma, we build inductively a sequence of CW-complexes $(X^m)_{m\geq n+2}$ such that $\pi_n(X^m) \simeq G$, $\pi_r(X^m) = 0$ for $r \leq m-1$, $r \neq n$ and X^m is obtained from X^{m-1} by attaching m-cells.

We take $X = \operatorname{colim} X^n$. Since $\pi_r(X) \simeq \pi_r(X^{r+1})$ for all $r \in \mathbb{N}$, it follows that X is an Eilenberg - MacLane space of type (G, n). □

Corollary 1.4.31. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of abelian groups. Then there exists a path-connected CW-complex X such that $\pi_n(X) = G_n$ for all $n \in \mathbb{N}$.*

Proof. Take $X = \prod_{n\in\mathbb{N}} X_n$, where X_n is an Eilenberg - MacLane space of type (G_n, n). □

An important result is that, for fixed G and n, Eilenberg - MacLane spaces of type (G, n) are unique up to homotopy equivalence.

Proposition 1.4.32. *The homotopy type of an Eilenberg - MacLane space of type (G, n) is uniquely determined by G and n.*

The proof is not difficult but technical and we omit it.

1.4.7 The Hurewicz theorem

As it was mentioned earlier, Hurewicz's theorem is one of the most important theorems of homotopy theory. It connects homotopy theory with homology theory via a map known as the Hurewicz map, which we recall right now.

Let X be a topological space and let $1 \in H_n(S^n) \simeq \mathbb{Z}$ be a fixed generator of $H_n(S^n)$. We define the *Hurewicz map* $h : \pi_n(X) \to H_n(X)$ by $h([\alpha]) = \alpha_*(1)$.

A relative version of the Hurewicz map can also be defined and this is done as follows. Let (X, A) be a topological pair and let $1 \in H_n(D^n, S^{n-1}) \simeq \mathbb{Z}$ be a fixed generator of $H_n(D^n, S^{n-1})$. We define $h : \pi_n(X) \to H_n(X)$ by $h([\alpha]) = \alpha_*(1)$.

Before stating Hurewicz's theorem we give a technical lemma which will be needed for the proof of the theorem.

Lemma 1.4.33. *Let $n \in \mathbb{N}$ and let X be a space obtained by attaching $(n+1)$-cells to $\bigvee\limits_{\alpha\in I} S^n$. Let Y be a path connected topological space and let $\varphi : \pi_n(X) \to \pi_n(Y)$ be a group homomorphism. Then there exists a continuous map $f : X \to Y$ such that the induced map $f_* : \pi_n(X) \to \pi_n(Y)$ coincides with φ.*

Proof. Let $i_\alpha : S^n \to X$ denote the composition of the inclusions $S^n \to \bigvee\limits_{i\in I} S^n$ and $\bigvee\limits_{i\in I} S^n \to X$, where the first one is the inclusion in the α-th copy, and let $\eta_\alpha : S^n \to Y$ be a continuous map such that $[\eta_\alpha] = \varphi([i_\alpha])$.

We define $\eta : \bigvee\limits_{\alpha\in I} S^n \to Y$ by $\eta = \underset{\alpha\in I}{+} \eta_\alpha$. Then $\eta_*([i_\alpha]) = [\eta \circ i_\alpha] = [\eta_\alpha] = \varphi([i_\alpha])$ for all $\alpha \in I$. Since $\{i_\alpha : \alpha \in I\}$ is a set of generators of $\pi_n(\bigvee\limits_{i\in I} S^n)$ we obtain that $\eta(\gamma) = \varphi(\gamma)$ for all $\gamma \in \pi_n(\bigvee\limits_{i\in I} S^n)$.

Let J be an index set for the $(n+1)$-cells of X. For each $\beta \in J$ let $g_\beta : S^n \to \bigvee\limits_{\alpha\in I} S^n$ be the attaching map of the cell e_β^{n+1}. Note that g_β is nullhomotopic in X since the corresponding characteristic map is an extension of $g_\beta : S^n \to X$ to the cone $\mathrm{C}S^n$.

Thus, $[\eta \circ g_\beta] = \eta_*([g_\beta]) = \psi([g_\beta]) = 0$ since $[g_\beta] = 0$ in $\pi_n(X)$. Hence, there exists continuous maps $\varphi_\beta : D^{n+1} \to Y$ such that $\phi_\beta|_{S^n} = \eta \circ g_\beta$.

We define $f : X \to Y$ as the dotted arrow in the diagram

$$\begin{tikzcd}
\coprod\limits_{\beta\in J} S^n \arrow[r, "\coprod_{\beta\in J} g_\beta"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & \bigvee\limits_{\alpha\in I} S^n \arrow[d, "\mathrm{inc}"] \arrow[ddr, bend left, "\eta"] & \\
\coprod\limits_{\beta\in J} D^{n+1} \arrow[r, "\coprod_{\beta\in J} f_\beta"'] \arrow[drr, bend right, "\coprod_{\beta\in J} \phi_\beta"'] & X \arrow[dr, dotted, "f"] & \\
 & & Y
\end{tikzcd}$$

Since the inclusion $\mathrm{inc} : X^n \to X$ induces an epimorphism $\mathrm{inc}_* : \pi_n(X^n) \to \pi_n(X)$ by 1.4.5, then $\{\mathrm{inc}_*([i_\alpha]) : \alpha \in I\}$ is a set of generators of $\pi_n(X)$. Note that $f_*(\mathrm{inc}_*([i_\alpha])) = \eta_*([i_\alpha]) = \varphi([i_\alpha])$. Hence, $f_* = \varphi$. □

Theorem 1.4.34 (Hurewicz's theorem)**.** *Let $n \in \mathbb{N}$, $n \geq 2$ and let X be an $(n-1)$-connected topological space. Then $\widetilde{H}_i(X) = 0$ for $i < n$ and the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism.*

Proof. By CW-approximation we may suppose that X is a CW-complex. Since X is $(n-1)$-connected, by 1.4.20 we may also suppose that $X^{n-1} = *$. Finally, since $\pi_i(X) = \pi_i(X^{n+1})$ for $i < n$ and $H_i(X) = H_i(X^{n+1})$ for $i < n$, we may suppose that $X = X^{n+1}$. Thus, X is in the hypothesis of the previous lemma and clearly $\widetilde{H}_i(X) = 0$ for $i < n$.

We consider the long exact sequence of homotopy groups associated to the pair (X, X^n):

$$\cdots \longrightarrow \pi_{n+1}(X, X^n) \xrightarrow{\partial_{n+1}} \pi_n(X^n) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, X^n) \longrightarrow \cdots$$

Since (X, X^n) is n-connected, $\pi_n(X) \simeq \operatorname{coker} \partial_{n+1}$. By 1.4.22,

$$\pi_{n+1}(X, X^n) = \pi_{n+1}(X/X^n) = \pi_{n+1}\Big(\bigvee_{\beta \in J_{n+1}} S^{n+1}\Big) = \bigoplus_{\beta \in J_{n+1}} \mathbb{Z}$$

It is not hard to prove that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(X, X^n) \arrow[r, "\partial_{n+1}"] \arrow[d, "h"] & \pi_n(X^n) \arrow[r] \arrow[d, "h"] & \pi_n(X) \arrow[r] \arrow[d, "h"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X, X^n) \arrow[r, "\partial_{n+1}"] & H_n(X^n) \arrow[r] & H_n(X) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and that the maps $h : \pi_{n+1}(X, X^n) \simeq \bigoplus_{\beta \in J_{n+1}} \mathbb{Z} \to H_{n+1}(X, X^n) \simeq \bigoplus_{\beta \in J_{n+1}} \mathbb{Z}$ and $h : \pi_n(X^n) \simeq \bigoplus_{\alpha \in J_n} \mathbb{Z} \to H_n(X^n) \simeq \bigoplus_{\alpha \in J_n} \mathbb{Z}$ are isomorphisms.

Since the rows are exact, the map $h : \pi_n(X) \to H_n(X)$ is also an isomorphism. □

We give now the relative version of Hurewicz's theorem.

Theorem 1.4.35 (Relative version of Hurewicz's theorem)**.** *Let $n \in \mathbb{N}$, $n \geq 2$ and let (X, A) be an $(n-1)$-connected topological pair such that A is simply-connected and non-empty. Then $\widetilde{H}_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \simeq H_n(X, A)$.*

Proof. By 1.4.19, we may suppose that (X, A) is a CW-pair. Let $q : (X, A) \to (X/A, *)$ be the quotient map and consider the following diagram

$$\begin{tikzcd}
\pi_n(X, A) \arrow[r, "q_*"] \arrow[d, "h"'] & \pi_n(X/A) \arrow[d, "h"] \\
H_n(X, A) \arrow[r, "q_*"'] & H_n(X/A)
\end{tikzcd}$$

It is not difficult to prove that this diagram commutes. We have that $q_* : \pi_n(X, A) \to \pi_n(X/A)$ is an isomorphism by 1.4.22 and that $q_* : H_n(X, A) \to H_n(X/A)$ is an isomorphism. Also $h : \pi_n(X/A) \to H_n(X/A)$ is an isomorphism by the previous version of the Hurewicz theorem. Hence, $h : \pi_n(X, A) \to H_n(X, A)$ is an isomorphism. □

From the relative Hurewicz theorem we obtain the homological version of Whitehead's theorem.

Theorem 1.4.36. *Let X and Y be simply connected CW-complexes and let $f : X \to Y$ be a continuous map such that $f_* : H_n(X) \to H_n(Y)$ is an isomorphism for all $n \in \mathbb{N}_0$. Then f is a homotopy equivalence.*

Proof. We may suppose that f is a cellular map. From the hypotheses of the theorem and from the long exact sequences of homotopy and homology groups associated to the pair (Z_f, X) we obtain that $\pi_1(Z_f, X) = 0$ and $H_n(Z_f, X) = 0$ for all $n \in \mathbb{N}$. Hence, $\pi_n(Z_f, X) = 0$ for all $n \in \mathbb{N}$ by the relative version of Hurewicz's theorem. Then the inclusion $i : X \to Z_f$ is a weak equivalence. Since X and Z_f are CW-complexes, i is a homotopy equivalence by Whitehead's theorem. But $f = ri$, where $r : Z_f \to Y$ is the standard strong deformation retraction. Hence, f is a homotopy equivalence. □

1.4.8 Homology decomposition

We study now homology decomposition of spaces, which will prove useful for our work.

Definition 1.4.37. Let Y be a topological space. A *homology decomposition of* Y is a CW-complex X together with a homotopy equivalence $f : X \to Y$ and a sequence $(X_n)_{n\in\mathbb{N}}$ of subcomplexes of X satisfying

(a) $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$.

(b) $X = \displaystyle\bigcup_{n\in\mathbb{N}} X_n$.

(c) X_1 is a Moore space of type $(H_1(Y), 1)$.

(d) For all $n \in \mathbb{N}$, X_{n+1} is the mapping cone of a cellular map $g_n : M_n \to X_n$, where M_n is a Moore space of type $(H_{n+1}(Y), n)$, and g_n is such that the induced map $(g_n)_* : H_n(M_n) \to H_n(X_n)$ is trivial.

Remark 1.4.38. The CW-complexes X_n satisfy $H_i(X_n) = H_i(Y)$ for $i \leq n$ and $H_i(X_n) = 0$ for $i > n$. Indeed, for $n = 1$ this holds by (c). Suppose that the statement is true for X_n. Consider the long exact sequence of homology associated to the pair (X_{n+1}, X_n):

$$\cdots \longrightarrow H_i(X_n) \longrightarrow H_i(X_{n+1}) \longrightarrow H_i(X_{n+1}, X_n) \xrightarrow{\partial_i} H_{i-1}(X_n) \longrightarrow \cdots$$

Since (X_{n+1}, X_n) is a CW-pair, $H_i(X_{n+1}, X_n) \simeq H_i(X_{n+1}/X_n) \simeq H_i(\Sigma M_n) \simeq H_{i-1}(M_n)$. It is easy to prove that under this isomorphism the boundary map ∂_{n+1} coincides with $(g_n)_*$ which is trivial. Since $H_{n+1}(X_n) = 0$ we obtain that $H_{n+1}(X_{n+1}) \simeq H_n(M_n) = H_{n+1}(Y)$. Since $H_i(X_{n+1}, X_n) \simeq H_{i-1}(M_n) = 0$ for $i \neq n+1$ and $\partial_{n+1} = 0$ we obtain that $H_i(X_n) \simeq H_i(X_{n+1})$ for $i \neq n+1$.

Theorem 1.4.39. *Every simply-connected CW-complex admits a homology decomposition.*

Proof. Let Y be a simply-connected CW-complex. We will build the CW-complexes X_n of the above definition inductively, together with maps $f_n : X_n \to Y$ inducing isomorphisms in H_i for $i \leq n$. Since Y is simply-connected we take $X_1 = *$ and $f_1 : X_1 \to Y$ any map.

Suppose we have constructed CW-complexes X_j and maps $f_j : X_j \to Y$ for $j \leq n$ such that the spaces X_j satisfy the conditions of the previous definition and f_j induces isomorphisms in H_i for $i \leq j$. Let Z_{f_n} be the mapping cylinder of f_n, let $i : X_n \to Z_{f_n}$ be the inclusion and let $r : Z_{f_n} \to Y$ be the standard strong deformation retraction.

Note that by the long exact sequence in homology associated to the pair (Z_{f_n}, X_n) we have that $H_i(Z_{f_n}, X_n) = 0$ for $i \leq n$ since f_n induces isomorphisms in H_i for $i \leq n$. By the Hurewicz theorem and the long exact sequence mentioned previously we obtain that $\pi_{n+1}(Z_{f_n}, X_n) \simeq H_{n+1}(Z_{f_n}, X_n) \simeq H_{n+1}(Z_{f_n}) \simeq H_{n+1}(Y)$.

As we have seen before, we can build a Moore space of type $(H_{n+1}(Y), n)$, M_n, as follows. We take a wedge of spheres S^n_λ corresponding to a set $\{g_\lambda : \lambda \in \Lambda\}$ of generators of $H_{n+1}(Y)$ and we attach $(n+1)$-cells according to certain relations $\{r_\alpha = \sum\limits_{\lambda\in\Lambda} m^{(\alpha)}_\lambda g_\lambda = 0 : \alpha \in J\}$ such that the group $H_{n+1}(Y)$ is the abelian group generated by elements $\{g_\lambda : \lambda \in \Lambda\}$ satisfying the relations $\{r_\alpha = \sum\limits_{\lambda\in\Lambda} m^{(\alpha)}_\lambda g_\lambda = 0 : \alpha \in J\}$.

Under the isomorphism $\pi_{n+1}(Z_{f_n}, X_n) \simeq H_{n+1}(Y)$, each generator g_λ corresponds to a map $f_\lambda : (D^{n+1}, S^n) \to (Z_{f_n}, X_n)$ which may be supposed cellular. Hence, from the relations above we get $\sum\limits_{\lambda\in\Lambda} m^{(\alpha)}_\lambda [f_\lambda] = 0$ in $\pi_{n+1}(Z_{f_n}, X_n)$ for all $\alpha \in J$. The corresponding homotopy H_α may be considered as a map $H_\alpha : (\mathrm{C}D^{n+1}, \mathrm{C}S^n) \to (Z_{f_n}, X_n)$ such that $H_\alpha \mathrm{inc} = \sum\limits_{\lambda\in\Lambda} m^{(\alpha)}_\lambda f_\lambda$, where $\mathrm{inc} : (D^{n+1}, S^n) \to (\mathrm{C}D^{n+1}, \mathrm{C}S^n)$ is the inclusion map and it also might be taken cellular.

For $\lambda \in \Lambda$, let $i_\lambda : S^n \to \bigvee\limits_{i\in\Lambda} S^n$ be the inclusion in the λ-th copy. We consider the following commutative diagram of solid arrows

$$\begin{tikzcd}
 & \bigsqcup\limits_{\alpha\in J} S^n \arrow[rr, "\bigsqcup\limits_{\alpha\in J}(\sum\limits_{\lambda\in\Lambda} m^{(\alpha)}_\lambda i_\lambda)"] \arrow[d] \arrow[ddl] \arrow[drr, phantom, "push"] & & \bigvee\limits_{i\in\Lambda} S^n \arrow[d] \arrow[ddl] \arrow[ddr, bend left, "\underset{\lambda\in\Lambda}{+} f_\lambda|_{S^n}"] & \\
 & \bigsqcup\limits_{\alpha\in J} D^{n+1} \arrow[rr] \arrow[ddl] \arrow[drrr, "\bigsqcup\limits_{\alpha\in J} H_\alpha|_{\mathrm{C}S^n}"'] & & M_n \arrow[ddl] \arrow[dr, dotted, "g_n"] & \\
\bigsqcup\limits_{\alpha\in J} \mathrm{C}S^n \arrow[rr, "\mathrm{C}(\bigsqcup\limits_{\alpha\in J}(\sum\limits_{\lambda\in\Lambda} m^{(\alpha)}_\lambda i_\lambda))"] \arrow[d, "\mathrm{inc}"'] \arrow[drr, phantom, "push"] & & \bigvee\limits_{i\in\Lambda} \mathrm{C}S^n \arrow[d] \arrow[drr, "\underset{\lambda\in\Lambda}{+} f_\lambda"] & & X_n \arrow[d, "\mathrm{inc}"] \\
\bigsqcup\limits_{\alpha\in J} \mathrm{C}D^{n+1} \arrow[rr] \arrow[rrrr, bend right, "\bigsqcup\limits_{\alpha\in J} H_\alpha"'] & & \mathrm{C}M_n \arrow[rr, dotted, "\varphi_n"] & & Z_{f_n}
\end{tikzcd}$$

and define the dotted arrows g_n and φ_n such that the whole diagram commutes. Note that g_n and φ_n are also cellular maps. Let X_{n+1} be the cone of the map g_n and let $f'_{n+1} : X_{n+1} \to Z_{f_n}$ be defined so as to make commutative the following diagram

$$\begin{tikzcd} M_n \arrow[r, "g_n"] \arrow[d] \arrow[dr, phantom, "push"] & X_n \arrow[d] \arrow[ddr, bend left, "\mathrm{inc}"] & \\ \mathrm{C}M_n \arrow[r] \arrow[drr, bend right, "\varphi_n"'] & X_{n+1} \arrow[dr, "f'_{n+1}"] & \\ & & Z_{f_n} \end{tikzcd}$$

By construction, it is not hard to prove that f'_{n+1} induces an isomorphism $(f'_{n+1})_* : H_{n+1}(X_{n+1}, X_n) \to H_{n+1}(Z_{f_n}, X_n)$. Then, by the five-lemma we obtain that f'_{n+1} induces isomorphisms $(f'_{n+1})_* : H_i(X_{n+1}) \to H_i(Z_{f_n})$ for $i \leq n$. We take $f_{n+1} = rf'_{n+1} : X_{n+1} \to Y$ which also induces isomorphisms in H_i for $i \leq n$ since r is a homotopy equivalence. Let $X = \bigcup\limits_{n \in \mathbb{N}} X_n$ and $f : X \to Y$ be defined by $f(x) = f_n(x)$ if $x \in X_n$. It is clear that f is well defined and continuous and that f induces isomorphisms in all homology groups. Since X and Y are simply connected CW-complexes it follows that f is a homotopy equivalence. $\square$

Chapter 2

Fibrations and spectral sequences

Spectral sequences constitute a powerful computational tool whose areas of application include algebra, topology and geometry. They where introduced by Leray in 1946 to compute sheaf cohomology, but some years later other mathematicians noticed that his idea could be applied to other settings. One of them was J.-P. Serre, who introduces a spectral sequence associated to fibration of topological spaces [17]. The Serre spectral sequence is a key tool in algebraic topology which has many applications including, for example, computations regarding homotopy groups of spheres and a generalization of Hurewicz's theorem.

In the mid fifties, Federer also applies the machinery of spectral sequences to study homotopy groups of spaces of maps and develops the spectral sequence named after him.

In this chapter we begin by recalling the definition of fibrations and some basic properties, such as the long exact homotopy sequence associated to a fibration. Then, we give the definition and construction of Postnikov towers, which can be thought as a kind of homotopy decomposition.

In the third section we give an algebraic approach to spectral sequences with some examples and results. We then introduce Serre spectral sequence in section 4, using in its proof the algebraic results given before. We also include important applications of it, such as the generalized version of Hurewicz's theorem and some others about homotopy group of spheres.

Section 5 deals with localization of topological spaces. The key theorem of this section is proved with the aid of Serre spectral sequence.

In the last section we give an alternative construction of Federer's spectral sequence [6] which is useful to obtain information about A-homotopy groups of spaces. We also use it to prove a generalization of Hopf-Whitney theorem (2.5.6).

2.1 Fibrations

Fibrations constitute a class of continuous maps of great importance in algebraic topology. Together with cofibrations and weak equivalences they form the basis of the classical homotopy theory and serve as models for abstract homotopy theories.

Before starting to work with fibrations we will define lifting properties so as to have language for later use.

Definition 2.1.1. Let $f : W \to X$ and $g : Y \to Z$ be continuous maps between topological spaces. We say that f *has the left lifting property with respect to* g or that g *has the right lifting property with respect to* f if for every continuous maps $h : W \to Y$ and $k : X \to Z$ such that $gh = kf$ there exists a map $\phi : X \to Y$ such that $\phi f = h$ and $g\phi = k$.

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "k"'] \arrow[ur, dotted, "\phi"] & Z \end{tikzcd}$$

Definition 2.1.2. Let $f : X \to Y$ be a continuous map and let $\mathscr{F}$ be a class of continuous maps between topological spaces. We say that f *has the left (resp. right) lifting property with respect to* $\mathscr{F}$ if f has the left (resp. right) lifting property with respect to g for all $g \in \mathscr{F}$.

Definition 2.1.3. Let $f : Y \to Z$ be a continuous map and let X be a topological space. We say that f *has the homotopy lifting property with respect to* X if f has the right lifting property with respect to $i_0 : X \to IX$, i.e. if for every continuous map $g : X \to Y$ and every homotopy $H : IX \to Z$ such that $Hi_0 = fg$ there exists a homotopy $\overline{H} : IX \to Y$ such that $\overline{H}i_0 = g$ and $f\overline{H} = H$.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i_0"'] & Y \arrow[d, "f"] \\ IX \arrow[r, "H"'] \arrow[ur, dotted, "\overline{H}"] & Z \end{tikzcd}$$

Now we are ready to define fibrations.

Definition 2.1.4. Let $p : E \to B$ be a continuous map. We say that p is a *fibration* if it has the homotopy lifting property with respect to every space X.

In this case, B is called the *base space of the firbation* p and E is called the *total space of* p. If $b_0 \in B$, the space $F_{b_0} = p^{-1}(b_0)$ is called the *fibre over* b_0.

Examples 2.1.5.

(a) If B and F are topological spaces, the projection $p : B \times F \to B$ is a fibration with fibre F.

(b) Fibre bundles over paracompact spaces are fibrations.

(c) The *Hopf map* $\eta : S^3 \to S^2$ is defined in the following way. We interpret $S^3 = \{(z, w) \in \mathbb{C}^2 \ / \ |z|^2 + |w|^2 = 1\}$ and $S^2 = \{(z, x) \in \mathbb{C} \times \mathbb{R} \ / \ |z|^2 + |x|^2 = 1\}$ and we define $\eta(z, w) = (2z\overline{w}, |z|^2 - |w|^2)$. Then $\eta : S^3 \to S^2$ is a fibration with fibre S^1.

(d) Let X be a topological space and let $x_0 \in X$. Let $PX = \{\gamma : [0, 1] \to X \mid \gamma(0) = x_0\}$ and let $p : PX \to X$ be defined by $p(\gamma) = \gamma(1)$. Then p is a fibration and $F_{x_0} = \Omega X$. The fibration p is called the *path-space fibration.*

(*e*) Related to the previous example is the following construction. Let $f : A \to B$ be a continuous map and let $E_f \subseteq A \times B^I$ be defined by $E_f = \{(a, \gamma) \in A \times B^I \mid \gamma(0) = f(a)\}$. Note that E_f can also be defined by the pullback diagram

$$\begin{tikzcd} E_f \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "pull"] & A \arrow[d, "f"] \\ B^I \arrow[r, "\mathrm{ev}_0"'] & B \end{tikzcd}$$

Then the map $p : E_f \to B$ defined by $p(a, \gamma) = \gamma(1)$ is a fibration.

The construction of the last example above is very important since it gives us a way to write any map as a composition of a fibration and a homotopy equivalence. Hence, if we are only interested in homotopy types, any map can be converted into a fibration.

Proposition 2.1.6. *Let $f : A \to B$ be a continuous map. Then there exists a factorization $f = pi$ with p a fibration and i a homotopy equivalence.*

Proof. Let E_f and $p : E_f \to B$ be defined as above and define $i : A \to E_f$ by $i(a) = (a, c_{f(a)})$, where $c_{f(a)}$ is the constant path at $f(a)$. It is easy to prove that i is a strong deformation retract, hence a homotopy equivalence. Clearly, $pi = f$. □

Definition 2.1.7. Let $p : E \to B$ be a continuous map. We say that p is a *Serre fibration* if it has the homotopy lifting property with respect to D^n for all $n \geq 0$.

Note that any fibration is a Serre fibration.

From the fact that there is a homeomorphism of topological pairs $(I^{n+1}, I^n \times \{0\}) \simeq (ID^n, D^n \times \{0\})$ it follows that a continuous map p is a Serre fibration if and only if it has the homotopy lifting property with respect to I^n for all $n \geq 0$.

We will now derive the long exact homotopy sequence associated to a Serre fibration $p : E \to B$. This is a very useful tool and constitutes one of the key homotopy properties of fibrations. It relates the homotopy groups of the base space, the total space and the fibre of a Serre fibration $p : E \to B$ and it is obtained in the following way.

Fix $b_0 \in B$ and $e_0 \in p^{-1}(b_0)$. Let $F = p^{-1}(b_0)$. We consider the long exact sequence of homotopy groups associated to the topological pair (E, F):

$$\cdots \longrightarrow \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \longrightarrow \pi_n(E, F, e_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \longrightarrow \cdots$$

We shall prove that $\pi_n(E, F, e_0) \simeq \pi_n(B, b_0)$. More precisely, we will prove that the Serre fibration p induces an isomorphism $p_* : \pi_n(E, F, e_0) \to \pi_n(B, \{b_0\}, b_0) \simeq \pi_n(B, b_0)$ for all $n \geq 1$.

In order to do this more easily, we will work with an alternative description of the homotopy groups. Let X be a topological space and let $x_0 \in X$ and $A \subseteq X$ such that $x_0 \in A$. When needed, for $n \geq 2$, we regard I^{n-1} as the subspace $I^{n-1} \times \{0\} \subseteq I^n$. Let

$$J^{n-1} = \bigcup_{i=1}^{n-1} \{(x_1, \ldots, x_n) \in I^n \;/\; x_i = 0 \text{ or } x_i = 1\} \cup \{(x_1, \ldots, x_n) \in I^n \;/\; x_n = 1\}$$

Note that $I^{n-1} \cup J^{n-1} = \partial I^n$.

It is known that $\pi_n(X, x_0)$ can also be defined as $[(I^n, \partial I^n), (X, x_0)]$ and that $\pi_n(X, A, x_0)$ can also be defined as $[(I^n, I^{n-1}, J^{n-1}), (X, A, x_0)]$. And it is easy to prove that we can make a slight modification in this last alternative definition to obtain that $\pi_n(X, A, x_0)$ is in bijection with $[(I^n, J^{n-1}, I^{n-1}), (X, A, x_0)]$.

We will see now that $p_* : \pi_n(E, F, e_0) \to \pi_n(B, \{b_0\}, b_0) \simeq \pi_n(B, b_0)$ is surjective. Let $[g] \in \pi_n(B, b_0)$ with $g : (I^n, \partial I^n) \to (B, b_0)$ and let $q : I^n \to I^n/I^{n-1}$ be the quotient map. Let $\psi : (I^n/I^{n-1}, \partial I^n/I^{n-1}, *) \to (I^n, \partial I^n, *)$ be a homeomorphism. There is a commutative diagram

$$\begin{tikzcd} I^{n-1} \arrow[r, "c_{e_0}"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ I^n \arrow[r, "g\psi q"'] & B \end{tikzcd}$$

Since the topological pair (I^n, I^{n-1}) is homeomorphic to $(ID^{n-1}, D^{n-1} \times \{0\})$ and p is a Serre fibration, there exists a continuous map $g' : I^n \to E$ such that $pg' = g\psi q$ and $g' i_0 = c_{e_0}$. Thus, there exists a continuous map $g'' : (I^n/I^{n-1}, \partial I^n/I^{n-1}, *) \to (E, F, e_0)$ such that $g'' q = g'$. Let $\overline{g} = g'' \psi^{-1}$. Note that $\overline{g} : (I^n, J^{n-1}, I^n) \to (E, F, e_0)$ and that $p\overline{g}\psi q = pg''\psi^{-1}\psi q = pg'' q = pg' = g\psi q$.

$$\begin{tikzcd} I^n \arrow[r, "q"] \arrow[dr, "g'"'] & I^n/I^{n-1} \arrow[r, "\psi"] \arrow[d, "g''"] & I^n \arrow[d, "g"] \arrow[dl, "\overline{g}"'] \\ & E \arrow[r, "p"'] & B \end{tikzcd}$$

But since q is surjective and ψ is a homeomorphism we conclude that $p\overline{g} = g$. Thus p_* is surjective.

We prove now that $p_* : \pi_n(E, F, e_0) \to \pi_n(B, \{b_0\}, b_0) \simeq \pi_n(B, b_0)$ is injective. Let $\alpha, \beta : (I^n, J^{n-1}, I^{n-1}) \to (E, F, \{e_0\})$ be continuous maps such that $p \circ \alpha \simeq p \circ \beta$ and let $H : I^n \times I \to B$ be a homotopy between $p \circ \alpha$ and $p \circ \beta$ relative to ∂I^n.

We define $\gamma : I^n \times \{0, 1\} \cup I^{n-1} \times \{0\} \times I \to E$ by

$$\gamma(x_1, x_2, \ldots, x_{n-1}, x_n, t) = \begin{cases} \alpha(x_1, \ldots, x_n) & \text{if } t = 0 \\ \beta(x_1, \ldots, x_n) & \text{if } t = 1 \\ e_0 & \text{if } x_n = 0 \end{cases}$$

To prove injectivity, we must demonstrate that there exists a lift

$$\begin{tikzcd} I^n \times \{0,1\} \cup I^{n-1} \times \{0\} \times I \arrow[r, "\gamma"] \arrow[d, "\mathrm{inc}"'] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "H"'] \arrow[ur, dotted, "\overline{H}"'] & B \end{tikzcd}$$

since if such a lift exists, then $\overline{H}(\partial I^n \times I) \subseteq F$ and $\overline{H}(I^{n-1} \times \{0\} \times I) = \{e_0\}$.

Note that the topological pair $(I^2, I \times \{0,1\} \cup \{0\} \times I)$ is homeomorphic to $(I^2, I \times \{0\})$. Taking product with I^{n-1} gives a homeomorphism $(I^{n+1}, I^{n-1} \times \{0,1\} \cup \{0\} \times I^{n-1}) \simeq (I^{n+1}, I^n \times \{0\})$.

Since the inclusion $i : I^n \to I^{n+1}$ has the left lifting property with respect to p, so does the inclusion $I^{n-1} \times \{0,1\} \cup \{0\} \times I^{n-1} \to I^{n+1}$. Hence, the desired lifting exists. Thus, p_* is injective.

In consequence, $p_* : \pi_n(E, F, e_0) \to \pi_n(B, b_0)$ is a bijection for $n \geq 1$ and we obtain a long exact sequence

$$\cdots \longrightarrow \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \longrightarrow \cdots$$

which is called the *exact homotopy sequence of the Serre fibration* $p : E \to B$.

As an example of an application, consider the exact homotopy sequence of the Hopf fibration $\eta : S^3 \to S^2$.

$$\cdots \to \pi_3(S^1) \xrightarrow{i_*} \pi_3(S^3) \xrightarrow{\eta_*} \pi_3(S^2) \xrightarrow{\partial} \pi_2(S^1) \xrightarrow{i_*} \pi_2(S^3) \xrightarrow{\eta_*} \pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1) \xrightarrow{i_*} \pi_1(S^3) \xrightarrow{\eta_*} \cdots$$

Since $\pi_1(S^3)$ and $\pi_2(S^3)$ are trivial by corollary 1.4.8, we obtain that $\pi_2(S^2) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.

Note that from the homology of S^n and the Hurewicz theorem we can also deduce that $\pi_n(S^n) \simeq \mathbb{Z}$ for all $n \geq 2$.

On the other hand, $\pi_2(S^1) = 0$ and $\pi_3(S^1) = 0$ because the universal covering of S^1 is contractible. Hence, from the exact sequence above we obtain that $\pi_3(S^2) \simeq \pi_3(S^3) \simeq \mathbb{Z}$.

Another important homotopy property of fibrations is that changing the base point in the base space we obtain homotopy equivalent fibers provided that the base space is path-connected.

Proposition 2.1.8. *Let $p : E \to B$ be a Serre fibration and let $b_0, b_1 \in B$. Let $\gamma : I \to B$ be a continuous map such that $\gamma(0) = b_0$ and $\gamma(1) = b_1$. Then γ induces a homotopy equivalence $\phi_\gamma : F_{b_0} \to F_{b_1}$.*

The proof can be found, for example, in [8].

2.1.1 Postnikov towers

In this subsection we recall the definition and construction of Postnikov towers.

Definition 2.1.9. Let X be a path-connected topological space. A *Postnikov tower for*

X is a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d, "p_3"] \\
 & X_2 \arrow[d, "p_2"] \\
X \arrow[r, "f_1"'] \arrow[ur, "f_2"'] \arrow[uur, "f_3"] & X_1
\end{tikzcd}$$

such that

(a) For all $j \geq 2$, the map $p_j : X_j \to X_{j-1}$ is a fibration.

(b) The induced map $(f_n)_* : \pi_i(X) \to \pi_i(X_n)$ is an isomorphism for $i \leq n$.

(c) $\pi_i(X_n) = 0$ for $i > n$.

Note that if F_n is the fibre of the fibration p_n, from the exact homotopy sequence for p_n it follows that F_n is an Eilenberg - MacLane space of type $(\pi_n(X), n)$.

Thus, the spaces X_n can be thought as 'homotopy approximations' of X, and we have better approximations as n increases.

We want to prove that any path-connected CW-complex admits a Postnikov tower, for which we need the following lemma.

Lemma 2.1.10. *Let (X, A) be a relative CW-complex and let Y be a path-connected topological space. Suppose that for all $n \in \mathbb{N}_0$ such that there exists at least one n-cell in $X - A$ we have that $\pi_{n-1}(Y) = 0$. Then every continuous map $f : A \to Y$ can be extended to X, i.e. for every continuous map $f : A \to Y$ there exists a continuous map $\overline{f} : X \to Y$ such that $\overline{f}|_A = f$.*

Its proof follows from the fact that a nullhomotopic map $S^{n-1} \to X$ can be extended to the disk D^n.

Theorem 2.1.11. *Let X be a path-connected CW-complex. Then X admits a Postnikov tower.*

Proof. By 1.4.17, we may build a CW-complex X_n such that (X_n, X) is an $(n+1)$-model of $(\mathrm{C}X, X)$. Moreover, we may construct X_n by attaching cells of dimension greater than $(n+1)$ to X. We take $f_n : X \to X_n$ to be the inclusion.

By the above lemma, the map $f_n : X \to X_n$ can be extended to a map $p_{n+1} : X_{n+1} \to X_n$ since X_{n+1} is obtained from X by attaching cells of dimension greater than $(n+2)$ and $\pi_i(X_n) = 0$ for $i > n$.

Now, we will turn the maps p_n into fibrations. We proceed by induction in j. For $j = 2$, consider a factorization $p_2 = p_2' i_2$ with $i_2 : X_2 \to X_2'$ a homotopy equivalence and

$p_2 : X_2' \to X_1$ a fibration. For $j \geq 3$, we write $p_j i_{j-1} = p_j' i_j$ with $i_j : X_j \to X_j'$ a homotopy equivalence and $p_j' : X_j' \to X_{j-1}'$ a fibration.

Thus, we obtain a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] \\
 & X_3 \arrow[r, "i_3"] \arrow[d, "p_3"] & X_3' \arrow[d, "p_3'"] \\
 & X_2 \arrow[r, "i_2"] \arrow[d, "p_2"] & X_2' \arrow[d, "p_2'"] \\
X \arrow[r, "f_1"'] \arrow[ur, "f_2"'] \arrow[uur, "f_3"] & X_1 \arrow[r, "i_1=\mathrm{Id}"] & X_1
\end{tikzcd}$$

and clearly, the spaces X_n', $n \in \mathbb{N}$, together with the maps $i_n f_n$, $n \in \mathbb{N}$, and the fibrations p_n', $n \in \mathbb{N}$, constitute a Postnikov tower for X. $\square$

Definition 2.1.12. Let $p : E \to B$ be a fibration with fibre F. We say that p is a *principal fibration* if there is a commutative diagram

$$\begin{tikzcd}
 & F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d] \\
\Omega B' \arrow[r] & F' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

where the bottom row is a fibration sequence (i.e F' is the fibre of the map $E' \to B'$) and where the vertical arrows are weak equivalences.

Theorem 2.1.13. *Let X be a path-connected CW-complex. Then X admits a Postnikov tower of principal fibrations if and only if $\pi_1(X)$ acts trivially on $\pi_n(X)$ for all $n > 1$.*

For its proof, see [8].

We end this section stating another result from [8] that we will need later.

Proposition 2.1.14. *Let X be a path-connected CW-complex and let*

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d, "p_3"] \\
 & X_2 \arrow[d, "p_2"] \\
X \arrow[r, "f_1"'] \arrow[ur, "f_2"'] \arrow[uur, "f_3"] & X_1
\end{tikzcd}$$

be a Postnikov tower for X. Then, the induced map $X \to \varprojlim X_n$ is a weak equivalence.

2.2 Spectral sequences

In this section we will recall the basic notions about spectral sequences and fix the corresponding notation for later use throughout this thesis. For a comprehensive exposition on spectral sequences and applications to algebra and topology, the reader might consult [10].

2.2.1 Definition

Definition 2.2.1. A *(homological) spectral sequence* (starting in $a \geq 0$) is a collection $(E^r, d^r)_{r\geq a}$ of $(\mathbb{Z}\times\mathbb{Z})$-graded modules $E^r = (E^r_{p,q})_{p,q\in\mathbb{Z}}$ together with morphisms $d^r_{p,q} : E_{p,q} \to E_{p-r,q+r-1}$ such that $d^r_{p,q}d^r_{p+r,q-r+1} = 0$ for all p, q, r, and explicit isomorphisms $E^{r+1}_{p,q} \simeq H_{p,q}(E^r) = \ker d^r_{p,q}/\operatorname{Im} d^r_{p+r,q-r+1}$.

The $(\mathbb{Z}\times\mathbb{Z})$-graded module (E^r, d^r) is called the *r-th page of E* (or the *r-th term of* E). Via the isomorphism $E^{r+1}_{p,q} \simeq H_{p,q}(E^r)$, E^{r+1} can be identified to a subquotient of E^r. We define the *total degree* of $E^r_{p,q}$ by $n = p + q$. Clearly, the boundary morphisms d^r have total degree -1.

In a similar way we define cohomological spectral sequences.

Definition 2.2.2. A *cohomological spectral sequence* (starting in $a \geq 0$) is a collection $(E_r, d_r)_{r\geq a}$ of $\mathbb{Z}\times\mathbb{Z}$-graded modules $E^r = (E^{p,q}_r)_{p,q}$ together with morphisms $d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ such that $d^r_{p,q}d^r_{p-r,q+r-1} = 0$ for all p, q, r, and explicit isomorphisms $E^{p,q}_{r+1} \simeq H_{p,q}(E_r) = \ker d^{p,q}_r/\operatorname{Im} d^{p-r,q+r-1}_r$.

For the following construction we will work with homological spectral sequences but the cohomological version can be done too.

Let E be a spectral sequence starting in page a. We define $Z^a_{p,q} = E^a_{p,q}$, $B^a_{p,q} = 0$, $Z^{a+1}_{p,q} = \ker d^a_{p,q}$, $B^{a+1}_{p,q} = \operatorname{Im} d^a_{p,q}$. Then $E^{a+1}_{p,q} = Z^{a+1}_{p,q}/B^{a+1}_{p,q}$. Let $\pi^{a+1}_{p,q} : Z^{a+1}_{p,q} \to E^{a+1}_{p,q}$ be the quotient map. We define $Z^{a+2}_{p,q} = (\pi^{a+1}_{p,q})^{-1}(\ker d^{a+1}_{p,q})$, $B^{a+2}_{p,q} = (\pi^{a+1}_{p,q})^{-1}(\operatorname{Im} d^{a+1}_{p,q})$. By the first isomorphism theorem, it is easy to verify that $Z^{a+2}_{p,q}/B^{a+2}_{p,q} = E^{a+2}_{p,q}$.

Thus, we define $\pi^{a+2}_{p,q} : Z^{a+2}_{p,q} \to E^{a+2}_{p,q}$ as the quotient map and repeat the above procedure. Inductively, for each $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ we construct sequences of modules $(B^j_{p,q})_{j\geq a}$ and $(Z^j_{p,q})_{j\geq a}$ such that

$$0 = B^a_{p,q} \subseteq B^{a+1}_{p,q} \subseteq \ldots \subseteq Z^{a+1}_{p,q} \subseteq Z^a_{p,q} = E^a_{p,q}$$

We define

$$B^\infty_{p,q} = \bigcup_{r=a}^{\infty} B^r_{p,q}\ , \qquad Z^\infty_{p,q} = \bigcap_{r=a}^{\infty} Z^r_{p,q} \qquad \text{and} \qquad E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}.$$

The bigraded module $E^\infty = (E^\infty_{p,q})_{p,q\in\mathbb{Z}}$ is called *limit* of the spectral sequence E.

We say that the spectral sequence E *converges* if for all $p, q \in \mathbb{Z}$ there exists $r(p,q) \geq a$ such that $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial for $r \geq r(p,q)$.

In this case, $E^{r+1}_{p,q}$ is isomorphic to a quotient of $E^r_{p,q}$ and $E^\infty_{p,q}$ is isomorphic to the direct limit

$$E^{r(p,q)}_{p,q} \to E^{r(p,q)+1}_{p,q} \to \ldots$$

Definition 2.2.3. A *first quadrant spectral sequence* is a spectral sequence E such that $E_{p,q} = 0$ if $p < 0$ or $q < 0$.

Note that if E is a first quadrant spectral sequence then E converges. Moreover, for all $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ there exists r_0 such that $E^r_{p,q} = E^\infty_{p,q}$ if $r \geq r_0$.

Definition 2.2.4. Let E be a spectral sequence. We say that E *collapses at page* k if $d^r_{p,q} = 0$ for $r \geq k$ and for all $p, q \in \mathbb{Z}$. In consequence, $E^r \simeq E^{r+1} \simeq \ldots \simeq E^\infty$.

Clearly, if the spectral sequence E collapses at page k then it converges.

We now recall the definition of a filtration of an R-module in order to complete the notion of convergence.

Definition 2.2.5. Let M be an R-module. A *filtration of* M is a sequence of R-modules $(F_n M)_{n\in\mathbb{Z}}$ such that $\{0\} \subseteq \ldots \subseteq F_n M \subseteq F_{n+1}M \subseteq \ldots \subseteq M$.

We will say that the filtration is *bounded from below* (resp. *bounded from above*) if there exists $a \in \mathbb{Z}$ such that $F_n M = 0$ for $n \leq a$ (resp. $F_n M = M$ for $n \geq a$). We say that a filtration is *bounded* if it is bounded from below and above.

Note that if $M = \bigoplus_{n\in\mathbb{N}} M_n$ is a graded module and $(F_n M)_{n\in\mathbb{Z}}$ is a filtration of M then $(F_n M \cap M_j)_{n\in\mathbb{Z}}$ is a filtration of M_j, for all $j \in \mathbb{N}$.

Definition 2.2.6. Let $E = (E^r, d^r)_{r\geq a}$ be a convergent spectral sequence. We say that E *converges to the graded module* $H = \bigoplus_{n\in\mathbb{Z}} H_n$ if there exists a filtration $(F_p H)_{p\in\mathbb{Z}}$ of H such that for all $p, q \in \mathbb{Z}$, $E^\infty_{p,q}$ is isomorphic to the quotient $(F_p H \cap H_{p+q})/(F_{p-1}H \cap H_{p+q})$.

2.2.2 Exact couples

Definition 2.2.7. An *exact couple* is a diagram of R-modules

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

which is exact at each module, i.e. $\ker i = \operatorname{Im} k$, $\ker j = \operatorname{Im} i$ and $\ker k = \operatorname{Im} j$. We will denote it (A, E, i, j, k).

Given an exact couple as above, we consider the map $d : E \to E$ defined by $d = j \circ k$. From exactness, it is clear that $d^2 = 0$. Hence, we may compute the homology of E with respect to d and define $E' = \ker d / \operatorname{Im} d$. We also define $A' = \operatorname{Im} i$ and morphisms $i' : A' \to A'$, $j' : A' \to E'$ and $k' : E' \to A'$ by $i' = i|_{A'}$, $j'(i(a)) = [j(a)]$ and $k'([e]) = k(e)$.

It is routine to check that k' and j' are well defined. Indeed, $j(a) \in \ker d$ since $d(j(a)) = (jkj)(a) = 0$ and if $i(a_1) = i(a_2)$ then $a_1 - a_2 \in \ker i = \operatorname{Im} k$, thus, $j(a_1) - j(a_2) \in \operatorname{Im} jk = \operatorname{Im} d$. Hence, j is well defined. Note also that if $[e] \in E'$ then $e \in \ker d$, hence $jk(e) = d(e) = 0$. Thus, $k(e) \in \ker j = \operatorname{Im} i = A'$. Then, $\operatorname{Im} k' \subseteq A'$ as desired. Moreover, if $[e] = 0$ in E' then $e \in \operatorname{Im} d \subseteq \operatorname{Im} j = \ker k$. In consequence, k' is well defined.

The diagram

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

is called the *derived couple* of the exact couple (A, E, i, j, k).

A crucial result is the following

Proposition 2.2.8. *The derived couple of an exact couple is an exact couple.*

The proof is easy and we omit it. It can be found in [10].

This result means that we can iterate the above process indefinitely. Given an exact couple (A, E, i, j, k), we will denote its n-th derived couple by $(A^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$.

Exact couples give rise to spectral sequences in the following way. Suppose that

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

is an exact couple where A and E are $(\mathbb{Z} \times \mathbb{Z})$-graded modules and where the maps i, j and k have bidegrees $(1, -1)$, $(0, 0)$ and $(-1, 0)$ respectively. Let (A', E', i', j', k') denote its derived couple. The modules A' and E' inherit a graduation from A and E since the morphisms i, j and k are bigraded. Moreover, it is clear that i' and k' have the same bidegrees as i and k, i.e. $(1, -1)$ and $(-1, 0)$ respectively. Regarding j', since $j'(i(a)) = [j(a)]$ it follows that $\deg(j') + \deg(i) = \deg(j)$. Hence, j' has bidegree $(-1, 1)$.

By induction, it is easy to prove that for all $n \in \mathbb{N}$, the morphisms $i^{(n)}$, $j^{(n)}$ and $k^{(n)}$ have bidegrees $(1, -1)$, $(-n, n)$ and $(-1, 0)$ respectively.

For $r \in \mathbb{N}$, let $E^r = E^{(r-1)}$ and $d^r = j^{(r-1)} \circ k^{(r-1)}$, where $(A^{(0)}, E^{(0)}, i^{(0)}, j^{(0)}, k^{(0)}) = (A, E, i, j, k)$ and $d^0 = j \circ k$. Hence, d^r has bidegree $(-r, r - 1)$. Thus, $(E^r, d^r)_{r \geq 1}$ is an spectral sequence.

In most topological applications one encounters an exact couple

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

where A and E are $(\mathbb{Z} \times \mathbb{Z})$-graded modules and where the maps i, j and k have bidegrees $(0, 1)$, $(0, 0)$ and $(-1, -1)$ respectively. In this case, to obtain the spectral sequence as above we will need to make a change of indexes. If $A = (A_{n,p})_{n,p}$ and $E = (E_{n,p})_{n,p}$, we call $q = n - p$ and consider $A = (A_{p,q})_{p,q}$ and $E = (E_{p,q})_{p,q}$. Since i, j and k have bidegrees $(0, 1)$, $(0, 0)$ and $(-1, -1)$ respectively in n and p, they have bidegrees $(1, -1)$, $(0, 0)$ and $(-1, 0)$ respectively in p and q. So, we are in the hypothesis of the previous construction.

A useful presentation of this case of bigraded exact couples is by means of the following 'staircase' diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\
\cdots \arrow[r, "k"] & A_{n,p} \arrow[r, "j"] \arrow[d, "i"] & E_{n,p} \arrow[r, "k"] & A_{n-1,p-1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & A_{n,p+1} \arrow[r, "j"] \arrow[d, "i"] & E_{n,p+1} \arrow[r, "k"] & A_{n-1,p} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r, "k"] & A_{n,p+2} \arrow[r, "j"] \arrow[d, "i"] & E_{n,p+2} \arrow[r, "k"] & A_{n-1,p+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\
 & \vdots & & \vdots &
\end{tikzcd}$$

Note that the sequences formed by arrows i, j and k in succession are exact and these are arranged in staircase form.

Example 2.2.9. Let X be a topological space X and let $\{X_p\}_{p\in\mathbb{Z}}$ be an increasing sequence of subspaces of X such that $\bigcup_{p\in\mathbb{Z}} X_p = X$. For example, X might be a CW-complex and X_p its p-skeleton. Take

$$A = \bigoplus_{n,p\in\mathbb{Z}} H_n(X_p) \quad \text{and} \quad E = \bigoplus_{n,p\in\mathbb{Z}} H_n(X_p, X_{p-1})$$

and let i, j and k be the maps defined by the long exact sequence of homology groups associated to the pairs (X_p, X_{p-1}) for $p \in \mathbb{N}$:

$$\cdots \xrightarrow{k} H_{n+1}(X_{p-1}) \xrightarrow{i} H_{n+1}(X_p) \xrightarrow{j} H_{n+1}(X_p, X_{p-1}) \xrightarrow{k} H_n(X_{p-1}) \xrightarrow{i} \cdots$$

This exact couple induces a spectral sequence which, under certain additional hypotheses on the filtration of X, will converge to the homology of X. Note that if X is a CW-complex and we consider the skeletal filtration, the morphism d is just the cellular boundary map. We will come back to this example later.

Now we will impose some extra conditions on the exact couple (A, E, i, j, k). As above, suppose that A and E are $(\mathbb{Z}\times\mathbb{Z})$-graded modules and that the maps i, j and k have bidegrees $(0,1)$, $(0,0)$ and $(-1,-1)$ respectively, and let (E^r, d^r) denote the induced spectral sequence.

(1) For fixed $n \in \mathbb{N}$ there are only a finite number of nontrivial modules $E_{n,p}$ i.e. in the staircase diagram above only finitely many terms in each E column are not 0. By exactness this is equivalent to saying that all but a finite number of maps in each A-column are isomorphisms.

Two important consequences arise from (1). Firstly, since the differential d^r has bidegree $(-r, r-1)$ (and total degree -1) it follows that the spectral sequence (E^r, d^r) converges. Secondly, for all $n \in \mathbb{N}$ there exists modules $A_{n,-\infty}$ and $A_{n,+\infty}$ and integers $a(n)$ and $b(n)$ such that $A_{n,p} \simeq A_{n,-\infty}$ for all $p \leq a(n)$ and $A_{n,p} \simeq A_{n,+\infty}$ for all $p \geq b(n)$.

Taking this into account, we can state the other conditions to consider:
(2) $A_{n,-\infty} = 0$ for all $n \in \mathbb{Z}$.
(3) $A_{n,+\infty} = 0$ for all $n \in \mathbb{Z}$.

Theorem 2.2.10. *Let (A, E, i, j, k) be an exact couple such that A and E are $(\mathbb{Z} \times \mathbb{Z})$-graded modules and that the maps i, j and k have bidegrees $(0,1)$, $(0,0)$ and $(-1,-1)$ respectively, and let (E^r, d^r) denote the induced spectral sequence. Then*

(a) If conditions (1) and (2) are satisfied then the induced spectral sequence converges to the graded module $\bigoplus_{n\in\mathbb{Z}} A_{n,+\infty}$. Moreover, for all $p, q \in \mathbb{Z}$, $E^{\infty}_{p,q}$ is isomorphic to the quotient F^p_n/F^{p-1}_n, where $n = p + q$ and F^p_n is the image of the map $A_{n,p} \to A_{n,+\infty}$.

(b) If conditions (1) and (3) are satisfied then the induced spectral sequence converges to the graded module $\bigoplus_{n\in\mathbb{Z}} A_{n,-\infty}$. Moreover, for all $p, q \in \mathbb{Z}$, $E^{\infty}_{p,q}$ is isomorphic to the quotient F^p_{n-1}/F^{p-1}_{n-1}, where $n = p + q$ and F^p_{n-1} is the kernel of the map $A_{n-1,-\infty} \to A_{n-1,p}$.

Proof. Let $(A^{(r)}, E^{(r)}, i^{(r)}, j^{(r)}, k^{(r)})$ denote the r-th derived couple of (A, E, i, j, k). For each $r \in \mathbb{N}$ there is an exact sequence

$$E^{(r)}_{n+1,p+r-1} \xrightarrow{k^{(r)}} A^{(r)}_{n,p+r-2} \xrightarrow{i^{(r)}} A^{(r)}_{n,p+r-1} \xrightarrow{j^{(r)}} E^{(r)}_{n,p} \xrightarrow{k^{(r)}} A^{(r)}_{n-1,p-1} \xrightarrow{i^{(r)}} A^{(r)}_{n-1,p} \xrightarrow{j^{(r)}} E^{(r)}_{n-1,p-r+1}$$

Fix n and p. If r is sufficiently large, then the first and the last terms of this sequence are 0 by condition (1). Since $A^r_{m,p} = (i^{(r-1)} \circ \ldots \circ i^{(1)} \circ i)(A_{m,p-r})$ then for sufficiently large r the last two A terms of this sequence are 0 by condition (2). Thus, $E^{(r)}_{n,p} = A^{(r)}_{n,p+r-1}/i^{(r)}(A^{(r)}_{n,p+r-2}) = (i^{(r-1)} \circ \ldots \circ i^{(1)} \circ i)(A_{n,p})/(i^{(r)} \circ i^{(r-1)} \circ \ldots \circ i^{(1)} \circ i)(A_{n,p-1})$. Hence $E^{\infty}_{n,p}$ is isomorphic to the quotient F^p_n/F^{p-1}_n.

We prove now the second statement. As before, if r is sufficiently large, then the first and the last terms of the sequence above as well as the two first A terms are 0. Hence, $E^{(r)}_{n,p} = \ker(i^{(r)} : A^{(r)}_{n-1,p-1} \to A^{(r)}_{n-1,p})$. We may suppose that $A_{n-1,p-r} = A_{n-1,p-r+1} = A_{n-1,-\infty}$, thus there are epimorphisms $i^{r-1} : A_{n-1,-\infty} = A_{n-1,p-r} \to A_{n-1,p-1}$ and $i^r : A_{n-1,-\infty} = A_{n-1,p-r} \to A_{n-1,p}$. There is a commutative triangle

$$\begin{tikzcd}
A_{n-1,-\infty} \arrow[r, "\beta=i^{r-1}"] \arrow[dr, "\gamma=i^r"'] & A^{(r)}_{n-1,p-1} \arrow[d, "i"] \\
 & A^{(r)}_{n-1,p}
\end{tikzcd}$$

Applying the first isomorphism theorem to $\beta|_{\ker\gamma} = \ker\gamma \to \ker i$ we obtain that $\ker i = \ker\gamma/\ker\beta = F^{n-1}_p/F^{n-1}_{p-1}$. $\square$

Examples 2.2.11.

(a) We return to example 2.2.9. Suppose, in addition, that $X_p = \varnothing$ for $p < 0$ and that, given n, the inclusion $X_p \hookrightarrow X$ induces isomorphisms in H_n for sufficiently large p. Then the exact couple defined there satisfies conditions (1) and (2) and the induced spectral sequence converges to the homology of X.

In particular, if we take X_p to be the p-skeleton of X for all $p \in \mathbb{N}_0$, then the conditions mentioned above are satisfied and hence the induced spectral sequence converges to the homology of X. In this case, the first page of this spectral sequence has (at most) one nontrivial row, namely $E^1_{p,0} = H_p(X_p, X_{p-1}) = \bigoplus_{p\text{-cells of } X} \mathbb{Z}$.

As was noted in example 2.2.9, the differential d is the cellular boundary map. Hence the nontrivial row of the induced spectral sequence coincides with the cellular chain complex of X. Since this spectral sequence converges to the singular homology of X it follows that cellular homology groups coincide with singular homology ones.

(b) We see now the cohomological version of example 2.2.9. As it was defined there, let X be a topological space X and let $\{X_p\}_{p\in\mathbb{Z}}$ be an increasing sequence of subspaces of X such that $\bigcup_{p\in\mathbb{Z}} X_p = X$. As above, suppose that $X_p = \varnothing$ for $p < 0$ and that for all $n \in \mathbb{N}$, the inclusion $X_p \hookrightarrow X$ induces isomorphisms in H^n for sufficiently large p.

Now define

$$A = \bigoplus_{n,p\in\mathbb{Z}} H^{-n}(X_{-p}) \quad \text{and} \quad E = \bigoplus_{n,p\in\mathbb{Z}} H^{-n}(X_{-p+1}, X_{-p})$$

and let i, j and k be the maps defined by the long exact sequence of cohomology groups associated to the pairs (X_{p+1}, X_p) for $p \in \mathbb{N}$:

$$\cdots \xrightarrow{j} H^n(X_{p+1}, X_p) \xrightarrow{k} H^n(X_{p+1}) \xrightarrow{i} H^n(X_p) \xrightarrow{j} H^{n+1}(X_{p+1}, X_p) \xrightarrow{k} \cdots$$

The exact couple (A, E, i, j, k) satisfies conditions (1) and (3) and the induced spectral sequence converges to the cohomology of X.

2.3 Serre spectral sequence

Let $p : E \to B$ be a Serre fibration with B path-connected and let $b_0 \in B$. By 2.1.8 all the fibres are homotopy equivalent to $F = F_{b_0}$. Moreover, if $\gamma \in \pi_1(B) = \pi_1(B, b_0)$ then γ induces a homotopy equivalence $L_\gamma : F \to F$ and hence isomorphisms $(L_\gamma)_* : H_n(F, G) \to H_n(F, G)$ for all $n \in \mathbb{N}$ and for all abelian groups G. Hence, $\pi_1(B)$ acts on $H_n(F, G)$ with action defined by $\gamma.x = (L_\gamma)_*(x)$ for $\gamma \in \pi_1(B)$ and $x \in H_n(F, G)$.

The following theorem is of great importance and is due to Serre ([17]). It gives a relation between the homology groups of the base space, the total space and the fibre of a Serre fibration by means of a spectral sequence: the Serre spectral sequence.

Theorem 2.3.1. *Let $f : X \to B$ a Serre fibration with fibre F, where B is a path-connected CW-complex and let G be an abelian group. If $\pi_1(B)$ acts trivially on $H_n(F;G)$ for all $n \in \mathbb{N}$ then there exists a homological spectral sequence $\{E^r_{p,q}, d_r\}$ which converges to $H_*(X;G)$ and such that $E^2_{p,q} \simeq H_p(B; H_q(F;G))$.*

Proof. We consider the filtration $\varnothing \subseteq \ldots \subseteq X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots$ of X given by the preimages of the skeletons of B, i.e. $X_p = f^{-1}(B^p)$ for $p \geq 0$ and $X_p = \varnothing$ for $p < 0$. This filtration induces an exact couple as in example 2.2.9 with

$$A = \bigoplus_{n,p\in\mathbb{Z}} H_n(X_p;G) \quad \text{and} \quad E = \bigoplus_{n,p\in\mathbb{Z}} H_n(X_p, X_{p-1};G).$$

Since (B, B^p) is p-connected and f is a fibration, from the homotopy lifting property of f it follows that (X, X_p) is also p-connected. Indeed, let $b_0 \in B^0$ be the base point of B and suppose we are given a continuous map $\alpha : (D^r, S^{r-1}) \to (X, X_p)$ with $r \leq n$. Since (B, B_p) is p-connected then $f\alpha : (D^r, S^{r-1}) \to (B, B^p)$ is nullhomotopic (as map of topological pairs). Hence, there exists a homotopy $H : D^r \times I \to B$ such that $Hi_0 = f\alpha$, $H(x, 1) = b_0$ for all $x \in D^n$ and $H(S^{r-1} \times I) \subseteq B^p$. Then, there is a commutative diagram of solid arrows

$$\begin{tikzcd}
D^r \arrow[r, "\alpha"] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\
D^r \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & B
\end{tikzcd}$$

Since f is a fibration, there exists a homotopy $\widetilde{H}$ making commutative the whole diagram. Note that $\widetilde{H}(S^{r-1} \times I) \subseteq f^{-1}(B^p) = X_p$ and $\widetilde{H}i_1(D^r) \subseteq f^{-1}(b_0) \subseteq X_p$. By, 1.4.5 $\alpha \simeq *$. Hence, (X, X_p) is p-connected.

Thus, by the Hurewicz theorem (1.4.34) it follows that $H_n(X, X_p) = 0$ if $p \geq \max\{1, n\}$. Note that the result holds even if the spaces are not path-connected since we can apply the Hurewicz theorem in each path-connected component. Then, the inclusion $X_p \hookrightarrow X$ induces isomorphisms in H_n if $p \geq \max\{1, n\}$. From the universal coefficient theorem (A.3) it follows that $X_p \hookrightarrow X$ induces isomorphisms in $H_n(\ ;G)$ if $p \geq \max\{1, n\}$.

Then, for fixed n, $H_n(X_p, X_{p-1};G)$ is nontrivial only for a finite number of p's. By 2.2.10, the induced spectral sequence converges to $H_*(X;G)$.

The proof of the fact that $E^2_{p,q} \simeq H_p(B; H_q(F;G))$ can be found in [9]. $\square$

We will give now two basic examples.

Example 2.3.2 (Homology of a $K(\mathbb{Z}, 2)$)**.** In this example we will compute the homology groups of an Eilenberg - MacLane space of type $(\mathbb{Z}, 2)$ by means of spectral sequences. A different way to do this is using the fact that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ and computing then its cellular homology.

Let B be a $K(\mathbb{Z}, 2)$ (note that B is simply connected). We consider the path space fibration $f : PB \to B$. Its fibre F is the loop space ΩB and hence it is a $K(\mathbb{Z}, 1)$. Then F is homotopy equivalent to $\mathbb{S}^1$ and we obtain $H_q(F;\mathbb{Z}) = \mathbb{Z}$ if $q = 0, 1$ and $H_q(F;\mathbb{Z}) = 0$

for $q \geq 2$. Let $(E^r, d^r)_{r \geq 1}$ be the spectral sequence associated to the fibration f. Then $E^2_{p,q} = H_p(B; (H_q(F; \mathbb{Z}))) = 0$ for $q \geq 2$. Hence, page E^2 has at most two nonzero rows:

$$\begin{tikzcd}[row sep=small, column sep=small]
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \mathbb{Z} & H_1(B) & H_2(B) & H_3(B) & H_4(B) & \cdots \\
\cdots & 0 & \mathbb{Z} & H_1(B) & H_2(B) \arrow[ull] & H_3(B) \arrow[ull] & H_4(B) \arrow[ull] & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{tikzcd}$$

Therefore, $E^3_{p,q} = E^\infty_{p,q}$, and since this spectral sequence converges to the homology of P, which is contractible, then $E^3_{p,q} = 0$ for $(p,q) \neq (0,0)$. Thus, the arrows of the previous diagram must be isomorphisms and $H_1(B) = 0$. In consequence,

$$H_n(B; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Example 2.3.3 (Homology of $\Omega \mathbb{S}^n$ for $n \geq 2$)**.** In this example we will compute the homology of $\Omega \mathbb{S}^n$ using the spectral sequence associated to the path space fibration $f : P\mathbb{S}^n \to \mathbb{S}^n$. Note that the fibre of f is $F = \Omega \mathbb{S}^n$ and that it is simply-connected.

Since $H_p(\mathbb{S}^n; \mathbb{Z}) = 0$ if $p \neq 0, n$, $H_0(\mathbb{S}^n; \mathbb{Z}) = \mathbb{Z}$ and $H_n(\mathbb{S}^n; \mathbb{Z}) = \mathbb{Z}$, by the universal coefficient theorem we obtain that $H_p(\mathbb{S}^n; H_q(\Omega \mathbb{S}^n; \mathbb{Z})) = 0$ if $p \neq 0, n$ and $H_p(\mathbb{S}^n; H_q(\Omega \mathbb{S}^n; \mathbb{Z})) = H_q(\Omega \mathbb{S}^n; \mathbb{Z})$ if $p = 0, n$.

Hence, the second page of the spectral sequence will have at most two nonzero columns:

$$\begin{tikzcd}[row sep=small, column sep=small]
\cdots & 0 & H_n(\Omega \mathbb{S}^n; \mathbb{Z}) & 0 & \cdots & 0 & H_n(\Omega \mathbb{S}^n; \mathbb{Z}) & 0 & \cdots \\
\cdots & 0 & H_{n-1}(\Omega \mathbb{S}^n; \mathbb{Z}) & 0 & \cdots & 0 & H_{n-1}(\Omega \mathbb{S}^n; \mathbb{Z}) & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & H_1(\Omega \mathbb{S}^n; \mathbb{Z}) & 0 & \cdots & 0 & H_1(\Omega \mathbb{S}^n; \mathbb{Z}) \arrow[uuullll] & 0 & \cdots \\
\cdots & 0 & H_0(\Omega \mathbb{S}^n; \mathbb{Z}) & 0 & \cdots & 0 & H_0(\Omega \mathbb{S}^n; \mathbb{Z}) \arrow[uuullll] & 0 & \cdots
\end{tikzcd}$$

It follows that $E^2 = E^3 = \ldots = E^n$ and $E^{n+1} = E^\infty$. But since P is contractible, then $E^{n+1}_{p,q} = 0$ for $(p,q) \neq (0,0)$. Hence, the arrows of the above diagram must be isomorphisms. Thus, for $n = 2$ we obtain that $H_q(\Omega \mathbb{S}^n; \mathbb{Z}) = \mathbb{Z}$ for all $q \geq 0$. If $n \geq 3$, since $\Omega \mathbb{S}^n$ is $(n-2)$-connected, by the Hurewicz theorem we obtain that $H_q(\Omega \mathbb{S}^n) = 0$ for $1 \leq q \leq n-2$. Hence,

$$H_q(\Omega \mathbb{S}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q \text{ is a multiple of } n-1 \\ 0 & \text{if } q \text{ is not a multiple of } n-1 \end{cases}$$

Now we turn our attention to an important example of application of Serre spectral sequence: a generalization of the Hurewicz theorem. This theorem is due to Serre himself, who also introduced Serre classes [17]. We begin by recalling its definition.

Definition 2.3.4. A nonempty class $\mathscr{C}$ of abelian groups will be called a *Serre class* if for all exact sequences of abelian groups of the form $A \to B \to C$ with $A, C \in \mathscr{C}$ we have that $B \in \mathscr{C}$.

Note that if $\mathscr{C}$ is a Serre class then it satisfies the following properties

(a) $0 \in \mathscr{C}$.

(b) If $A \in \mathscr{C}$ and A' is isomorphic to A, then $A' \in \mathscr{C}$.

(c) If $A \subseteq B$ and $B \in \mathscr{C}$ then $A \in \mathscr{C}$.

(d) If $A \subseteq B$ and $B \in \mathscr{C}$ then $B/A \in \mathscr{C}$.

(e) If $0 \to A \to B \to C \to 0$ is a short exact sequence with $A, C \in \mathscr{C}$ then $B \in \mathscr{C}$.

Moreover, if a nonempty class of abelian groups satisfies (e) then it also satisfies (a), (b), (c) and (d). From this, we get that a nonempty class $\mathscr{C}$ of abelian groups is a Serre class if and only if it satisfies

(e') If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups then $A, C \in \mathscr{C}$ if and only if $B \in \mathscr{C}$.

Indeed, it is clear that a Serre class satisfies this property. So, suppose $\mathscr{C}$ is a nonempty class of abelian groups satisfying (e') and let $A \xrightarrow{i} B \xrightarrow{j} C$ be a short exact sequence with $A, C \in \mathscr{C}$. We consider the induced short exact sequence $0 \to A/\ker i \xrightarrow{i} B \xrightarrow{j} \operatorname{Im} j \to 0$. Since $A, C \in \mathscr{C}$ then $A/\ker i \in \mathscr{C}$ and $\operatorname{Im} j \in \mathscr{C}$ by (d) and (c) respectively. Thus, $B \in \mathscr{C}$ by (e').

It is easy to prove that the following classes of abelian groups are Serre classes.

- The class of finitely generated abelian groups.
- The class of finite abelian groups.
- The class $\mathcal{T}_{\mathcal{P}}$ of torsion abelian groups whose elements have orders which are divisible only by primes in a set $\mathcal{P}$ of prime numbers.
- The class of finite groups in $\mathcal{T}_{\mathcal{P}}$.
- The class of trivial groups.

These will prove to be very interesting and useful examples.

Definition 2.3.5. Let $\mathscr{C}$ be a Serre class and let $f : G \to H$ be a morphism between abelian groups.

- We say that f is a *$\mathscr{C}$-monomorphism* if $\ker f \in \mathscr{C}$.
- We say that f is a *$\mathscr{C}$-epimorphism* if $\operatorname{coker} f \in \mathscr{C}$.
- We say that f is a *$\mathscr{C}$-isomorphism* if it is a $\mathscr{C}$-monomorphism and a $\mathscr{C}$-epimorphism.

For example, note that if $\mathscr{C}$ is the class of trivial groups then $\mathscr{C}$-monomorphisms, $\mathscr{C}$-epimorphisms and $\mathscr{C}$-isomorphisms are just monomorphisms, epimorphisms and isomorphisms of groups respectively.

Suppose now that $\mathscr{C}$ is the class of 2-torsion abelian groups. Then the trivial map from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ and the multiplication by 2 map from $\mathbb{Z}_6$ to $\mathbb{Z}_6$ are $\mathscr{C}$-isomorphisms.

The following proposition helps us to understand $\mathscr{C}$-isomorphisms in case $\mathscr{C} = \mathcal{T}_{\mathcal{P}}$.

Proposition 2.3.6. *Let $\mathcal{P}$ be a set of prime numbers and let $f : G \to H$ be a morphism between torsion abelian groups. We denote by $\mathcal{T}_{\mathcal{P}}(G)$ and $\mathcal{T}_{\mathcal{P}}(H)$ the subgroups of torsion elements of G and H respectively such that their elements have orders which are divisible only by primes in $\mathcal{P}$. Then f is a $\mathcal{T}_{\mathcal{P}}$-isomorphism if and only if the induced map $\overline{f} : G/\mathcal{T}_{\mathcal{P}}(G) \to H/\mathcal{T}_{\mathcal{P}}(H)$ is an isomorphism.*

Proof. Suppose first that f is a $\mathcal{T}_{\mathcal{P}}$-isomorphism, i.e. $\ker f \in \mathcal{T}_{\mathcal{P}}$ and $\operatorname{coker} f \in \mathcal{T}_{\mathcal{P}}$. Thus, $\ker f \subseteq \mathcal{T}_{\mathcal{P}}(G)$.

Let $x \in G$ such that $\overline{f}(\overline{x}) = 0$ in $H/\mathcal{T}_{\mathcal{P}}(H)$. Then $f(x) \in \mathcal{T}_{\mathcal{P}}(H)$, i.e. there exists $m \in \mathbb{N}$ divisible only by primes in $\mathcal{P}$ such that $mf(x) = 0$. Hence, $mx \in \ker f \subseteq \mathcal{T}_{\mathcal{P}}(G)$. Thus, $x \in \mathcal{T}_{\mathcal{P}}(G)$ and $\overline{x} = 0$ in $G/\mathcal{T}_{\mathcal{P}}(G)$. So, $\overline{f}$ is a monomorphism.

Now, let $\overline{y} \in H/\mathcal{T}_{\mathcal{P}}(H)$ and let $[y]$ denote the class of y in $H/\operatorname{Im} f = \operatorname{coker} f$. Then, there exists $m \in \mathbb{N}$ divisible only by primes in $\mathcal{P}$ such that $m[y] = 0$ in $\operatorname{coker} f$. Thus, $my \in \operatorname{Im} f$, i.e. there exists $x \in G$ such that $f(x) = my$. Hence, $\overline{f}(\overline{x}) = m\overline{y}$.

From group theory we know that if A is a torsion abelian group and p is a prime number such that A has no elements of order p, then the map $\mu_p : A \to A$ defined by multiplication by p is an isomorphism. Indeed, it is clear that μ_p is a monomorphism. We will prove that it is also an epimorphism. Let $a \in A$ and let $k = \operatorname{ord}(a)$. Let $S = \{ja : 0 \leq j \leq k-1\} \subseteq A$. Then $\mu_p : S \to S$ is a monomorphism. Since S is finite, μ_p is an isomorphism. Thus, $a \in \operatorname{Im} \mu_p$. Hence, μ_p is an epimorphism.

Returning to the above situation, note that $G/\mathcal{T}_{\mathcal{P}}(G)$ and $H/\mathcal{T}_{\mathcal{P}}(H)$ are torsion groups whose elements have orders which are divisible only by primes not in $\mathcal{P}$. Since $m \in \mathbb{N}$ is divisible only by primes in $\mathcal{P}$ it follows that $\mu_m : G/\mathcal{T}_{\mathcal{P}}(G) \to G/\mathcal{T}_{\mathcal{P}}(G)$ and $\mu_m : H/\mathcal{T}_{\mathcal{P}}(H) \to H/\mathcal{T}_{\mathcal{P}}(H)$ are isomorphisms. Then we obtain $(\mu_m)^{-1}\overline{f}(\overline{x}) = (\mu_m)^{-1}(m\overline{y}) = y$. Thus, $\overline{f}((\mu_m)^{-1}(\overline{x})) = y$. Hence, $\overline{f}$ is an epimorphism.

Conversely, suppose $\overline{f}$ is an isomorphism. Let $q : G \to /\mathcal{T}_{\mathcal{P}}(G)$ and $q' : H \to H/\mathcal{T}_{\mathcal{P}}(H)$ denote the quotient maps. Then $\ker f \subseteq \ker(q'f) = \ker(\overline{f}q) = \ker q = \mathcal{T}_{\mathcal{P}}(G)$. Thus, f is a $\mathcal{T}_{\mathcal{P}}(G)$-monomorphism.

Now let $h \in H$. Then $q'(h) = \overline{f}(q(g))$ for some $g \in G$. Hence, $q'(h) = q'(f(g))$ and thus $q'(h - f(g)) = 0$. Then $h - f(g) \in \mathcal{T}_{\mathcal{P}}(H)$. In consequence, there exists $m \in \mathbb{N}$ divisible only by primes in $\mathcal{P}$ such that $m(h - f(g)) = 0$. Thus, $mh = mf(g) = f(mg) \in \operatorname{Im} f$. Hence, $m[h] = [mh] = 0$ in $H/\operatorname{Im} f = \operatorname{coker} f$. Therefore, $\operatorname{coker} f \in \mathcal{T}_{\mathcal{P}}$. $\square$

Note that this proposition does not hold if G and H are not torsion groups. For example, the multiplication by 2 map from $\mathbb{Z}$ to $\mathbb{Z}$ is a $\mathcal{T}_{\{2\}}$-isomorphism, but $\overline{f} : \mathbb{Z} \to \mathbb{Z}$ coincides with f which is not an isomorphism.

Definition 2.3.7. Let X be a topological space. We say that X is *$\mathscr{C}$-acyclic* if $H_n(X;\mathbb{Z}) \in \mathscr{C}$ for all $n \in \mathbb{N}$.

Later on, we will need that the product of $\mathscr{C}$-acyclic spaces is again a $\mathscr{C}$-acyclic space. This does not hold for all Serre classes and hence leads to the following definition.

Definition 2.3.8. Let $\mathscr{C}$ be a Serre class. We say that $\mathscr{C}$ is a *ring of abelian groups* if for all $A, B \in \mathscr{C}$ we have that $A \otimes B \in \mathscr{C}$ and $\mathrm{Tor}(A, B) \in \mathscr{C}$.

Note that if $\mathscr{C}$ is a ring of abelian groups then by the Künneth formula the product of $\mathscr{C}$-acyclic spaces is a $\mathscr{C}$-acyclic space.

It is easy to verify that the examples of Serre classes given above are also rings of abelian groups.

Lemma 2.3.9. *Let $\mathscr{C}$ be a ring of abelian groups and let $f : X \to B$ be a fibration between path-connected spaces with path-connected fibre F such that $\pi_1(B)$ acts trivially on $H_*(F)$. Then, any two of the followings three conditions imply the third*

(a) $H_n(F) \in \mathscr{C}$ for all $n > 0$.

(b) $H_n(X) \in \mathscr{C}$ for all $n > 0$.

(c) $H_n(B) \in \mathscr{C}$ for all $n > 0$.

Proof. Let $(E^r, d^r)_{r\in\mathbb{N}}$ be the Serre spectral sequence associated to the fibration f.

Suppose first that (a) and (c) hold. Then, by the universal coefficient theorem $E^2_{p,q} = H_p(B; H_q(F)) \simeq H_p(B) \otimes H_q(F) \oplus \mathrm{Tor}(H_{p-1}(B), H_q(F))$. Since $\mathscr{C}$ is a ring of abelian groups, we obtain that $E^2_{p,q} \in \mathscr{C}$ for $(p, q) \neq (0, 0)$. By induction on r, it follows that $E^r_{p,q} \in \mathscr{C}$ for $(p, q) \neq (0, 0)$ and for all $r \geq 2$. Indeed, we have just proved the case $r = 2$, and if we suppose that $E^r_{p,q} \in \mathscr{C}$ for some $p, q \in \mathbb{Z}$, then its subgroups $\ker d^r$ and $\mathrm{Im}\, d^r$ are also in $\mathscr{C}$ and hence their quotient $E^{r+1}_{p,q}$ also belongs to $\mathscr{C}$.

But given $p, q \in \mathbb{Z}$, there exists $r \in \mathbb{N}$ such that $E^\infty_{p,q} = E^r_{p,q}$. Therefore, $E^\infty_{p,q} \in \mathscr{C}$ for $(p, q) \neq (0, 0)$. But since the spectral sequence $(E^r, d^r)_{r\in\mathbb{N}}$ converges to $H_*(X)$ we know that $E^\infty_{p,n-p}$ are the succesive quotients in a filtration

$$0 \subseteq F_0H_n(X) \subseteq \ldots \subseteq F_nH_n(X) = H_n(X).$$

Hence, by induction we obtain that $F_iH_n(X) \in \mathscr{C}$ for $0 \leq i \leq n$. In particular, $H_n(X) \in \mathscr{C}$.

Now suppose that (a) and (b) hold. Since B and F are path-connected we obtain that $E^2_{p,0} = H_p(B)$ for $p > 0$ and $E^2_{0,q} = H_q(F) \in \mathscr{C}$ for $q > 0$. By hypothesis, $H_n(X) \in \mathscr{C}$ for all $n > 0$, hence $E^\infty_{p,n-p} \in \mathscr{C}$ for $n > 0$ and $0 \leq p \leq n$ since they are the succesive quotients in a filtration of $H_n(X)$.

We will prove by induction that $H_n(B) \in \mathscr{C}$ for all $n > 0$. Thus, suppose that $H_i(B) \in \mathscr{C}$ for $0 < i < n$. As before, by the universal coefficient theorem $E^2_{p,q} = H_p(B; H_q(F)) \simeq H_p(B) \otimes H_q(F) \oplus \mathrm{Tor}(H_{p-1}(B), H_q(F))$. Hence, $E^2_{p,q} \in \mathscr{C}$ for $0 \leq p < n$ and $(p, q) \neq (0, 0)$. Thus, $E^r_{p,q} \in \mathscr{C}$ for all $r > 0$ and for the same values of p and q.

Since $(E^r, d^r)_{r\in\mathbb{N}}$ is a first cuadrant spectral sequence then $E^{r+1}_{n,0} = \ker d^r_{n,0} \subseteq E^r_{n,0}$. Thus, there is a short exact sequence

$$0 \longrightarrow E^{r+1}_{n,0} \longrightarrow E^r_{n,0} \xrightarrow{d^r_{n,0}} \mathrm{Im}\, d^r_{n,0} \longrightarrow 0$$

Since $\operatorname{Im} d^r_{n,0} \subseteq E^r_{n-r,r-1} \in \mathscr{C}$ we obtain that $\operatorname{Im} d^r_{n,0} \in \mathscr{C}$. Thus, from the short exact sequence above we get that $E^{r+1}_{n,0} \in \mathscr{C}$ if and only if $E^r_{n,0} \in \mathscr{C}$. Since $E^\infty_{n,0} \in \mathscr{C}$ and $E^\infty_{n,0} = E^k_{n,0}$ for sufficiently large k, then by an inductive process we get that $E^2_{n,0} = H_n(B) \in \mathscr{C}$ as desired.

The third case of the proof is analogous to the previous one, so we omit it. □

Moreover, with the same proof as above we obtain the following stronger result which will be needed later.

Lemma 2.3.10. *Let $\mathscr{C}$ be a ring of abelian groups and let $f : X \to B$ be a fibration between path-connected spaces with path-connected fibre F such that $\pi_1(B)$ acts trivially on $H_*(F)$. Let $n \in \mathbb{N}$. Then*

(a) If $H_i(F) \in \mathscr{C}$ for $1 \leq i \leq n+1$ and $H_i(X) \in \mathscr{C}$ for $1 \leq i \leq n$ then $H_i(B) \in \mathscr{C}$ for $1 \leq i \leq n$.

(b) If $H_i(B) \in \mathscr{C}$ for $1 \leq i \leq n+1$ and $H_i(X) \in \mathscr{C}$ for $1 \leq i \leq n$ then $H_i(F) \in \mathscr{C}$ for $1 \leq i \leq n$.

(c) If $H_i(F) \in \mathscr{C}$ for $1 \leq i \leq n$ and $H_i(B) \in \mathscr{C}$ for $1 \leq i \leq n$ then $H_i(X) \in \mathscr{C}$ for $1 \leq i \leq n$.

One of the key ingredients needed for the proof of the generalized version of Hurewicz Theorem is that Eilenberg - MacLane spaces of type (G, n) are $\mathscr{C}$-acyclic.

Definition 2.3.11. Let $\mathscr{C}$ be a Serre class. We say that $\mathscr{C}$ is *acyclic* if for all $G \in \mathscr{C}$, Eilenberg - MacLane spaces of type $(G, 1)$ are $\mathscr{C}$-acyclic.

Note that if $\mathscr{C}$ is an acyclic ring of abelian groups (i.e. a ring of abelian groups and an acyclic Serre class) and $G \in \mathscr{C}$ then, for all $n \in \mathbb{N}$, any Eilenberg - MacLane space of type (G, n) is $\mathscr{C}$-acyclic. Indeed, if Z is an Eilenberg - MacLane space of type (G, n) with $n \geq 2$ then its loop space is an Eilenberg - MacLane space of type $(G, n-1)$. Hence, if we consider the path-space fibration $\Omega Z \to PZ \to Z$, since PZ is contractible, by the above lemma we obtain that Z is $\mathscr{C}$-acyclic if and only if ΩZ is $\mathscr{C}$-acyclic. Since any Eilenberg - MacLane space of type $(G, 1)$ is $\mathscr{C}$-acyclic, it follows that for all $n \in \mathbb{N}$, any Eilenberg - MacLane space of type (G, n) is $\mathscr{C}$-acyclic.

An important result is that the examples of Serre classes given above are also acyclic rings of abelian groups. This will allow us to apply the generalized version of Hurewicz Theorem to that classes.

Before stating the theorem, we recall the definition of abelian spaces.

Definition 2.3.12. Let X be a path-connected topological space. We say that X is *abelian* (or *simple*) if $\pi_1(X)$ acts trivially on $\pi_n(X)$ for all $n \in \mathbb{N}$.

For example, simply-connected spaces are abelian. Note also that if X is an abelian topological space then $\pi_1(X)$ acts trivially on $\pi_1(X)$ and hence it is an abelian group.

Theorem 2.3.13 (Generalized Hurewicz's theorem)**.** *Let X be an abelian topological space and let $\mathscr{C}$ be an acyclic ring of abelian groups. Then the following are equivalent:*

(a) $\pi_i(X) \in \mathscr{C}$ *for* $1 \leq i \leq n-1$.

(b) $H_i(X) \in \mathscr{C}$ *for* $1 \leq i \leq n-1$.

Moreover, any of them imply that the Hurewicz morphism $h : \pi_n(X) \to H_n(X)$ *is a* $\mathscr{C}$*-isomorphism.*

Proof. By CW-approximation, we may suppose that X is a CW-complex. Moreover, we may suppose that X has only one 0-cell since any path-connected CW-complex is homotopy equivalent to a CW-complex with only one 0-cell. Let this 0-cell be the base point of X and let

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d, "p_3"] \\
 & X_2 \arrow[d, "p_2"] \\
X \arrow[r, "f_1"'] \arrow[ur, "f_2"'] \arrow[uur, "f_3"] & X_1
\end{tikzcd}$$

be a Postnikov tower of principal fibrations for X where, for all $j \in \mathbb{N}$, X_j is a CW-complex built by attaching cells of dimension greater than $j+1$ to X.

Suppose first that $\pi_i(X) \in \mathscr{C}$ for $1 \leq i \leq n-1$. Then $\pi_i(X_k) \in \mathscr{C}$ for $k \leq n-1$ and for all $i \in \mathbb{N}$. Since X_1 is a $K(\pi_1(X), 1)$ and $\pi_1(X) \in \mathscr{C}$, it follows that $H_i(X_1) \in \mathscr{C}$ for all $i \in \mathbb{N}$. From the fibration sequences $F_k \to X_k \to X_{k-1}$, since F_k is an Eilenberg - MacLane space of type $(\pi_k(X), k)$, by the previous lemma we obtain inductively that $H_i(X_j) \in \mathscr{C}$ for $j \leq n-1$ and for all $i \in \mathbb{N}$.

In particular, $H_i(X_{n-1}) \in \mathscr{C}$ for all $i \in \mathbb{N}$. Since X_{n-1} is built by attaching cells of dimension greater than n to X, $H_i(X_{n-1}) = H_i(X)$ for $i \leq n-1$ and hence $H_i(X) \in \mathscr{C}$ for $1 \leq i \leq n-1$.

Conversely, suppose that $H_i(X) \in \mathscr{C}$ for $1 \leq i \leq n-1$. Since X_j is built by attaching cells of dimension greater than $j+1$ to X, then $H_i(X_j) \simeq H_i(X)$ for $1 \leq i \leq j$ and there is an epimorphism $H_{j+1}(X) \to H_{j+1}(X_j)$. Thus, $H_i(X_j) \in \mathscr{C}$ for $1 \leq i \leq j \leq n-1$. We will prove inductively that $\pi_i(X) \in \mathscr{C}$ and X_i is $\mathscr{C}$-acyclic for $1 \leq i \leq n-1$.

Since X is abelian, $\pi_1(X) = H_1(X) \in \mathscr{C}$ and hence X_1 is $\mathscr{C}$-acyclic because it is an Eilenberg - MacLane space of type $(\pi_1(X), 1)$.

Suppose that $\pi_i(X) \in \mathscr{C}$ and X_i is $\mathscr{C}$-acyclic for $1 \leq i \leq k-1 \leq n-2$. Consider the fibration sequence $F_k \to X_k \to X_{k-1}$. Since X_{k-1} is $\mathscr{C}$-acyclic and $H_i(X_k) \in \mathscr{C}$ for $1 \leq i \leq k$, by the previous lemma $H_i(F_k) \in \mathscr{C}$ for $1 \leq i \leq k$. In particular $H_k(F_k) \in \mathscr{C}$, but by the Hurewicz theorem $H_k(F_k) = \pi_k(F_k) = \pi_k(X)$, hence $\pi_k(X) \in \mathscr{C}$. Thus, F_k is $\mathscr{C}$-acyclic. Then, applying again the previous lemma to the fibration sequence $F_k \to X_k \to X_{k-1}$ we obtain that X_k is $\mathscr{C}$-acyclic. Therefore, we have proved that $\pi_i(X) \in \mathscr{C}$ for $1 \leq i \leq n-1$.

To prove the last statement it suffices to show that the hurewicz map $h : \pi_n(X_n) \to H_n(X_n)$ is a $\mathscr{C}$-isomorphism. Let $(E^r, d^r)_{r\geq 1}$ be the spectral sequence associated to the fibration $X_n \to X_{n-1}$. Since its fibre F_n is an Eilenberg - MacLane space of type $(\pi_n(X), n)$, $E^2_{p,q} = 0$ for $1 \leq q \leq n-1$. Hence, $E^{n+1}_{0,n} = H_n(F_n)$ and $E^{n+1}_{n+1,0} = H_{n+1}(X_{n-1})$.

Since $(E^r, d^r)_{r\geq 1}$ converges to $H_*(X)$ there exists a filtration $0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = H_n(X)$ of $H_n(X)$ with $F_i/F_{i-1} \simeq E^\infty_{i,n-i}$ for $0 \leq i \leq n$. Since $E^\infty_{i,n-i} = 0$ for $0 < i < n$ we obtain that $E^\infty_{0,n} = F_0 = F_1 = \ldots = F_{n-1}$. Hence, there is a short exact sequence

$$0 \longrightarrow F_0 \longrightarrow F_n \xrightarrow{q} F_n/F_0 \longrightarrow 0$$

Recall that $F_n = H_n(X)$ and $F_n/F_0 = E^\infty_{n,0}$. Moreover, by 2.2.10 and by the proof of 2.3.1 we know that F_0 is the image of the map $H_n(X^0_n) \to H_n(X_n)$, where $(X^i_n)_{i\in\mathbb{N}_0}$ is the filtration of X_n obtained by taking preimages of the skeletons of X_{n-1}. But since the 0-skeleton of X_{n-1} is the base point, it follows that $X^0_n = F_n$. Hence, F_0 is the image of the map $\mathrm{in}_* : H_n(F_n) \to H_n(X_n)$ induced by the inclusion.

It is clear that $E^\infty_{0,n} = E^{n+2}_{0,n} = \operatorname{coker} d^{n+1}_{n+1,0}$ and that $E^{n+1}_{0,n} = H_n(F_n)$, therefore there is a short exact sequence

$$0 \longrightarrow H_{n+1}(X_{n-1}) \xrightarrow{d^{n+1}_{n+1,0}} H_n(F_n) \longrightarrow E^\infty_{0,n} \longrightarrow 0$$

Combining this short exact sequence with the previous one we obtain an exact sequence

$$H_{n+1}(X_{n-1}) \xrightarrow{d^{n+1}_{n+1,0}} H_n(F_n) \xrightarrow{\mathrm{inc}_*} H_n(X) \xrightarrow{q} E^\infty_{n,0} \longrightarrow 0$$

(note that $\ker q = F_0 = \mathrm{Im}\,(\mathrm{inc}_*)$).

Consider the commutative square

$$\begin{tikzcd}
\pi_n(F_n) \arrow[r, "\mathrm{inc}_*"] \arrow[d, "h"'] & \pi_n(X_n) \arrow[d, "h"] \\
H_n(F_n) \arrow[r, "\mathrm{inc}_*"'] & H_n(X_n)
\end{tikzcd}$$

Since $\pi_{n+1}(X_{n-1}) = 0$ and $\pi_n(X_{n-1}) = 0$, from the long exact sequence of homotopy groups associated to the fibration $X_n \to X_{n-1}$ we obtain that the upper map is an isomorphism. Also, the left-hand vertical map is an isomorphism by the Hurewicz theorem.

Now, if we assume that $\pi_i(X) \in \mathscr{C}$ for $1 \leq i \leq n-1$ then $\pi_i(X_{n-1}) \in \mathscr{C}$ for all $i \in \mathbb{N}$ and hence, by the first part of this theorem $H_i(X_{n-1}) \in \mathscr{C}$ for all $i \in \mathbb{N}$. Thus, the first and fourth terms of the exact sequence above belong to the class $\mathscr{C}$ and hence the map $\mathrm{inc}_* : H_n(F_n) \to H_n(X_n)$ is a $\mathscr{C}$-isomorphism.

Therefore, the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is a $\mathscr{C}$-isomorphism. $\square$

As a first example of application when $\mathscr{C}$ is the class of finitely generated abelian groups we obtain that all the homotopy groups of spheres are finitely generated. However, we shall prove later a much stronger result.

Note also that if $\mathscr{C}$ is the class of trivial groups, then we obtain the classical version of Hurewicz Theorem.

Now we turn to the cohomological version of Serre spectral sequence. In this case we require that $\pi_1(B)$ acts trivially on $H^n(F;G)$ for all $n \in \mathbb{N}$, where this action is defined in a similar way as its homological counterpart.

Theorem 2.3.14. *Let $f : X \to B$ a Serre fibration with fibre F, where B is a path-connected CW-complex and let G be an abelian group. If $\pi_1(B)$ acts trivially on $H^n(F;G)$ for all $n \in \mathbb{N}$ then there exists a cohomological spectral sequence $\{E_r^{p,q}, d_r\}$ which converges to $H^*(X;G)$ and such that $E_2^{p,q} \simeq H^p(B;H^q(F;G))$.*

The proof is similar to the homological one, thus we will omit it.

A big advantage of this cohomological version is that cohomology can be given a ring structure by means of cup product and this product behaves wonderfully with respect to (cohomological) Serre spectral sequence, as is shown by the following theorem.

Theorem 2.3.15. *Let $f : X \to B$ a Serre fibration with fibre F, where B is a path-connected CW-complex and let R be a ring. Suppose that $\pi_1(B)$ acts trivially on $H^n(F;R)$ for all $n \in \mathbb{N}$ and let $\{E_r^{p,q}, d_r\}$ be the (cohomological) Serre spectral sequence associated to the fibration f. Then there exists bilinear products $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ for $r \in \mathbb{N} \cup \{\infty\}$ such that:*

(a) The differentials d_r are derivations, i.e. $d_r(xy) = d(x)y + (-1)^{p+q}x(dy)$ for all $x \in E_r^{p,q}$, $y \in E_r^{s,t}$, $r \in \mathbb{N}$.

(b) The product in E_{r+1} is the product induced by the product in E_r (note that since the differential d_r is a derivation, the product in E_r induces one in cohomology).

(c) The product in E_∞ is induced by the product in the faces E_r for finite r.

(d) The product in E_2 is induced by the composition

$$\begin{array}{c} E_2^{p,q} \times E_2^{s,t} = H^p(B;H_q(F;R)) \times H^s(B;H_t(F;R)) \\ \downarrow \\ H^p(B;H_*(F;R)) \times H^s(B;H_*(F;R)) \\ \downarrow{\smile'} \\ H^{p+s}(B;H_*(F;R)) \end{array}$$

where the product in $H_*(F;R)$ is the standard cup product and where the second arrow is defined by $\alpha \smile' \beta = (-1)^{qs}\alpha \smile \beta$ with $\smile$ denoting cup product.

(e) If for $n \geq 0$, $\{F_p^n = F^pH^n(X)\}_{p\in\mathbb{Z}}$ (with $F_{p+1}^n \subseteq F_p^n$ for all p) denotes the filtration of $H_n(X)$ given by the definition of convergence of $\{E_r^{p,q}, d^r\}$, then the cup product in $H^*(X;R)$ restricts to maps $F_p^n \times F_s^m \to F_{p+s}^{n+m}$. Moreover, these maps induce maps $F_p^n/F_{p+1}^n \times F_s^m/F_{s+1}^m \to F_{p+s}^{n+m}/F_{p+s+1}^{n+m}$ which coincide with the products $E_\infty^{p,n-p} \times E_\infty^{s,m-s} \to E_\infty^{p+s,n+m-p-s}$.

The proof of this theorem con be found in [9].

We will give first an easy example of application and then we will show how this results con be applied for instance to obtain interesting information about homotopy groups of spheres.

Example 2.3.16 (Computation of $H^*(K(\mathbb{Z},2)),\mathbb{Z})$**.** Let B be an Eilenberg - MacLane space of type $(\mathbb{Z},2)$. Consider the pathspace fibration $f : PB \to B$. Its fibre F is ΩZ, and hence an Eilenberg - MacLane space of type $(\mathbb{Z},1)$. Thus F is homotopy equivalent to S^1. Let $(E_r^{p,q}, d_r)_{r\in\mathbb{Z}}$ be the cohomological Serre spectral sequence associated to the fibration f. Since B is simply-connected, $E_2^{p,q} \simeq H^p(B; H^q(S^1;\mathbb{Z}))$.

Hence, the E_2 page has at most two nonzero rows:

$$
\begin{tikzcd}[row sep=small, column sep=small]
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \mathbb{Z} \arrow[drr] & H^1(B) \arrow[drr] & H^2(B) \arrow[drr] & H^3(B) & H^4(B) & \cdots \\
\cdots & 0 & \mathbb{Z} & H^1(B) & H^2(B) & H^3(B) & H^4(B) & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{tikzcd}
$$

In a similar way as in 2.3.2, we obtain that

$$H^n(B;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Let a denote a generator of $E_2^{0,1} \simeq \mathbb{Z}$ and for even i, let x_i denote a generator of $E_2^{i,0} \simeq \mathbb{Z}$. Since we consider homology with coefficients in $\mathbb{Z}$ which is a unital ring, the ring $H^*(B;\mathbb{Z})$ has also an identity element which is $1 \in \mathbb{Z} \simeq H^0(B;\mathbb{Z})$. Therefore, the product $E_2^{0,q} \times E_2^{s,t} \to E_2^{s,q+t}$ is just multiplication of coefficients. Hence, ax_i is a generator of $E_2^{i,1} \simeq \mathbb{Z}$.

$$
\begin{tikzcd}[row sep=small, column sep=small]
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \mathbb{Z}a \arrow[drr] & 0 & \mathbb{Z}ax_2 \arrow[drr] & 0 & \mathbb{Z}ax_4 \arrow[drr] & 0 & \mathbb{Z}ax_6 & \cdots \\
\cdots & 0 & \mathbb{Z} & 0 & \mathbb{Z}x_2 & 0 & \mathbb{Z}x_4 & 0 & \mathbb{Z}x_6 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{tikzcd}
$$

Since the arrows shown above are isomorphisms, $d_2 a$ is a generator of $\mathbb{Z}x_2$. Thus, we may suppose that $x_2 = d_2 a$. Then,

$$d_2(ax_{2i}) = d_2(a)x_{2i} - ad_2(x_2 i) = d_2(a)x_{2i} = x_2 x_{2i}.$$

Now, since $d_2(ax_{2i})$ is a generator of $\mathbb{Z}x_{2i}$ we may assume that $x_2 x_{2i} = x_{2i+2}$.

Thus, $H^*(B;\mathbb{Z})$ is the polynomial ring $\mathbb{Z}[x]$, where $x = x_2$.

We give now the first application of the cohomological Serre spectral sequence to computation of homotopy groups of spheres.

Proposition 2.3.17. *Let p be a prime number. Then, the p-torsion subgroup of $\pi_i(S^3)$ is 0 for $i < 2p$ and $\mathbb{Z}_p$ for $i = 2p$.*

Proof. Note that if we apply the generalized Hurewicz theorem directly to S^3, we will not be able to see beyond π_3 since $\pi_3(S^3) = \mathbb{Z}$ is not a p-torsion group. Hence, the idea is to construct a space F which has the same homotopy groups as S^3, except for $\pi_3(F)$ which will be trivial and then apply the generalized Hurewicz theorem to F.

Let Y be an Eilenberg - MacLane space of type $(\mathbb{Z}, 3)$ and let $g : S^3 \to Y$ be a continuous map such that $g_* : \pi_3(S^3) \to \pi_3(Y)$ is an isomorphism. Let $S^3 \xrightarrow{i} Z \xrightarrow{p} Y$ be a factorization of g into a homotopy equivalence followed by a fibration and let F be the fibre of p. Note that p induces an isomorphism in π_3. Hence, from the homotopy exact sequence associated to the fibration p it follows that the inclusion inc : $F \to S^3$ induces isomorphisms $\mathrm{inc}_* : \pi_i(F) \to \pi_i(S^3)$ for $i > 3$ and that F is 3-connected.

Let $F \xrightarrow{i'} X \xrightarrow{p'} S^3$ be a factorization of inc : $F \to S^3$ into a homotopy equivalence followed by a fibration and let F' be the fibre of p'. It follows that $(p')_* : \pi_i(X) \to \pi_i(S^3)$ is an isomorphism for $i > 3$ and that X is 3-connected. From the homotopy exact sequence associated to the fibration p' it follows that F' is an Eilenberg - MacLane space of type $(\mathbb{Z}, 2)$.

Let $(E_r, d_r)_{r \geq 1}$ be the cohomological Serre spectral sequence associated to the fibration p'. By the example above we know that $H^*(F')$ is isomorphic to the polynomial ring $\mathbb{Z}[x]$, with x corresponding to an element $a \in H^2(F')$. Hence, page E_2 looks like

$$\begin{tikzcd}[row sep=small, column sep=small]
0 & \mathbb{Z} \arrow[ddrrr, "d_3^{0,6}"] & 0 & 0 & \mathbb{Z} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} \arrow[ddrrr, "d_3^{0,4}"] & 0 & 0 & \mathbb{Z} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} \arrow[ddrrr, "d_3^{0,2}"] & 0 & 0 & \mathbb{Z} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \cdots
\end{tikzcd}$$

Note that $E_3 = E_2$. Let $z = d_3(a)$. Since X is 3-connected, the differential $d_3^{0,2} : \mathbb{Z} \to \mathbb{Z}$ must be an isomorphism. Then z is a generator of $E_3^{3,0} \simeq \mathbb{Z}$. Thus, as in the previous example, $a^i z$ is a generator of $E_3^{3,2i} \simeq \mathbb{Z}$ for all $i \in \mathbb{N}$.

It follows that $d_3(a^2) = d_3(a)a + a d_3(a) = 2ax$, and by an inductive argument we obtain that $d_3(a^n) = n a^{n-1} x$. Therefore, $E_4^{0,q} = 0$ for $q \in \mathbb{N}$ and $E_4^{3,2k} = \mathbb{Z}_{k+1}$ for $k \in \mathbb{N}$. Since $E_4 = E_\infty$ we obtain that

$$H^i(X; \mathbb{Z}) = \begin{cases} 0 & \text{if } i = 1 \text{ or } i = 3 \\ \mathbb{Z}_{k+1} & \text{if } i = 2k+3 \text{ for some } k \in \mathbb{N} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

Hence, by the universal coefficient theorem for cohomology groups we get

$$H_i(X; \mathbb{Z}) = \begin{cases} 0 & \text{if } i = 2 \\ \mathbb{Z}_k & \text{if } i = 2k \text{ for some } k \in \mathbb{N},\ k \geq 2 \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

Applying the generalized version of Hurewicz's theorem with $\mathscr{C}$ the class of p-torsion abelian groups gives the desired result. $\square$

Corollary 2.3.18.

(a) $\pi_4(S^3) = \mathbb{Z}_2$.

(b) The groups $\pi_i(S^3)$ *are nonzero for infinitely many values of* i.

For the proof of the last theorem of this section we need the following lemma. Its proof involves the cohomological Serre spectral sequence and similar arguments to those in the example and proposition above, but we will not give the details here. A sketch of the proof can be found in [9].

Lemma 2.3.19. *Let* $n \in \mathbb{N}$ *and let* X *be an Eilenberg - MacLane space of type* $(\mathbb{Z}, n)$*. Then* $H^*(X;\mathbb{Q}) \simeq \mathbb{Q}[x]$ *for* n *even and* $H^*(X;\mathbb{Q}) \simeq \mathbb{Q}[x]/\langle x^2\rangle$ *for* n *odd, where* x *corresponds to an element of* $H^n(X;\mathbb{Q})$*.*

Now we state the last theorem of this section, which is a strong result about homotopy groups of spheres. To prove it, we will work with rational coefficients, so as kill torsion in homology groups and retain nontorsion information. The idea of the proof is analogous to that of the previous proposition: we will kill certain homotopy groups leaving the others unchanged and then we will apply the generalized version of Hurewicz's theorem.

Theorem 2.3.20. *The groups* $\pi_i(S^n)$ *are finite for* $i > n$*, except for* $\pi_{4n-1}(S^{2n})$ *which is the direct sum of* $\mathbb{Z}$ *with a finite group.*

Proof. We may assume that $n \geq 2$, since the result clearly holds for $n = 1$.

Let B be an Eilenberg - MacLane space of type $(\mathbb{Z}, n)$ and let $g : S^n \to B$ be a continuous map which induces an isomorphism on π_n. Let $S^n \xrightarrow{i} X \xrightarrow{f} B$ be a factorization of g into a homotopy equivalence followed by a fibration and let F be the fibre of f. From the homotopy exact sequence associated to the fibration f it follows that F is n-connected and $\pi_i(F) \simeq \pi_i(S^n)$ for $i > n$.

Let $(E_r, d_r)_{r\in\mathbb{N}}$ be the cohomological Serre spectral sequence with coefficients in $\mathbb{Q}$ associated to the fibration f.

We suppose first that n is odd. Since by the previous lemma $H^*(B;\mathbb{Q}) \simeq \mathbb{Q}[x]/\langle x^2\rangle$ (where x corresponds to an element of $H^n(X;\mathbb{Q})$), then the E_2 page has at most two nonzero columns, namely columns $p = 0$ and $p = n$. Moreover, $E_2^{0,q} \simeq E_2^{n,q}$ for all $q \in \mathbb{Z}$ and $E_2^{0,0} \simeq E_2^{n,0} \simeq \mathbb{Q}$. Since F is n-connected, $E_2^{0,q} \simeq 0$ for $1 \leq q \leq n$. Suppose there exists $k \in \mathbb{N}$ such that $E_2^{0,k}$ is not a trivial group and let m be the minimum of such k's. Then $m \geq n+1$ and hence $E_2^{0,m} \simeq E_\infty^{0,m}$. Thus, $H^m(X) \simeq E_\infty^{0,m}$ and therefore it is not trivial, which entails a contradiction since X is homotopy equivalent to S^n.

Therefore, $E_2^{0,k} \simeq 0$ for all $k \in \mathbb{N}$. Hence, $H^k(F;\mathbb{Q}) \simeq 0$ for all $k \in \mathbb{N}$. From the universal coefficient theorem for cohomology it follows that $\operatorname{Hom}(H_k(F;\mathbb{Z});\mathbb{Q}) = 0$ for all $k \in \mathbb{N}$. Then $H_k(F;\mathbb{Z})$ is a torsion group for all $k \in \mathbb{N}$. Now since F is n-connected and $\pi_i(F) \simeq \pi_i(S^n)$ for $i > n$, it follows that $\pi_i(F)$ is finitely generated for all $i \in \mathbb{N}$. Hence, by the generalized Hurewicz theorem $H_i(F;\mathbb{Z})$ is finitely generated for all $i \in \mathbb{N}$. But since

$H_i(F;\mathbb{Z})$, $i \in \mathbb{N}$ are torsion groups we get that $H_i(F;\mathbb{Z})$ is a finite group for all $i \in \mathbb{N}$. Applying the generalized Hurewicz theorem again we obtain that $\pi_i(F;\mathbb{Z})$ is a finite group for all $i \in \mathbb{N}$ and hence $\pi_i(S^n)$ is a finite group for $i > n$.

Now suppose that n is even. Then, by the previous lemma $H^*(B;\mathbb{Q}) \simeq \mathbb{Q}[x]$, where x corresponds to an element $z \in H^n(X;\mathbb{Q})$. Thus,

$$E_2^{p,0} = \begin{cases} \mathbb{Q} & \text{if } p = kn \text{ for some } k \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases}$$

Since F is n-connected, $E_2^{0,q} \simeq 0$ for $1 \le q \le n$. Suppose that $E_2^{0,q}$ is not trivial for some $q < 2n-1$ and let m be the minimum q with this property. Then $m > n$ and $E_\infty^{0,m} \simeq E_2^{0,m}$. Hence $H^m(X;\mathbb{Q})$ is not trivial, but X is homotopy equivalent to S^n, which entails a contradiction. Hence, $H^q(F;\mathbb{Q}) \simeq E_2^{0,q} \simeq 0$ for $1 \le q < 2n-1$. Then $E_2^{p,q} \simeq 0$ for $1 \le q < 2n-1$ and for all $p \in \mathbb{N}$.

In consequence, $E_n^{0,2n-1} \simeq E_2^{0,2n-1}$, $E_n^{2n,0} \simeq E_2^{2n,0} \simeq \mathbb{Q}$ and the differential $d_{2n}^{0,2n-1}$: $E_n^{0,2n-1} \to E_n^{2n,0} \simeq \mathbb{Q}$ must be an isomorphism because otherwise either $E_\infty^{0,2n-1}$ or $E_\infty^{2n,0}$ would be nontrivial. Hence we have that $H^{2n-1}(F;\mathbb{Q}) \simeq \mathbb{Q}$ and $H^q(F;\mathbb{Q}) \simeq 0$ for $1 \le q < 2n-1$. From the universal coefficient theorem for cohomology, since $H_i(F;\mathbb{Z})$ is finitely generated for all $i \in \mathbb{N}$, by an inductive argument and proceeding as in the case above it is not difficult to prove that $H_i(F;\mathbb{Z})$ is a finite group for $1 \le i < 2n-1$ and $H_{2n-1}(F;\mathbb{Z})$ is the direct sum of $\mathbb{Z}$ with a finite group. Applying the generalized Hurewicz theorem with $\mathscr{C}$ the class of finite groups we obtain that the same holds for the groups $\pi_i(F)$, $1 \le i \le 2n-1$. Therefore, $\pi_i(S^n)$ is a finite group for $1 \le i < 2n-1$ and $\pi_{2n-1}(F;\mathbb{Z})$ is the direct sum of $\mathbb{Z}$ with a finite group.

Now, let $a \in E_n^{0,2n-1}$ be such that $d_n(a) = z^2 \in E_n^{2n,0} \simeq H^{2n}(X;\mathbb{Q})$. Thus, by the derivation property $d_n^{ni,2n-1}(az^i) = z^{i+2}$ for $i \in \mathbb{N}$. Then $d_n^{ni,2n-1}$ is an isomorphism for all $i \in \mathbb{N}$. With a similar argument as in the previous case we obtain that $E_2^{0,k} \simeq 0$ for all $k \ge 2n$, since the first nontrivial entry would survive to E_∞. Thus, $H^*(F;\mathbb{Q}) \simeq \mathbb{Q}[x]/\langle x^2 \rangle$, where x corresponds to an element of $H^{2n-1}(X;\mathbb{Q})$.

Let F' be obtained from F by attaching cells of dimension greater than $2n-1$ and such that $\pi_i(F') = 0$ for $i \ge 2n-1$. Let $\mathrm{inc} : F \to F'$ denote the inclusion map and let $F \xrightarrow{i'} X' \xrightarrow{f'} F'$ be a factorization of inc into a homotopy equivalence followed by a fibration. Let F'' be the fibre of f'. From the homotopy exact sequence for the fibration f' it follows that $\pi_i(F'') \simeq \pi_i(F) \simeq \pi_i(S^n)$ for $i \ge 2n-1$.

Since F' has finite homotopy groups, the same holds for its reduced homology groups and hence $H^n(F',\mathbb{Q}) \simeq 0$ for $n \in \mathbb{N}$. From the cohomological Serre spectral sequence for the fibration f' we obtain that $H^*(F'';\mathbb{Q}) \simeq H^*(X';\mathbb{Q}) \simeq H^*(F;\mathbb{Q}) \simeq \mathbb{Q}[x]/\langle x^2 \rangle$, where x corresponds to an element of $H^{2n-1}(F'';\mathbb{Q})$.

Let Y be an Eilenberg - MacLane space of type $(\mathbb{Z}, 2n-1)$ and let $F'' \to Y$ be a map inducing an isomorphism on the nontorsion in π_{2n-1}. Now, we consider factorization of this map into a homotopy equivalence followed by a fibration and with a similar argument as in the case n odd we obtain that $\pi_i(F'') \simeq \pi_i(S^n)$ is finite for $i > 2n-1$. $\square$

2.4 Localization of CW-complexes

In this section we will study localization of CW-complexes. We begin by recalling some basic facts about algebraic localization of rings and modules.

For the whole section, $\mathcal{P}$ will denote a subset of the set of prime numbers. If G is an abelian group and $a \in \mathbb{Z}$, we denote by $\mu_a : G \to G$ the group homomorphism defined by $\mu_a(g) = ag$.

We denote
$$\mathbb{Z}_{\mathcal{P}} = \left\{ \frac{a}{b} \in \mathbb{Q} \ / \ (b : p) = 1 \text{ for all } p \in \mathcal{P} \right\}$$
Note that $\mathbb{Z}_{\mathcal{P}}$ is torsionfree and that $\mathbb{Z}_{\varnothing} = \mathbb{Q}$. If the subset $\mathcal{P}$ consists of just one prime number p we write $\mathbb{Z}_{\mathcal{P}} = \mathbb{Z}_{(p)}$.

If G is an abelian group, its *localization at* $\mathcal{P}$ is defined as $G \otimes \mathbb{Z}_{\mathcal{P}}$. The induced morphism $G \to G \otimes \mathbb{Z}_{\mathcal{P}}$ is called *localization map*. Clearly, given a map $f : G \to H$, there is an induced map $f \otimes \mathbb{Z}_{\mathcal{P}} : G \otimes \mathbb{Z}_{\mathcal{P}} \to H \otimes \mathbb{Z}_{\mathcal{P}}$ and a commutative diagram

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d] & H \arrow[d] \\
G \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r, "f \otimes \mathbb{Z}_{\mathcal{P}}"'] & H \otimes \mathbb{Z}_{\mathcal{P}}
\end{tikzcd}$$

where the vertical arrows are the localization maps.

We know that $G \otimes \mathbb{Z}_{\mathcal{P}}$ is the abelian group generated by $\{g \otimes r \ / \ g \in G \text{ and } r \in \mathbb{Z}_{\mathcal{P}}\}$. Note that a finite sum $g_1 \otimes r_1 + \ldots + g_n \otimes r_n$ can be written in the form $g \otimes \frac{1}{m}$, with m not divisible by primes in $\mathcal{P}$. Indeed, we proceed by taking m the least common multiple of the denominators of $r_1, \ldots, r_n$ and we write $g_i \otimes r_i = g'_i \otimes \frac{1}{m}$.

Note also that the map $G \to G \otimes \mathbb{Z}_{\mathcal{P}}$ is an isomorphism if and only if G can be given a (compatible) $\mathbb{Z}_{\mathcal{P}}$-module structure. The following lemma states an equivalent condition.

Lemma 2.4.1. *Let G be an abelian group and let $\mathcal{P}$ be a subset of the set of prime numbers. Then G can be given a (compatible) $\mathbb{Z}_{\mathcal{P}}$-module structure if and only if for all prime numbers p such that $p \notin \mathcal{P}$ the map $\mu_p : G \to G$ is an isomorphism.*

Proof. For the first implication, note that if $p \notin \mathcal{P}$ then $\frac{1}{p} \in \mathbb{Z}_{\mathcal{P}}$. Hence, the map $\nu_p : G \to G$ defined by $\nu_p(g) = \frac{1}{p} g$ is the inverse of μ_p.

Conversely, suppose that for all prime numbers p such that $p \notin \mathcal{P}$ the map $\mu_p : G \to G$ is an isomorphism. Thus, if $b \in \{m \in \mathbb{Z} \ / \ (m : p) = 1 \text{ for all } p \in \mathcal{P}\}$, the map μ_b is an isomorphism. Hence, for $g \in G$ and $\frac{a}{b} \in \mathbb{Z}_{\mathcal{P}}$ (with b not divisible by prime numbers in $\mathcal{P}$), we define $\frac{a}{b}.g = \mu_b^{-1} \mu_a g$. It is easy to prove that this defines a $\mathbb{Z}_{\mathcal{P}}$-module structure on G. $\square$

For example, $\mathbb{Z}_3$ can be given a compatible $\mathbb{Z}_{(3)}$-module structure and can not be given a (compatible) $\mathbb{Z}_{(2)}$-module structure. In general, given a prime number p and $n \in \mathbb{N}$, $\mathbb{Z}_{p^n}$ can be given a (compatible) $\mathbb{Z}_{\mathcal{P}}$-module structure if and only if $p \in \mathcal{P}$.

In the following proposition we give some results that will be needed later.

Proposition 2.4.2. *Let $\mathcal{P}$ be a subset of the set of prime numbers.*

(a) *If* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E$ *is an exact sequence of abelian groups and* A, B, D *and* E *can be given compatible* $\mathbb{Z}_{\mathcal{P}}$*-module structures then the same holds for* C.

(b) *The* $\mathcal{P}$*-localization functor is exact, that is, it takes exact sequences to exact sequences.*

Proof. (a) Let p be a prime number such that $p \notin \mathcal{P}$ and consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mu_p^A"] & B \arrow[r, "g"] \arrow[d, "\mu_p^B"] & C \arrow[r, "h"] \arrow[d, "\mu_p^C"] & D \arrow[r, "k"] \arrow[d, "\mu_p^D"] & E \arrow[d, "\mu_p^E"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & D \arrow[r, "k"'] & E \end{tikzcd}$$

where the vertical arrows are the maps induced by multiplication by p. By the previous lemma, the maps μ_p^A, μ_p^B, μ_p^D and μ_p^E are isomorphisms. Since the rows are exact, by the five lemma the map μ_p^C is also an isomorphism. Hence, C can be given a compatible $\mathbb{Z}_{\mathcal{P}}$-module structure.

(b) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of abelian groups. We want to prove that the sequence $A \otimes \mathbb{Z}_{\mathcal{P}} \xrightarrow{f \otimes \mathbb{Z}_{\mathcal{P}}} B \otimes \mathbb{Z}_{\mathcal{P}} \xrightarrow{g \otimes \mathbb{Z}_{\mathcal{P}}} C \otimes \mathbb{Z}_{\mathcal{P}}$ is exact. It is clear that $g \otimes \mathbb{Z}_{\mathcal{P}} \circ f \otimes \mathbb{Z}_{\mathcal{P}} = 0$. Suppose that $b \otimes \frac{1}{m} \in \ker(g \otimes \mathbb{Z}_{\mathcal{P}})$, i.e. $g(b) \otimes \frac{1}{m} = 0$ in $C \otimes \mathbb{Z}_{\mathcal{P}}$. Thus, $g(b)$ has finite order k not divisible by primes in $\mathcal{P}$. Hence, $kb \in \ker(g) = \mathrm{Im}\,(f)$. Then, $kb = f(a)$ for some $a \in A$ and $(f \otimes \mathbb{Z}_{\mathcal{P}})(a \otimes \frac{1}{km}) = b \otimes \frac{1}{m}$. $\square$

Now we turn to the topological setting.

Definition 2.4.3. Let X be an abelian space. We say that X is $\mathcal{P}$*-local* if $\pi_n(X)$ is a $\mathbb{Z}_{\mathcal{P}}$-module for all $n \in \mathbb{N}$.

For example, if $3 \in \mathcal{P}$ and $n \in \mathbb{N}$, an Eilenberg-MacLane space of type $(\mathbb{Z}_3, n)$ is $\mathcal{P}$-local.

Definition 2.4.4. Let X and Y be abelian spaces and let $f : X \to Y$ be a continuous map. We say that f is a $\mathcal{P}$*-localization map* if Y is $\mathcal{P}$-local and for all $n \in \mathbb{N}$, the induced map $f_* \otimes \mathbb{Z}_{\mathcal{P}} : \pi_n(X) \otimes \mathbb{Z}_{\mathcal{P}} \to \pi_n(Y) \otimes \mathbb{Z}_{\mathcal{P}}$ is an isomorphism.

A $\mathcal{P}$*-localization of* X consists of a $\mathcal{P}$-local space Y together with a $\mathcal{P}$-localization map $f : X \to Y$.

Note that since $\mathbb{Z}_{\mathcal{P}}$ is torsion-free, from the universal coefficient theorem (A.3) we obtain that $\pi_n(X) \otimes \mathbb{Z}_{\mathcal{P}} \simeq \pi_n(X; \mathbb{Z}_{\mathcal{P}})$.

The following lemma will be used to prove an equivalent homological definition of $\mathcal{P}$-localization.

Lemma 2.4.5. *Let* $F \to E \to B$ *be a fibration sequence of path-connected spaces such that* $\pi_1(B)$ *acts trivially on* $H_*(F, \mathbb{Z}_p)$ *for all* $p \notin \mathcal{P}$*. Then, any two of the following statements imply the third.*

(a) $\widetilde{H}_n(F)$ *is a* $\mathbb{Z}_{\mathcal{P}}$*-module for all* $n \in \mathbb{N}$.

(b) $\widetilde{H}_n(E)$ *is a* $\mathbb{Z}_{\mathcal{P}}$*-module for all* $n \in \mathbb{N}$.

(c) $\widetilde{H}_n(B)$ *is a* $\mathbb{Z}_{\mathcal{P}}$*-module for all* $n \in \mathbb{N}$.

Proof. Consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0$, where the second arrow is multiplication by p. For any space X, taking tensor product with $C_n(X)$, where $(C_*(X), d)$ is the singular chain complex of X, we obtain a short exact sequence of chain complexes, which induces a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(X,\mathbb{Z}) \xrightarrow{\mu_p} \widetilde{H}_n(X,\mathbb{Z}) \longrightarrow \widetilde{H}_n(X,\mathbb{Z}_p) \longrightarrow \widetilde{H}_{n-1}(X,\mathbb{Z}) \xrightarrow{\mu_p} \widetilde{H}_{n-1}(X,\mathbb{Z}) \longrightarrow \cdots$$

Hence, applying 2.4.1 we obtain that $\widetilde{H}_n(X)$ is a $\mathbb{Z}_{\mathcal{P}}$-module for all $n \in \mathbb{N}$ if and only if $\widetilde{H}_n(X, \mathbb{Z}_p) = 0$ for all $n \in \mathbb{N}$ and for all prime numbers $p \notin \mathcal{P}$.

The result then follows from the Serre spectral sequence. □

Proposition 2.4.6. *Let* X *and* Y *be abelian spaces and let* $f : X \to Y$ *be a* $\mathcal{P}$*-localization map. Then for all* $n \in \mathbb{N}$, $\widetilde{H}_n(Y)$ *can be given a compatible* $\mathbb{Z}_{\mathcal{P}}$*-module structure and the induced map* $f_* \otimes \mathbb{Z}_{\mathcal{P}} : \widetilde{H}_n(X) \otimes \mathbb{Z}_{\mathcal{P}} \to \widetilde{H}_n(Y) \otimes \mathbb{Z}_{\mathcal{P}}$ *is an isomorphism.*

Proof. We consider first the particular case in which X is an Eilenberg - MacLane space of type (G, n). Hence, Y is an Eilenberg - MacLane space of type $(G \otimes \mathbb{Z}_{\mathcal{P}}, n)$.

We proceed by induction on n, starting with the case $n = 1$. If $G = \mathbb{Z}$ then Y is an Eilenberg - MacLane space of type $(\mathbb{Z}_{\mathcal{P}}, 1)$. But a Moore space of type $(\mathbb{Z}_{\mathcal{P}}, 1)$ can be constructed as a mapping telescope of a sequence of maps $S^2 \to S^2$ of appropiate degrees, as in [8], p. 312. From this construction it is not difficult to verify that a Moore space of type $(\mathbb{Z}_{\mathcal{P}}, 1)$ is also an Eilenberg - MacLane space of type $(\mathbb{Z}_{\mathcal{P}}, 1)$. Hence, Y is a Moore space of type $(\mathbb{Z}_{\mathcal{P}}, 1)$ and thus, for all $n \in \mathbb{N}$, $\widetilde{H}_n(Y)$ can be given a compatible $\mathbb{Z}_{\mathcal{P}}$-module structure.

The naturality of the Hurewicz map gives us a commutative square

$$\begin{tikzcd}
\pi_1(X) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r, "f_* \otimes \mathbb{Z}_{\mathcal{P}}"] \arrow[d, "h \otimes \mathbb{Z}_{\mathcal{P}}"'] & \pi_1(Y) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[d, "h \otimes \mathbb{Z}_{\mathcal{P}}"] \\
H_1(X) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r, "f_* \otimes \mathbb{Z}_{\mathcal{P}}"'] & H_1(Y) \otimes \mathbb{Z}_{\mathcal{P}}
\end{tikzcd}$$

The vertical arrows are the abelianization maps, and hence isomorphisms in this case. The upper arrow is also an isomorphism since $f : X \to Y$ is a $\mathcal{P}$-localization map. Thus, $f_* \otimes \mathbb{Z}_{\mathcal{P}} : \widetilde{H}_n(X) \otimes \mathbb{Z}_{\mathcal{P}} \to \widetilde{H}_n(Y) \otimes \mathbb{Z}_{\mathcal{P}}$ is an isomorphism.

If $G = \mathbb{Z}_{p^m}$ with $p \notin \mathcal{P}$ and $m \in \mathbb{N}$ then $G \otimes \mathbb{Z}_{\mathcal{P}} = G$. Since $f : X \to Y$ induces an isomorphism on π_1 and X and Y are Eilenberg - MacLane space of type (G, n) it follows that f is a homotopy equivalence. Hence, the result holds in this case.

If $G = \mathbb{Z}_{p^m}$ with $p \in \mathcal{P}$ and $m \in \mathbb{N}$ then $G \otimes \mathbb{Z}_{\mathcal{P}} = 0$. Hence, Y is contractible and the result follows.

If G is finitely generated, the result follows from the cases above applying the Künneth formula (A.11).

For an arbitrary group G, we know that G is the direct limit of its finitely generated subgroups. Hence, the result holds since homology commutes with direct limits.

Now suppose that $n \geq 2$ and that the statement holds for $n-1$. Consider the commutative diagram

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & PX \arrow[r] \arrow[d] & X \arrow[d] \\ \Omega Y \arrow[r] & PY \arrow[r] & Y \end{tikzcd}$$

where the rows are the path-space fibration sequences for X and Y and where the vertical arrows are induced by the morphism $G \to G \otimes \mathbb{Z}_{\mathcal{P}}$. Note that ΩX and ΩY are Eilenberg-MacLane spaces of type $(G, n-1)$ and $(G \otimes \mathbb{Z}_{\mathcal{P}}, n-1)$ respectively.

By 2.4.5 and the inductive hypothesis it follows that $H_i(Y)$ can be given a compatible $\mathbb{Z}_{\mathcal{P}}$-module structure for all $i \in \mathbb{N}$. Moreover, the commutative diagram above induces a map between the Serre spectral sequences associated to both fibration sequences. Since the induced maps $H_i(\Omega X, \mathbb{Z}_{\mathcal{P}}) \to H_i(\Omega Y, \mathbb{Z}_{\mathcal{P}})$ and $H_i(PX, \mathbb{Z}_{\mathcal{P}}) \to H_i(PY, \mathbb{Z}_{\mathcal{P}})$ are isomorphisms for all $i \in \mathbb{N}$, then by proposition 1.12 of [9], the induced map $H_i(X, \mathbb{Z}_{\mathcal{P}}) \to H_i(Y, \mathbb{Z}_{\mathcal{P}})$ is an isomorphism for all $i \in \mathbb{N}$.

Hence, the statement is proved in case X is an Eilenberg-MacLane space.

For the general case, the $\mathcal{P}$-localization $f : X \to Y$ induces a map of Postnikov towers. For $n \in \mathbb{N}$, let X_n and Y_n denote the n-th stage of the Postnikoy towers of X and Y respectively. Then, there is a commutative diagram

$$\begin{tikzcd} F_n \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ F'_n \arrow[r] & Y_n \arrow[r] & Y_{n-1} \end{tikzcd}$$

where F_n and F'_n are the fibres of $X_n \to X_{n-1}$ and $Y_n \to Y_{n-1}$ respectively. Hence F_n is an Eilenberg - MacLane space of type $(\pi_n(X), n)$ and $F'n$ is an Eilenberg - MacLane space of type $(\pi_n(Y), n)$. Since X and Y are abelian, it is not difficult to prove that $\pi_1(X_{n-1})$ acts trivially on $H_r(F_n)$ for all r and that a similar statement holds for the fibration sequence $F'_n \to Y_n \to Y_{n-1}$.

By induction on n and applying naturality of spectral sequences (cf. [9]) it follows that $H_n(Y)$ can be given a compatible $\mathbb{Z}_{\mathcal{P}}$-module structure and the induced map $f_* \otimes \mathbb{Z}_{\mathcal{P}} : \widetilde{H}_n(X) \otimes \mathbb{Z}_{\mathcal{P}} \to \widetilde{H}_n(Y) \otimes \mathbb{Z}_{\mathcal{P}}$ is an isomorphism for all $n \in \mathbb{N}$. $\square$

Theorem 2.4.7. *Let X be an abelian space. Then there exists a $\mathcal{P}$-localization $X \to Y$.*

Proof. By 2.1.13, X admits a Postnikov tower of principal fibrations. For $n \in \mathbb{N}$ let X_n denote the n-th stage of this Postnikov tower. Let X'_1 be an Eilenberg - MacLane space of type $(\pi_1(X) \otimes \mathbb{Z}_{\mathcal{P}}, 1)$. Since X_1 is an Eilenberg - MacLane space of type $(\pi_1(X), 1)$, the natural morphism $\pi_1(X) \to \pi_1(X) \otimes \mathbb{Z}_{\mathcal{P}}$ induces a continuous map $i_1 : X_1 \to X'_1$ which turns out to be a $\mathcal{P}$-localization. Moreover, replacing X'_1 by Z_{i_1} and applying CW-approximation we may suppose that i_1 is an inclusion map of CW-complexes.

Since $X_2 \to X_1$ is a principal fibration with fibre an Eilenberg - MacLane space of type $(\pi_2(X), 2)$, we may suppose that there exists an Eilenberg - MacLane space Z of type $(\pi_2(X), 3)$ and a fibration $k_1 : X_1 \to Z$.

Let Z' be an Eilenberg - MacLane space of type $(\pi_2(X) \otimes \mathbb{Z}_{\mathcal{P}}, 3)$ and let $j : Z \to Z'$ be the induced map which, as before, is a $\mathcal{P}$-localization.

Since $i_1 : X_1 \to X_1'$ is a $\mathcal{P}$-localization, the induced map $(i_1)_* : H_n(X_1; \mathbb{Z}_{\mathcal{P}}) \to H_n(X_1'; \mathbb{Z}_{\mathcal{P}})$ is an isomorphism for all $n \in \mathbb{N}$ by 2.4.6. Applying the universal coefficient theorem over $\mathbb{Z}_{\mathcal{P}}$ we obtain that $(i_1)^* : H^n(X_1'; A) \to H^n(X_1; A)$ is also an isomorphism for any $\mathbb{Z}_{\mathcal{P}}$-module A and for all $n \in \mathbb{N}$. Hence $H^n(X_1', X_1; A) = 0$ for all $n \in \mathbb{N}$. Thus, by obstruction theory, the composition $jk_1 : X_1 \to Z'$ can be extended to a continuous map $k_1' : X_1' \to Z'$.

We may suppose that k_1' is a fibration by writing it as a composition of a homotopy equivalence with a fibration. Let X_2' be the fibre of k_1' and let $i_2 : X_2 \to X_2'$ be the induced map. Note that $\pi_1(X_2')$ is abelian and that X_2' admits a Postnikov tower of principal fibrations by construction. Hence, by 2.1.13 we obtain that X_2' is abelian. Also, from the long exact sequence of homotopy groups associated to the fibration $k_1' : X_1' \to Z'$ and applying 2.4.2 we conclude that X_2' is $\mathcal{P}$-local. Also, from the five-lemma and item (b) of 2.4.2 it follows that $i_2 : X_2 \to X_2'$ is a $\mathcal{P}$-localization.

Hence, we may construct inductively a Postnikov tower of principal fibrations $X_n' \to X_{n-1}'$ together with $\mathcal{P}$-localizations $i_n : X_n \to X_n'$. Let X' be a CW-approximation to $\varprojlim X_n'$. Since by 2.1.14 there is a weak equivalence $X \to \varprojlim X_n'$, we obtain the desired $\mathcal{P}$-localization $X \to \varprojlim X_n' \to X'$. $\square$

Now we can complete the equivalent homological definition of localization with the following theorem. Note that it aids us in checking that a map is a $\mathcal{P}$-isomorphism, since it is often easier to work with homology groups than with homotopy groups.

Theorem 2.4.8. *Let X and Y be abelian spaces and let $f : X \to Y$ be a continuous map. Then f is a $\mathcal{P}$-localization map if and only if for all $n \in \mathbb{N}$, $\widetilde{H}_n(Y)$ can be given a compatible $\mathbb{Z}_{\mathcal{P}}$-module structure and the induced map $f_* \otimes \mathbb{Z}_{\mathcal{P}} : \widetilde{H}_n(X) \otimes \mathbb{Z}_{\mathcal{P}} \to \widetilde{H}_n(Y) \otimes \mathbb{Z}_{\mathcal{P}}$ is an isomorphism.*

Proof. The first implication is proposition 2.4.6. For the converse, let $X \to X'$ be a $\mathcal{P}$-localization of X. We may suppose that (Y, X) is a CW-pair, replacing Y by Z_f and then taking a CW-approximation. Hence, $H_n(Y, X, \mathbb{Z}_{\mathcal{P}}) = 0$ for all $n \in \mathbb{N}$, and thus $H^n(Y, X, \mathbb{Z}_{\mathcal{P}}) = 0$ for all $n \in \mathbb{N}$ by the universal coefficient theorem over $\mathbb{Z}_{\mathcal{P}}$. By obstruction theory, the $\mathcal{P}$-localization map $X \to X'$ can be extended to $g : Y \to X'$.

By the first implication, $\widetilde{H}_n(X')$ can be given a compatible $\mathbb{Z}_{\mathcal{P}}$-module structure and the induced map $\widetilde{H}_n(X; \mathbb{Z}_{\mathcal{P}}) \to \widetilde{H}_n(X'; Z_{\mathcal{P}})$ is an isomorphism. Hence, the induced map $g_* : \widetilde{H}_n(Y; \mathbb{Z}_{\mathcal{P}}) \to \widetilde{H}_n(X'; Z_{\mathcal{P}})$ is also an isomorphism. But since $\widetilde{H}_n(Y; \mathbb{Z}_{\mathcal{P}}) \simeq \widetilde{H}_n(Y)$ and $\widetilde{H}_n(X'; \mathbb{Z}_{\mathcal{P}}) \simeq \widetilde{H}_n(X')$ for all $n \in \mathbb{N}$ we obtain that $g_* : \widetilde{H}_n(Y) \to \widetilde{H}_n(X')$ is an isomorphism for all $n \in \mathbb{N}$.

Since Y and X are abelian, applying 1.4.36 we get that $g : Y \to X'$ is a homotopy equivalence. Thus, $X \to Y$ is a $\mathcal{P}$-localization map. $\square$

Proposition 2.4.9. *Let $f : X \to Y$ and $f' : X' \to Y'$ be $\mathcal{P}$-localizations and let $g : X \to X'$ be a continuous map. Then, there exists a continuous map $h : Y \to Y'$ such that $hf = f'g$. Moreover, if $g' : X \to X'$ and $h' : Y \to Y'$ are continuous maps such that $h'f = f'g'$ and $g \simeq g'$ then $h \simeq h'$.*

The proof of this proposition involves obstruction theory and similar arguments as those above and will not be given here. From it we obtain the following important corollary.

Corollary 2.4.10. *Let X be an abelian space. Then the $\mathcal{P}$-localization of X is unique up to homotopy equivalence, i.e. given $\mathcal{P}$-localizations $f : X \to Y$ and $f' : X \to Y'$ there exists a homotopy equivalence $h : Y \to Y'$ such that $hf = f'$.*

As a corollary of 2.4.7 and 2.4.9 we obtain a homological caracterization of $\mathcal{P}$-local spaces.

Corollary 2.4.11. *Let X be an abelian space. Then X is $\mathcal{P}$-local if and only if for all $n \in \mathbb{N}$, $\widetilde{H}_n(Y)$ can be given a compatible $\mathbb{Z}_{\mathcal{P}}$-module structure.*

2.5 Federer spectral sequence

In this section we will recall the Federer spectral sequence [6] and exhibit an alternative proof of his result. It is worth mentioning that our approach differs from Federer's since we are interested in A-homotopy groups of spaces rather than in homotopy groups of function spaces. Of course they are isomorphic if, for example, A is a locally compact CW-complex.

Definition 2.5.1. Let A and X be pointed topological spaces and let $n \in \mathbb{N}_0$. We define the *n-th A-homotopy group* of X as $\pi_n^A(X) = [\Sigma^n A, X]$. These are groups for $n \geq 1$ and abelian groups if $n \geq 2$.

For example, $\pi_n^{S^m}(X) = \pi_{n+m}(X)$ and $\pi_n^{D^m}(X) = 0$ for all $n, m \in \mathbb{N}_0$ and for all topological spaces X.

Definition 2.5.2. Let $i : A \to B$ be a cofibration. The *cofibre* of i is the space B/A defined by the pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & * \arrow[d] \\ B \arrow[r] & B/A \end{tikzcd}$$

The sequence $A \to B \to B/A$ is called a *cofibration sequence.*

Note that this notion is dual to that of the *fibre* of a fibration.

Dual to the long exact sequence of homotopy groups associated the fibration, there is a long exact sequence associated to a cofibration sequence.

Let $A \xrightarrow{i} B \xrightarrow{q} B/A$ be a cofibration sequence and let X be a topological space. Then there is a long exact sequence

$$\cdots \to \pi_n^{B/A}(X) \xrightarrow{q^*} \pi_n^B(X) \xrightarrow{i^*} \pi_n^A(X) \xrightarrow{\partial} \pi_{n-1}^{B/A}(X) \to \cdots \to \pi_0^{B/A}(X) \xrightarrow{q^*} \pi_0^B(X) \xrightarrow{i^*} \pi_0^A(X)$$

Let Y be a topological space with abelian fundamental group and let A be a finite dimensional CW-complex. Suppose, in addition that A has only one 0-cell. Note that this is not a homotopical restriction since any path-connected CW-complex is homotopy equivalent to a CW-complex (of the same dimension) with only one 0-cell. We define $A^r = *$ for $r \leq -1$.

For $r \in \mathbb{N}$, let J_r be an index set for the r-cells of A. For $\alpha \in J_r$ let g^r_α be the attaching map of the cell e^r_α.

The cofiber sequences $A^{r-1} \to A^r \to \bigvee_{J_r} S^r$ induce the corresponding long exact sequences which may be extended as follows

$$\cdots \to \pi_1^{A^{r-1}}(Y) \xrightarrow{\partial_r} \pi_0^{\bigvee S^r}(Y) \xrightarrow{q} \pi_0^{\bigvee S^r}(Y)/\mathrm{Im}\,\partial_r \xrightarrow{0} \pi_0^{\bigvee S^{r-1}}(Y)/\mathrm{Im}\,\partial_{r-1} \xrightarrow{\mathrm{id}} \pi_0^{\bigvee S^{r-1}}(Y)/\mathrm{Im}\,\partial_{r-1} \to 0$$

where q is the quotient map.

These extended exact sequences yield an exact couple (A_0, E_0, i, j, k) where the bigraded groups $A_0 = \bigoplus_{p,q\in\mathbb{Z}} A_{p,q}$ and $E_0 = \bigoplus_{p,q\in\mathbb{Z}} E^1_{p,q}$ are defined by

$$A_{p,q} = \begin{cases} \pi_{p+q+1}^{A^{-p-1}}(Y) & \text{if } p+q \geq 0 \\ \pi_0^{\bigvee S^{-p-1}}(Y)/\mathrm{Im}\,\partial & \text{if } p+q = -1 \\ 0 & \text{if } p+q \leq -2 \end{cases}$$

and

$$E^1_{p,q} = \begin{cases} \pi_{p+q}^{\bigvee S^{-p}}(Y) & \text{if } p+q \geq 0 \\ \pi_0^{\bigvee S^{-p-1}}(Y)/\mathrm{Im}\,\partial & \text{if } p+q = -1 \\ 0 & \text{if } p+q \leq -2 \end{cases}$$

Note that all these groups are abelian, except perhaps for $\pi_1^{A^r}(Y)$, $r \in \mathbb{N}$. We will prove now that $\pi_1^{A^r}(Y)$ is also an abelian group for all $r \in \mathbb{N}$. We know that $\pi_1^{A^r}(Y) = [\Sigma A^r, Y] \simeq [A^r, \Omega Y]$ and that ΩY is an H-group, where the multiplication map is given by standard path composition. Since $\pi_1(Y)$ is abelian, it follows that this multiplication is commutative and hence ΩY is an abelian H-group. Therefore, $\pi_1^{A^r}(Y)$ is an abelian group for all $r \in \mathbb{N}$.

Hence, the exact couple (A_0, E_0, i, j, k) induces a spectral sequence which, since A is finite dimensional, converges to $\pi_n^A(Y)$ for $n \geq 1$ by 2.2.10.

Note that
$$E^1_{p,q} = \pi_{p+q}^{\bigvee S^{-p}}(Y) = \prod_{J_{-p}} \pi_q(Y) \simeq C^{-p}(A; \pi_q(Y))$$

for $p+q \geq 0$ and $p \leq -1$, where $C^*(A; \pi_q(Y))$ denotes the cellular cohomology complex associated to A with coefficients in $\pi_q(Y)$.

The isomorphism $\gamma : E^1_{p,q} = \pi_{p+q}^{\bigvee S^{-p}}(Y) \to C^{-p}(A; \pi_q(Y))$ is given by

$$\gamma([f])(e_\alpha^{-p}) = [f\Sigma^{p+q} i_\alpha]$$

where $i_\alpha : S^{-p} \to \bigvee S^{-p}$ denotes the inclusion in the α-th copy. Note also that $E^2_{p,q} = 0$ if $p+q \leq -1$ or $p \geq 0$.

We wish to prove now that $E^2_{p,q} \simeq H^{-p}(A; \pi_q(Y))$ for $p+q \geq 1$ and $p \leq -1$. We will need the following lemma.

Lemma 2.5.3. *Let X be a CW-complex, let $r \geq 2$ and let $g : S^r \to X^r$ be a continuous map. For $r \in \mathbb{N}$, let J_r be an index set for the r-cells of X and for each $\beta \in J_r$ let $i_\beta : S^r \to \bigvee_{J_r} S^r$ be the inclusion in the β-th copy and let $q_\beta : X^r \to S^r$ be the quotient map which collapses $X^r - \overset{\circ}{e}{}^r_\beta$ to a point. Let $q : X^r \to X^r/X^{r-1} = \bigvee_{J_r} S^r$ be the quotient map.*

Then

$$[qg] = \sum_{\beta \in J_r} [i_\beta q_\beta g]$$

in $\pi_r(\bigvee_{J_r} S^r)$.

Proof. For each $\beta \in J_r$ let $q'_\beta : \bigvee_{\gamma \in J_r} S^r_\gamma \to S^r_\beta$ be the quotient map which collapses all but the β-th copy of S^r to a point. It is easy to see that $\bigoplus_{\beta \in J_r} (q'_\beta)_*$ is the inverse of the isomorphism $\bigoplus_{\beta \in J_r} (i_\beta)_* : \bigoplus_{\beta \in J_r} \pi_r(S^r) \to \pi_r(\bigvee_{J_r} S^r)$.

Thus,

$$[qg] = \bigoplus_{\beta \in J_r} (i_\beta)_*(\bigoplus_{\beta \in J_r} (q'_\beta)_*([qg])) = \bigoplus_{\beta \in J_r} (i_\beta)_*(\{[q_\beta g]\}_\beta) = \sum_{\beta \in J_r} [i_\beta q_\beta g].$$

$\square$

Now we consider $\delta : E^1_{p,q} \simeq C^{-p}(A; \pi_q(Y)) \to E^1_{p-1,q} \simeq C^{-p+1}(A; \pi_q(Y))$ induced from the spectral sequence. We will prove that $\delta = d^*$ for $n = p + q \geq 1$ and $p \leq -1$, where d is the cellular boundary map. This is equivalent to saying that the following diagram commutes

$$\begin{tikzcd}
\pi_n^{\bigvee S^{p'}}(Y) \arrow[r, "q^*"] \arrow[d, "\gamma"', "\simeq"] & \pi_n^{A^{p'}}(Y) \arrow[r, "(\underset{J_{p'}}{+} g_\beta^{p'+1})^*"] & \pi_{n-1}^{\bigvee S^{p'+1}}(Y) \arrow[d, "\gamma"', "\simeq"] \\
C^{p'}(A; \pi_{n+p'}(Y)) \arrow[rr, "d^*"] & & C^{p'+1}(A; \pi_{n+p'}(Y))
\end{tikzcd}$$

Here $p' = -p$.

If $[h] \in \pi_n^{\bigvee S^{p'}}(Y)$ and $e^{p'+1}_\alpha$ is a $(p'+1)$-cell of A, then

$$\left(\gamma(\bigvee_\beta g_\beta^{p'+1})^* q^*(h)\right)(e_\alpha^{p'+1}) = \gamma(h\Sigma^n q \bigvee_\beta \Sigma^n g_\beta^{p'+1})(e_\alpha^{p'+1}) = [h\Sigma^n q \Sigma^n g_\alpha^{p'+1}].$$

On the other hand,

$$\begin{aligned}
d^*(\gamma([h]))(e_\alpha^{p'+1}) &= (\gamma([h]))(d(e_\alpha^{p'+1})) = \sum_{\beta \in J_{p'}} \deg(q_\beta g_\alpha^{p'+1})(\gamma([h]))(e_\beta^{p'}) = \\
&= \sum_{\beta \in J_{p'}} \deg(q_\beta g_\alpha^{p'+1})[h\Sigma^n i_\beta].
\end{aligned}$$

Applying 2.5.3 we get

$$\begin{aligned}[h\Sigma^n q\Sigma^n g_\alpha^{p'+1}] &= h_*([\Sigma^n q g_\alpha^{p'+1}]) = h_*(\Sigma^n(\sum_{\beta\in J_{p'}}[i_\beta q_\beta g_\alpha^{p'+1}])) = \sum_{\beta\in J_{p'}}[h\Sigma^n(i_\beta q_\beta g_\alpha^{p'+1})] = \\ &= \sum_{\beta\in J_{p'}}\deg(\Sigma^n(q_\beta g_\alpha^{p'+1}))[h\Sigma^n i_\beta] = \sum_{\beta\in J_{p'}}\deg(q_\beta g_\alpha^{p'+1})[hi_\beta].\end{aligned}$$

It follows that $E^2_{p,q} \simeq H^{-p}(A;\pi_q(Y))$ for $p+q\geq 1$ and $p\leq -1$.
Hence, we have proved

Theorem 2.5.4. *Let Y be a topological space with abelian fundamental group and let A be a finite dimensional CW-complex. Then there exists a homological spectral sequence $\{E^a_{p,q}\}_{a\geq 1}$, with $E^2_{p,q}$ satisfying*

- $E^2_{p,q} \simeq H^{-p}(A;\pi_q(Y))$ *for* $p+q\geq 1$ *and* $p\leq -1$.
- $E^2_{p,q} = 0$ *if* $p+q<0$ *or* $p\geq 0$.

which converges to $\pi^A_{p+q}(Y)$ for $p+q\geq 1$.

We will call $\{E^a_{p,q}\}_{a\geq 1}$ the *Federer spectral sequence associated to A and Y.*

Example 2.5.5. If A is a Moore space of type (G,m) (with G finitely generated) and X is a path-connected abelian topological space, in the Federer spectral sequence we get

$$E^2_{-p,q} = \begin{cases} \mathrm{Hom}(G,\pi_q(X)) & \text{if } p=m \\ \mathrm{Ext}(G,\pi_q(X)) & \text{if } p=m+1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } -p+q\geq 1.$$

Hence, from the corresponding filtrations, we deduce that, for $n\geq 1$, there are short exact sequences of groups

$$0 \longrightarrow \mathrm{Ext}(G,\pi_{n+m+1}(X)) \longrightarrow \pi^A_n(X) \longrightarrow \mathrm{Hom}(G,\pi_{n+m}(X)) \longrightarrow 0$$

As a corollary, if G is a finite group of exponent r then $\alpha^{2r}=0$ for every $\alpha\in\pi^A_n(X)$. For example, if X is a path-connected and abelian topological space, then every element in $\pi^{\mathbb{P}^2}_n(X)$ $(n\geq 1)$ has order 1, 2 or 4.

Now, we will apply the previous theorem to give an extension to the Hopf-Whitney theorem, which is not only interesting for its own sake but also will be useful for us later.

Theorem 2.5.6. *Let K be a path-connected CW-complex of dimension $n\geq 2$ and let Y be $(n-1)$-connected. Then there exists a bijection $[K,Y]\leftrightarrow H^n(K;\pi_n(Y))$.*

In addition, if K is the suspension of a path-connected CW-complex (or if Y is a loop space), then the groups $[K,Y]$ and $H^n(K;\pi_n(Y))$ are isomorphic.

The first part is the Hopf-Whitney theorem [16]. The second part can be proved easily by means of the Federer spectral sequence. Concretely, suppose that $K = \Sigma K'$ with K' path-connected. Let $\{E^a_{p,q}\}$ denote the Federer spectral sequence associated to K' and Y. Then $E^2_{p,q} = 0$ for $q \leq n-1$ since Y is $(n-1)$-connected, and $E^2_{p,q} = 0$ for $p \leq -n$ since $\dim K' = n-1$. Hence, $E^2_{-(n-1),n} = H^{n-1}(K'; \pi_n(Y))$ survives to E^∞. As it is the only nonzero entry in the diagonal $p+q=1$ of E^2 it follows that

$$[K, Y] = \pi_1^{K'}(Y) = E^2_{-(n-1),n} = H^{n-1}(K'; \pi_n(Y)) = H^n(K; \pi_n(Y)).$$

Chapter 3

Definition of CW(A)-complexes and first results

In this chapter, we introduce CW-complexes of type A, or CW(A)-complexes for short, generalizing CW-complexes, which turn out to be CW(S^0)-complexes. As mentioned in the introduction, there exist other generalizations of CW-complexes in the literature, but our approach is quite different from them and keeps the geometric and combinatorial nature of Whitehead's original theory. Thus, it also gives us a deeper insight onto the classical theory of CW-complexes.

We also mention that many results given in this chapter are completely new, while others are generalization of well-known properties of CW-complexes. Among these latter ones, we find that some proofs can be generalized without difficulty, while others need a different argument.

In the first section of this chapter, we give the constructive definition of CW(A)-complexes, analyse some of their topological properties and generalize known results for CW-complexes. We study basic constructions such as cylinders, cones and suspensions of CW(A)-complexes which are useful when dealing with homotopy and homology of these spaces.

Of course, some classical results are no longer true for general cores A. For example, the notion of dimension of a space (as a CW(A)-complex) is not always well defined. Recall that in the classical case, the good definition of dimension is deduced from the famous invariance of dimension theorem. By a similar argument, we can prove that in particular cases (for example when the core A is itself a finite dimensional CW-complex) the dimension of a CW(A)-complex is well defined. We study this and other invariants and exhibit many examples and counterexamples to clarify the main concepts.

Although our definition of CW(A)-complexes is constructive, we also give a descriptive definition and compare them. In the classical theory of CW-complexes it is well known that both definitions coincide, but for an arbitrary chosen core A they may differ, as we shall see.

We also study the relationship between different decompositions and analyse the change from a core A to a core B via a map $\alpha : A \to B$.

From now on we will work in the category of pointed topological spaces. Hence, maps will be pointed and cylinders, cones and suspensions will always be reduced.

3.1 The constructive approach

Let A be a fixed pointed topological space.

Definition 3.1.1. We say that a (pointed) space X *is obtained from a (pointed) space* B *by attaching an* n*-cell of type* A (or simply, an A-n-*cell*) if there exists a pushout diagram

$$\begin{tikzcd} \Sigma^{n-1}A \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & B \arrow[d] \\ \mathrm{C}\Sigma^{n-1}A \arrow[r, "f"'] & X \end{tikzcd}$$

The A-*cell* is the image of f. The map g is the *attaching map* of the cell, and f is its *characteristic map.*

We say that X *is obtained from* B *by attaching a* 0*-cell of type* A if $X = B \vee A$.

Note that attaching an S^0-n-cell is the same as attaching an n-cell in the usual sense, and that attaching an S^m-n-cell means attaching an $(m+n)$-cell in the usual sense.

The reduced cone $\mathrm{C}A$ of A is obtained from A by attaching an A-1-cell. In particular, D^2 is obtained from D^1 by attaching a D^1-1-cell. Also, the reduced suspension ΣA can be obtained from the singleton $*$ by attaching an A-1-cell.

Of course, we can attach many n-cells at the same time by taking various copies of $\Sigma^{n-1}A$ and $\mathrm{C}\Sigma^{n-1}A$.

$$\begin{tikzcd} \bigvee\limits_{\alpha\in J} \Sigma^{n-1}A \arrow[r, "\underset{\alpha\in J}{+} g_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & B \arrow[d] \\ \bigvee\limits_{\alpha\in J} \mathrm{C}\Sigma^{n-1}A \arrow[r, "\underset{\alpha\in J}{+} f_\alpha"'] & X \end{tikzcd}$$

Definition 3.1.2. A *CW-structure with base* A *on a space* X, or simply a *CW(A)-structure on* X, is a sequence of spaces $* = X^{-1}, X^0, X^1, \dots, X^n, \dots$ such that, for $n \in \mathbb{N}_0$, X^n is obtained from X^{n-1} by attaching n-cells of type A, and X is the colimit of the diagram

$$* = X^{-1} \to X^0 \to X^1 \to \dots \to X^n \to \dots$$

We call X^n the n-*skeleton* of X. The base point $*$ will be regarded as a (-1)-cell.

We say that the space X is a *CW(A)-complex* (or simply a *CW(A)*), if it admits some CW(A)-structure. In this case, the space A will be called the *core* or the *base space* of the structure.

Note that a CW(A)-complex may admit many different CW(A)-structures.

Examples 3.1.3.

(1) A CW(S^0)-complex is just a CW-complex and a CW(S^n)-complex is a CW-complex with no cells of dimension less than n, apart from the base point. Moreover, any path-connected CW-complex is homotopy equivalent to a CW(S^0)-complex. Indeed, if X is a path-connected CW-complex, then X is homotopy equivalent to a CW-complex with only one 0-cell and any attaching map is homotopic to a base-point preserving map. Hence, the result follows applying 1.1.19 and 1.1.22.

(2) The space D^n admits several different CW(D^1)-structures. For instance, we can take $X^r = D^{r+1}$ for $0 \leq r \leq n-1$ since $\mathrm{C}D^r = D^{r+1}$. We may also take $X^0 = \ldots = X^{n-2} = *$ and $X^{n-1} = D^n$ since there is a pushout

$$\begin{tikzcd}
\Sigma^{n-2}D^1 = D^{n-1} \arrow[rr] \arrow[d, "i"'] \arrow[drr, phantom, "push"] & & * \arrow[d] \\
\mathrm{C}\Sigma^{n-2}D^1 = \mathrm{C}D^{n-1} \arrow[r] & \Sigma D^{n-1} = D^n &
\end{tikzcd}$$

As in the classical case, instead of starting attaching cells from a base point $*$, we can start attaching cells on a pointed space B.

Definition 3.1.4. A *relative CW(A)-complex* is a pair (X, B) such that X is the colimit of a diagram

$$B = X_B^{-1} \to X_B^0 \to X_B^1 \to \ldots \to X_B^n \to \ldots$$

where X_B^n is obtained from X_B^{n-1} by attaching n-cells of type A.

One can also build a space X by attaching cells (of some type A) without requiring them to be attached in such a way that their dimensions form an increasing sequence. That means, for example, that a 2-cell may be attached on a 5-cell.

In general, those spaces might not admit a CW(A)-structure and they will be called *generalized CW(A)-complexes* (see 3.1.6). If the core A is itself a CW-complex, then a generalized CW(A)-complex has the homotopy type of a CW-complex. This generalizes the well-known fact that a *generalized CW-complex* has the homotopy type of a CW-complex.

Before giving the formal definition we show an example of a generalized CW-complex which is not a CW-complex.

Example 3.1.5. We build X as follows. We start with a 0-cell and we attach a 1-cell by the identity map obtaining the interval $[-1;1]$. We regard 1 as the base point. Now, for each $n \in \mathbb{N}$ we define $g_n : S^0 \to [-1,1]$ by $g_n(1) = 1$, $g_n(-1) = 1/n$. We attach 1-cells by the maps g_n. This space X is an example of a generalized CW-complex (with core S^0).

It is not hard to verify that it is not a CW-complex. To prove it, suppose that X admits a CW-complex structure. We will prove that the points $1/n$ must be 0-cells, but they have a cluster point which is not possible for 0-cells of a CW-complex. Fix n and call $p = 1/n$. The point p must be in the interior of some cell. By a dimension argument it is easy to see that p can't belong to the interior of an r-cell for $r \geq 1$ because the neighbourhoods of p are not homeomorphic to the r-disk. Thus, p must belong to the interior of a 0-cell, and hence it is a 0-cell.

We shall see that if A is a CW-complex then a generalized CW(A)-complex has the homotopy type of a CW-complex. In particular, a generalized CW-complex (that is, a generalized CW(S^0)) has the homotopy type of a CW-complex.

Definition 3.1.6. We say that X *is obtained from* B *by attaching cells* (of different dimensions) *of type* A if there is a pushout

$$\begin{tikzcd}
\bigvee\limits_{\alpha\in J} \Sigma^{n_\alpha -1}A \arrow[rr, "\underset{\alpha\in J}{+} g_\alpha"] \arrow[d, "i"'] \arrow[drr, phantom, "push"] & & B \arrow[d] \\
(\bigvee\limits_{\alpha\in J_0} A) \vee (\bigvee\limits_{\alpha\in J} C\Sigma^{n_\alpha -1}A) \arrow[rr, "\underset{\alpha\in J}{+} f_\alpha"'] & & X
\end{tikzcd}$$

where $n_\alpha \in \mathbb{N}$ for all $\alpha \in J$. We say that X is a *generalized CW(A)-complex* if X is the colimit of a diagram

$$* = X^0 \to X^1 \to X^2 \to \ldots \to X^n \to \ldots$$

where X^n is obtained from X^{n-1} by attaching cells (of different dimensions) of type A.

We call X^n the *n-th layer* of X.

One can also define *generalized relative CW(A)-complexes* in the obvious way.

For standard CW-complexes, by the classical *Invariance of Dimension Theorem*, one can prove that the notion of dimension is well defined. Any two different structures of a CW-complex must have the same dimension.

For a general core A this is no longer true. However, we shall prove later that for particular cases (for example when A is a finite dimensional CW-complex) the notion of dimension of a CW(A)-complex is well defined.

Definition 3.1.7. Let X be a CW(A)-complex. We consider X endowed with a particular CW(A)-structure $\mathcal{K}$. We say that the *dimension* of $\mathcal{K}$ is n if $X^n = X$ and $X^{n-1} \neq X$, and we write $\dim(\mathcal{K}) = n$. We say that $\mathcal{K}$ is *finite dimensional* if $\dim(\mathcal{K}) = n$ for some $n \in \mathbb{N}_0$.

Important remark 3.1.8. A CW(A)-complex may admit CW(A)-structures of different dimensions. For example, let $A = \bigvee\limits_{n\in\mathbb{N}} S^n$ and let $X = \bigvee\limits_{j\in\mathbb{N}} A$. Then X has a zero-dimensional CW(A)-structure. But we can see that $X = (\bigvee\limits_{j\in\mathbb{N}} A) \vee \Sigma A$, which induces a 1-dimensional structure. Note that $\bigvee\limits_{j\in\mathbb{N}} A = (\bigvee\limits_{j\in\mathbb{N}} A) \vee \Sigma A$ since both spaces consist of countably many copies of S^n for each $n \in \mathbb{N}$.

Another example is the following. It is easy to see that if B is a topological space with the indiscrete topology then its reduced cone and suspension also have the indiscrete topology. So, let A be an indiscrete topological space with $1 \leq \#A \leq c$. If A is just a point then its reduced cone and suspension are also singletons, so $*$ can be given a CW($*$) structure of any dimension. If $\#A \geq 2$ then $\#(\Sigma^n A) = c$ for all n, and $\Sigma^n A$ are all indiscrete spaces. Since they have all the same cardinality and they are indiscrete then all of them are homeomorphic. But each $\Sigma^n A$ has an obvious CW(A)-structure of dimension n. Thus, the homeomorphisms between $\Sigma^n A$ and $\Sigma^m A$, for all m, allow us to give $\Sigma^n A$ a CW(A)-structure of any dimension (greater than zero).

Given a CW(A)-complex X, we define the *boundary* of an n-cell e^n by $\overset{\bullet}{e}{}^n = e^n \cap X^{n-1}$ and the *interior* of e^n by $\overset{\circ}{e}{}^n = e^n - \overset{\bullet}{e}{}^n$.

A cell e^m_β is called an *immediate face of* e^n_α if $\overset{\circ}{e}{}^m_\beta \cap e^n_\alpha \neq \varnothing$, and a cell e^m_β is called a *face of* e^n_α if there exists a finite sequence of cells

$$e^m_\beta = e^{m_0}_{\beta_0}, e^{m_1}_{\beta_1}, e^{m_2}_{\beta_2}, \ldots, e^{m_k}_{\beta_k} = e^n_\alpha$$

such that $e^{m_j}_{\beta_j}$ is an immediate face of $e^{m_{j+1}}_{\beta_{j+1}}$ for $0 \leq j < k$.

Finally, we call a cell *principal* if it is not a face of any other cell.

Remark 3.1.9. Note that $\overset{\circ}{e}{}^n_\alpha \cap \overset{\circ}{e}{}^m_\beta \neq \varnothing$ if and only if $n = m$, $\alpha = \beta$. Thus, if e^m_β is a face of e^n_α and $e^m_\beta \neq e^n_\alpha$ then $m < n$.

As in the classical case, we can define subcomplexes and A-cellular maps in the obvious way. A-cellular maps will often be called simply cellular when there is no risk of ambiguity.

Remark 3.1.10. If X is a CW(A), then $X = \bigcup\limits_{n,\alpha} \overset{\circ}{e}{}^n_\alpha$.

Proof. Let $x \in X$. Then there exist m, β such that $x \in e^m_\beta$, and we may choose a cell with minimum m. If $x \in \overset{\circ}{e}{}^m_\beta$ we are done. If not, $x \in \overset{\bullet}{e}{}^m_\beta \subseteq X^{m-1}$, then x belongs to a cell with dimension less than m, a contradiction. □

Proposition 3.1.11. *Let X be a CW(A)-complex and suppose that the base point of A is closed in A. Then the interiors of the n-cells are open in the n-skeleton. In particular, X^{n-1} is a closed subspace of X^n.*

Proof. For $n = -1$ and $n = 0$, the statement is clear. Let $n \geq 1$. There is a pushout diagram

$$\begin{tikzcd}
\bigvee\limits_{\alpha \in J} \Sigma^{n-1} A \arrow[r, "\underset{\alpha\in J}{+} g_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & X^{n-1} \arrow[d] \\
\bigvee\limits_{\alpha \in J} \mathrm{C}\Sigma^{n-1} A \arrow[r, "\underset{\alpha\in J}{+} f_\alpha"'] & X^n = X^{n-1} \cup \bigcup\limits_{\alpha} e^n_\alpha
\end{tikzcd}$$

Consider a cell e^n_β. In order to verify that $\overset{\circ}{e}{}^n_\beta$ is open in X^n we have to prove that $(+f_\beta)^{-1}(\overset{\circ}{e}{}^n_\beta)$ is open in $\bigvee\limits_{\alpha \in J} \mathrm{C}\Sigma^{n-1}A$. Since $(+f_\beta)^{-1}(\overset{\circ}{e}{}^n_\beta) = \mathrm{C}\Sigma^{n-1}A - \Sigma^{n-1}A$ is open in $\mathrm{C}\Sigma^{n-1}A$, then $\overset{\circ}{e}{}^n_\beta$ is open in X^n. □

The previous proposition does not hold if the base point is not closed in A. For example, let A be an indiscrete space with $\#A \geq 2$ and $X = \mathrm{C}A$, obtained by attaching an A-1-cell to A by the identity map. Then X has also the indiscrete topology and hence, the interior of the A-1-cell is not open in X.

Recall that a topological space Y is T1 if the points are closed in X.

Proposition 3.1.12. *Let A be a pointed T1 topological space, let X be a CW(A)-complex and let $K \subseteq X$ be a compact subspace. Then K meets only a finite number of interiors of cells.*

Proof. Let $\Lambda = \{\alpha /\ K \cap \overset{\circ}{e}{}^{n_\alpha}_\alpha \neq \varnothing\}$. For each $\alpha \in \Lambda$ choose $x_\alpha \in K \cap \overset{\circ}{e}{}^{n_\alpha}_\alpha$. We want to show that for any $\alpha \in \Lambda$ there exists an open subspace $U_\alpha \subseteq X$ such that $U_\alpha \supseteq \overset{\circ}{e}{}^{n_\alpha}_\alpha$ and $x_\beta \notin U_\alpha$ for any $\beta \neq \alpha$.

For each n, let J_n be the index set of the n-cells. We denote by g^n_α the attaching map of e^n_α and by f^n_α its characteristic map.

Fix $\beta \in \Lambda$. Take $U_1 = \overset{\circ}{e}{}^{n_\beta}_\beta$, which is open in X^{n_β}. If $n_\beta = -1$, we take $U_2 = (\bigvee\limits_{\alpha \in J_0 \cap \Lambda} A - \{x_\alpha\}) \vee (\bigvee\limits_{\alpha \in J_0 - \Lambda} A)$, which is open in the 0-skeleton.

Now, for $n_\beta + n - 1 \geq 1$ we construct inductively open subspaces U_n of $X^{n_\beta + n - 1}$ with $U_{n-1} \subseteq U_n$, $U_n \cap X^{n_\beta + n - 2} = U_{n-1}$ and such that $x_\alpha \notin U_n$ if $\alpha \neq \beta$.

If the base point $a_0 \notin U_{n-1}$, we take

$$U_n = U_{n-1} \cup \bigcup_{\alpha \in J_{n_\beta + n - 1}} f^{n_\alpha}_\alpha((g^{n_\alpha}_\alpha)^{-1}(U_{n-1}) \times (1 - \varepsilon_\alpha, 1])$$

with $0 < \varepsilon_\alpha < 1$ chosen in such a way that $x_\alpha \notin U_n$ if $\alpha \neq \beta$. Note that U_n is open in $X^{n_\beta + n - 1}$.

If $a_0 \in U_{n-1}$ we take

$$U_n = U_{n-1} \cup \bigcup_{\alpha \in J_{n_\beta + n - 1}} f^{n_\alpha}_\alpha(((g^{n_\alpha}_\alpha)^{-1}(U_{n-1}) \times (1 - \varepsilon_\alpha, 1]) \cup (W_{x_\alpha} \times I) \cup (\Sigma^{n_\beta + n - 1} A \times [0, \varepsilon'_\alpha)))$$

with $W_{x_\alpha} = V_{x_\alpha} \cap (g^{n_\alpha}_\alpha)^{-1}(U_{n-1})$, where $V_{x_\alpha} \subseteq \Sigma^{n_\beta + n - 1} A$ is an open neighbourhood of the base point not containing x'_α (where $x_\alpha = f^{n_\alpha}_\alpha(x'_\alpha, t_\alpha)$), and $0 < \varepsilon_\alpha < 1$, $0 < \varepsilon'_\alpha < 1$, chosen in such a way that $x_\alpha \notin U_n$ if $\alpha \neq \beta$. Note that U_n is open in $X^{n_\beta + n - 1}$.

We set $U_\beta = \bigcup\limits_{n \in \mathbb{N}} U_n$. Thus $K \subseteq \bigcup\limits_{\alpha \in \Lambda} \overset{\circ}{e}{}^{n_\alpha}_\alpha \subseteq \bigcup\limits_{\alpha \in \Lambda} U_\alpha$, and $x_\alpha \notin U_\beta$ if $\alpha \neq \beta$. Since $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open covering of K which does not admit a proper subcovering, Λ must be finite. $\square$

Lemma 3.1.13. *Let A and B be Hausdorff spaces and suppose X is obtained from B by attaching cells of type A. Then X is Hausdorff.*

Proof. Let $x, y \in X$. If x, y lie in the interior of some cell, then it is easy to choose the open neighbourhoods. If one of them belongs to B and the other to the interior of a cell, let us say $x \in e^{n_\alpha}_\alpha$, we work as in the previous proof. Explicitly, if $x = f_\alpha(a, t)$ with $a \in \Sigma^{n_\alpha - 1} A$, $t \in I$ then we take $U' \subseteq \Sigma^{n_\alpha - 1} A$ open set such that $a \in U'$ and

$a_0 \notin U'$, where a_0 is the basepoint of $\Sigma^{n_\alpha - 1}A$. We define $U = f_\alpha(U' \times (t/2, (1+t)/2))$, and $V = X - f_\alpha(\overline{U'} \times [t/2, (1+t)/2])$.

If $x, y \in B$, since B is Hausdorff there exist $U', V' \subseteq B$ open disjoint sets such that $x \in U'$ and $y \in V'$. However, U' and V' need not be open in X. Suppose first that x, y are both different from the base point. So we may suppose that neither U' nor V' contain the base point. We take

$$U = U' \cup \bigcup_{\alpha \in J} f_\alpha((g_\alpha)^{-1}(U') \times (1/2; 1])$$

$$V = V' \cup \bigcup_{\alpha \in J} f_\alpha((g_\alpha)^{-1}(V') \times (1/2; 1])$$

If x is the base point then we take

$$U = U' \cup \bigcup_{\alpha \in J} f_\alpha(((g_\alpha)^{-1}(U') \times I) \cup (\Sigma^{n_\alpha - 1}A \times [0; 1/2)))$$

□

Proposition 3.1.14. *Let A be a Hausdorff space and let X be a CW(A)-complex. Then X is a Hausdorff space.*

Proof. By the previous lemma and induction we have that X^n is a Hausdorff space for all $n \geq -1$. Given $x, y \in X$, choose $m \in \mathbb{N}$ such that $x, y \in X^m$. As X^m is a Hausdorff space, there exist disjoint sets U_0 and V_0, which are open in X^m, such that $x \in U_0$ and $y \in V_0$. Proceeding in a similar way as we did in the previous results we construct inductively sets U_k, V_k for $k \in \mathbb{N}$ such that $U_k, V_k \subseteq X^{m+k}$ are open sets, $U_k \cap V_k = \varnothing$, $U_k \cap X^{m+k-1} = U_{k-1}$ and $V_k \cap X^{m+k-1} = V_{k-1}$ for all $k \in \mathbb{N}$. We take $U = \bigcup U_k$, $V = \bigcup V_k$. □

Remark 3.1.15. Let X be a CW(A)-complex and $S \subseteq X$ a subspace. Then S is closed in X if and only if $S \cap e^n_\alpha$ is closed in e^n_α for all n, α.

Proposition 3.1.16. *Let A be a finite dimensional CW-complex, $A \neq *$, and let X be a CW(A)-complex. Let $\mathcal{K}$ and $\mathcal{K}'$ be CW(A)-structures in X and let $n, m \in \mathbb{N}_0 \cup \{\infty\}$ denote their dimensions. Then $n = m$.*

Proof. We suppose first that $\mathcal{K}$ and $\mathcal{K}'$ are finite dimensional and $n \geq m$.

Let $k = \dim(A)$ and let e^n_α be an n-cell of $\mathcal{K}$. There is a homeomorphism $\mathring{e}^n_\alpha \simeq \mathrm{C}\Sigma^{n-1}A - \Sigma^{n-1}A$, and $\mathring{e}^n_\alpha$ is open in X. Let e be a cell of maximum dimension of the CW-complex $\mathrm{C}\Sigma^{n-1}A$ and let $U = \mathring{e}$. Thus U is open in X and homeomorphic to $\mathring{D}^{n+k}$.

Now, U intersects some interiors of cells of type A of $\mathcal{K}'$. Let e_0 be one of those cells with maximum dimension. Suppose e_0 is an m'-cell, with $m' \leq m$. Then $\mathring{e}_0$ is open in the m'-skeleton of X with the $\mathcal{K}'$ structure. It is not hard to see that $V = U \cap \mathring{e}_0$ is open in U, extending $\mathring{e}_0$ to an open subset of X as in the proof of 3.1.12. Indeed, let U_0 be an open subset of X such that $U_0 \cap X^{m'} = \mathring{e}_0$. Note that $U \subseteq X^{m'}$. Hence, $U_0 \cap U \subseteq U_0 \cap X^{m'} = \mathring{e}_0$.

Then, $U_0 \cap U \subseteq \overset{\circ}{e_0} \cap U$. On the other hand, $\overset{\circ}{e_0} \subseteq U_0$ and hence $\overset{\circ}{e_0} \cap U \subseteq U_0 \cap U$. Thus, $V = \overset{\circ}{e_0} \cap U \subseteq U_0 \cap U$ and therefore V is open in U.

In a similar way, $\overset{\circ}{e_0} \simeq C\Sigma^{m'-1}A - \Sigma^{m'-1}A$, and V meets some interiors of cells of the CW-complex $C\Sigma^{m'-1}A$. We take e_1 a cell (of type S^0) of maximum dimension among those cells and we denote $k' = \dim(e_1)$. Then $\overset{\circ}{e_1}$ is homeomorphic to $\overset{\circ}{D^{k'}}$. Let $W = V \cap \overset{\circ}{e_1}$. One can check that W is open in $\overset{\circ}{e_1} \simeq \overset{\circ}{D^{k'}}$ and that it is also open in $U \simeq \overset{\circ}{D^{n+k}}$.

Indeed, let $W = V \cap \overset{\circ}{e_1}$. Again, W is open in $\overset{\circ}{e_1}$ and since $\overset{\circ}{e_1}$ is open in the k'-th skeleton of e_0, there exists $U_2 \subseteq \overset{\circ}{e_0}$ open subset with $U_2 \cap (e_0)^{k'} = \overset{\circ}{e_1}$. Then $W = V \cap \overset{\circ}{e_1} = V \cap U_2$. Now, $V \subseteq \overset{\circ}{e_0}$, so W is open in V. Thus W is open in $\overset{\circ}{e_1}$ and in V and hence in U. So, W is open in $\overset{\circ}{e_1} \simeq \overset{\circ}{D^{k'}}$ and W is open in $U \simeq \overset{\circ}{D^{n+k}}$.

Hence, by the invariance of dimension theorem, $n+k = k'$, but also $k' \leq m+k \leq n+k$. Thus $n = m$.

It remains to be shown that if $m = \infty$ then $n = \infty$. Suppose that $m = \infty$ and $n \neq \infty$. Let $k = \dim(A)$. We choose e^l an l-cell of $\mathcal{K}'$ with $l > n+k$. Then $\overset{\circ}{e^l}$ is open in the l-skeleton $(\mathcal{K}')^l$. As in the proof of 3.1.12, we can extend $\overset{\circ}{e^l}$ to an open subset U of X with $U \cap (\mathcal{K}')^{l-1} = \varnothing$. Now we take a cell e_1 of $\mathcal{K}$ such that $\overset{\circ}{e_1} \cap U \neq \varnothing$ and with the property of being of maximum dimension among the cells of $\mathcal{K}$ whose interior meets U. Let $r = \dim(e_1)$. We have that $U \subseteq \mathcal{K}^r$. As before, we extend $\overset{\circ}{e_1}$ to an open subset V of X with $V \cap \mathcal{K}^{r-1} = \varnothing$, $V \cap \mathcal{K}^r = \overset{\circ}{e_1}$. So $U \cap \overset{\circ}{e_1} = U \cap V$ is open in X. Proceeding similarly, since $\overset{\circ}{e_1} \simeq C\Sigma^{r-1}A - \Sigma^{r-1}A$, we can choose a cell e_2 of e_1 (of type S^0) with maximum dimension such that $W = \overset{\circ}{e_2} \cap (U \cap \overset{\circ}{e_1}) \neq \varnothing$. Again, W is open in X. Let $s = \dim e_2$. So W is open in $\overset{\circ}{e_2} \simeq \overset{\circ}{D^s}$ and $s \leq r+k \leq n+k < l$. On the other hand, W must meet the interior of some cell of type S^0 belonging to one of the cells of $\mathcal{K}'$ with dimension greater than or equal to l (since $U \cap (\mathcal{K}')^{l-1} = \varnothing$). So, a subset of W is homeomorphic to an open set of $\overset{\circ}{D^q}$ with $q \geq l$, a contradiction. $\square$

Lemma 3.1.17. *Let X and Y be CW(A)-complexes, let $B \subseteq X$ be a subcomplex, and let $f : B \to Y$ be an A-cellular map. Then the pushout*

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & Y \arrow[d] \\ X \arrow[r] & X \underset{B}{\cup} Y \end{tikzcd}$$

is a CW(A)-complex.

Proof. We denote by $\{e^n_{X,\alpha}\}_{\alpha \in J_n}$ the n-cells (of type A) of the relative CW(A)-complex (X, B) and by $\{e^n_{Y,\alpha}\}_{\alpha \in J'_n}$ the n-cells of Y. We will construct $X \underset{B}{\cup} Y$ attaching the cells of Y with the same attaching maps and at the same time we will attach the cells of (X, B) using the map $f : B \to Y$.

Let $J''_0 = J_0 \cup J'_0$ and $Z^0 = \bigvee_{\alpha\in J''_0} A$. We define $f_0 : X^0 \to Z^0$ by $f_0|_{B^0} = f|_{B^0}$ and $f_0|_{\bigcup_{\alpha\in J_0} e^0_{X,\alpha}}$ the inclusion.

Suppose that Z^{n-1} and $f_{n-1} : X^{n-1} \to Z^{n-1}$ with $f_{n-1}|_{B^{n-1}} = f$ are defined. We define Z^n by the following pushout.

$$\begin{tikzcd}
\bigvee_{\alpha\in J''_n} \Sigma^{n-1}A \arrow[r, "+_{\alpha\in J''_n} g''_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & Z^{n-1} \arrow[d, "i_{n-1}"] \\
\bigvee_{\alpha\in J''_n} \mathrm{C}\Sigma^{n-1}A \arrow[r, "+_{\alpha\in J''_n} f''_\alpha"'] & Z^n
\end{tikzcd}$$

where $J''_n = J_n \cup J'_n$ and

$$g''_\alpha = \begin{cases} f_{n-1}\circ g_\alpha & \text{if } \alpha\in J_n \\ g'_\alpha & \text{if } \alpha \in J'_n \end{cases}$$

where g_α and g'_α are the attaching maps. We define $f_n : X^n \to Z^n$ by $f_n|_{B^n} = f|_{B^n}$, $f_n|_{X^{n-1}} = f_{n-1}$ and $f_n|_{\bigcup_{\alpha\in J_n} e^n_{X,\alpha}} = f''_\alpha$ (i.e. $f_n(f_\alpha(x)) = f''_\alpha(x)$). Note that f_n is well defined.

Let Z be the colimit of the Z^n. By construction it is not difficult to verify that Z satisfies the universal property of the pushout. $\square$

Corollary 3.1.18. *Let X be a CW(A)-complex and $B \subseteq X$ a subcomplex. Then X/B is a CW(A)-complex.*

Theorem 3.1.19. *Let X be a CW(A)-complex. Then the reduced cone $\mathrm{C}X$ and the reduced suspension ΣX are CW(A)-complexes. Moreover, X is a subcomplex of both of them.*

Proof. By the previous lemma, it suffices to prove the result for $\mathrm{C}X$.

Let e^n_α be the n-cells of X and, for each n, let J_n be the index set of the n-cells. We denote by g^n_α the attaching maps and by f^n_α the characteristic maps. Let $i_{n-1} : X^{n-1} \to X^n$ be the inclusions. We construct $Y = \mathrm{C}X$ as follows.

Let $Y^0 = \bigvee_{\alpha\in J_0} A = X^0$.

We construct Y^1 from Y^0 and from the 0-cells and the 1-cells of X by the pushout

$$\begin{tikzcd}
\bigvee_{\alpha\in J'_1} A \arrow[r, "+_{\alpha\in J'_1} g'_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & Y^0 \arrow[d, "i'_0"] \\
\bigvee_{\alpha\in J'_1} \mathrm{C}A \arrow[r, "+_{\alpha\in J'_1} f'_\alpha"'] & Y^1
\end{tikzcd}$$

where $J_1' = J_0 \sqcup J_1$. The maps g'_α, for $\alpha \in J_1'$, are defined as

$$g'_\alpha = \begin{cases} i_\alpha & \text{if } \alpha \in J_0 \\ g_\alpha & \text{if } \alpha \in J_1 \end{cases}$$

and $i_\alpha : A \to \bigvee\limits_{\alpha\in J_0} A$ is the inclusion of A in the α-th copy. Note that X^1 is a subcomplex of Y^1.

Note also that the 1-cells of Y are divided into two sets. The ones with $\alpha \in J_1$ are the 1-cells of X, and the others are the cone of the 0-cells of X.

Inductively, suppose we have constructed Y^{n-1}. We define Y^n as the pushout

$$\begin{tikzcd}
\bigvee\limits_{\alpha\in J_n'} \Sigma^{n-1}A \arrow[r, "\underset{\alpha\in J_n'}{+} g'_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & Y^{n-1} \arrow[d, "i'_{n-1}"] \\
\bigvee\limits_{\alpha\in J_n'} \mathrm{C}\Sigma^{n-1}A \arrow[r, "\underset{\alpha\in J_n'}{+} f'_\alpha"'] & Y^n
\end{tikzcd}$$

where $J_n' = J_{n-1} \sqcup J_n$ and

$$g'_\alpha = \begin{cases} g_\alpha & \text{for } \alpha \in J_n \\ f_\alpha \cup \mathrm{C}g_\alpha & \text{for } \alpha \in J_{n-1}\ . \end{cases}$$

We prove now that $Y^n = \mathrm{C}X^{n-1} \cup \bigcup\limits_{\alpha} e^n_\alpha$. We have the following commutative diagram.

$$\begin{tikzcd}
\bigvee\limits_{\alpha\in J_n'} \Sigma^{n-1}A \arrow[r, "(\underset{\alpha\in J_{n-1}}{+} g'_\alpha)\vee \mathrm{Id}"] \arrow[d, "\bigvee\limits_{\alpha\in J_n'} i"'] & Y^{n-1} \vee \bigvee\limits_{\alpha\in J_n} \Sigma^{n-1}A \arrow[r, "\mathrm{Id}+(\underset{\alpha\in J_n}{+} g'_\alpha)"] \arrow[d, "i_{n-1}\vee \bigvee\limits_{\alpha\in J_n} i"] \arrow[dr, phantom, "push"] & Y^{n-1} \arrow[d] \\
\bigvee\limits_{\alpha\in J_n'} \mathrm{C}\Sigma^{n-1}A \arrow[r, "(\underset{\alpha\in J_{n-1}}{+} f'_\alpha)\vee \mathrm{Id}"'] & \mathrm{C}X^{n-1} \vee \bigvee\limits_{\alpha\in J_n} \mathrm{C}\Sigma^{n-1}A \arrow[r, "\mathrm{Id}+(\underset{\alpha\in J_n}{+} f'_\alpha)"'] & \mathrm{C}X^{n-1} \cup \bigcup\limits_{\alpha} e^n_\alpha
\end{tikzcd}$$

The right square is clearly a pushout. To prove that the left square is also a pushout it suffices to verify that the following is also a pushout.

$$\begin{tikzcd}
\bigvee\limits_{\alpha\in J_{n-1}} \Sigma^{n-1}A \arrow[r, "\underset{\alpha\in J_{n-1}}{+} g'_\alpha"] \arrow[d, "\bigvee\limits_{\alpha\in J_{n-1}} i"'] & Y^{n-1} = \mathrm{C}X^{n-2} \cup \bigcup\limits_{\alpha\in J_{n-1}} e^{n-1}_\alpha \arrow[d, "\mathrm{inc}"] \\
\bigvee\limits_{\alpha\in J_{n-1}} \mathrm{C}\Sigma^{n-1}A \arrow[r, "\underset{\alpha\in J_{n-1}}{+} f'_\alpha"'] & \mathrm{C}X^{n-1}
\end{tikzcd}$$

For simplicity, we will prove this in the case that there is only one A-$(n-1)$-cell. Let

$$\begin{aligned}
&j : \Sigma^{n-1}A \to \mathrm{C}\Sigma^{n-1}A \\
&i_1 : \mathrm{C}(\Sigma^{n-1}A) \times \{1\} \to \mathrm{CC}\Sigma^{n-1}A \\
&i_2 : (\Sigma^{n-1}A) \times \{1\} \times I/\sim \to \mathrm{CC}\Sigma^{n-1}A \\
&i : \Sigma^n A = \mathrm{C}\Sigma^{n-1}A \underset{A}{\cup} \mathrm{C}\Sigma^{n-1}A \to \mathrm{C}\Sigma^n A
\end{aligned}$$

be the corresponding inclusions.

Let $\varphi : \mathrm{CC}(\Sigma^{n-1}A) \to \mathrm{C}\Sigma(\Sigma^{n-1}A)$ be a homeomorphism, such that $\varphi^{-1}i = i_1 + i_2$. Note that $\mathrm{C}j = i_2$. There are pushout diagrams

$$\begin{tikzcd}
\Sigma^{n-1}A \arrow[r, "g"] \arrow[d, "j"'] \arrow[dr, phantom, "push"] & X^{n-1} \arrow[d, "\mathrm{inc}"] \\
\mathrm{C}\Sigma^{n-1}A \arrow[r, "f"'] & X^n = X^{n-1} \cup e^n
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{C}\Sigma^{n-1}A \arrow[r, "\mathrm{C}g"] \arrow[d, "\mathrm{C}j=i_2"'] \arrow[dr, phantom, "push"] & \mathrm{C}X^{n-1} \arrow[d, "\mathrm{Cinc}"] \\
\mathrm{CC}\Sigma^{n-1}A \arrow[r, "\mathrm{C}f"'] & \mathrm{C}X^n
\end{tikzcd}$$

It is not hard to check that the diagram

$$\begin{tikzcd}
\Sigma^n A = \mathrm{C}\Sigma^{n-1}A \underset{A}{\cup} \mathrm{C}\Sigma^{n-1}A \arrow[r, "f+\mathrm{C}g"] \arrow[d, "i"'] & \mathrm{C}X^{n-1} \cup e^n \arrow[d, "\mathrm{inc}"] \\
\mathrm{C}\Sigma^n A \arrow[r, "(\mathrm{C}f)\varphi^{-1}"'] & \mathrm{C}X^n
\end{tikzcd}$$

satisfies the universal property of pushouts. Indeed, $\mathrm{C}f\varphi^{-1}i = \mathrm{C}f(i_1 + i_2) = f + \mathrm{C}g$, so the diagram commutes. Suppose we have $\alpha : \mathrm{C}X^{n-1} \cup e^n \to Z$ and $\beta : \mathrm{C}\Sigma^n A \to Z$ such that $\beta i = \alpha(f + \mathrm{C}g)$. Then $\beta\varphi i_1 = \alpha \mathrm{in} f$ and $\beta\varphi i_2 = \alpha \mathrm{in}\mathrm{C}g = \alpha|_{\mathrm{C}X^{n-1}}$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{C}\Sigma^{n-1}A \arrow[r, "\mathrm{C}g"] \arrow[d, "\mathrm{C}j=i_2"'] \arrow[dr, phantom, "push"] & \mathrm{C}X^{n-1} \arrow[d] \arrow[ddr, bend left, "\alpha|_{\mathrm{C}X^{n-1}}"] & \\
\mathrm{CC}\Sigma^{n-1}A \arrow[r, "\mathrm{C}f"'] \arrow[drr, bend right, "\beta\varphi"'] & \mathrm{C}X^n \arrow[dr, dotted, "\gamma"] & \\
& & Z
\end{tikzcd}$$

Then, there exists a (unique) map γ such that $\gamma \mathrm{in} = \alpha|_{\mathrm{C}X^{n-1}}$ and $\gamma\mathrm{C}f = \beta\varphi$. So, $\gamma\mathrm{C}f\varphi^{-1} = \beta$. We must see that $\gamma\mathrm{in} = \alpha$, so it remains to be proved that this holds in e^n. But $\gamma\mathrm{C}f = \beta\varphi$, then $\gamma\mathrm{C}fi_1 = \beta\varphi i_1 = \alpha\mathrm{in}f$. Thus $\gamma\mathrm{in}f = \alpha\mathrm{in}f$, and since f is surjective this implies that $\gamma = \alpha$ in e^n. The uniqueness of γ is clear. We have proved that the diagram at the beginning is a pushout.

Now we take Y to be the colimit of Y^n, which satisfies the desired properties. □

Remark 3.1.20.

(1) The standard proof of the previous theorem for a CW-complex X uses the fact that the reduced cylinder IX is also a CW-complex. For general cores A, it is not always true

that IX is a CW(A)-complex when X is. For example, take $A = X = S^0$. The reduced cylinder IX has two path-connected components: the base point and the unit interval. But in any CW(S^0), each of the path connected components which do not contain the base point consists of only one point. Thus, IX is not a CW(S^0).

However, we will see below that if A is the suspension of a locally compact and Hausdorff space then the reduced cylinder of a CW(A)-complex is also a CW(A)-complex.

(2) It is easy to see that if X is a CW(A), then ΣX is a CW(A)-complex. Just apply the Σ functor to each of the pushout diagrams used to construct X. In this way we give ΣX a CW(A)-structure in which each of the cells is the reduced suspension of a cell of X. This is a simple and useful structure. However, it does not have the property of having X as a subcomplex.

Note that $(IX) \wedge A = I(X \wedge A)$ for any spaces A and X since $IZ = (I \sqcup *) \wedge Z$ for every topological space Z (here $\sqcup$ denotes disjoint union).

Lemma 3.1.21. *Let A be a loccaly compact and Hausdorff space and let X be a topological space. Then $(\mathrm{C}X) \wedge A = \mathrm{C}(X \wedge A)$, $(X \vee Y) \wedge A = (X \wedge A) \vee (Y \wedge A)$ and $(\Sigma X) \wedge A = \Sigma(X \wedge A)$.*

Proof. Since A is locally compact and Hausdorff, by the exponential law we get that the functor $-\wedge A$ is left adjoint to the functor $\mathrm{Hom}(A, -)$. Hence $-\wedge A$ commutes with colimits. The result follows applying the $- \wedge A$ functor to the pushouts

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "push"] & IX \arrow[d] \\ * \arrow[r] & \mathrm{C}X \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "push"] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd} \qquad \begin{tikzcd} X \vee X \arrow[r] \arrow[d] \arrow[dr, phantom, "push"] & IX \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

□

Proposition 3.1.22. *Let A be a locally compact and Hausdorff space and let X be a CW-complex. Then $X \wedge A$ is a CW(A)-complex. Moreover, the CW(A)-complex structure of $X \wedge A$ is induced by the CW-complex structure of X.*

Proof. As it was said in the proof of the previous lemma, the functor $- \wedge A$ commutes with colimits. The result follows using the previous lemma and applying $- \wedge A$ functor to the pushouts which define the A-skeletons by attaching cells and to the colimit of the A-skeletons. □

Lemma 3.1.23. *Let $\nu : S^1 \to S^1 \vee S^1$ be the usual map inducing the comultiplication in S^1. Then there is a pushout*

$$\begin{tikzcd} S^1 \arrow[r, "\nu"] \arrow[d] \arrow[dr, phantom, "push"] & S^1 \vee S^1 \arrow[d] \\ \mathrm{C}S^1 \arrow[r] & IS^1 \end{tikzcd}$$

Proof. Note that the pushout of the diagram

$$\begin{tikzcd} S^1 \arrow[r, "\nu"] \arrow[d] & S^1 \vee S^1 \\ \mathrm{C}S^1 & \end{tikzcd}$$

is $D^2/\{(-1,0),(1,0)\}$. There are homeomorphisms

$$IS^1 = S^1 \times I/(\{(1,0)\} \times I) \simeq I \times I/(\{0,1\} \times I) \simeq D^2/\{(-1,0),(1,0)\}$$

and hence, the result follows. □

Proposition 3.1.24. *Let A' be a locally compact and Hausdorff space and let $A = \Sigma A'$. Let X be a CW(A)-complex. Then the reduced cylinder IX is a CW(A)-complex. Moreover, $i_0(X)$ and $i_1(X)$ are CW(A)-subcomplexes of IX.*

Proof. For $n \in \mathbb{N}$ let J_n be an index set for the A-n-cells of X. We proceed by induction in the A-skeletons of X. For the initial case we have that $X^0 = \bigvee\limits_{\alpha \in J_0} A$. Then $IX^0 = \bigvee\limits_{\alpha \in J_0} IA$. But IA is a CW(A)-complex since applying $- \wedge A'$ to the pushout of the previous lemma gives a pushout

$$\begin{tikzcd} \Sigma A' = A \arrow[r] \arrow[d] \arrow[dr, phantom, "push"] & \Sigma A' \vee \Sigma A' = A \vee A \arrow[d] \\ \mathrm{C}\Sigma A' = \mathrm{C}A \arrow[r] & I\Sigma A' = IA \end{tikzcd}$$

Now suppose that IX^{n-1} is a CW(A)-complex with $i_0(X^{n-1})$ and $i_1(X^{n-1})$ are CW(A)-subcomplexes. We consider

$$\begin{tikzcd} \bigvee\limits_{\alpha \in J_{n-1} \sqcup J_{n-1}} \Sigma^{n-1} A \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "push"] & IX^{n-1} \arrow[d] \\ \bigvee\limits_{\alpha \in J_{n-1} \sqcup J_{n-1}} \mathrm{C}\Sigma^{n-1} A \arrow[r, "+_\alpha f^n_\alpha"'] & Y_n \end{tikzcd} \qquad \begin{tikzcd} \bigvee\limits_{\alpha \in J_n} \Sigma^n A \arrow[r, "+_\alpha G^n_\alpha"] \arrow[d] \arrow[dr, phantom, "push"] & Y_n \arrow[d] \\ \bigvee\limits_{\alpha \in J_n} \mathrm{C}\Sigma^n A \arrow[r] & Z_n \end{tikzcd}$$

where $g = \underset{\alpha \in J_{n-1}}{+} i_0 g^n_\alpha + \underset{\alpha \in J_{n-1}}{+} i_1 g^n_\alpha$ with $(g^n_\alpha)_\alpha$ the adjunction maps of the A-n-cells of X, and where the maps G_α are defined as the composition

$$\Sigma(\Sigma^{n-1}A) \longrightarrow \mathrm{C}(\Sigma^{n-1}A) \bigcup_{\Sigma^{n-1}A} I(\Sigma^{n-1}A) \bigcup_{\Sigma^{n-1}A} \mathrm{C}(\Sigma^{n-1}A) \xrightarrow{f^n_\alpha \cup F_\alpha \cup f^n_\alpha} Y_n$$

where the first map is a homeomorphism and F_α is the composition

$$I(\Sigma^{n-1}A) \xrightarrow{Ig^n_\alpha} IX^{n-1} \xrightarrow{\text{inc}} Y_n$$

We wish to prove that Z_n is homeomorphic to IX^n. Note that $Y_n = X^n \bigcup\limits_{X^{n-1}} IX^{n-1} \bigcup\limits_{X^{n-1}} X^n$.

We have that

$$\begin{array}{ccccc}
\bigvee\limits_{\alpha\in J_{n-1}\sqcup J_{n-1}} \Sigma^{n-1}A & \xrightarrow{g} & X^{n-1}\times\{0\}\vee X^{n-1}\times\{1\} & \xrightarrow{\text{inc}} & IX^{n-1} \\
\downarrow & push & \downarrow & push & \downarrow \\
\bigvee\limits_{\alpha\in J_{n-1}\sqcup J_{n-1}} \mathrm{C}\Sigma^{n-1}A & \xrightarrow[\underset{\alpha}{+} f^n_\alpha]{} & W_n & \longrightarrow & Y_n
\end{array}$$

and clearly $W_n = X^n\times\{0\}\vee X^n\times\{1\}$.

Now, the homeomorphism $\Sigma(\Sigma^{n-1}A) \longrightarrow \mathrm{C}(\Sigma^{n-1}A) \bigcup\limits_{\Sigma^{n-1}A} I(\Sigma^{n-1}A) \bigcup\limits_{\Sigma^{n-1}A} \mathrm{C}(\Sigma^{n-1}A)$ extends to a homeomorphism $\mathrm{C}(\Sigma^{n-1}A) \longrightarrow I\mathrm{C}(\Sigma^{n-1}A)$. Indeed, this follows applying $-\wedge A$ to the homeomorphism of topological pairs $\psi : (D^{n+1}, S^n) \to (ID^n, \mathrm{C}S^{n-1} \underset{S^{n-1}}{\cup} IS^{n-1} \underset{S^{n-1}}{\cup} \mathrm{C}S^{n-1})$

Then, we have

$$\begin{array}{ccccc}
\bigvee\limits_{\alpha\in J_n} (\mathrm{C}\Sigma^{n-1}A \bigcup\limits_{\Sigma^{n-1}A} I\Sigma^{n-1}A \bigcup\limits_{\Sigma^{n-1}A} \mathrm{C}\Sigma^{n-1}A) & \longrightarrow & \bigvee\limits_{\alpha\in J_n} \Sigma^n A & \xrightarrow{\underset{\alpha}{+} G^n_\alpha} & Y_n \\
\downarrow & push & \downarrow & push & \downarrow \\
\bigvee\limits_{\alpha\in J_n} I\mathrm{C}\Sigma^{n-1}A & \longrightarrow & \bigvee\limits_{\alpha\in J_n} \mathrm{C}\Sigma^n A & \longrightarrow & Z_n
\end{array}$$

where the first square is a pushout since it commutes and its two horizontal arrows are homeomorphisms.

Note now that the top horizontal composition is $f^n_\alpha \cup F_\alpha \cup f^n_\alpha$ and that $Z_n = IX^n$ since $F_\alpha = \text{inc}\circ Ig^n_\alpha$. The result follows. $\square$

Lemma 3.1.25. *Let A be a topological space and let (X,B) be a relative CW(A)-complex (resp. a generalized relative CW(A)-complex). Let Y be a topological space, and let $f : B\to Y$ be a continuous map. We consider the pushout diagram*

$$\begin{array}{ccc}
B & \xrightarrow{f} & Y \\
{\scriptstyle i}\downarrow & push & \downarrow \\
X & \longrightarrow & X \underset{B}{\cup} Y
\end{array}$$

Then $(X\underset{B}{\cup} Y, Y)$ is a relative CW(A)-complex (resp. a generalized relative CW(A)-complex).

Moreover, if (X,B) has a CW(A)-stucture of dimension $n\in\mathbb{N}_0$ (resp. a CW(A)-structure with a finite number of layers) then $(X\underset{B}{\cup} Y, Y)$ can also be given a CW(A)-stucture of dimension n (resp. a CW(A)-structure with a finite number of layers).

The proof is easy and we omit it.

Theorem 3.1.26. *Let A be a CW(B)-complex of finite dimension and let X be a generalized CW(A)-complex. Then X is a generalized CW(B)-complex. In particular, if A is a CW-complex of finite dimension then X is a generalized CW-complex.*

Proof. Let
$$* = X^0 \to X^1 \to \ldots \to X^n \to \ldots$$
be a generalized CW(A)-structure on X. Then, for each $n \in \mathbb{N}$ we have a pushout diagram

$$\begin{tikzcd}
C_n = \bigvee\limits_{\alpha\in J} \Sigma^{n_\alpha -1} A \arrow[r, "\underset{\alpha\in J}{+} g_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & X^{n-1} \arrow[d] \\
D_n = (\bigvee\limits_{\alpha\in J_0} A) \vee (\bigvee\limits_{\alpha\in J} \mathrm{C}\Sigma^{n_\alpha -1} A) \arrow[r, "\underset{\alpha\in J}{+} f_\alpha"'] & X^n
\end{tikzcd}$$

where $n_\alpha \in \mathbb{N}$ for all $\alpha \in J$.

We have that (D_n, C_n) is a relative CW(B)-complex by 3.1.19, and it has finite dimension since A does. So, by 3.1.25, (X^n, X^{n-1}) is a relative CW(B)-complex of finite dimension. Then, for each $n \in \mathbb{N}$, there exist spaces Y_n^j for $0 \le j \le m_n$, with $m_n \in \mathbb{N}$ such that Y_n^j is obtained from Y_n^{j-1} by attaching cells of type B of dimension j and $Y_n^{-1} = X^{n-1}$, $Y_n^{m_n} = X^n$. Thus, there exists a diagram
$$* = X^0 = Y_1^{-1} \to Y_1^0 \to Y_1^1 \to \ldots \to Y_1^{m_1} = X^1 = Y_2^{-1} \to \ldots \to Y_2^{m_2} = X^2 = Y_3^{-1} \to \ldots$$
where each space is obtained from the previous one by attaching cells of type B. It is clear that X, the colimit of this diagram, is a generalized CW(B)-complex. $\square$

In the following example we exhibit a space X which is not a CW-complex but is a CW(A), with A a CW-complex.

Example 3.1.27. Let $A = [0,1] \cup \{2\}$, with 0 as the base point. We build X as follows. We attach two 0-cells to get $A \vee A$. We will denote the points in $A \vee A$ as (a, j), where $a \in A$ and $j = 1, 2$. We define now, for each $n \in \mathbb{N}$, maps $g_n : A \to A \vee A$ in the following way. We set $g_n(a) = (a, 1)$ if $a \in [0,1]$ and $g_n(2) = (1/n, 2)$. We attach 1-cells of type A by means of the maps g_n. By a similar argument to the one in 3.1.5, the space X obtained in this way is not a CW-complex.

If A is a finite dimensional CW-complex and X is a generalized CW(A), the previous theorem says that X is a generalized CW-complex, and so it has the homotopy type of a CW-complex. The following result asserts that the last statement is also true for any CW-complex A.

Proposition 3.1.28. *If A is a CW-complex and X is a generalized CW(A)-complex then X has the homotopy type of a CW-complex.*

Proof. Let
$$* \subseteq X^1 \subseteq X^2 \subseteq \ldots \subseteq X^n \subseteq \ldots$$
be a generalized CW(A)-structure on X. We may suppose that all the 0-cells are attached in the first step, that is,
$$X^1 = \bigvee_\beta A \vee \bigvee_\alpha \Sigma^{n_\alpha} A$$
with $n_\alpha \in \mathbb{N}$. It is clear that X^1 is a CW complex.

We will construct inductively a sequence of CW-complexes Y_n for $n \in \mathbb{N}$ with $Y_{n-1} \subseteq Y_n$ subcomplex and homotopy equivalences $\phi_n : X^n \to Y_n$ such that $\phi_n|_{X^{n-1}} = \phi_{n-1}$.

We take $Y_1 = X^1$ and ϕ_1 the identity map. Suppose we have already constructed $Y_1, \ldots, Y_k$ and $\phi_1, \ldots, \phi_k$ satisfying the conditions mentioned above. We consider the following pushout diagram.

$$\begin{tikzcd}
\bigvee_\alpha \Sigma^{n_\alpha -1} A \arrow[r, "\underset{\alpha}{+} g_\alpha"] \arrow[d, "\bigvee_\alpha i"'] \arrow[dr, phantom, "push"] & X^k \arrow[r, "\phi_k"] \arrow[d, "i_k"] \arrow[dr, phantom, "push"] & Y_k \arrow[d, "\gamma'_k"] \\
\bigvee_\alpha C\Sigma^{n_\alpha -1} A \arrow[r, "\underset{\alpha}{+} f_\alpha"'] & X^{k+1} \arrow[r, "\beta"'] & Y'_{k+1}
\end{tikzcd}$$

Note that β is a homotopy equivalence since i_k is a closed cofibration and ϕ_k is a homotopy equivalence.

We deform $\phi_k \circ (\underset{\alpha}{+} g_\alpha)$ to a cellular map ψ and we define Y_{k+1} as the pushout

$$\begin{tikzcd}
\bigvee_\alpha \Sigma^{n_\alpha -1} A \arrow[r, "\psi"] \arrow[d, "\bigvee_\alpha i"'] \arrow[dr, phantom, "push"] & Y_k \arrow[d, "\gamma_k"] \\
\bigvee_\alpha C\Sigma^{n_\alpha -1} A \arrow[r] & Y_{k+1}
\end{tikzcd}$$

There exists a homotopy equivalence $k : Y'_{k+1} \to Y_{k+1}$ with $k|_{Y_k} = \mathrm{Id}$. Let $i_k : X^k \to X^{k+1}$ be the inclusion. Then $k\beta i_k = k\gamma'_k\phi_k$ and $k\gamma'_k = \gamma_k$ is the inclusion. Let $\phi_{k+1} = k\beta$. Then, ϕ_{k+1} is a homotopy equivalence and $\phi_{k+1}|_{X^k} = \phi_k$.

We take Y to be the colimit of the Y_n's. Then Y is a CW-complex. As the inclusions i_k, γ_k are closed cofibrations, by 1.1.22, it follows that X is homotopy equivalent to Y. □

We prove now a variant of theorem 3.1.26.

Theorem 3.1.29. *Let A be a generalized CW(B)-complex with B compact, and let X be a generalized CW(A)-complex. If A and B are T1 then X is a generalized CW(B)-complex.*

Proof. Let
$$* = X^0 \to X^1 \to \ldots \to X^n \to \ldots$$
be a generalized CW(A)-structure on X. Let C_n, D_n be as in the proof of 3.1.26.

We have that (D_n, C_n) is a relative CW(B)-complex by 3.1.19. By 3.1.25, (X^n, X^{n-1}) is also a relative CW(B)-complex, but it need not be finite dimensional, so we can not continue with the same argument as in the proof of 3.1.26. But using the compactness of B, we will show that the cells of type B may be attached in a certain order to obtain spaces Z^n for $n \in \mathbb{N}$ such that X is the colimit of the Z^n's.

Let J denote the set of all cells of type B belonging to some of the relative CW(B)-complexes (X^n, X^{n-1}) for $n \in \mathbb{N}$. We associate an ordered pair $(a, b) \in (\mathbb{N}_0)^2$ to each cell in J in the following way. Note that each cell of type B is included in exactly one cell of type A. The number a will be the smallest number of layer in which that A-cell lies. In a similar way, if we regard that A-cell as a relative CW(B)-complex $(\mathrm{C}\Sigma^{n-1}A, \Sigma^{n-1}A)$ (or more precisely, the image of this by the characteristic map), we set b to be the smallest number of layer (in $(\mathrm{C}\Sigma^{n-1}A, \Sigma^{n-1}A)$) in which the B-cell lies. If e is the cell, we denote $\varphi(e) = (a, b)$.

We will consider in $(\mathbb{N}_0)^2$ the lexicographical order with the first coordinate greater than the second one.

Now we set the order in which the B-cells are attached. Let J_1 be the set of all the cells whose attaching map is the constant. We define inductively J_n for $n \in \mathbb{N}$ to be the set of all the B-cells whose attaching map has image contained in the union of all the cells in J_{n-1}. Clearly $J_{n-1} \subseteq J_n$. We wish to attach first the cells of J_1, then those of $J_2 - J_1$, etc. This can be done because of the construction of the J_n. We must verify that there are no cells missing, i.e. that $J = \bigcup_{n\in\mathbb{N}} J_n$.

Suppose there exists one cell in J, which we call e_1, which is not in any of the J_n. The image of its attaching map, denoted K, is compact, since B is compact and therefore it meets only a finite number of interiors of A-cells. For each of these cells e_A we consider the relative CW(B)-complex $(\overline{e_A}, \overline{e_A} - \mathring{e}_A)$, where e_A is the cell of type A.

Then $K \cap \overline{e_A}$ is closed in K and hence compact, so it meets only a finite number of interiors of B-cells of the relative CW(B)-complex $(\overline{e_A}, \overline{e_A} - \mathring{e}_A)$.

Thus K meets only a finite number of interiors of B-cells in J.

This implies that K, which is the image of the attaching map of e_1, meets the interior of some cell e_2 which does not belong to any of the J_n, because of the finiteness condition.

Recall that e_2 is an immediate face of e_1, which easily implies that $\varphi(e_2) < \varphi(e_1)$.

Applying the same argument inductively we get a sequence of cells $(e_n)_{n\in\mathbb{N}}$ such that $\varphi(e_{n+1}) < \varphi(e_n)$ for all n.

But this induces an infinite decreasing sequence for the lexicographical order, which is impossible. Hence, $J = \bigcup_{n\in\mathbb{N}} J_n$.

Let $Z^n = \bigcup_{e\in J_n} e$. It is clear that (Z^n, Z^{n-1}) is a relative CW(B)-complex.

Since colimits commute, we prove that $X = \operatorname{colim} Z^n$ is a generalized CW(B)-complex.

$\square$

3.2 The descriptive approach

In this section we introduce the descriptive definition of CW(A)-complexes, which will be used to prove some results. It also gives a different approach to CW(A)-complexes and generalizes the usual definition of CW-complexes. We will compare the constructive definition of CW(A)-complexes given in the previous section with the descriptive one giving conditions for each of them to imply the other, providing counterexamples if these conditions do not hold. This shows once more which intrinsic properties of S^0 are used in the usual theory of CW-complexes giving us the chance to study it in depth.

As before, let A be a fixed pointed topological space.

Definition 3.2.1. Let X be a pointed topological space (with base point x_0). A *cellular complex structure of type A on X* is a collection $\mathcal{K} = \{e^n_\alpha : n \in \mathbb{N}_0, \alpha \in J_n\}$ of subsets of X, which are called the *cells (of type A)*, such that $x_0 \in e^n_\alpha$ for all n and α, and satisfying conditions (1), (2) and (3) below.

Let $\mathcal{K}^n = \{e^r_\alpha, r \leq n, \alpha \in J_r\}$ for $n \in \mathbb{N}_0$, $\mathcal{K}^{-1} = \{\{x_0\}\}$. $\mathcal{K}^n$ is called the *n-skeleton of* $\mathcal{K}$. Let $|\mathcal{K}^n| = \bigcup\limits_{\substack{r\leq n\\ \alpha\in J_r}} e^r_\alpha$, $|\mathcal{K}^n| \subseteq X$ a subspace.

We call $\overset{\bullet}{e}{}^n_\alpha = e^n_\alpha \cap |\mathcal{K}^{n-1}|$ the *boundary of the cell* e^n_α and $\overset{\circ}{e}{}^n_\alpha = e^n_\alpha - \overset{\bullet}{e}{}^n_\alpha$ the *interior of the cell* e^n_α.

The collection $\mathcal{K}$ must satisfy the following properties.

(1) $X = \bigcup\limits_{n,\alpha} e^n_\alpha = |\mathcal{K}|$

(2) $\overset{\circ}{e}{}^n_\alpha \cap \overset{\circ}{e}{}^m_\beta \neq \varnothing \Rightarrow m = n, \alpha = \beta$

(3) For every cell e^n_α with $n \geq 1$ there exists a continuous map

$$f^n_\alpha : (\mathrm{C}\Sigma^{n-1}A, \Sigma^{n-1}A, a_0) \to (e^n_\alpha, \overset{\bullet}{e}{}^n_\alpha, x_0)$$

such that f^n_α is surjective and $f^n_\alpha : \mathrm{C}\Sigma^{n-1}A - \Sigma^{n-1}A \to \overset{\circ}{e}{}^n_\alpha$ is a homeomorphism. For $n = 0$, there is a homeomorphism $f^0_\alpha : (A, a_0) \to (e^0_\alpha, x_0)$.

The *dimension of* $\mathcal{K}$ is defined as $\dim \mathcal{K} = \sup\{n : J_n \neq \varnothing\}$.

Definition 3.2.2. Let $\mathcal{K}$ be a cellular complex structure of type A in a topological space X. We say that $\mathcal{K}$ *is a cellular CW-complex with base A* if it satisfies the following conditions.

(C) Every compact subspace of X intersects only a finite number of interiors of cells.

(W) X has the weak (final) topology with respect to the cells.

In this case we will say that X is a *descriptive CW(A)*.

We study now the relationship between both approaches.

Theorem 3.2.3. *Let A be a T1 space. If X is a constructive CW(A)-complex, then it is a descriptive CW(A)-complex.*

Proof. Let $\mathcal{K} = \{e^n_\alpha\}_{n,\alpha} \cup \{\{x_0\}\}$. It is not difficult to verify that $\mathcal{K}$ defines a cellular complex structure on X.

It remains to prove that it satisfies conditions (C) and (W). Note that condition (C) follows from 3.1.12, while (W) follows from 3.1.15. □

Note that the hypothesis of T1 on A is necessary. For example, take $A = \{0, 1\}$ with the indiscrete topology and 0 as base point. Let $X = \bigvee\limits_{j\in\mathbb{N}} A$. The space X also has the indiscrete topology and it is a constructive CW(A)-complex. If it were a descriptive CW(A), it could only have cells of dimension 0 since X is countable. But X is not finite, then it must have infinite many cells, but it is a compact space. This implies that (C) does not hold, thus X is not a descriptive CW(A)-complex.

Theorem 3.2.4. *Let A be a compact space and let X be a descriptive CW(A)-complex. If X is Hausdorff then it is a constructive CW(A)-complex.*

Proof. We will prove that $|\mathcal{K}^n|$ can be obtained from $|\mathcal{K}^{n-1}|$ by attaching A-n-cells. For $n = 0$ this is clear since we have a homeomorphism $\bigvee\limits_{\alpha\in J_0} f^0_\alpha : \bigvee\limits_{\alpha\in J_0} A \to |\mathcal{K}^0|$.

For any $n \in \mathbb{N}$, there is a pushout

$$\begin{tikzcd}
\bigvee\limits_{\alpha\in J_n} \Sigma^{n-1}A \arrow[r, "\underset{\alpha\in J_n}{+} f^n_\alpha|_{\Sigma^{n-1}A}"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & |\mathcal{K}^{n-1}| \arrow[d] \\
\bigvee\limits_{\alpha\in J_n} \mathrm{C}\Sigma^{n-1}A \arrow[r, "\underset{\alpha\in J_n}{+} f^n_\alpha"'] & |\mathcal{K}^n|
\end{tikzcd}$$

The topology of $|\mathcal{K}^n|$ coincides with the pushout topology since X is Hausdorff and A is compact. Indeed, suppose $F \subseteq |\mathcal{K}^n|$ is closed in the pushout topology. Then $F \cap |\mathcal{K}^{n-1}|$ is closed in $|\mathcal{K}^{n-1}|$, and so $F \cap e^m_\beta$ is closed in e^m_β for all $m < n$. Since $(f^n_\alpha)^{-1}(F \cap e^n_\alpha)$ is closed in $\mathrm{C}\Sigma^{n-1}A$, then $(f^n_\alpha)^{-1}(F \cap e^n_\alpha)$ is compact. This implies that $F \cap e^n_\alpha$ is compact and, since $|\mathcal{K}^n|$ is Hausdorff, it is closed in e^n_α. Therefore, F is closed in $|\mathcal{K}^n|$ with the subspace topology. □

It is interesting to see that 3.2.4 need not be true if X is not Hausdorff. For example, take $A = S^0$ with the usual topology, and $X = [-1, 1]$ with the following topology. The proper open sets are $[-1, 1)$, $(-1, 1]$ and the subsets $U \subseteq (-1, 1)$ which are open in $(-1, 1)$ with the usual topology. It is easy to see that X is a descriptive CW(A)-complex. We denote $D^1 = [-1, 1]$ with the usual topology. Take $e^0 = \{-1, 1\}$, $e^1 = X$. Let $f^0 : A \to \{-1, 1\}$ and $f^1 : CA = D^1 \to e^1$ be the identity maps on the underlying sets. Both maps are continuous and surjective. The maps f^0 and $f^1|_{\mathring{D}^1} : \mathring{D}^1 \to \mathring{e}^1$ are homeomorphisms. So conditions (1), (2) and (3) of the definition of cellular complex are satisfied. Condition (C) is obvious, and (W) follows from the fact that $e^1 = X$. So X is a descriptive CW(A)-complex. But it is not a constructive CW(A)-complex because it is not Hausdorff.

In a similar way one can define the notion of *descriptive generalized CW(A)-complex.* The relationship between the constructive and descriptive approachs of generalized CW(A)-complexes is analogous to the previous one.

3.3 Changing cores

Suppose we have two spaces A and B and maps $\alpha : A \to B$ and $\beta : B \to A$. Let X be a CW(A)-complex. We want to construct a CW(B)-complex out of X, using the maps α and β.

We shall consider two special cases. First, we consider the case $\beta\alpha = \mathrm{Id}_A$, that is, A is a retract of B. In this case, we construct a CW(B)-complex Y such that X is a retract of Y.

Let J_n be an index set for the A-n-cells of X. For $\alpha \in J_n$ let g^n_γ be the attaching map of the cell e^n_α and let f^n_γ be its characteristic map. Let $Y^0 = \bigvee\limits_{\gamma\in J_0} B$ and let $\varphi_0 : X^0 \to Y^0$ be the map $\vee\alpha$ and let $\psi_0 : Y^0 \to X^0$ be the map $\vee\beta$. Clearly $\psi_0\varphi_0 = \mathrm{Id}_{X^0}$.

By induction suppose we have constructed Y^{n-1} and maps $\varphi_{n-1} : X^{n-1} \to Y^{n-1}$ and $\psi_{n-1} : Y^{n-1} \to X^{n-1}$ such that $\psi_{n-1}\varphi_{n-1} = \mathrm{Id}_{X^{n-1}}$ and such that φ_k, ψ_k extend φ_{k-1}, ψ_{k-1} for all $k \leq n-1$. We define Y^n by the following pushout.

$$\begin{tikzcd}
\bigvee\limits_{\gamma\in J_n} \Sigma^{n-1}B \arrow[rr, "\varphi_{n-1}(\underset{\gamma\in J_n}{+} g^n_\gamma\Sigma^{n-1}\beta)"] \arrow[d, "\vee i"'] \arrow[drr, phantom, "push"] & & Y^{n-1} \arrow[d, "j"] \\
\bigvee\limits_{\gamma\in J_n} \mathrm{C}\Sigma^{n-1}B \arrow[rr, "\underset{\gamma\in J_n}{+} h^n_\gamma"'] & & Y^n
\end{tikzcd}$$

Since

$$\begin{aligned}
(\underset{\gamma\in J_n}{+} f^n_\gamma \mathrm{C}\Sigma^{n-1}\beta)(\vee i) &= \underset{\gamma\in J_n}{+}(f^n_\gamma \mathrm{C}\Sigma^{n-1}\beta i) = \underset{\gamma\in J_n}{+}(f^n_\gamma i\Sigma^{n-1}\beta) = \underset{\gamma\in J_n}{+}(\mathrm{inc} g^n_\gamma \Sigma^{n-1}\beta) = \\
&= \mathrm{inc}\psi_{n-1} \underset{\gamma\in J_n}{+}(\varphi_{n-1} g^n_\gamma \Sigma^{n-1}\beta)
\end{aligned}$$

there exists a map $\psi_n : Y^n \to X^n$ extending ψ_{n-1} such that $\psi_n \underset{\gamma\in J_n}{+} h^n_\gamma = \underset{\gamma\in J_n}{+}(f^n_\gamma \mathrm{C}\Sigma^{n-1}\beta)$ and $\psi_n j = \mathrm{inc}\psi_{n-1}$.

On the other hand we have the following commutative diagram

$$\begin{tikzcd}
\bigvee\limits_{\gamma\in J_n}\Sigma^{n-1}A \arrow[rrr, "\underset{\gamma\in J_n}{+} g^n_\gamma"] \arrow[dr, "\vee\Sigma^{n-1}\alpha"] \arrow[dd, "\vee i"'] & & & X^{n-1} \arrow[dr, "\varphi_{n-1}"] \arrow[dd, "\mathrm{inc}"] & \\
& \bigvee\limits_{\gamma\in J_n}\Sigma^{n-1}B \arrow[rrr, "\varphi_{n-1}(\underset{\gamma\in J_n}{+} g^n_\gamma\Sigma^{n-1}\beta)"] \arrow[dd, "\vee i"] & & & Y^{n-1} \arrow[dd, "j"] \\
\bigvee\limits_{\gamma\in J_n}\mathrm{C}\Sigma^{n-1}A \arrow[rrr, "\underset{\gamma\in J_n}{+} f^n_\gamma"'] \arrow[dr, "\vee\mathrm{C}\Sigma^{n-1}\alpha"] & & & X^n \arrow[dr, dotted, "\varphi_n"] & \\
& \bigvee\limits_{\gamma\in J_n}\mathrm{C}\Sigma^{n-1}B \arrow[rrr, "\underset{\gamma\in J_n}{+} h^n_\gamma"'] & & & Y^n
\end{tikzcd}$$

where the front and back faces are pushouts. Then the dotted arrow exists and we have $\varphi_n = j\varphi_{n-1} + (\underset{\gamma\in J_n}{+} h^n_\gamma \mathrm{C}\Sigma^{n-1}\alpha)$. Also, $\psi_n\varphi_n = \mathrm{Id}_{X^n}$, since

$$\begin{aligned}
\psi_n\varphi_n &= \psi_n j\varphi_{n-1} + (\underset{\gamma\in J_n}{+} \psi_n h^n_\gamma \mathrm{C}\Sigma^{n-1}\alpha) = \mathrm{inc}\psi_{n-1}\varphi_{n-1} + (\underset{\gamma\in J_n}{+} f^n_\gamma \mathrm{C}\Sigma^{n-1}\beta\mathrm{C}\Sigma^{n-1}\alpha) = \\
&= \mathrm{inc} + (\underset{\gamma\in J_n}{+} f^n_\gamma) = \mathrm{Id}_{X^n}
\end{aligned}$$

Let $Y = \mathrm{colim}\, Y^n$. Then there exist maps $\varphi : X \to Y$ and $\psi : Y \to X$ induced by the ψ_n's and φ_n's and they satisfy $\psi\varphi = \mathrm{Id}_X$. So, X is a retract of Y.

The second special case we consider is the following. Suppose A and B have the same homotopy type, that is, there exists a homotopy equivalence $\beta : B \to A$ with homotopy inverse α. Suppose, in addition, that the base points of A and B are closed. Let X be a CW(A)-complex. We will construct a CW(B)-complex which is homotopy equivalent to X.

Again we take $Y^0 = \bigvee\limits_{\gamma\in J_0} B$. Let $\varphi_0 : X^0 \to Y^0$ be the map $\vee\alpha$. So, φ_0 is a homotopy equivalence.

Now, let $n \in \mathbb{N}$ and suppose we have constructed Y^{n-1} and a homotopy equivalence $\varphi_{n-1} : X^{n-1} \to Y^{n-1}$. We define Y^n as in the first case. Consider the commutative diagrams

$$\begin{tikzcd}
\bigvee\limits_{\gamma\in J_n} \Sigma^{n-1}B \arrow[rr, "\underset{\gamma\in J_n}{+} g^n_\gamma \Sigma^{n-1}\beta"] \arrow[dr, "\mathrm{Id}"] \arrow[dd, "i_B"'] & & X^{n-1} \arrow[dr, "\varphi_{n-1}"] \arrow[dd, "\mathrm{inc}"] & \\
& \bigvee\limits_{\gamma\in J_n} \Sigma^{n-1}B \arrow[rr, "\varphi_{n-1}(\underset{\gamma\in J_n}{+} g^n_\gamma \Sigma^{n-1}\beta)"] \arrow[dd, "i_B"] & & Y^{n-1} \arrow[dd, "j"] \\
\bigvee\limits_{\gamma\in J_n} \mathrm{C}\Sigma^{n-1}B \arrow[rr] \arrow[dr, "\mathrm{Id}"] & & X^{n-1}\cup e^n_B \arrow[dr, dotted, "p_1"] & \\
& \bigvee\limits_{\gamma\in J_n} \mathrm{C}\Sigma^{n-1}B \arrow[rr, "\underset{\gamma\in J_n}{+} h^n_\gamma"'] & & Y^n
\end{tikzcd}$$

$$\begin{tikzcd}
\bigvee\limits_{\gamma\in J_n} \Sigma^{n-1}B \arrow[rr, "\underset{\gamma\in J_n}{+} g^n_\gamma \Sigma^{n-1}\beta"] \arrow[dr, "\vee\Sigma^{n-1}\beta"] \arrow[dd, "i_B"'] & & X^{n-1} \arrow[dr, "\mathrm{Id}"] \arrow[dd, "\mathrm{inc}"] & \\
& \bigvee\limits_{\gamma\in J_n} \Sigma^{n-1}A \arrow[rr, "\underset{\gamma\in J_n}{+} g^n_\gamma"] \arrow[dd, "i_A"] & & X^{n-1} \arrow[dd, "\mathrm{inc}"] \\
\bigvee\limits_{\gamma\in J_n} \mathrm{C}\Sigma^{n-1}B \arrow[rr] \arrow[dr, "\vee\mathrm{C}\Sigma^{n-1}\beta"'] & & X^{n-1}\cup e^n_B \arrow[dr, dotted, "p_2"] & \\
& \bigvee\limits_{\gamma\in J_n} \mathrm{C}\Sigma^{n-1}A \arrow[rr, "\underset{\gamma\in J_n}{+} f^n_\gamma"'] & & X^n
\end{tikzcd}$$

Since the front and rear faces of both cubical diagrams are pushouts, the dotted arrows p_1 and p_2 exist. Now φ_{n-1}, $\vee\Sigma^{n-1}\beta$ and $\vee\mathrm{C}\Sigma^{n-1}\beta$ are homotopy equivalences and i_A and i_B are closed cofibrations. Then, by 1.1.20, p_1 and p_2 are homotopy equivalences. We have the following commutative diagram.

$$\begin{tikzcd}
Y^{n-1} \arrow[d, "i"] & X^{n-1} \arrow[l, "\varphi_{n-1}"'] \arrow[r, "\mathrm{Id}"] \arrow[d, "j"] & X^{n-1} \arrow[d, "k"] \\
Y^n & X^{n-1}\cup e^n_B \arrow[l, "p_1"'] \arrow[r, "p_2"] & X^n
\end{tikzcd}$$

where i, j and k are the inclusions. Let p_2^{-1} be a homotopy inverse of p_2. Then $p_1p_2^{-1}k = p_1p_2^{-1}p_2j \simeq p_1j = i\varphi_{n-1}$. Since $k : X^{n-1} \to X^n$ is a cofibration, φ_{n-1} extends to some $\varphi_n : X^n \to Y^n$ and φ_n is homotopic to $p_1p_2^{-1}$, and thus, it is a homotopy equivalence.

Again, we take $Y = \operatorname{colim} Y^n$. Then the maps φ_n for $n \in \mathbb{N}$ induce a map $\varphi : X \to Y$ which is a homotopy equivalence by 1.1.22.

We summarize the previous results in the following theorem.

Theorem 3.3.1. *Let A and B be pointed topological spaces. Let X be a CW(A)-complex, and let $\alpha : A \to B$ and $\beta : B \to A$ be continuous maps.*

i. *If $\beta\alpha = \mathrm{Id}_A$, then there exists a CW(B)-complex Y and maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi\varphi = \mathrm{Id}_X$.*

ii. *Suppose A and B have closed base points. If β is a homotopy equivalence, then there exists a CW(B)-complex Y and a homotopy equivalence $\varphi : X \to Y$.*

iii. *Suppose A and B have closed base points. If $\beta\alpha = \mathrm{Id}_A$ and $\alpha\beta \simeq \mathrm{Id}_A$ then there exists a CW(B)-complex Y and maps $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi\varphi = \mathrm{Id}_X$ and $\varphi\psi \simeq \mathrm{Id}_Y$.*

Note that item (iii) follows by a similiar argument.

The previous theorem has an easy but interesting corollary.

Corollary 3.3.2. *Let A be a contractible space (with closed base point) and let X be a CW(A)-complex. Then X is contractible.*

This corollary also follows from a result analogous to Whitehead's Theorem which we prove in the next chapter.

3.4 Localization

In this section the core A will be assumed to be an abelian CW-complex.

Remark 3.4.1. Let $\gamma : A \to A_{\mathcal{P}}$ be a $\mathcal{P}$-localization map. Note that $\mathrm{C}_\gamma : \mathrm{C}A \to \mathrm{C}A_{\mathcal{P}}$ is also a $\mathcal{P}$-localization since $(\mathrm{C}A)_{\mathcal{P}}$ and $\mathrm{C}(A_{\mathcal{P}})$ are contractible.

Moreover, $\Sigma(A_{\mathcal{P}})$ is a $\mathcal{P}$-local space since $A_{\mathcal{P}}$ is. By theorem 2.4.8 applied to γ, we deduce that $\Sigma\gamma : \Sigma A \to \Sigma(A_{\mathcal{P}})$ induces isomorphisms $H_*(\Sigma A)\otimes\mathbb{Z}_{\mathcal{P}} \to H_*(\Sigma(A_{\mathcal{P}}))\otimes\mathbb{Z}_{\mathcal{P}} \simeq H_*(\Sigma(A_{\mathcal{P}}))$. Hence, by the mentioned theorem we obtain that $\Sigma\gamma$ is a $\mathcal{P}$-localization.

Theorem 3.4.2. *Let A be a simply connected CW-complex and let X be an abelian CW(A)-complex. Let $\mathcal{P}$ be a set of prime numbers. Given a $\mathcal{P}$-localization $A \to A_{\mathcal{P}}$ there exists a $\mathcal{P}$-localization $X \to X_{\mathcal{P}}$ with $X_{\mathcal{P}}$ a CW($A_{\mathcal{P}}$)-complex. Moreover, the CW($A_{\mathcal{P}}$)-complex structure of $X_{\mathcal{P}}$ is obtained by localizing the adjunction maps of the CW(A)-complex structure of X.*

Proof. We proceed by induction in the A-n-skeletons of X. For X^0 the result follows immediately from 2.4.8. Suppose now that the result holds for X^{n-1}. Consider the following diagram

$$\begin{tikzcd}
\bigvee_{\alpha\in J} \Sigma^{n-1}A \arrow[r, "g=\underset{\alpha\in J}{+} g_\alpha"] \arrow[d, "i"'] & X^{n-1} \\
\bigvee_{\alpha\in J} \mathrm{C}\Sigma^{n-1}A &
\end{tikzcd}$$

By $\mathcal{P}$-localizing it we obtain a diagram

$$\begin{tikzcd}
\bigvee\limits_{\alpha\in J}\Sigma^{n-1}A \arrow[rr, "g=\underset{\alpha\in J}{+}g_\alpha"] \arrow[dr, "\gamma_1"] \arrow[dd, "i"'] & & X^{n-1} \arrow[dr, "\gamma_2"] & \\
& \bigvee\limits_{\alpha\in J}\Sigma^{n-1}A_{\mathcal{P}} \arrow[rr, "g_{\mathcal{P}}"] \arrow[dd, "i_{\mathcal{P}}"'] & & (X^{n-1})_{\mathcal{P}} \\
\bigvee\limits_{\alpha\in J}\mathrm{C}\Sigma^{n-1}A \arrow[dr, "\gamma_3"] & & & \\
& \bigvee\limits_{\alpha\in J}\mathrm{C}\Sigma^{n-1}A_{\mathcal{P}} & &
\end{tikzcd}$$

where γ_1, γ_2 and γ_3 are the localization maps. We consider the pushouts of the original and localized diagrams and obtain a commutative cube

$$\begin{tikzcd}
\bigvee\limits_{\alpha\in J}\Sigma^{n-1}A \arrow[rr, "g=\underset{\alpha\in J}{+}g_\alpha"] \arrow[dr, "\gamma_1"] \arrow[dd, "i"'] & & X^{n-1} \arrow[dr, "\gamma_2"] \arrow[dd] & \\
& \bigvee\limits_{\alpha\in J}\Sigma^{n-1}A_{\mathcal{P}} \arrow[rr, "g_{\mathcal{P}}"] \arrow[dd, "i_{\mathcal{P}}"'] & & (X^{n-1})_{\mathcal{P}} \arrow[dd] \\
\bigvee\limits_{\alpha\in J}\mathrm{C}\Sigma^{n-1}A \arrow[dr, "\gamma_3"] \arrow[rr] & & X^n \arrow[dr, "\gamma"] & \\
& \bigvee\limits_{\alpha\in J}\mathrm{C}\Sigma^{n-1}A_{\mathcal{P}} \arrow[rr] & & Z
\end{tikzcd}$$

We will prove that $\gamma : X^n \to Z$ is a $\mathcal{P}$-localization map. By 2.4.8, it suffices to prove that Z is a $\mathcal{P}$-local space and that γ induces isomorphisms $H_*(X^n)\otimes\mathbb{Z}_{\mathcal{P}} \to H_*(Z)\otimes\mathbb{Z}_{\mathcal{P}} \simeq H_*(Z)$.

It is clear that Z is a $\mathcal{P}$-local space since we have a cofibration sequence $(X^{n-1})_{\mathcal{P}} \to Z \to Z/(X^{n-1})_{\mathcal{P}} = \bigvee\limits_{\alpha\in J}\Sigma^n A_{\mathcal{P}}$ and $(X^{n-1})_{\mathcal{P}}$ and $\bigvee\limits_{\alpha\in J}\Sigma^n A_{\mathcal{P}}$ are $\mathcal{P}$-local spaces.

Now, considering the exact sequences in homology associated to the topological pairs (X^n, X^{n-1}) and $(Z, (X^{n-1})_{\mathcal{P}})$ and tensoring them with $\mathbb{Z}_{\mathcal{P}}$, by the naturality of localization we obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] \\
H_{r+1}(X^n, X^{n-1}) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r, "\simeq"] \arrow[d] & H_{r+1}(Z, (X^{n-1})_{\mathcal{P}}) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[d] \\
H_r(X^{n-1}) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r, "\simeq"] \arrow[d] & H_r((X^{n-1})_{\mathcal{P}}) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[d] \\
H_r(X^n) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r] \arrow[d] & H_r(Z) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[d] \\
H_r(X^n, X^{n-1}) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r, "\simeq"] \arrow[d] & H_r(Z, (X^{n-1})_{\mathcal{P}}) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[d] \\
H_{r-1}(X^{n-1}) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r, "\simeq"] \arrow[d] & H_{r-1}((X^{n-1})_{\mathcal{P}}) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[d] \\
\cdots & \cdots
\end{tikzcd}$$

Note that the second and fifth horizontal arrows are isomorphisms by inductive hypothesis. Also, the first and fourth horizontal arrows are isomorphisms by the previous remark since $H_j(X^n, X^{n-1}) \simeq H_j(X^n/X^{n-1}) \simeq H_j(\bigvee_{\alpha\in J} \Sigma^n A)$ and $H_j(Z, (X^{n-1})_{\mathcal{P}}) \simeq H_j(Z/(X^{n-1})_{\mathcal{P}}) \simeq H_j(\bigvee_{\alpha\in J} \Sigma^n A_{\mathcal{P}})$ for all j.

Hence, the inductive step is finished and the theorem is proved in case X is a finite dimensional CW(A)-complex.

For the general case note that $H_n(X) \simeq H_n(X^n)$ and that by the above construction $(X_{\mathcal{P}})^n$ is a $\mathcal{P}$-localization of X^n. The theorem follows from commutativity of the square

$$\begin{tikzcd}
H_n(X^n) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & H_r((X_{\mathcal{P}})^n) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[d, "\simeq"] \\
H_n(X) \otimes \mathbb{Z}_{\mathcal{P}} \arrow[r] & H_n((X)_{\mathcal{P}}) \otimes \mathbb{Z}_{\mathcal{P}}
\end{tikzcd}$$

where the vertical arrows are induced by the inclusion maps. $\square$

Chapter 4

Homotopy theory of CW(A)-complexes

In this chapter we start to develop the homotopy theory of CW(A)-complexes. We define degrees of A-connectedness, A-n-equivalences and A-weak equivalences, all of them related to A-homotopy groups of spaces. Then we study degrees of connectedness and A-connectedness of CW(A)-complexes.

The main result of this chapter is theorem 4.2.4 which generalizes the famous Whitehead Theorem.

4.1 A-connectedness and A-homotopy groups

In this section we prove various homotopical results concerning CW(A)-complexes which will be needed later and are also interesting for their own sake.

Let X be a (pointed) topological space and let $r \in \mathbb{N}_0$. Recall that the sets $\pi_r^A(X)$ are defined by $\pi_r^A(X) = [\Sigma^r A, X]$, the homotopy classes of maps from $\Sigma^r A$ to X. It is well known that these are groups for $r \geq 1$ and abelian for $r \geq 2$.

Similarly, for $B \subseteq X$ one defines $\pi_r^A(X, B) = [(\mathrm{C}\Sigma^{r-1}A, \Sigma^{r-1}A), (X, B)]$ for $r \in \mathbb{N}$, which are groups for $r \geq 2$ and abelian for $r \geq 3$.

Note that $\pi_r^{S^0}(X) = \pi_r(X)$ and $\pi_r^{S^n}(X) = \pi_{r+n}(X)$. Note also that $\pi_r^A(X)$ are trivial if A is contractible.

Definition 4.1.1. Let (X, B) be a pointed topological pair. The pair (X, B) is called *A-0-connected* if for any given continuous function $f : A \to X$ there exists a map $g : A \to B$ such that $ig \simeq f$, where $i : B \to X$ is the inclusion.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & B \arrow[d, "i"] \\ A \arrow[ur, "g"] \arrow[r, "f"'] \arrow[r, phantom, shift left=3, "\simeq"] & X \end{tikzcd}$$

Definition 4.1.2. Let $n \in \mathbb{N}$. The pointed topological pair (X, B) is called *A-n-connected* if it is A-0-connected and $\pi_r^A(X, B) = 0$ for $1 \leq r \leq n$.

Definition 4.1.3. Let $f : X \to Y$ be a continuous map, and let A be a topological space. The map f is called an *A-0-equivalence* if for any given continuous function $g : A \to Y$, there exists a map $h : A \to X$ such that $fh \simeq g$.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A \arrow[r, "g"'] \arrow[ur, "h"] \arrow[ur, phantom, "\simeq"', shift right=2] & Y \end{tikzcd}$$

Given $n \in \mathbb{N}$, the map f is called an *A-n-equivalence* if it induces isomorphisms $f_* : \pi_r^A(X, x_0) \to \pi_r^A(Y, f(x_0))$ for $0 \leq r < n$ and an epimorphism for $r = n$.
Also, f is called an *A-weak equivalence* if it is an A-n-equivalence for all $n \in \mathbb{N}$.

Remark 4.1.4. Let $f : X \to Y$ be map and let $n \in \mathbb{N}$. We denote by Z_f the mapping cylinder of f. Then f is an *A-n-equivalence* if and only if the topological pair (Z_f, X) is A-n-connected.

Lemma 4.1.5. *Let X, S, B be pointed topological spaces, $S \subseteq X$ a subspace, $x_0 \in S$ and $b_0 \in B$ the base points. Let $f : (\mathrm{C}B, B) \to (X, S)$ be a continuous map. Then the following are equivalent.*

i) There exists a base point preserving homotopy $H : (\mathrm{C}B \times I, B \times I) \to (X, S)$ such that $Hi_0 = f$, $Hi_1(x) = x_0$ $\forall x \in \mathrm{C}B$.

ii) There exists a (base point preserving) homotopy $G : \mathrm{C}B \times I \to X$, relative to B, such that $Gi_0 = f$, $Gi_1(\mathrm{C}B) \subseteq S$.

iii) There exists a (base point preserving) homotopy $G : \mathrm{C}B \times I \to X$, such that $Gi_0 = f$, $Gi_1(\mathrm{C}B) \subseteq S$.

Proof. *i)* $\Rightarrow$ *ii)* Define G as follows.

$$G([x,s],t) = \begin{cases} H([x, \frac{2s}{2-t}], t) & \text{if } 0 \leq s \leq 1 - \frac{t}{2} \\ H([x,1], 2-2s) & \text{if } 1 - \frac{t}{2} \leq s \leq 1 \end{cases}$$

It is clear that G is well defined and continuous. Note that

$$\begin{array}{ll} Gi_0([x,s]) = H([x, \frac{2s}{2}], 0) = H([x,s],0) = f(x,s) & \\ Gi_1([x,s]) = H([x,2s],1) = x_0 \in S & \text{if } s \leq \frac{1}{2} \\ Gi_1([x,s]) = H([x,1], 2-2s) \in S & \text{if } s \geq \frac{1}{2} \end{array}$$

since $H(B \times I) \subseteq S$.
ii) $\Rightarrow$ *iii)* Obvious.
iii) $\Rightarrow$ *i)* We define H by

$$H([x,s],t) = \begin{cases} G([x,s],2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ Gi_1([x, s(2-2t)]) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$\square$

Lemma 4.1.6. *Let X, Y be pointed topological spaces and let $f : X \to Y$ be an A-n-equivalence. Let $r \in \mathbb{N}$, $r \leq n$ and let $i_A : \Sigma^{r-1}A \to \mathrm{C}\Sigma^{r-1}A$ be the inclusion. Suppose that $g : \Sigma^{r-1}A \to X$ and $h : \mathrm{C}\Sigma^{r-1}A \to Y$ are continuous maps such that $hi_A = fg$. Then, there exists a continuous map $k : \mathrm{C}\Sigma^{r-1}A \to X$ such that $ki_A = g$ and $fk \simeq h$ rel $\Sigma^{r-1}A$.*

$$\begin{tikzcd}
\Sigma^{r-1}A \arrow[r, "g"] \arrow[d, "i_A"'] \arrow[dr, phantom, "|||", near start] & X \arrow[d, "f"] \\
\mathrm{C}\Sigma^{r-1}A \arrow[r, "h"'] \arrow[ur, "k"] \arrow[ur, phantom, "\simeq", near end, shift right=3] & Y
\end{tikzcd}$$

Proof. Consider the inclusions $i : X \to Z_f$ and $j : Y \to Z_f$. Let $r : Z_f \to Y$ be the usual retraction. Note that there is a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma^{r-1}A \arrow[r, "g"] \arrow[d, "i_A"'] & X \arrow[d, "i"] \\
\mathrm{C}\Sigma^{r-1}A \arrow[r, "jh"'] & Z_f
\end{tikzcd}$$

Let $H : \Sigma^{r-1}A \times I \to Z_f$ be the homotopy from jhi_A to ig defined by $H(a,t) = [g(a), t]$ for $a \in \Sigma^{r-1}A$, $t \in I$. Consider the commutative diagram of solid arrows

$$\begin{tikzcd}
\Sigma^{r-1}A \arrow[r, "i_0"] \arrow[d, "i_A"'] & \Sigma^{r-1}A \times I \arrow[d] \arrow[ddr, bend left, "H"] & \\
\mathrm{C}\Sigma^{r-1}A \arrow[r, "i_0"'] \arrow[drr, bend right, "jh"'] & \mathrm{C}\Sigma^{r-1}A \times I \arrow[dr, dotted, "H'"] & \\
 & & Z_f
\end{tikzcd}$$

Since i_A is a cofibration there exists a map H' such that the whole diagram commutes, which induces a commutative diagram

$$\begin{tikzcd}
\Sigma^{r-1}A \arrow[r, "g"] \arrow[d, "i_A"'] & X \arrow[d, "i"] \\
\mathrm{C}\Sigma^{r-1}A \arrow[r, "H'i_1"'] & Z_f
\end{tikzcd}$$

The pair (Z_f, X) is A-n-connected, so by lemma 4.1.5 there exists a continuous function $k : \mathrm{C}\Sigma^{r-1}A \to X$ such that $ki_A = g$, $ik \simeq H'i_1$ rel $\Sigma^{r-1}A$. Then

$$fk = rik \simeq rH'i_1 \simeq rH'i_0 = rjh = h$$

Note that the homotopy is relative to $\Sigma^{r-1}A$, thus $fk \simeq h$ rel $\Sigma^{r-1}A$. $\square$

Lemma 4.1.7. *Let A be an l-connected CW-complex, let B be a topological space, and suppose X is obtained from B by attaching a 1-cell of type A. Then (X, B) is $(l+1)$-connected.*

Proof. Let g be the attaching map of the cell and f its characteristic map. Since A is an l-connected CW-complex, $(\mathrm{C}A, A)$ is a relative CW-complex which is $(l+1)$-connected. Then there exists a relative CW-complex (Z, A') such that A is a strong deformation retract of A', $\mathrm{C}A$ is a strong deformation retract of Z and $(Z_{A'})^{l+1} = A'$. Let $r : A' \to A$ be the retraction and let $i_X : B \to X$ be the inclusion. Consider the pushout

$$\begin{tikzcd} A' \arrow[r, "gr"] \arrow[d, "i_{A'}"'] \arrow[dr, phantom, "push"] & B \arrow[d, "i_Y"] \\ Z \arrow[r, "f'"'] & Y \end{tikzcd}$$

Then (Y, B) is a relative CW-complex with $(Y_B)^{l+1} = B$, and hence it is $(l+1)$-connected. The inclusions $i : A \to A'$ and $j : \mathrm{C}A \to Z$ and the identity map of B induce a map $\varphi : X \to Y$ with $\varphi i_X = i_Y \mathrm{Id}_B$. Now, i_A, $i_{A'}$ are closed cofibrations and i, j and Id_B are homotopy equivalences, then by 1.1.20, φ is a homotopy equivalence. Thus, (X, B) is $(l+1)$-connected. $\square$

Note that the previous lemma can be applied when attaching a cell of any positive dimension, since attaching an A-n-cell is the same as attaching a $(\Sigma^{n-1}A)$-1-cell. The following lemma deals with the case in which we attach an A-0-cell. The proof is similar to the previous one.

Lemma 4.1.8. *Let A be an l-connected CW-complex, B a topological space, and suppose X is obtained from B by attaching a 0-cell of type A (i.e. $X = B \vee A$). Then (X, B) is l-connected.*

Now, using both lemmas we are able to prove the following proposition.

Proposition 4.1.9. *Let A be an l-connected CW-complex, and let X be a CW(A)-complex. Then the pair (X, X^n) is $(n+l+1)$-connected. In particular, X is l-connected.*

Proof. Let $r \leq n+l+1$ and $f : (D^r, S^{r-1}) \to (X^{n+1}, X^n)$. We want to construct a map $f' : (D^r, S^{r-1}) \to (X^{n+1}, X^n)$ such that $f'(D^r) \subseteq X^n$, and $f \simeq f'$ rel S^{r-1}. Since $f(D^r)$ is compact, it intersects only a finite number of interiors of $(n+1)$-cells (note that A is T1). By an inductive argument, we may suppose that we are attaching just one $(n+1)$-cell of type A, which is equivalent to attaching a 1-cell of type $\Sigma^n A$. Since $\Sigma^n A$ is $(n+l)$-connected, (X^{n+1}, X^n) is $(n+l+1)$-connected. The result of the proposition follows. $\square$

Proposition 4.1.10. *Let A be an l-connected CW-complex, with $\dim(A) = k \in \mathbb{N}_0$, and let X be a CW(A)-complex. Then the pair (X, X^n) is A-$(n-k+l+1)$-connected.*

Proof. We prove first the A-0-connectedness in case $k \leq n+l+1$. We have to find a dotted arrow in a diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "|||"] & X^n \arrow[d, "i"] \\ A \arrow[r, "f"'] \arrow[ur, dotted] \arrow[ur, phantom, "\simeq", shift right=2] & X \end{tikzcd}$$

This map exists because A is a CW-complex with $\dim(A) = k$ and (X, X^n) is $(n+l+1)$-connected.

Now we prove the A-r-connectedness in case $1 \leq r \leq n-k+l+1$. By lemma 4.1.5, it suffices to find a dotted arrow in a diagram

$$\begin{tikzcd}
\Sigma^{r-1}A \arrow[r] \arrow[d] \arrow[dr, phantom, "\equiv", pos=0.3] & X^n \arrow[d, "i"] \\
\mathrm{C}\Sigma^{r-1}A \arrow[r, "f"'] \arrow[ur, dotted] \arrow[ur, phantom, "\simeq", pos=0.7, shift right=2ex] & X
\end{tikzcd}$$

This map exists because $(\mathrm{C}\Sigma^{r-1}A, \Sigma^{r-1}A)$ is a CW-complex of dimension $r+k$, (X, X^n) is $(n+l+1)$-connected, and $r+k \leq n+l+1$. □

Corollary 4.1.11. *Let A be an l-connected CW-complex, with $\dim(A) = k \in \mathbb{N}_0$, and let X be a CW(A)-complex. Let $i : X^n \to X$ denote the inclusion of the A-n-skeleton. Then $i_* : \pi_r^A(X^n) \to \pi_r^A(X)$ is an isomorphism for $r \leq n-k+l$ and an epimorphism for $r = n-k+l+1$. In consequence, $\pi_r^A(X)$ depends only on the A-$(r+k-l)$-skeleton of X.*

Proposition 4.1.12. *Let A be a CW-complex of dimension k. Let X be an n-connected CW-complex and Y be an m-connected CW-complex and let $i_X : X \to X \vee Y$ and $i_Y : Y \to X \vee Y$ denote the inclusions. Suppose either X or Y is locally compact. Then the map $((i_X)_*, (i_Y)_*) : \pi_r^A(X) \bigoplus \pi_r^A(Y) \to \pi_r^A(X \vee Y)$ is an isomorphism for $2 \leq r \leq n+m-k$.*

Proof. We know that the topological pair $(X \times Y, X \vee Y)$ is $(n+m+1)$-connected since $(X \times Y, X \vee Y)^{(n+m+1)} = X \vee Y$. Then $(X \times Y, X \vee Y)$ is A-$(n+m+1-k)$-connected. Hence, from the long exact sequence of A-homotopy groups for the topological pair $(X \times Y, X \vee Y)$ it follows that the inclusion $i : X \vee Y \to X \times Y$ induces isomorphisms $i_* : \pi_r^A(X \vee Y) \to \pi_r^A(X \times Y)$ for $1 \leq r \leq n+m-k$.

Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the projections. It is clear that the map $((p_X)_*, (p_Y)_*) : \pi_r^A(X \times Y) \to \pi_r^A(X) \times \pi_r^A(Y)$ is an isomorphism for all $r \in \mathbb{N}$. □

Corollary 4.1.13. *Let A be an l-connected CW-complex of dimension k with $k \leq 2l+2$ and let $m \in \mathbb{N}_{\geq 2}$. For $1 \leq \alpha \leq m$ we denote by $i_\alpha : \Sigma^r A \to \bigvee_{\alpha=1}^{m} \Sigma^r A$ the inclusion in the α-th copy. Then the map*

$$\bigoplus_{\alpha=1}^{m} (i_\alpha)_* : \bigoplus_{\alpha=1}^{m} \pi_r^A(\Sigma^r A) \to \pi_r^A(\bigvee_{\alpha=1}^{m} \Sigma^r A)$$

is an isomorphism.

Proof. We proceed by induction in m. For $m = 2$, since $\Sigma^r A$ is an $r+l$-connected CW-complex of dimension $k+r$, by the previous proposition we get an isomorphism $((i_1)_*, (i_2)_*) : \pi_r^A(\Sigma^r A) \bigoplus \pi_r^A(\Sigma^r A) \to \pi_r^A(\Sigma^r A \vee \Sigma^r A)$ for $2 \leq r \leq (l+r)+(l+r)-k$, which is equivalent to $r \geq 2$ and $k \leq 2l+r$, that holds by hypothesis.

The inductive step is similar. □

The following corollary shows that the same result holds for an infinite wedge if A is compact. The proof is formally identical to that of corollary 6.37 of [20].

Corollary 4.1.14. *Let A be an l-connected and compact CW-complex of dimension k with $k \leq 2l+2$ and let J be an index set. For $\alpha \in J$ we denote by $i_\alpha : \Sigma^r A \to \bigvee_{\alpha \in J} \Sigma^r A$ the inclusion in the α-th copy. Then the map*

$$\bigoplus_{\alpha \in J} (i_\alpha)_* : \bigoplus_{\alpha \in J} \pi_r^A(\Sigma^r A) \to \pi_r^A(\bigvee_{\alpha \in J} \Sigma^r A)$$

is an isomorphism for $r \geq 2$.

Proof. Note that every map $f : \Sigma^r A \to \bigvee_{\alpha \in J} \Sigma^r A$ has compact image. Then there is a finite set $L = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ such that $\operatorname{Im} f \subseteq \bigvee_{\alpha \in L} i_\alpha(\Sigma^r A)$. From the commutative square

$$\begin{tikzcd}
\bigoplus_{\alpha \in L} \pi_r^A(\Sigma^r A) \arrow[r] \arrow[d, "\bigoplus_{\alpha \in L} (i_\alpha)_*"', "\simeq"] & \bigoplus_{\alpha \in J} \pi_r^A(\Sigma^r A) \arrow[d, "\bigoplus_{\alpha \in J} (i_\alpha)_*"] \\
\pi_r^A(\bigvee_{\alpha \in L} \Sigma^r A) \arrow[r, "\mathrm{in}_*"'] & \pi_r^A(\bigvee_{\alpha \in J} \Sigma^r A)
\end{tikzcd}$$

we get that $f \in \operatorname{Im} \bigoplus_{\alpha \in J} (i_\alpha)_*$. Hence $\bigoplus_{\alpha \in J} (i_\alpha)_*$ is surjective.

Similarly, any homotopy $H : \Sigma^r A \times I \to \bigvee_{\alpha \in J} \Sigma^r A$ has compact image, so $\bigoplus_{\alpha \in J} (i_\alpha)_*$ is injective. □

Lemma 4.1.15. *Let A be a finite dimensional CW-complex, let X be a CW(A)-complex and let B be a CW(A)-subcomplex of X. Then there exists a CW-pair (X', B') homotopy equivalent to (X, B).*

Its proof is not difficult and is analogous to that of 3.1.28.

Proposition 4.1.16. *Let A be an l-connected CW-complex of dimension k. Let X be a CW(A)-complex and let B be a CW(A)-subcomplex of X such that B is m-connected and (X, B) is n-connected, with $n \geq 1$. Let $p : (X, B) \to (X/B, *)$ be the projection. Then $p_* : \pi_r^A(X, B) \to \pi_r^A(X/B)$ is an isomorphism for $2 \leq r \leq m+n-k$ and an epimorphism for $r = m+n-k+1$.*

Proof. We proceed inductively in the skeletons of A. As usual we may suppose that A has only one 0-cell (the base point) since any connected CW-complex has the homotopy type of a CW-complex with just a single 0-cell. Hence for A^0 the result trivially holds.

Now suppose the result holds for A^{j-1} and consider the following commutative diagram made up by two long exact sequences associated to the cofibration sequence $A^{j-1} \hookrightarrow A^j \to A^j/A^{j-1} = \bigvee S^j$.

$$\begin{array}{ccccccccccccc}
\cdots \to & \pi_{r+1}^{A^{j-1}}(X,B) & \to & \pi_r^{\bigvee S^j}(X,B) & \to & \pi_r^{A^j}(X,B) & \to & \pi_r^{A^{j-1}}(X,B) & \to & \pi_{r-1}^{\bigvee S^j}(X,B) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \pi_{r+1}^{A^{j-1}}(X/B) & \to & \pi_r^{\bigvee S^j}(X/B) & \to & \pi_r^{A^j}(X/B) & \to & \pi_r^{A^{j-1}}(X/B) & \to & \pi_{r-1}^{\bigvee S^j}(X/B) & \to \cdots
\end{array}$$

Now observe that the first and second vertical arrows are isomorphisms for $2 \leq r \leq m+n-j-1$ and an epimorphism for $r = m+n-j$ by inductive hypothesis and 6.22 of [20] respectively (since (X,B) has the homotopy type of a CW-pair). Similarly, the fourth and fifth vertical arrows are isomorphisms for $2 \leq r \leq m+n-j$. Hence, by the five lemma the third vertical arrow is an $2 \leq r \leq m+n-j-1$ and an epimorphism for $r = m+n-j$. The result follows. □

Now, we turn our attention to stable homotopy groups.

Definition 4.1.17. Let A and X be pointed topological spaces. We define the *stable A-homotopy groups of X* by $\pi_n^{A,\mathrm{st}}(X) = \operatorname{colim}_j \pi_{n+j}^A(\Sigma^j X)$.

Proposition 4.1.18. *Let A be a CW-complex of dimension k. Let X be a CW(A)-complex and let B be a CW(A)-subcomplex of X. Then there are exact sequences $\pi_n^{A,st}(B) \to \pi_n^{A,st}(X) \to \pi_n^{A,st}(X/B)$ for all $n \in \mathbb{Z}$.*

Proof. Fix $n \in \mathbb{Z}$. For each $j \in \mathbb{N}$ there are exact sequences

$$\pi_{n+j}^A(\Sigma^j B) \to \pi_{n+j}^A(\Sigma^j X) \to \pi_{n+j}^A(\Sigma^j X, \Sigma^j B)$$

which are natural in j since $\mathrm{C}\Sigma Y$ is homeomorphic to $\Sigma \mathrm{C} Y$ for every Y.

It is clear that the CW-pair $(\Sigma^j X, \Sigma^j B)$ is j-connected. Then the quotient map $(\Sigma^j X, \Sigma^j B) \to (\Sigma^j X/\Sigma^j B)$ induces an isomorphism $\pi_{n+j}^A(\Sigma^j X, \Sigma^j B) \to \pi_{n+j}^A(\Sigma^j X/\Sigma^j B)$ if $2 \leq n+j \leq j+j-k$, or equivalently $j \geq \max\{2, n+k\}$.

Hence there are exact sequences

$$\pi_{n+j}^A(\Sigma^j B) \to \pi_{n+j}^A(\Sigma^j X) \to \pi_{n+j}^A(\Sigma^j (X/B))$$

for $j \geq \max\{2, n+k\}$.

Taking colimit in j, by proposition 7.50 of [20], we obtain a short exact sequence

$$\pi_n^{A,\mathrm{st}}(B) \to \pi_n^{A,\mathrm{st}}(X) \to \pi_n^{A,\mathrm{st}}(X/B).$$

□

4.2 Whitehead's theorem

In this section we prove a generalization of Whitehead's theorem. It is interesting to comment that the proof of Whitehead's theorem given in [20] can be generalized to our setting with almost no difficulties. This is the proof we give here. We also point out that in proposition 4.2.3 the standard proof of injectivity of the induced map f_* uses the fact that the cylinder of a CW-complex is a CW-complex, which does not hold for CW(A)-complexes. However, surjectivity of f_* suffices for the proof of Whitehead's theorem, as we shall see.

Theorem 4.2.1. *Let $f : X \to Y$ be an A-n-equivalence ($n = \infty$ is allowed) and let (Z, B) be a relative CW(A)-complex which admits a CW(A)-structure of dimension less than or equal to n. Let $g : B \to X$ and $h : Z \to Y$ be continuous functions such that $h|_B = fg$. Then there exists a continuous map $k : Z \to X$ such that $k|_B = g$ and $fk \simeq h$ rel B.*

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "i"'] \arrow[dr, phantom, "\equiv" near start] & X \arrow[d, "f"] \arrow[dl, phantom, "\simeq" near start] \\ Z \arrow[r, "h"'] \arrow[ur, dotted, "k"] & Y \end{tikzcd}$$

Proof. Let

$$\begin{aligned} S \;=\; & \{(Z', k', K')/B \subseteq Z' \subseteq Z \ A-\text{subcomplex}\ ,\ k' : Z' \to Z \text{ with } k'|_B = g \text{ and} \\ & K' : Z' \times I \to Y, K' : fk' \simeq h|_{Z'} \text{ rel } B\} \end{aligned}$$

It is clear that $S \neq \varnothing$. We define a partial order in S in the following way.

$$(Z', k', K') \leq (Z'', k'', K'') \text{ if and only if } Z' \subseteq Z'', k''|_{Z'} = k' \ \mathrm{K}''|_{\mathrm{Z}' \times \mathrm{I}} = \mathrm{K}'$$

It is clear that every chain has an upper bound since Z has the weak topology. Then, by Zorn's lemma, there exists a maximal element (Z', k', K'). We want to prove that $Z' = Z$. Suppose $Z' \neq Z$, then there exist some A-cells in Z which are not in Z'. Choose e an A-cell with minimum dimension. We want to extend the maps k' and K' to $Z' \cup e$. If e is an A-0-cell this is easy to do since f is an A-0-equivalence and all homotopies are relative to the base point. Suppose then that $\dim e \geq 1$. Let $\phi : (\mathrm{C}\Sigma^{r-1}A, \Sigma^{r-1}A) \to (Z, Z')$ be the characteristic map of e, let $\psi = \phi|_{\Sigma^{r-1}A}$, and let $Z'' = Z' \cup e$. We have the following diagram.

$$\begin{tikzcd} \Sigma^{r-1}A \arrow[r, "\psi"] \arrow[d, "i_A"'] \arrow[dr, phantom, "\equiv"] & Z' \arrow[r, "k'"] \arrow[d, "i_{Z'}"'] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "f"] \\ \mathrm{C}\Sigma^{r-1}A \arrow[r, "\phi"'] & Z'' \arrow[r, "h|_{Z''}"'] & Y \end{tikzcd}$$

Here, the homotopy of the right square is relative to B. Let $\alpha : I \to I$ be defined by $\alpha(t) = 1 - t$. Since $i_{Z'}$ is a cofibration we can extend $K'(\mathrm{Id} \times \alpha)$ to some $H : Z'' \times I \to Y$, and then we obtain a commutative diagram

$$\begin{tikzcd} \Sigma^{r-1}A \arrow[r, "\psi"] \arrow[d, "i_A"'] \arrow[dr, phantom, "\equiv"] & Z' \arrow[r, "k'"] \arrow[d, "i_{Z'}"'] \arrow[dr, phantom, "\equiv"] & X \arrow[d, "f"] \\ \mathrm{C}\Sigma^{r-1}A \arrow[r, "\phi"'] & Z'' \arrow[r, "Hi_1"'] & Y \end{tikzcd}$$

By the previous lemma, there exists $l : C\Sigma^{r-1}A \to X$ such that $li_A = k'\psi$ and $fl \simeq Hi_1\phi$ rel $\Sigma^{r-1}A$. Let G denote this homotopy.

Now, since the left square is a pushout, there is a map $\gamma : Z'' \to X'$ such that $\gamma\phi = l$, $\gamma i_{Z'} = k'$. So γ extends k'. We want now to define a homotopy $K'' : f\gamma \simeq h|_{Z''}$ extending K'. We consider $C\Sigma^{r-1}A \times [0,2]/\sim$ where we identify $(b,t) \sim (b,t')$ for $b \in \Sigma^{r-1}A$, $t, t' \in [1,2]$. There is a homeomorphism $\beta : C\Sigma^{r-1}A \times [0,2]/\sim \to C\Sigma^{r-1}A \times I$ defined by

$$\beta([a,s],t) = \begin{cases} ([a,s], \frac{t}{2-s}) & \text{if } 0 \leq t \leq 1 \\ ([a,s], \frac{1-s}{2-s}\, t + \frac{s}{2-s}) & \text{if } 1 \leq t \leq 2 \end{cases}$$

We have the following commutative diagram.

$$\begin{tikzcd}
\Sigma^{r-1}A \times I \arrow[r, "\psi\times \mathrm{Id}_I"] \arrow[d, "i_A\times \mathrm{Id}_I"'] \arrow[dr, phantom, "\text{push}"] & Z' \times I \arrow[d, "i_{Z'}\times \mathrm{Id}_I"] \arrow[dd, bend left, "K'(\mathrm{Id}\times\alpha)"] \\
C\Sigma^{r-1}A \times I \arrow[r, "\phi\times \mathrm{Id}_I"'] \arrow[dr, bend right, "(H(\phi\times \mathrm{Id}_I)+G(\mathrm{Id}\times\alpha))\beta^{-1}"'] & Z'' \times I \arrow[d, dotted, "\widetilde{K}"] \\
 & Y
\end{tikzcd}$$

Note that

$$\begin{aligned}
&(H(\phi \times \mathrm{Id}_I) + G(\mathrm{Id} \times \alpha))\beta^{-1}(i_A \times \mathrm{Id}_I) = H(\phi \times \mathrm{Id}_I)(i_A \times \mathrm{Id}_I) = \\
&= H(i_{Z'} \times \mathrm{Id}_I)(\psi \times \mathrm{Id}_I) = K'(\mathrm{Id} \times \alpha)(\psi \times \mathrm{Id}_I)
\end{aligned}$$

Then, the map $\widetilde{K}$ exists. We take $K'' = \widetilde{K}(\mathrm{Id} \times \alpha)$. □

Remark 4.2.2. If (Y, B) is a relative CW(A)-complex which is A-n-connected for all $n \in \mathbb{N}$ then $i : B \to Y$ is an A-n-equivalence for all $n \in \mathbb{N}$ and we have

$$\begin{tikzcd}
B \arrow[r, "\mathrm{Id}_B"] \arrow[d, "i"'] & B \arrow[d, "i"] \\
Y \arrow[r, "\mathrm{Id}_Y"'] \arrow[ur, dotted, "r"] \arrow[ur, phantom, "|||", near end] \arrow[ur, phantom, "\simeq"', near start] & Y
\end{tikzcd}$$

Thus B is a strong deformation retract of Y. In particular, if X is a CW(A)-complex with $\pi_n^A(X) = 0$ for all $n \in \mathbb{N}_0$, then X is contractible.

The following proposition follows immediately from 4.2.1.

Proposition 4.2.3. *Let $f : Z \to Y$ be an A-n-equivalence ($n = \infty$ is allowed) and let X be a CW(A)-complex which admits a CW(A)-structure of dimension less than or equal to n. Then, the map $f_* : [X, Z] \to [X, Y]$ is surjective.*

Finally we obtain a generalization of Whitehead's theorem.

Theorem 4.2.4. *Let X and Y be CW(A)-complexes and let $f : X \to Y$ be a continuous map. Then f is a homotopy equivalence if and only if it is an A-weak equivalence.*

Proof. Suppose f is an A-weak equivalence. We consider $f_* : [Y, X] \to [Y, Y]$. By the previous proposition, f_* is surjective, then there exists $g : Y \to X$ such that $fg \simeq \mathrm{Id}_Y$. Then g is also an A-weak equivalence, so applying the above argument, there exists an $h : X \to Y$ such that $gh \simeq \mathrm{Id}_X$. Then $f \simeq fgh \simeq h$, and so, $gf \simeq gh \simeq \mathrm{Id}_X$. Thus f is a homotopy equivalence. $\square$

Chapter 5

Homology of CW(A)-complexes

In this chapter we start investigating the homology theory of CW(A)-complexes. Our main goal is to develop tools and techniques which allow us to compute the singular homology of these spaces out of the homology of the core A and the CW(A)-structure of the space. The tools we work with in this chapter are generalizations of classical cellular homology.

Note that the (reduced) homology of S^0 (with coefficients in $\mathbb{Z}$) has two significant properties: it is concentrated in one degree (degree zero) and it is free (as an abelian group). Keeping this in mind, we study two cases: when the reduced homology of A is concentrated in a certain degree and when the homology groups of A are free.

When the homology of the core A is neither concentrated nor free, the homology of X is more difficult to compute. Example 5.2.8 of Section 3 shows that, in that case, the homology of X cannot be computed from an A-cellular complex as in the other cases. However, in next chapter, we will study the general case by means of spectral sequences.

In the last section of this chapter we define and investigate the A-Euler characteristic χ_A of CW(A)-complexes, which is a homotopy invariant if A is a CW-complex with $\chi(A) \neq 0$. We also define the *multiplicative Euler characteristic* when the core A has finite homology (see Theorem 5.3.8 below).

Throughout this chapter, homology will mean reduced homology with coefficients in $\mathbb{Z}$.

5.1 Easy computations

As we claimed in the introduction, our aim is to compute the singular homology groups of CW(A)-complexes out of the homology of A and the CW(A)-structure of the space.

Remark 5.1.1. Recall that if A and X are (pointed) CW-complexes and $g : A \to X$ is a continuous (cellular) map there is a long exact sequence

$$\cdots \longrightarrow H_n(A,*) \xrightarrow{g_*} H_n(X,*) \xrightarrow{i_*} H_n(\mathrm{C}g,*) \xrightarrow{q_*} H_{n-1}(A,*) \xrightarrow{g_*} \cdots$$

which induces short exact sequences

$$0 \longrightarrow \mathrm{Coker}\ g_* \xrightarrow{i_*} H_n(\mathrm{C}g,*) \xrightarrow{q_*} \mathrm{Ker}\ g_* \longrightarrow 0$$

Here, $\mathrm{C}g$ denotes the mapping cone of g. This has an evident analogy with the chain complex $\mathrm{C}g_*$, where g_* is the induced map in the singular chain complexes.

In case all these short exact sequences split, the homology of $\mathrm{C}g$ can be computed in the following way. The map g induces a morphism of chain complexes $g_* : H_*(A) \to H_*(X)$. The homology of the cone of this morphism

$$\cdots \longrightarrow H_{n+1}(X) \oplus H_n(A) \xrightarrow{\begin{pmatrix} 0 & g_* \\ 0 & 0 \end{pmatrix}} H_n(X) \oplus H_{n-1}(A) \xrightarrow{\begin{pmatrix} 0 & g_* \\ 0 & 0 \end{pmatrix}} H_{n-1}(X) \oplus H_{n-2}(A) \longrightarrow \cdots$$

is clearly the homology of $\mathrm{C}g$.

The well-known remark above will be our starting point to compute the singular homology of finite CW(A)-complexes. Consider the following example. Define $\mathbb{D}^2_4$ as the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{g_4} & S^1 \\ \downarrow & \mathit{push} & \downarrow \\ D^2 & \longrightarrow & \mathbb{D}^2_4 \end{array}$$

where g_4 is a map of degree 4. Let the core A be $\mathbb{D}^2_4$ and let $g : D^2 \subseteq \mathbb{C} \to D^2$ be the map $g(z) = z^2$. The map g induces a well defined cellular map $g' : A \to A$. Let X be the CW(A)-complex of dimension one defined by the following pushout

$$\begin{array}{ccc} A & \xrightarrow{g'} & A \\ \downarrow & \mathit{push} & \downarrow \\ \mathrm{C}A & \longrightarrow & X \end{array}$$

Note that $H_1(A) = \mathbb{Z}_4$ and $H_r(A) = 0$ for $r \neq 1$. Also, the induced map $g'_* : H_1(A) \to H_1(A)$ is given by multiplication by 2. The cone of g' is in this case

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}_4 \xrightarrow{g'_*} \mathbb{Z}_4 \longrightarrow 0$$

where the group $\mathbb{Z}_4$ appears in degrees 1 and 2. Note that in the short exact sequences as above one gets $\ker g_* = 0$ or $\operatorname{coker} g_* = 0$. It follows that $H_r(X) = \mathbb{Z}_2$ for $r = 1, 2$ and $H_r(X) = 0$ for $r \neq 1, 2$.

The previous idea can also be applied to prove the following.

Proposition 5.1.2. *Let A be a CW-complex and let $n \in \mathbb{N}$. Let X be a CW(A)-complex with the property that, for every $r \in \mathbb{N}_0$, $H_{n-r}(A) = 0$ whenever X has at least one A-r-cell. Then $H_n(X) = 0$.*

Proof. Since A is a CW-complex, by cellular approximation we may suppose that X is also a standard CW-complex.

Since all (standard) cells of dimension less than or equal to $n+1$ lie in the A-$(n+1)$-skeleton X^{n+1}, it suffices to prove that $H_n(X^{n+1}) = 0$.

We proceed by induction in the A-skeletons X^k.

For $k = 0$ the result is clear. Suppose it holds for X^{k-1} and X has A-k-cells. We denote by $g_\alpha : \Sigma^{k-1}A \to X^{k-1}$, $\alpha \in \Lambda$, the corresponding attaching maps. Consider the long exact sequence

$$\cdots \longrightarrow H_n(X^{k-1}) \xrightarrow{i_*} H_n(X^k, *) \xrightarrow{q_*} \bigoplus_{\alpha\in\Lambda} H_{n-1}(\Sigma^{k-1}A) = \bigoplus_{\alpha\in\Lambda} H_{n-k}(A) \xrightarrow{+(g_\alpha)_*} \cdots$$

By hypothesis, $H_n(X^{k-1}) = 0$ and since X^k has an A-k-cell, $H_{n-k}(A) = 0$. Hence, $H_n(X^k, *) = 0$. □

As an easy consequence we obtain the following.

Corollary 5.1.3. *Let A be a CW-complex with homology concentrated in degree r and let X be a CW(A)-complex. If X does not have any A-n-cells, then $H_{n+r}(X) = 0$.*

5.2 A-cellular chain complex

Given a CW(A)-complex X, our aim is to construct a suitable chain complex whose homology coincides with the homology of X. We investigate two particular cases: when the homology of the core A is concentrated in one degree and when the homology groups of A are all free. The constructions and results that we obtain in both cases generalize the standard results on cellular homology of CW-complexes.

We begin with the first case. Suppose $H_n(A) = 0$ for $n \neq r$, i.e. the (reduced) homology of A is concentrated in degree r.

In this case, given a CW(A)-complex X, we define the *A-cellular chain complex* (C_*, d_*) *of* X as follows. Take

$$C_n = \bigoplus_{A\text{-}(n-r)\text{-cells}} H_r(A)$$

and define $d_{n+r} : C_{n+r} \to C_{n+r-1}$ in the following way. Given e^n_α and e^{n-1}_β A-cells of dimensions n and $n-1$ respectively we consider $g_\alpha : \Sigma^{n-1}A \to X^{n-1}$ the attaching map of e^n_α (where X^{n-1} denotes the A-n-skeleton of X) and the quotient map

$$q_\beta : X^{n-1} \to X^{n-1}/(X^{n-1} - \overset{\circ}{e}{}^{n-1}_\beta) = \Sigma^{n-1}A.$$

The map $q_\beta g_\alpha : \Sigma^{n-1}A \to \Sigma^{n-1}A$ induces

$$(q_\beta g_\alpha)_* : H_{n+r-1}(\Sigma^{n-1}A) = H_r(A) \to H_{n+r-1}(\Sigma^{n-1}A) = H_r(A).$$

Finally, d_n is induced by the maps $d_n^{\alpha,\beta} = (q_\beta g_\alpha)_*$ from the α-th copy of $H_r(A)$ to the β-th copy of $H_r(A)$ (recall that $H_k(A) = 0$ if $k \neq r$).

Note that this chain complex is very similar to the standard (cellular) one. In fact, to prove that (C_*, d_*) is actually a chain complex one may proceed as in the classical case, but replacing S^{n-1} by $\Sigma^{n-1}A$ and D^n by $\mathrm{C}\Sigma^{n-1}A$. Explicitly, consider the following commutative diagram

$$\begin{tikzcd}
H_{n+1}(\mathrm{C}\Sigma^{n-r}A,\Sigma^{n-r}A) \arrow[r, "\cong"] \arrow[d, "(f_\alpha)_*"'] & H_n(\Sigma^{n-r}A)=H_r(A) \arrow[rr, "d_n^{\alpha,\beta}"] \arrow[d, "(g_\alpha)_*"'] & & H_n(\Sigma^{n-r}A)=H_r(A) \\
H_{n+1}(X^{n+1-r},X^{n-r}) \arrow[r, "\partial"] \arrow[dr, "d'"'] & H_n(X^{n-r}) \arrow[rr, "q_*"] \arrow[urr, "(q_\beta)_*"] \arrow[d, "j"] & & H_n(X^{n-r}/X^{n-r-1}) \arrow[u, "(q'_\beta)_*"'] \arrow[d, "\simeq"] \\
& H_n(X^{n-r},X^{n-r-1}) \arrow[urr, "\varphi_n", "\simeq"'] \arrow[rr, "\cong"] & & H_n(X^{n-r}/X^{n-r-1},X^{n-r-1}/X^{n-r-1})
\end{tikzcd}$$

(compare with the analogous diagram in [8] page 141) and let J_n and J_{n-1} be index sets for the A-$(n-r)$-cells and A-$(n-r-1)$-cells of X respectively.

Since the map $(f_\alpha)_*$ corresponds to the inclusion $H_r(A)\to\bigoplus\limits_{J_{n-r}}H_r(A)$ into the α-th coordinate and the map $(q_\beta)'_*$ corresponds to the projection $\bigoplus\limits_{J_{n-r-1}}H_r(A)\to H_r(A)$ onto the β-th copy, it follows that $d_n=j\partial$ up to isomorphisms. Therefore $d_nd_{n+1}=0$ since the maps j and ∂ come from exact sequences as the diagram below shows.

$$\begin{tikzcd}
& H_n(X^{n-r}) \arrow[dr, "j"] & & & \\
H_{n+1}(X^{n-r+1},X^{n-r}) \arrow[ur, "\partial"] \arrow[rr, "d'"] & & H_n(X^{n-r},X^{n-r-1}) \arrow[rr, "d'"] \arrow[dr, "\partial"'] & & H_{n-1}(X^{n-r-1},X^{n-r-2}) \\
& & & H_{n-1}(X^{n-r-1}) \arrow[ur, "j"] &
\end{tikzcd}$$

More precisely, let $j_\alpha:\Sigma^{n-r}A\to X^{n-r}/X^{n-r-1}=\bigvee\limits_{J_{n-r}}\Sigma^{n-r}A$ be the inclusion in the α-th copy and let $q':\mathrm{C}\Sigma^{n-r}A\to\Sigma^{n-r+1}A$ be the quotient map. Consider the boundary map $\partial':H_{n+1}(\mathrm{C}\Sigma^{n-r+1}A,\Sigma^{n-r+1}A)\to H_n(\Sigma^{n-r}A)$, which is an isomorphism in this case. Let $\varepsilon':H_{n+1}(\Sigma^{n-r+1}A)\to H_n(\Sigma^{n-r}A)$ be the isomorphism given by $\varepsilon'=\partial'(q'_*)^{-1}$.

$$\begin{tikzcd}
H_{n+1}(\mathrm{C}\Sigma^{n-r+1}A,\Sigma^{n-r+1}A) \arrow[r, "\partial'", "\simeq"'] \arrow[d, "q'_*"', "\simeq"] & H_n(\Sigma^{n-r}A) \\
H_{n+1}(\Sigma^{n-r+1}A) \arrow[ur, "\varepsilon'", "\simeq"'] &
\end{tikzcd}$$

Let $\varepsilon_{n-r}:H_n(\Sigma^{n-r}A)\to H_r(A)$ be the isomorphism given by $\varepsilon_{n-r}=(\varepsilon')^{n-r}$. The inclusion maps j_α induce isomorphisms $\bigoplus\limits_{J_{n-r}}(j_\alpha)_*:\bigoplus\limits_{J_{n-r}}H_n(\Sigma^{n-r}A)\to H_n(\bigvee\limits_{J_{n-r}}A)$ and $\bigoplus\limits_{J_{n-r}}(j_\alpha)_*\varepsilon_{n-r}^{-1}:\bigoplus\limits_{J_{n-r}}H_r(A)\to H_n(\bigvee\limits_{J_{n-r}}A)=H_n(X^{n-r}/X^{n-r-1})$.

For each $n\in\mathbb{N}_0$, let

$$\phi_n:H_n(X^{n-r},X^{n-r-1})\longrightarrow\bigoplus_{A\text{-}(n-r)\text{-cells}}H_r(A)$$

be the isomorphism defined by $\phi_n = (\bigoplus_{\alpha \in J_{n-r}} (j_\alpha)_* \varepsilon_{n-r}^{-1})^{-1} \varphi_n$. Then, there is a commutative diagram

$$\begin{tikzcd}
H_n(X^{n-r}, X^{n-r-1}) \arrow[r, "\phi_n", "\simeq"'] \arrow[dr, "\simeq", "\varphi_n"'] & \bigoplus\limits_{A\text{-}(n-r)\text{-cells}} H_r(A) \arrow[r, "p_\beta"] \arrow[d, "\bigoplus_{J_{n-r}} (j_\alpha)_* \varepsilon_{n-r}^{-1}", "\simeq"'] & H_r(A) \\
& H_n(X^{n-r}/X^{n-r-1}) \arrow[r, "(q_\beta)_*"'] & H_n(\Sigma^{n-r} A) \arrow[u, "\varepsilon_{n-r}", "\simeq"']
\end{tikzcd}$$

Note that the right square commutes since for a in the α-th copy of $H_r(A)$ in $\bigoplus_{J_{n-r}} H_r(A)$ we have

$$(q_\beta)_*(\bigoplus_{\gamma \in J_{n-r}} (j_\gamma)_* \varepsilon_{n-r}^{-1})(a) = (q_\beta)_*(j_\alpha)_* \varepsilon_{n-r}^{-1}(a) = \begin{cases} \varepsilon_{n-r}^{-1}(a) & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha \end{cases}$$

thus, $\varepsilon_{n-r}(q_\beta)_*(\bigoplus_{\gamma \in J_{n-r}} (j_\gamma)_* \varepsilon_{n-r}^{-1}) = p_\beta$.

Hence $\varepsilon_{n-r}(q_\beta)_* \varphi_n = p_\beta \phi_n$.

We want to prove now that $\phi_{n+1}(f_\alpha)_* = i_\alpha \varepsilon_{n-r} \partial'$ in the following diagram

$$\begin{tikzcd}
H_{n+1}(\mathrm{C}\Sigma^{n-r} A, \Sigma^{n-r} A) \arrow[r, "\partial'", "\simeq"'] \arrow[d, "(f_\alpha)_*"'] & H_n(\Sigma^{n-r} A) \arrow[r, "(\varepsilon')^{-1}", "\simeq"'] \arrow[d, "i_\alpha \varepsilon_{n-r}"] \arrow[dr, phantom, "|||"] & H_{n+1}(\Sigma^{n-r+1} A) \arrow[dd, "(j_\alpha)_*"] \\
H_{n+1}(X^{n-r+1}, X^{n-r}) \arrow[r, "\simeq", "\phi_{n+1}"'] \arrow[drr, bend right, "\varphi_{n+1} = q_*"'] & \bigoplus\limits_{A\text{-}(n-r+1)\text{-cells}} H_r(A) \arrow[dr, "\bigoplus_{\gamma \in J_{n-r+1}} (j_\gamma)_* \varepsilon_{n-r+1}^{-1}", "\simeq"'] \arrow[d, phantom, "|||"] & \\
& \, & H_{n+1}(X^{n-r+1}/X^{n-r})
\end{tikzcd}$$

Since the triangle and the right square commute and $\bigoplus_{\gamma \in J_{n-r+1}} (j_\gamma)_* \varepsilon_{n-r+1}^{-1}$ is an isomorphism it suffices to prove that $q_*(f_\alpha)_* = (j_\alpha)_*(\varepsilon')^{-1} \partial'$. But there is a commutative diagram

$$\begin{tikzcd}
(\mathrm{C}\Sigma^{n-r} A, \Sigma^{n-r} A) \arrow[r, "f_\alpha"] \arrow[d, "q'"'] & (X^{n-r+1}, X^{n-r}) \arrow[d, "q"'] \\
\Sigma^{n-r+1} A \arrow[r, "j_\alpha"'] & X^{n-r+1}/X^{n-r} = \bigvee\limits_{J_{n-r+1}} \Sigma^{n-r+1} A
\end{tikzcd}$$

Hence, $q_*(f_\alpha)_* = (j_\alpha)_*(q')_* = (j_\alpha)_*(\varepsilon')^{-1} \partial'$ as we wanted to prove.

Summing up, we have shown that $\varepsilon_{n-r}(q_\beta)_* \varphi_n = p_\beta \phi_n$ and $\phi_{n+1}(f_\alpha)_* = i_\alpha \varepsilon_{n-r} \partial'$. We consider now the following diagram

$$\begin{tikzcd}
H_{n+1}(X^{n-r+1}, X^{n-r}) \arrow[r, "d'=j\partial"] \arrow[d, "\phi_{n+1}"'] & H_n(X^{n-r}, X^{n-r-1}) \arrow[d, "\phi_n"'] \arrow[dd, bend left, "(q_\beta)_*\varphi_n"] \\
\bigoplus_{A\text{-}(n-r+1)\text{-cells}} H_r(A) \arrow[r, "d"] & \bigoplus_{A\text{-}(n-r)\text{-cells}} H_r(A) \arrow[d, "p_\beta"] \\
H_r(A) \arrow[u, "i_\alpha"] \arrow[r, "d^{\alpha,\beta}"] \arrow[uu, bend left, "(f_\alpha)_*(\partial')^{-1}(\varepsilon_{n-r})^{-1}"] & H_r(A)
\end{tikzcd}$$

The lower square commutes by definition and the two triangles commute because of what we proved previously. Also, $d^{\alpha,\beta} = (q_\beta)_*\varphi_n d'(f_\alpha)_*(\partial')^{-1}(\varepsilon_{n-r})^{-1}$ as we have proved at the beginning. But ϕ_n and ϕ_{n+1} are isomorphisms, hence $d = \phi_n d'(\phi_{n+1})^{-1}$ and $d^2 = 0$ follows. Note that to prove that $d = \phi_n d'(\phi_{n+1})^{-1}$ it suffices to check that $p_\beta d i_\alpha = p_\beta \phi_n d'(\phi_{n+1})^{-1} i_\alpha$ for all α, β, and this follows from the last diagram.

Theorem 5.2.1. *Let A be a CW-complex with homology concentrated in degree r and let X be a CW(A)-complex. Then, the homology of the A-cellular chain complex defined as above coincides with the singular homology of X.*

Proof. We proceed by induction in the A-n-skeleton X^n. For $n = 0$ the result is clear.

Suppose the result holds for X^{n-1}. For simplicity, we assume that X is obtained from X^{n-1} by attaching only one A-n-cell. The general case is similar.

Let (C'_*, d'_*) be the A-cellular chain complex of X^{n-1}. By hypothesis, the homology of (C'_*, d'_*) coincides with the singular homology of X^{n-1}. Hence, by 5.1.1, the singular homology of X^n can be computed as the homology of the chain complex

$$\cdots \longrightarrow H_{n+1}(C'_*) \bigoplus H_n(\Sigma^{n-1}A) \xrightarrow{\begin{pmatrix} 0 & g_* \\ 0 & 0 \end{pmatrix}} H_n(C'_*) \bigoplus H_{n-1}(\Sigma^{n-1}A) \longrightarrow \cdots$$

where $g : \Sigma^{n-1}A \to X^{n-1}$ is the attaching map of the A-n-cell.

We want to prove that this complex has the same homology as the A-cellular complex of X, namely

$$\cdots \longrightarrow 0 \longrightarrow H_{n+r-1}(\Sigma^{n-1}A) \xrightarrow{+(q_\beta g)_*} C'_{n+r-1} \xrightarrow{d'_{n+r-1}} \cdots$$

By the long exact sequence of the homology of the cone, it suffices to prove that $+q_\beta g_*$ induces the map g_* in homology. But this follows from the commutativity of the diagram

$$\begin{tikzcd}
H_{n+r-1}(\Sigma^{n-1}A) \arrow[r, "g_*"] & H_{n+r-1}(X^{n-1}) \arrow[dl, "\simeq"'] \arrow[d, "+(q_\beta)_*"] \\
\ker d'_{n+r-1} \arrow[r, "\mathrm{inc}"'] & \bigoplus H_{n+r-1}(\Sigma^{n-1}A)
\end{tikzcd}$$

where the isomorphism $H_{n+r-1}(X^{n-1}) \to \ker d'_{n+r-1}$ is induced by the map $+(q_\beta)_*$. $\square$

Remark 5.2.2. The previous construction generalizes the classical one for cellular homology of CW-complexes. Note that the S^0-cellular chain complex of X is the standard cellular chain complex.

Remark 5.2.3. Note that $C_{n+r} = \bigoplus_{n\text{-cells}} H_r(A) = H_{n+r}(X^n/X^{n-1})$. As in the classical cellular setting, we could have obtained a chain complex as above in the following way. Consider the following commutative diagram made up of pieces of the long exact sequences associated to the topological pairs (X^k, X^{k-1}), $k \in \mathbb{N}$.

$$\begin{tikzcd}
0 \arrow[dr] & & H_{n+r-1}(X^n) & & & & \cdots \\
& H_{n+r-1}(X^{n-1}) \arrow[ur] \arrow[dr, "q_*"] & & & & H_{n+r-3}(X^{n-3}) \arrow[ur] \arrow[dr, "q_*"] & \\
H_{n+r}(X^n/X^{n-1}, *) \arrow[ur, "\partial"] \arrow[rr] & & H_{n+r}(X^{n-1}/X^{n-2}, *) \arrow[rr] \arrow[dr, "\partial"] & & H_{n+r}(X^{n-2}/X^{n-3}, *) \arrow[rr] \arrow[ur, "\partial"] & & \cdots \\
& & & H_{n+r-2}(X^{n-2}) \arrow[ur, "q_*"] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & \cdots & &
\end{tikzcd}$$

Define the differentials as $q_*\partial$. It can be proved that the chain complex that we obtain in this way has the same homology groups as X (cf. [8] pages 137-140). Note that, with this construction, the differentials are not explicitly computed.

The following corollary is an example of one possible application of theorem 5.2.1.

Corollary 5.2.4. *Let G and H be finite abelian groups with relatively prime orders. Let A and B be CW-complexes with homology concentrated in certain degrees n and m respectively, and with $H_n(A) = G$ and $H_m(B) = H$. Let X be a simply connected CW(A)-complex and let Y be a simply connected CW(B)-complex. Then X and Y have the same homotopy type if and only if both of them are contractible.*

Proof. By the hypothesis on the order of the elements, a quotient of $\bigoplus G$ different from 0 cannot be isomorphic to any quotient of $\bigoplus H$. It follows that if X and Y have the same homotopy type, then all their singular homology groups must vanish. $\square$

We investigate now the second case, i.e. when the homology groups $H_n(A)$ are free for all n. The following lemma plays a key role in the proof of 5.2.6. Since its proof is standard, we only sketch the main ideas.

Lemma 5.2.5. *Let (C_*, d_*) and (D_*, d'_*) be chain complexes of $\mathbb{Z}$-modules, with C_n free for every n. Given morphisms $f_n : H_n(C_*) \to H_n(D_*)$, $n \in \mathbb{N}$, there exists a morphism of chain complexes $g : (C_*, d_*) \to (D_*, d'_*)$ which induces the maps f_n in homology.*

Proof. Since C_0 is projective, there exists a map $g_0 : C_0 \to D_0$ inducing f_0 in homology. Suppose that we have already defined $g_0, \dots, g_{n-1}$ and they commute with the differentials and induce $f_0, \dots, f_{n-1}$ in homology. Since $\ker d_n$ is projective there exists a map β in a commutative diagram

$$\begin{tikzcd} \ker d_n \arrow[r, "q_n^C"] \arrow[d, "\beta"'] & \ker d_n/\operatorname{Im} d_{n+1} \arrow[d, "f_n"] \\ \ker d'_n \arrow[r, "q_n^D"] & \ker d'_n/\operatorname{Im} d'_{n+1} \end{tikzcd}$$

Note that $C_n \simeq \ker d_n \oplus \operatorname{Im} d_n$. We define $g_n = \beta$ in $\ker d_n$. Since $\operatorname{Im} d_n$ is projective, we can define g_n in $\operatorname{Im} d_n$ such that $g_{n-1}(y) = d'_n g_n(y)$ for all $y \in \operatorname{Im} d_n$. It is easy to check that $d'_n g_n = g_{n-1} d_n$ and that g_n induces the map f_n. □

Theorem 5.2.6. *Let A be a CW-complex with free homology groups and let X be a finite dimensional CW(A)-complex. Then there exists a chain complex of $\mathbb{Z}$-modules (C_*, d) whose homology is the singular homology of X, where*

$$C_n = \bigoplus_r H_{n-r}(A)^{\#r-cells}.$$

Proof. We proceed by induction in the dimension of X. If X has dimension zero, the result is trivial. If X has dimension one, the result follows from remark 5.1.1. Suppose that the proposition is true for X' and that X is obtained from X' by attaching A-n-cells. For simplicity, we will suppose that only one A-n-cell is attached, and we call g its attaching map. We denote by $H_*(\Sigma^{n-1}A)$ and $H_*(X')$ the chain complexes of the homology of $\Sigma^{n-1}A$ and X' respectively with all differentials equal to zero, and by $C(X')$ the chain complex of X' of the inductive step. By remark 5.1.1, the homology of X can be computed as the homology of the chain complex

$$\mathrm{C}g_* \qquad H_*(\Sigma^{n-1}A) \xrightarrow{g_*} H_*(X').$$

By lemma 5.2.5, there exists a morphism $\varphi : H_*(\Sigma^{n-1}A) \to C(X')$ inducing g_* in homology. It is easy to prove that the homology of $C\varphi$ coincides with the homology of $\mathrm{C}g_*$ which is the homology of X. □

Example 5.2.7. Let A be a CW-complex such that $H_r(A) = \mathbb{Z}$ for $r = 1, 4$ and 0 otherwise. Let X be a CW(A)-complex having n A-0-cells and m A-2-cells. Note that all the maps in the chain complex of the previous theorem are 0 and hence

$$H_r(X) = \begin{cases} \mathbb{Z}^n & \text{for } r = 1, 4 \\ \mathbb{Z}^m & \text{for } r = 3, 6 \\ 0 & \text{otherwise} \end{cases}$$

We can generalize this situation in the following way.

Let A be a CW-complex such that $H_r(A) = \mathbb{Z}$ for $r = 1, 4$ and 0 otherwise. Let X be a CW(A)-complex satisfying the following condition: 'For all $r \in \mathbb{N}_0$, if $\#\{A\text{-}r\text{-cells of } X\} \neq$

0 then $\#\{A\text{-}(r+1)\text{-cells of } X\} = 0$ and $\#\{A\text{-}(r+4)\text{-cells of } X\} = 0$'. Then all the maps in the chain complex of the previous theorem are 0 and therefore

$$H_n(X) = (\bigoplus_{A\text{-}(n-1)\text{-cells}} \mathbb{Z}) \bigoplus (\bigoplus_{A\text{-}(n-4)\text{-cells}} \mathbb{Z})$$

Important example 5.2.8. This example shows that theorem 5.2.6 may not hold if the hypothesis are not satisfied. Concretely, for the core $A = \mathbb{D}^2_4 \vee \Sigma\mathbb{D}^2_4$ (see page 133) we exhibit a CW(A)-complex X whose homology cannot be computed with a chain complex as in 5.2.6. Note that the homology of A is not concentrated in any degree and that its homology groups are not free.

The space X will consist of 3 A-cells, one of each dimension 0, 1 and 2. It will be also a CW-complex because the attaching maps will be cellular maps. The attaching maps are defined as follows.

For each $n \in \mathbb{Z}$, let $g'_n : D^2 \subseteq \mathbb{C} \to D^2$ be the map $g'_n(z) = z^n$. The map g'_n induces a well defined cellular map $g_n : \mathbb{D}^2_4 \to \mathbb{D}^2_4$. We also denote $g'_n = g'_n|_{S^1} : S^1 \to S^1$.

Let X^1 be the CW(A)-complex of dimension one defined by attaching an A-1-cell to A by the map $* \vee \Sigma g_2$. We obtain X by attaching an A-2-cell to X^1 by the map $\beta \vee *$, where $\beta : \Sigma\mathbb{D}^2_4 \to X^1$ is the unique map induced by γ and δ in the following pushout

$$\begin{tikzcd}
S^2 \arrow[r, "\Sigma g'_4"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "push"] & S^2 \arrow[d, "\mathrm{in}_1"] \arrow[ddr, bend left, "\gamma"] & \\
D^3 \arrow[r, "\mathrm{in}_2"'] \arrow[drr, bend right, "\delta"'] & \Sigma\mathbb{D}^2_4 \arrow[dr, "\beta"] & \\
 & & X^1
\end{tikzcd}$$

The map γ is defined as the composition

$$S^2 \xrightarrow{\Sigma g'_{-2}} S^2 \xrightarrow{\mathrm{in}_1} \Sigma\mathbb{D}^2_4 \xrightarrow{\mathrm{in}_3} X^1$$

(where in_3 is the canonical inclusion in the pushout) and $\delta = (\delta_1 \vee \delta_2) \circ q$, where δ_1, δ_2 and q are defined as follows. The map $q : D^2 \to D^2 \vee D^2$ is the quotient map that collapses the equator to a point. The map δ_1 is the composition

$$D^3 \xrightarrow{\mathrm{C}\Sigma g'_{-1}} D^3 \xrightarrow{\mathrm{in}_2} \Sigma\mathbb{D}^2_4 \xrightarrow{\mathrm{in}_3} X^1$$

and the map δ_2 is the composition

$$D^3 \xrightarrow{\mathrm{C}\Sigma g'_{-2}} D^3 \xrightarrow{\mathrm{Cin}_1} \mathrm{C}\Sigma\mathbb{D}^2_4 \xrightarrow{\mathrm{in}_4} X^1$$

The map in_4 is the canonical map induced in the pushout

$$\begin{tikzcd}
A = \mathbb{D}^2_4 \vee \Sigma\mathbb{D}^2_4 \arrow[r, "* \vee \Sigma g_2"] \arrow[d] \arrow[dr, phantom, "push"] & A = \mathbb{D}^2_4 \vee \Sigma\mathbb{D}^2_4 \arrow[d, "\mathrm{in}'_3 \vee \mathrm{in}_3"] \\
\mathrm{C}A = \mathrm{C}\mathbb{D}^2_4 \vee \mathrm{C}\Sigma\mathbb{D}^2_4 \arrow[r, "\mathrm{in}'_4 \vee \mathrm{in}_4"'] & X^1
\end{tikzcd}$$

Let $\nu : S^2 \to S^2 \vee S^2$ be the suspension of the quotient map $S^1 \to S^1/S^0 \simeq S^1 \vee S^1$. We obtain that

$$\begin{aligned}
\delta\mathrm{inc} &= (\delta_1 \vee \delta_2)q\mathrm{inc} = ((\delta_1\mathrm{inc}) \vee (\delta_2\mathrm{inc})) \circ \nu = (\mathrm{in}_3\mathrm{in}_2\mathrm{inc}\Sigma g'_{-1} \vee \mathrm{in}_4\mathrm{in}_1\Sigma g'_{-2}) \circ \nu = \\
&= (\mathrm{in}_3\mathrm{in}_1\Sigma g'_4\Sigma g'_{-1} \vee \mathrm{in}_3\Sigma g_2\mathrm{in}_1\Sigma g'_{-2}) \circ \nu = (\mathrm{in}_3\mathrm{in}_1\Sigma g'_{-4} \vee \mathrm{in}_3\mathrm{in}_1\Sigma g'_2\Sigma g'_{-2}) \circ \nu = \\
&= (\mathrm{in}_3\mathrm{in}_1\Sigma g'_{-4} \vee \mathrm{in}_3\mathrm{in}_1\Sigma g'_{-4}) \circ \nu = \mathrm{in}_3\mathrm{in}_1\Sigma g'_{-8} = \mathrm{in}_3\mathrm{in}_1\Sigma g'_{-2}\Sigma g'_4 = \gamma\Sigma g'_4.
\end{aligned}$$

Hence, $\delta\mathrm{inc} = \gamma\Sigma g'_4$.

Since the attaching maps are cellular, it follows that X is a CW-complex. We will show that $H_3(X) = \mathbb{Z}_8 \oplus \mathbb{Z}_4$. Hence, its homology cannot be computed with a chain complex as in 5.2.6 because $H_3(X)$ has an element of order 8.

Note that X, as a standard CW-complex, has 1 0-cell, 1 1-cell, 3 2-cells, 4 3-cells, 3 4-cells and 1 5-cell. Moreover, by construction, the rightmost part of its cellullar chain complex is the following:

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 4 & 0 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & -2 \\ 0 & 4 & 0 & 0 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 4 & 0 & 0 \end{pmatrix}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

It follows that $H_3(X) = \mathbb{Z}_8 \oplus \mathbb{Z}_4$ with an element of order 8 being the class of $(0, 0, 1, 1)$.

The key fact about this example is the following: a 3-dimensional cell belonging to the A-2-cell is attached onto a 2-dimensional cell which belongs to an A-0-cell. This attaching map must be taken into account when computing cellular homology groups, but will not be considered at all in an A-cellular chain complex as above. This is the reason why an A-cellular chain complex might not work in the general case.

To sum up, while for cellular homology groups the way that an n-cell is attached to an $(n-2)$-cell is irrelevant, in CW(A)-complexes this is not so, because A-cells might not be 'dimensionally homogeneous'.

Important remark 5.2.9. Note that X is a generalized CW$(\mathbb{D}^4_2)$-complex which does not have the homotopy type of a CW$(\mathbb{D}^4_2)$-complex. Indeed, if there existed a CW$(\mathbb{D}^4_2)$-complex Z, homotopy equivalent to X, then by theorem 5.2.1, $\mathbb{Z}_8 = H_3(X) = H_3(Z)$ would be a subquotient of $\bigoplus \mathbb{Z}_4$, which is impossible.

5.3 A-Euler characteristic and multiplicative characteristic

Let X be a pointed finite CW-complex. Recall that the reduced Euler characteristic of X is defined by

$$\chi(X) = \sum_{j \geq 0} (-1)^j \alpha_j$$

where α_j is the number of j-cells and where the base point does not count as a 0-cell. In this way the reduced Euler characteristic differs in 1 from the standard (unreduced) one.

Definition 5.3.1. Let A be a CW-complex and let X be a CW(A)-complex with a finite number of A-cells. We define the *A-Euler characteristic of* X by

$$\chi_A(X) = \sum_{j \geq 0} (-1)^j \alpha_j^A$$

where α_j^A is the number of A-j-cells of X.

Note that if $A = S^0$ then the A-Euler characteristic of X is the reduced Euler characteristic in the usual sense. Also, if $A = S^n$ then $\chi_A(X) = (-1)^n \chi(X)$. Recall that a CW($S^n$)-complex is a CW-complex with no cells of dimension less than n, apart from the base point.

The A-Euler characteristic gives useful information about the space. For example, proposition 5.3.2 will show that if the core A is a finite CW-complex and X is a finite CW(A)-complex then $\chi(X)$ can be computed from $\chi(A)$ and $\chi_A(X)$. Note that $\chi(X)$ is well defined since X has the homotopy type of a finite CW-complex. When $\chi(A) \neq 0$, the A-Euler characteristic is a homotopical invariant. In case $\chi(A) = 0$, it might not be invariant by homotopy equivalences or even homeomorphisms, as the following example shows.

Take the core A as D^1 (with 1 as base point). The disk D^2 is homeomorphic to $\mathrm{C}A$ and ΣA. We know that $\mathrm{C}A$ is obtained from A by attaching an A-1-cell, hence $\chi_A(\mathrm{C}A) = 0$. On the other hand, ΣA is obtained from $*$ by attaching a A-1-cell, so $\chi_A(\Sigma A) = -1$. Note that there are A-cellular approximations to the identity map of D^2 between these two different A-cellular structures, and that the homology of D^2 can be computed from the A-cellular complex by 5.2.1. But in this case the A-Euler characteristic cannot be computed from the A-cellular complex since, in contrast to the classical situation where the cellular complex has a copy of $\mathbb{Z}$ for each cell, the A-cellular complex has a trivial group for each A-cell of D^2.

Nevertheless the A-Euler characteristic gives us very useful information about the space. The following proposition shows that if the core A is a finite CW-complex and X is a finite CW(A)-complex then $\chi(X)$ can be computed from $\chi(A)$ and $\chi_A(X)$. Note that $\chi(X)$ is well defined since X has the homotopy type of a finite CW-complex.

Proposition 5.3.2. *Let A be a finite CW-complex and let X be a finite CW(A)-complex. Then $\chi(X) = \chi_A(X)\chi(A)$.*

Proof. The proposition follows from the fact that, for all $n \in \mathbb{N}_0$ the relative CW-complexes $(\mathrm{C}\Sigma^n A, \Sigma^n A)$ have exactly the same cells as A but shifted in dimension. Note also that X has the homotopy type of a CW-complex X' which is obtained by approximating the attaching maps of X by cellular maps. $\square$

Corollary 5.3.3. *If $\chi(A) \neq 0$ and $\chi_A(X) \neq 0$ then X is not contractible.*

Note that in case $A = S^n$ the corollary does not say anything new. But, for example, if A is a torus ($\chi(A) = -1$) and X is a CW(A)-complex with an odd number of cells, then X is not contractible. Also, in this case, if X has any number of cells but only in even dimensions, it cannot be contractible.

We study now another interesting case: when the homology of A is a *finite graded group.*

We say that graded group $(G_n)_{n\in\mathbb{N}_0}$ is *finite* if only a finite number of groups are non trivial and all of them are finite. In a similar way we say that a chain complex of abelian groups is *finite* if the underlying graded group is finite.

Definition 5.3.4. Let $\mathcal{G} = (G_n)_{n\in\mathbb{N}_0}$ be a finite graded group. We define the *multiplicative Euler characteristic of* $\mathcal{G}$ as

$$\chi_m(\mathcal{G}) = \prod_{n\geq 0} \#(G_n)^{(-1)^n}$$

Let $C = (C_*, d_*)$ be a chain complex of abelian groups whose underlying graded group is finite. Since $H_n(C) = \ker d_n/\operatorname{Im} d_{n+1}$ and $C_n/\ker d_n = \operatorname{Im} d_n$, then

$$\#H_n(C) = \#\ker d_n/\#\operatorname{Im} d_{n+1} \quad \text{and} \quad \#C_n = \#\ker d_n.\#\operatorname{Im} d_n.$$

It follows that

$$\prod_{n\geq 0} \#(H_n(C))^{(-1)^n} = \prod_{n \text{ even}} \#\ker d_n.\#\operatorname{Im} d_n / \prod_{n \text{ odd}} \#\ker d_n.\#\operatorname{Im} d_n = \prod_{n\geq 0} \#(C_n)^{(-1)^n}$$

Therefore, the multiplicative Euler characteristic of C coincides with the multiplicative Euler characteristic of the graded group $H_*(C)$. In particular, the multiplicative Euler characteristic is invariant by quasi isomorphisms.

Example 5.3.5. Let (C_*, d_*) be a chain complex with $C_n = \bigoplus_{i\in I_n} \mathbb{Z}_4$ for all n (where I_n is any index set). Let (D_*, d'_*) be a chain complex with $H_k(D) = \mathbb{Z}_2$ for some k and $H_r(D) = 0$ for $r \neq k$. Then C and D are not quasi isomorphic, because $\chi_m(C) = 4^m$ for some $m \in \mathbb{Z}$, while $\chi_m(D) = 2$ or $\chi_m(D) = \frac{1}{2}$.

We may also ask whether the converse is true. Namely, given finite abelian groups $G_1, \dots, G_k$ with $\prod_{n\geq 0} \#(G_n)^{(-1)^n} = 4^m$ for some m, can we find a chain complex of abelian groups (C_*, d_*) with $C_n = \bigoplus_{i\in I_n} \mathbb{Z}_4$ for all n such that $H_n(C) = G_n$ for $n = 1, \dots, k$ and $H_n(C) = 0$ in other case? For example, given $m \in \mathbb{N}$, can we construct a chain complex (C_*, d_*) with $C_n = \bigoplus_{j\in J_n} \mathbb{Z}_4$ for all n such that $H_k(C) = \mathbb{Z}_{4^m}$ for some k and $H_r(C) = 0$ for $r \neq k$?

The answer to this question is negative. For instance, if $H_k(C) = \mathbb{Z}_{16}$, then it contains an element of order 16 which cannot be obtained by taking a quotient of a subspace of $\bigoplus_{j\in J_n} \mathbb{Z}_4$.

Remark 5.3.6. Let $C = (C_n)_{n\in\mathbb{N}_0}$, $D = (D_n)_{n\in\mathbb{N}_0}$ and $E = (E_n)_{n\in\mathbb{N}_0}$ be finite graded groups. Suppose that for each $n \in \mathbb{N}_0$ there exists a short exact sequence

$$0 \to C_n \to D_n \to E_n \to 0.$$

Then, $\#D_n = \#C_n.\#E_n$. Hence, $\chi_m(D) = \chi_m(C)\chi_m(E)$.

The same holds in case there is an exact sequence

$$\dots \to E_{n+1} \to C_n \to D_n \to E_n \to C_{n-1} \to \dots$$

for if we call this complex B, we have $\chi_m(B) = \chi_m(H_*(B)) = 1$ and clearly $\chi_m(B) = \chi_m(C)\chi_m(E)/\chi_m(D)$.

Definition 5.3.7. Let X be a topological space with finite homology. We define the *multiplicative Euler characteristic of* X as the multiplicative Euler characteristic of $H_*(X)$.

Theorem 5.3.8. *Let A be a CW-complex with finite homology and let X be a finite CW(A)-complex. Then*

$$\chi_m(X) = \prod_{n\geq 0} \chi_m(A)^{(-1)^n \# A\text{-}n\text{-}cells} = \chi_m(A)^{\chi_A(X)}$$

Proof. We proceed by induction in the number of cells of X. If X has only one cell the theorem trivially holds. Suppose the result is true for X' and suppose X is obtained from X' by attaching an A-r-cell. There exists a long exact sequence

$$\cdots \longrightarrow H_n(\Sigma^{r-1}A, *) \longrightarrow H_n(X', *) \longrightarrow H_n(X, *) \longrightarrow H_{n-1}(\Sigma^{r-1}A, *) \longrightarrow \cdots$$

Then, by 5.3.6,

$$\begin{aligned} \chi_m(X') &= \chi_m(H_*(X')) = \chi_m(H_*(\Sigma^{r-1}A))\chi_m(H_*(X)) = \\ &= \chi_m(H_*(A))^{(-1)^{r-1}}\chi_m(H_*(X)) = \chi_m(A)^{(-1)^{r-1}}\chi_m(X) \end{aligned}$$

So, $\chi_m(X) = \chi_m(X')\chi_m(A)^{(-1)^r}$. □

Example 5.3.9. Let A be a CW-complex with $H_1(A) = \mathbb{Z}_4$ and $H_r(A) = 0$ for $r \neq 1$. Let X be a topological space with $H_k(A) = \mathbb{Z}_2$ for some k and $H_r(A) = 0$ for $r \neq k$. Then X does not have the homotopy type of a CW(A)-complex.

The next result follows immediately from 5.3.8.

Proposition 5.3.10. *Let A and B be CW-complexes with finite homology. Let X be a topological space with finite homology such that $\chi_m(X) \neq 1$. Suppose, in addition, that X can be given both CW(A) and CW(B) structures. Then there exist $k, l \in \mathbb{Z} - \{0\}$ such that $\chi_m(A)^k = \chi_m(B)^l$.*

Example 5.3.11 (Moore spaces)**.** Fix a core A. Some questions that arise naturally are the following. For which abelian groups G and $n \in \mathbb{N}$ does there exist a CW(A)-complex X such that $H_n(X) = G$ and $H_r(X) = 0$ if $r \neq n$? Or more generally, for which sequences of abelian groups $(G_n)_{n\in\mathbb{N}_0}$ does there exist a CW(A)-complex X such that $H_n(X) = G_n$ for all n?

For example, if the core A is a simply-connected CW-complex with $H_r(A) = \mathbb{Z}$ for $r = n$ and $H_r(A) = 0$ in other case, then A is homotopy equivalent to S^n. We know that for any abelian group G and for any $k \geq n$ there exists a CW-complex Z such that

$H_k(Z) = G$ and $H_r(Z) = 0$ if $r \neq k$. Hence, by 3.3.1, there exists a CW(A)-complex X such that X has the same homology groups as Z.

Therefore, in this particular case, for any sequence of abelian groups $(G_j)_{j \geq n}$ there exists a CW(A)-complex X such that $H_j(X) = G_j$ for all $j \geq n$.

If A is a CW-complex with finite homology, the results above provide necessary conditions for the required CW(A)-complex X to exist. For instance, 5.3.8 settles an easy-to-check necessary condition, as example 5.3.9 shows. In the case $A = \mathbb{D}^2_4$ (see page 133), we cannot construct a CW(A)-complex X such that $H_n(X) = \mathbb{Z}_5$ for some $n \in \mathbb{N}$ since, by 5.2.1, $H_n(X)$ must be a quotient of a subgroup of $\bigoplus \mathbb{Z}_4$.

We will continue studying these questions in chapter 7, where we will give some very interesting partial anwers to them.

Chapter 6

Applications of spectral sequences to CW(A)-complexes

One basic and fundamental idea in topology is to study homotopical and homological properties of a topological space by decomposing it into smaller parts. Since a CW(A)-complex is built up out of A-cells, one expects that its homotopical properties will depend heavily on the homotopy type of A. Our aim in this chapter is to develop methods to compute homotopy and homology groups of a CW(A)-complex X from those of A and the CW(A)-structure of X generalizing those in the previous chapter. It is evident that the skeletal filtration plays an important role and it seems quite natural that spectral sequences are the right tool to work with. For example, from the skeletal filtration of a CW(A)-complex X we will construct a spectral sequence which converges to the singular homology groups of X.

Moreover, as one might expect, the homotopy groups of a CW(A)-complex X are strongly related to those of A, although in general explicit computation seems to be hard work. However, we obtain results which show how the homotopy groups of X depend on those of A.

In the first section, given a CW-complex A we define a reduced homology theory on the category of CW-complexes, called A-homology, which coincides with classical singular homology in case $A = S^0$. One of the most interesting results is a generalization of the Hurewicz theorem (6.1.5), which gives a relationship between A-homology groups and A-homotopy groups.

In section 2 we study homotopy and homology of CW-complexes by means of spectral sequences and Serre classes obtaining many interesting results. Moreover, we introduce a generalization of Serre classes which is suitable for working with CW(A)-complexes.

Finally, in the last section we derive some applications to real projective spaces.

6.1 A-homology and A-homotopy

In this section we will define an A-*shaped* homology theory, which we call A-homology. This is a reduced homology theory which not only generalizes the classical singular homology theory, but also satisfies nice properties such as a Hurewicz-type theorem (6.1.5).

Definition 6.1.1. Let $p:(E,e_0)\to(B,b_0)$ be a continuous map and let $F=p^{-1}(b_0)$. We say that p is an *A-quasi-fibration* if it induces isomorphisms $p_*:\pi_i^A(E,F,e_0)\to\pi_i^A(B,b_0)$ for all $i\in\mathbb{N}$, where $\pi_i^A(E,F,e_0)=[(\mathrm{C}\Sigma^{i-1}A,\Sigma^{i-1}A,a_0),(E,F,e_0)]$.

Let c_{e_0} and c_{b_0} denote the constant loops at e_0 and b_0 respectively. Recall that, for any space A, $\pi_i^A(B,b_0)=\pi_{i-1}^A(\Omega B,c_{b_0})$ and $\pi_i^A(E,F,e_0)=\pi_{i-1}^A(P(E,e_0,F),c_{e_0})$. Since a weak equivalence is also an A-weak equivalence if A is a CW-complex, we deduce the following.

Lemma 6.1.2. *If p is a quasifibration and A is a CW-complex, then p is an A-quasi-fibration.*

We define the A-homology groups inspired by the Dold-Thom theorem.

Definition 6.1.3. Let A be a CW-complex and let X be a topological space. For $n\in\mathbb{N}_0$ we define the *n-th A-homology group of X* as

$$H_n^A(X)=\pi_n^A(SP(X))$$

where $SP(X)$ denotes the infinite symmetric product of X.

Theorem 6.1.4. *The functor $H_*^A(-)$ defines a reduced homology theory on the category of (path-connected) CW-complexes.*

Proof. It is clear that $H_*^A(-)$ is a homotopy functor. If (X,B,x_0) is a pointed CW-pair, then by the Dold-Thom theorem, the quotient map $q:X\to X/B$ induces a quasifibration $\hat{q}:SP(X)\to SP(X/B)$ whose fiber is homotopy equivalent to $SP(B)$. Then there is a long exact sequence

$$\cdots\longrightarrow\pi_n^A(SP(B))\longrightarrow\pi_n^A(SP(X))\longrightarrow\pi_n^A(SP(X/B))\longrightarrow\pi_{n-1}^A(SP(B))\longrightarrow\cdots$$

It remains to show that there is a natural isomorphism $H_n^A(X)\simeq H_{n+1}^A(\Sigma X)$ and that $H_n^A(X)$ are abelian groups for $n=0,1$. The natural isomorphism follows from the long exact sequence above applied to the CW-pair $(\mathrm{C}X,X)$. Note that $H_n^A(\mathrm{C}X)=0$ since $\mathrm{C}X$ is contractible and H_n^A is a homotopy functor. The second part follows immediately, since $H_0^A(X)\simeq H_1^A(\Sigma X)\simeq H_2^A(\Sigma^2X)$. The group structure on $H_0^A(X)$ is induced from the one on $H_1^A(X)$ by the corresponding natural isomorphism. □

Theorem 6.1.5. *Let A be a path-connected CW-complex of dimension $k\geq 1$ and let X be an n-connected topological space (with $n\geq k$). Then $H_r^A(X)=0$ for $r\leq n-k$ and $\pi_{n-k+1}^A(X)\simeq H_{n-k+1}^A(X)$.*

Proof. By Hurewicz, $H_r(X)=0$ for $r\leq n$ and $H_{n+1}(X)\simeq\pi_{n+1}(X)$. Hence $SP(X)$ is n-connected. Since $\dim A=k$, it follows that $SP(X)$ is A-$(n-k)$-connected. Thus, $H_r^A(X)=0$ for $r\leq n-k$. Also,

$$\begin{aligned}\pi_{n-k+1}^A(X) &= [\Sigma^{n-k+1}A,X]\simeq H^{n+1}(\Sigma^{n-k+1}A,\pi_{n+1}(X))\simeq\\ &\simeq H^{n+1}(\Sigma^{n-k+1}A,H_{n+1}(X))\simeq H^{n+1}(\Sigma^{n-k+1}A,\pi_{n+1}(SP(X)))\simeq\\ &\simeq [\Sigma^{n-k+1}A,SP(X)]=\pi_{n-k+1}^A(SP(X))=H_{n-k+1}^A(X)\end{aligned}$$

where the first and fourth isomorphisms hold by 2.5.6. □

Federer's spectral sequence provides a first method of computation of A-homology groups. Given CW-complexes A and X with A finite and $H_1(X) = 0$, the associated Federer spectral sequence $\{E^a_{p,q}\}$ converges to the A-homotopy groups of $SP(X)$ (note that $SP(X)$ is simply-connected). In this case we have $E^2_{p,q} = H^{-p}(A, \pi_q(SP(X))) = H^{-p}(A, H_q(X))$ if $p + q \geq 1$ and $p \leq -1$.

We exhibit now some examples.

Example 6.1.6. If A is a finite CW-complex and X is a Moore space of type (G, n) then $SP(X)$ is an Eilenberg-MacLane space of the same type. Note that $SP(X)$ is abelian. Hence, by the Federer spectral sequence

$$H^A_r(X) = \pi^A_r(SP(X)) = H^{n-r}(A, \pi_n(SP(X))) = H^{n-r}(A, G) \qquad \text{for } r \geq 1.$$

In particular, $H^A_r(S^n) = H^{n-r}(A, \mathbb{Z})$.

We also deduce that if X is a Moore space of type (G, n) and A is $(n-1)$-connected, then $H^A_r(X) = 0$ for all $r \geq 1$.

Example 6.1.7. Let A be a Moore space of type (G, m) (with G finitely generated) and let X be a path-connected abelian topological space. As in example 2.5.5, for $n \geq 1$, there are short exact sequences of abelian groups

$$0 \longrightarrow \operatorname{Ext}(G, H_{n+m+1}(X)) \longrightarrow H^A_n(X) \longrightarrow \operatorname{Hom}(G, H_{n+m}(X)) \longrightarrow 0$$

As a consequence, if G is a finite group of exponent r, $\alpha^{2r} = 0$ for every $\alpha \in H^A_n(X)$.

Using 6.1.6 we will show now an explicit formula to compute A-homology groups.

Proposition 6.1.8. *Let A be a finite CW-complex and let X be a connected CW-complex. Then for every $n \in \mathbb{N}_0$, $H^A_n(X) = \bigoplus_{j \in \mathbb{N}} H^{j-n}(A, H_j(X))$.*

Proof. By corollary 4K.7 of [8], $SP(X)$ has the weak homotopy type of $\prod_{n \in \mathbb{N}} K(H_n(X), n)$. Also, since A is a CW-complex, a weak equivalence is also an A-weak equivalence. Hence,

$$\begin{aligned} H^A_n(X) &= \pi^A_n(SP(X)) = \pi^A_n(\prod_{j \in \mathbb{N}} K(H_j(X), j)) = \prod_{j \in \mathbb{N}} \pi^A_n(K(H_j(X), j)) = \\ &= \prod_{j \in \mathbb{N}} H^{j-n}(A, H_j(X)) = \bigoplus_{j \in \mathbb{N}} H^{j-n}(A, H_j(X)) \end{aligned}$$

where the first equality of the second line follows from 6.1.6 since for every group G and $m \in \mathbb{N}$, $SP(M(G, m))$ is a $K(G, m)$. $\square$

Now we show that, in case A is compact, H^A_* satisfies the wedge axiom. This can be proved in two different ways: using the definition of A-homotopy groups or using the above formula. We choose the first one.

Proposition 6.1.9. *Let A be a finite CW-complex, and let $\{X_i\}_{i\in I}$ be a collection of CW-complexes. Then*

$$H_n^A\left(\bigvee_{i\in I} X_i\right) = \bigoplus_{i\in I} H_n^A(X_i).$$

Proof. It is known that $SP(\bigvee_{i\in I} X_i) = \prod_{i\in I}^{w} SP(X_i)$ with the weak product topology, i.e. the colimit of the products of finitely many factors. Since A is compact, $\pi_n^A(\prod_{i\in I}^{w} SP(X_i)) = \bigoplus_{i\in I} \pi_n^A(SP(X_i))$ and the result follows. □

Henceforward, $\mathscr{C}$ will denote a Serre class of abelian groups unless specified otherwise.

Proposition 6.1.10. *Let A be a finite CW-complex and let $k \in \mathbb{N}$. Let X be a topological space such that $\pi_n(X) \in \mathscr{C}$ for all $n \geq k$. Then $\pi_n^A(X) \in \mathscr{C}$ for all $n \geq k$.*

Proof. We proceed by induction in the number of cells of A. If A has only one cell, the result trivially holds. Suppose A is obtained from A' attaching an m-cell. The cofibration $A' \hookrightarrow A$ induces a long exact sequence

$$\cdots \longrightarrow \pi_n^{S^m}(X) \longrightarrow \pi_n^A(X) \longrightarrow \pi_n^{A'}(X) \longrightarrow \pi_{n-1}^{S^m}(X) \longrightarrow \cdots$$

By hypothesis, $\pi_i^{S^m}(X) \in \mathscr{C}$ and $\pi_i^{A'}(X) \in \mathscr{C}$ for $i \geq k$. Then $\pi_i^A(X) \in \mathscr{C}$ for $i \geq k$. □

6.2 Homology and homotopy of CW(A)-complexes

In this section we present a variety of results which give information about the homotopy groups of a CW(A)-complex showing that they depend strongly on the homology and homotopy groups of A.

Let A be a CW-complex. Let X be a (generalized) CW(A)-complex and let $B \subseteq X$ be a (generalized) A-subcomplex of X. Replacing (X, B) by a homotopy equivalent CW-pair (X', B') we obtain that $H_n(X, B) \simeq H_n(X', B') \simeq H_n(X'/B') \simeq H_n(X/B)$, where the last isomorphism holds since a homotopy equivalence of pairs $\phi : (X, B) \to (X', B')$ induces a homotopy equivalence $\overline{\phi} : X/B \to X'/B'$ by 1.1.20.

Proposition 6.2.1. *Let A be a $\mathscr{C}$-acyclic CW-complex and let B be a topological space. Suppose X is obtained from B by attaching a finite number of A-cells (in one step). Then the inclusion $i : B \to X$ induces $\mathscr{C}$-isomorphisms $i_* : H_n(B) \to H_n(X)$ for all n.*

Proof. Let J be an index set for the A-cells attached, and for each $j \in J$ let m_j denote the dimension of the j-th A-cell. Consider the long exact sequence

$$\cdots \xrightarrow{\partial} H_n(B) \xrightarrow{i_*} H_n(X) \longrightarrow H_n(X, B) \xrightarrow{\partial} H_{n-1}(B) \longrightarrow \cdots$$

Note that $\ker(i_*) \in \mathscr{C}$ because it is isomorphic to a quotient of $H_{n+1}(X, B)$. On the other hand, $\ker \partial = \operatorname{coker}(i_*) \in \mathscr{C}$ because it is a subgroup of $H_n(X, B) = H_n(X/B) = H_n(\bigvee_{j\in J} \Sigma^{m_j} A) = \bigoplus_{j\in J} H_{n-m_j}(A)$. Then i_* is a $\mathscr{C}$-isomorphism. □

The following proposition gives a relation between the homology groups of a CW(A)-complex X and those of A. Of course, we need X to have a finite number of A-cells.

Proposition 6.2.2. *Let A be a $\mathscr{C}$-acyclic CW-complex and let X be a finite generalized CW(A)-complex. Then X is also $\mathscr{C}$-acyclic.*

Proof. We proceed by induction in the number of A-cells of X. If X has only one A-cell the result follows. For the inductive step, suppose X is obtained from B by attaching only one A-cell (say of dimension m). Consider the short exact sequences

$$0 \longrightarrow \operatorname{coker} \partial \longrightarrow H_n(X) \longrightarrow \ker \partial \longrightarrow 0$$

associated to the long exact sequence

$$\cdots \xrightarrow{\partial} H_n(B) \longrightarrow H_n(X) \longrightarrow H_n(X,B) \xrightarrow{\partial} H_{n-1}(B) \longrightarrow \cdots$$

Note that $H_n(X,B) = H_n(X/B) = H_n(\Sigma^m A) = H_{n-m}(A)$. Also, $\operatorname{coker} \partial \in \mathscr{C}$ because it is a quotient of $H_n(B)$ and $\ker \partial \in \mathscr{C}$ because it is a subgroup of $H_n(X,B) = H_{n-m}(A)$. Then $H_n(X) \in \mathscr{C}$. □

Using the generalized Hurewicz theorem (2.3.13) and the previous proposition we obtain the following.

Corollary 6.2.3. *Let A be a $\mathscr{C}$-acyclic CW-complex and let X be a finite generalized CW(A)-complex. Suppose, in addition, that X is simply connected and that $\mathscr{C}$ is an acyclic ring of abelian groups. Then $\pi_n(X) \in \mathscr{C}$ for all $n \in \mathbb{N}$.*

We turn now our attention to A-homotopy groups. From 6.2.3 and 6.1.10 we deduce

Corollary 6.2.4. *Let A be a finite CW-complex and let X be a finite generalized CW(A)-complex. Suppose that A is $\mathscr{C}$-acyclic (with $\mathscr{C}$ an acyclic ring of abelian groups) and that X is simply connected. Then $\pi_n^A(X) \in \mathscr{C}$ for all $n \in \mathbb{N}$.*

We propose now a slight modification of Serre classes and rings of abelian groups to get rid of the finiteness hypothesis in the previous results.

Definition 6.2.5. Let $\mathscr{C}'$ be a nonempty class of abelian groups. We say that $\mathscr{C}'$ is a *special Serre class* if the following conditions are satisfied

(i) For any three-term exact sequence of abelian groups $A \to B \to C$ if $A, C \in \mathscr{C}'$ then $B \in \mathscr{C}'$.

(ii) For any collection of abelian groups $\{A_i\}_{i\in I}$ if $A_i \in \mathscr{C}'$ for all i, then $\bigoplus_{i\in I} A_i \in \mathscr{C}'$.

(iii) If $\{G_i\}_{i\in\Lambda}$ is a direct system of abelian groups, all of which belong to $\mathscr{C}'$, then $\operatorname{colim}_{i\in\Lambda} G_i \in \mathscr{C}'$

Note that a special Serre class is, in particular, a Serre class.

An interesting example for our purposes is the following. Let $\mathcal{P}$ be a set of prime numbers. Then, the class $\mathcal{T}_{\mathcal{P}}$ of torsion abelian groups whose elements have orders which are divisible only by primes in $\mathcal{P}$ is a special Serre class.

If A is a CW-complex and X is a generalized CW(A)-complex, then every compact subspace of X intersects only a finite number of interiors of A-cells (cf. 3.1.12). Hence, if X^l denotes the l-th layer of X, we have $\operatorname{colim}_l H_n(X^l) = H_n(X)$. From this we can deduce the following result which is an interesting variation of 6.2.2, 6.2.3 and 6.2.4.

Proposition 6.2.6. *Let $\mathscr{C}'$ be a special Serre class, let A be a $\mathscr{C}'$-acyclic CW-complex and let X be a generalized CW(A)-complex. Then:*

(a) X is $\mathscr{C}'$-acyclic.

(b) If, in addition, X is simply connected and $\mathscr{C}'$ is an acyclic ring of abelian groups, then $\pi_n(X) \in \mathscr{C}'$ for all $n \in \mathbb{N}$.

(c) If A is finite and X is simply connected and $\mathscr{C}'$ is an acyclic ring of abelian groups, then $\pi_n^A(X) \in \mathscr{C}'$ for all $n \in \mathbb{N}$.

The next result provides a more concrete description of the singular homology of a CW(A)-complex, in case that A is a finite dimensional CW-complex.

Given a CW(A)-complex X, for each $n \in \mathbb{N}_0$, let J_n be an index set for the A-n-cells of X. For $\alpha \in J_n$ and $\beta \in J_{n-1}$ let g_α and q_β be defined as in section 2.

Proposition 6.2.7. *Let A be a finite dimensional CW-complex and let X be a CW(A)-complex. Then there exists a spectral sequence $\{E^a_{p,q}\}_{p,q\in\mathbb{Z}}$ with $E^1_{p,q} = \bigoplus_{A-p-cells} H_q(A)$ which converges to $H_*(X)$.*

Moreover, the differentials $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ are given by $d^1_{p,q} = \bigoplus_{\substack{\alpha\in J_p \\ \beta\in J_{p-1}}} (q_\beta g_\alpha)_*$.

Proof. Let $E^1_{p,q} = H_{p+q}(X^p_A, X^{p-1}_A)$ and $A^1_{p,q} = H_{p+q}(X^p_A)$. From the long exact sequences in homology associated to the pairs (X^m_A, X^{m-1}_A), $m \in \mathbb{N}_0$, we obtain an exact couple (E^1, A^1, i, j, k), which induces a spectral sequence $\{E^a_{p,q}\}_{p,q\in\mathbb{Z}}$. Since A is finite dimensional, it follows that this spectral sequence converges to $H_*(X)$.

Moreover, the differential $d^1_{p,q}$ is given by the composition

$$H_{p+q}(X^p_A, X^{p-1}_A) \xrightarrow{\partial} H_{p+q-1}(X^{p-1}_A) \xrightarrow{j} H_{p+q-1}(X^{p-1}_A, X^{p-2}_A)$$

where these maps come from the long exact sequences mentioned above. Under the isomorphisms $H_{p+q}(X^p_A, X^{p-1}_A) \simeq \bigoplus_{A-p-\text{cells}} H_q(A)$ and $H_{p+q-1}(X^{p-1}_A, X^{p-2}_A) \simeq \bigoplus_{A-(p-1)-\text{cells}} H_q(A)$, the map ∂ corresponds to $\bigoplus_{\alpha\in J_p} (g_\alpha)_*$ and the map j corresponds to q_*, where $q : X^{p-1}_A \to X^{p-1}_A / X^{p-2}_A$ is the quotient map.

Hence, the result follows. □

Remark 6.2.8. The previous proposition generalizes 5.2.1.

The following theorem generalizes theorem 6.2.7 above and its proof is formally identical.

Theorem 6.2.9. *Let A and B be CW-complexes, with A finite and B such that $H_r(B) \neq 0$ only for a finite number of r's (this holds, for example, if B is finite dimensional). Let X be a CW(B)-complex. Then there exists a spectral sequence $\{E^a_{p,q}\}$ with $E^1_{p,q} = \bigoplus_{A-p-cells} H^A_{p+q}(\Sigma^p B)$ which converges to $H^A_*(X)$.*

It is well known that if a CW-complex does not have cells of a certain dimension j, then its j-th homology group vanishes. The following proposition heads towards that direction, giving, in several cases, a range of dimensions outside of which the A-homology groups are trivial.

Proposition 6.2.10. *Let A be an l-connected CW-complex of dimension k and let X be a topological space such that $SP(X)$ is abelian (this holds, for example, if $H_1(X) = 0$).*

(a) If X is an abelian CW-complex of dimension m, then $H^A_r(X) = 0$ for $r \geq m - l$.

(b) If X is an abelian CW(A)-complex of dimension m, then $H^A_r(X) = 0$ for $r \geq m + k - l$.

(c) If X is an abelian CW(A)-complex without cells of dimension less than m', then $H^A_r(X) = 0$ for $r \leq m' + l - k$.

The proof follows immediately from the Federer spectral sequence applied to the space $SP(X)$.

6.3 Examples on real projective spaces

We exhibit now some examples concerning real projective spaces.

It follows from 2.5.5 that if X is a path-connected abelian topological space, every element in $\pi^{\mathbb{P}^2}_n(X)$ ($n \geq 1$) has order 1, 2 or 4. This can be generalized to $\mathbb{P}^l$ (for any dimension l) in the following way. By 2.5.4 we know that there is a spectral sequence $\{E^a_{p,q}\}$ which converges to $\pi^{\mathbb{P}^l}_{p+q}(X)$ for $p + q \geq 1$. If l is even then, for $p + q \geq 1$ and $p \leq -1$, we get

$$E^2_{p,q} \simeq H^{-p}(\mathbb{P}^l; \pi_q(X)) = \begin{cases} \pi_q(X)/2\pi_q(X) & \text{if } p \text{ is even and } -p \leq l \\ \{\alpha \in \pi_q(X) / \operatorname{ord}(\alpha) = 1 \text{ or } 2\} & \text{if } p \text{ is odd and } -p \leq l \end{cases}$$

It follows that if $\beta \in \pi^{\mathbb{P}^l}_n(X)$ (and $n \geq 1$) then $\operatorname{ord}(\beta) | 2^l$.

It is worth mentioning that if the homotopy groups of X are finite and do not contain elements of order 2, then $H^{-p}(\mathbb{P}^l; \pi_q(X)) = 0$ for $p+q \geq 1$ and $p \leq -1$. Thus, $\pi^{\mathbb{P}^l}_n(X) = 0$ for $n \geq 1$.

On the other hand, if l is odd then, for $p + q \geq 1$ and $p \leq -1$, we get

$$E^2_{p,q} \simeq H^{-p}(\mathbb{P}^l; \pi_q(X)) = \begin{cases} \pi_q(X)/2\pi_q(X) & \text{if } p \text{ is even and } -p \leq l-1 \\ \{\alpha \in \pi_q(X) / \operatorname{ord}(\alpha) = 1 \text{ or } 2\} & \text{if } p \text{ is odd and } -p \leq l-1 \\ \pi_q(X) & \text{if } p = -l \end{cases}$$

It follows that, for $n \geq 1$, there exists a short exact sequence of groups

$$0 \longrightarrow \pi_{n+l}(X) \longrightarrow \pi_n^{\mathbb{P}^l}(X) \longrightarrow G \longrightarrow 0$$

where G is such that $\exp(G)|2^{l-1}$. Note that G turns out to be solvable if $n = 1$, and abelian for $n \geq 2$.

Now, we turn our attention to $\mathbb{P}^2$-homology. From example 6.1.6 we get

$$H_r^{\mathbb{P}^2}(\Sigma\mathbb{P}^2) = H^{2-r}(\mathbb{P}^2, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } r = 1 \\ 0 & \text{if } r \geq 2 \end{cases}$$

Also, from example 6.1.7 it follows that if X is a (path-connected) CW-complex, then every element in $H_n^{\mathbb{P}^2}(X)$ $(n \geq 1)$ has order 1, 2 or 4.

From 6.2.6 we deduce the following relationship with Eilenberg-MacLane spaces.

Example 6.3.1. There is no generalized CW($\mathbb{P}^2$)-complex X such that X is a $K(\mathbb{Z}, n)$ for some $n \geq 2$.

Example 6.3.2. For $n \geq 2$, there does not exist any CW($\mathbb{P}^2$)-complex which is a $K(\mathbb{Z}_4, n)$. Indeed, if X is a CW($\mathbb{P}^2$)-complex and a $K(\mathbb{Z}_4, n)$ then by Hurewicz, $H_n(X) = \mathbb{Z}_4$. But by 5.2.1 $H_n(X)$ must be a subquotient of $\bigoplus \mathbb{Z}_2$, which entails a contradiction.

Given $n \geq 2$, we construct now a generalized CW($\mathbb{P}^2$)-complex which is a $K(\mathbb{Z}_2, n)$. We need first the following lemma.

Lemma 6.3.3. *Let $f : S^1 \to X$ be a continuous map. Then $[f]^2 = 0$ in $\pi_1(X)$ if and only if f can be extended to $\mathbb{P}^2$.*

This lemma can be proved easily considering the pushout

$$\begin{tikzcd} S^1 \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "push"] & S^1 \arrow[d] \\ D^2 \arrow[r] & \mathbb{P}^2 \end{tikzcd}$$

where g is a map of degree 2. This result can also be deduced from the cofibration sequence $S^1 \hookrightarrow \mathbb{P}^2 \to S^2$. Clearly, this can be generalized for maps g of any degree.

Proposition 6.3.4. *Given $n \in \mathbb{N}$, $n \geq 2$, there exists a generalized CW($\mathbb{P}^2$)-complex which is an Eilenberg-MacLane space of type $(\mathbb{Z}_2, n)$.*

Proof. We start with the singleton and attach a $\mathbb{P}^2$-$(n-1)$-cell to obtain $\Sigma^{n-1}\mathbb{P}^2$. Clearly, $\pi_r(\Sigma^{n-1}\mathbb{P}^2) = 0$ for $r \leq n-1$ and $\pi_n(\Sigma^{n-1}\mathbb{P}^2) = \mathbb{Z}_2$. Moreover, by the generalized Hurewicz theorem we know that the groups $\pi_r(\Sigma^{n-1}\mathbb{P}^2)$ must be finite and of 2-torsion for all $r \in \mathbb{N}$. Then, there exists $l \in \mathbb{N}$ such that $\exp(\pi_{n+1}(\Sigma^{n-1}\mathbb{P}^2)) = 2^l$.

We attach now $\mathbb{P}^2$-$(n+1)$-cells to $\Sigma^{n-1}\mathbb{P}^2$ to kill π_{n+1}. We proceed inductively in l. If $l \geq 1$, let J be a set of generators of the elements of order 2 in $\pi_{n+1}(\Sigma^{n-1}\mathbb{P}^2)$. For each $\alpha \in J$ we will attach a $\mathbb{P}^2$-$(n+1)$-cell in the following way. By the previous lemma, α

can be extended to some $\overline{\alpha} : \Sigma^n \mathbb{P}^2 \to \Sigma^{n-1}\mathbb{P}^2$, which will be the attaching map of the $\mathbb{P}^2$-$(n+1)$-cell.

Let Y be the space obtained in this way. It follows that $\exp(\pi_{n+1}(Y)) \leq 2^{l-1}$. Thus, by induction, we construct a generalized CW($\mathbb{P}^2$)-complex X_{n+1} such that $\pi_n(X_{n+1}) = \mathbb{Z}_2$ and $\pi_r(X_{n+1}) = 0$ for $r \leq n+1$, $r \neq n$. By 6.2.3, $\pi_r(X_{n+1})$ must be finite and of 2-torsion for all $r \in \mathbb{N}$, so the previous argument may be applied again and the result follows. $\square$

Example 6.3.5. Let X be a $CW(\mathbb{P}^3)$ with no cells in adjacent dimensions. By theorem 6.2.7 there exist short exact sequences

$$0 \longrightarrow \bigoplus_{J_{n-3}} \mathbb{Z} \longrightarrow H_n(X) \longrightarrow \bigoplus_{J_{n-1}} \mathbb{Z}_2 \longrightarrow 0$$

where J_k is an index set for the A-k-cells of X. In the same way, if X is a $CW(\mathbb{P}^l)$ (with l odd) with no cells in adjacent dimensions, we obtain short exact sequences

$$0 \longrightarrow \bigoplus_{J_{n-l}} \mathbb{Z} \longrightarrow H_n(X) \longrightarrow G \longrightarrow 0$$

where G is an abelian group such that $\exp(G) \,|\, 2^{\frac{l-1}{2}}$.

Chapter 7

CW(A)-approximations when A is a Moore space

In this chapter we give CW(A)-approximation theorems for topological spaces in case A is a Moore space. As corollaries, we obtain homotopy classification theorems for CW(A)-complexes.

7.1 First case: A is a $M(\mathbb{Z}_p, r)$ with p prime

Proposition 7.1.1. *Let A be a Moore space of type $(\mathbb{Z}_p, r)$ with p prime, and let X be a simply-connected topological space. Then there exists a CW(A)-complex Z and a weak equivalence $f : Z \to X$ if and only if $H_i(X) = 0$ for $1 \leq i \leq \max\{r-1, 1\}$ and $H_i(X) = \bigoplus\limits_{J_i} \mathbb{Z}_p$ for all $i \geq \max\{r, 2\}$.*

Proof. Suppose first that there exists a CW(A)-complex Z and a weak equivalence $f : Z \to X$. By 5.2.1 we know that, for all $n \in \mathbb{N}$, $H_n(Z)$ is a subquotient of $\bigoplus\limits_{A\text{-}(n-1)\text{-cells}} \mathbb{Z}_p$, hence it is isomorphic to $\bigoplus\limits_{J_n} \mathbb{Z}_p$ for some index set J_n. Note that Z is $(r-1)$-connected by 4.1.9. Since f is a weak equivalence, it induces isomorphisms in all homology groups. Thus $H_i(X) = 0$ for $1 \leq i \leq \max\{r-1, 1\}$ and $H_i(X) = H_i(Z) = \bigoplus\limits_{J_i} \mathbb{Z}_p$ for all $i \geq \max\{r, 2\}$.

For the converse we will analyse first the case $r = 1$ for simplicity. So, suppose that $H_i(X) = 0$ for $1 \leq i \leq 1$ and $H_i(X) = \bigoplus\limits_{J_i} \mathbb{Z}_p$ for all $i \geq 2$. Let $\varphi : X' \to X$ be a CW-approximation of X. Since $\bigvee\limits_{J_i} \Sigma^i A$ is a Moore space of type $(\bigoplus\limits_{J_i} \mathbb{Z}_p, i+1)$, we may take a homology decomposition $f' : Z \to X'$ of X' such that $Z = \operatorname*{colim}\limits_{n \in \mathbb{N}} Z_n$, with $Z_1 = *$ and Z_n the mapping cone of a map $\bigvee\limits_{J_n} \Sigma^{n-2} A \to Z_{n-1}$ for $n \geq 2$. Hence, Z is a CW(A)-complex and $f = \varphi \circ f' : Z \to X$ is a weak equivalence.

Now we study the case $r \geq 2$, which is similar to the previous one. Suppose that $H_i(X) = 0$ for $1 \leq i \leq r-1$ and $H_i(X) = \bigoplus\limits_{J_i} \mathbb{Z}_p$ for all $i \geq r$. Let $\varphi : X' \to X$ be a

CW-approximation of X. Since $\bigvee_{J_i} \Sigma^i A$ is a Moore space of type $(\bigoplus_{J_i} \mathbb{Z}_p, r+i)$, we may take a homology decomposition $f' : Z \to X'$ of X' such that $Z = \operatorname*{colim}_{n \in \mathbb{N}} Z_n$, with $Z_n = *$ for $1 \leq n \leq r-1$, $Z_r = \bigvee_{J_r} A$ and Z_n the mapping cone of a map $\bigvee_{J_n} \Sigma^{n-r-1} A \to Z_{n-1}$ for $n \geq r+1$. Hence, Z is a CW(A)-complex and $f = \varphi \circ f' : Z \to X$ is a weak equivalence. □

Applying Whitehead's theorem we get the following corollary.

Theorem 7.1.2. *Let A be a Moore space of type $(\mathbb{Z}_p, r)$ with p prime, and let X be a simply-connected topological space having the homotopy type of a CW-complex. Then X has the homotopy type of a CW(A)-complex if and only if $H_i(X) = 0$ for $1 \leq i \leq \max\{r-1, 1\}$ and $H_i(X) = \bigoplus_{J_i} \mathbb{Z}_p$ for all $i \geq \max\{r, 2\}$.*

We want to obtain now a homotopy classification theorem for generalized CW(A)-complexes. We will need the following proposition which states that, under certain hypotheses, given a chain complex (C_*, d_*) one can construct a CW-complex such that its cellular chain complex is (C_*, d_*). Other results of this kind can be found in [23] (cf. theorem 7.2.1 below).

Proposition 7.1.3. *Let (C_*, d_*) be a chain complex such that $C_0 = C_1 = \mathbb{Z}$, $d_1 : C_1 \to C_0$ is the trivial map, $C_n = 0$ for $n \geq 4$ and $C_n = \bigoplus_{J_n} \mathbb{Z}$ for $2 \leq n \leq 3$, where J_2 and J_3 are index sets. Then there exists a CW-complex X such that its cellular chain complex is (C_*, d_*).*

Proof. We fix the following notation. For $\alpha \in J_n$ let $1_\alpha \in \bigoplus_{J_n} \mathbb{Z}$ be defined by $(1_\alpha)_\beta = 0$ if $\beta \neq \alpha$ and $(1_\alpha)_\alpha = 1$.

For $\alpha \in J_2$ let $g^2_\alpha : S^1 \to S^1$ be a map of degree $d_2(1_\alpha)$ in $\pi_1(X^1)$. We define X^2 by

$$\begin{tikzcd} \bigvee_{J_2} S^1 \arrow[r, "\underset{\alpha \in J_2}{+} g_\alpha"] \arrow[d, "i"'] \arrow[dr, phantom, "push"] & S^1 \arrow[d, "\mathrm{inc}"] \\ \bigvee_{J_2} D^2 \arrow[r, "f"'] & X^2 \end{tikzcd}$$

Note that the cellular chain complex of X^2 is

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0.$$

Now, for each $m \in \mathbb{Z}$, let $F_m : S^1 \to S^1$ be a map of degree m, and for $\beta \in J_2$, let $\mathrm{inc}_\beta : S^2 \to \bigvee_{J_2} S^2$ denote the inclusion in the β-th copy of S^2.

For $\alpha \in J_3$ we define $g_\alpha : S^2 \to X^2$ as follows. Let $A_\alpha = \{\beta \in J_2 \ / \ (d_3(1_\alpha))_\beta \neq 0\}$. Note that A_α is a finite set. We define $G_\alpha : \bigvee_{\beta \in A_\alpha} D^2 \to \bigvee_{\beta \in A_\alpha} D^2$ by $G_\alpha = \underset{\beta \in A_\alpha}{+} \mathrm{inc}_\beta \mathrm{C}(F_{(d_3(1_\alpha))_\beta})$.

Now let $i : \bigvee_{\beta\in A_\alpha} D^2 \to S^2$ be a subspace map. Note that the boundary of $i(\bigvee_{\beta\in A_\alpha} D^2) \subseteq S^2$ is homeomorphic to $\bigvee_{\beta\in A_\alpha} S^1$. It is not difficult to prove that $S^2 - i(\bigvee_{\beta\in A_\alpha} D^2)$ is homeomorphic to $\overset{\circ}{D^2}$ and that its closure in S^2 is homeomorphic to D^2/Z, where $Z \subseteq S^1 \subseteq D^2$ is a finite set with $\#Z = \#A_\alpha$.

Thus, there exists a pushout diagram

$$\begin{tikzcd}
\bigvee_{\beta\in A_\alpha} S^1 \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "push"] & \bigvee_{\beta\in A_\alpha} D^2 \arrow[d, "i"] \\
D^2/Z \arrow[r] & S^2
\end{tikzcd}$$

where j is the inclusion map.

Now let $\nu' : S^1 \to \bigvee_{\beta\in A_\alpha} S^1$ be the map induced by taking standard comultiplication $\#(A_\alpha) - 1$ times. For $\beta \in J_2$ let f^2_β be the characteristic map of e^2_β.

Note that the composition

$$\bigvee_{\beta\in A_\alpha} S^1 \xrightarrow{\ j\ } \bigvee_{\beta\in A_\alpha} D^2 \xrightarrow{\ G_\alpha\ } \bigvee_{\beta\in A_\alpha} D^2 \xrightarrow{\underset{\beta\in A_\alpha}{+} f^2_\beta} X^2$$

coincides with $\mathrm{inc} \circ \left(\underset{\beta\in A_\alpha}{+} g^2_\beta F_{(d_3(1_\alpha))_\beta}\right)$. Hence,

$$\begin{aligned}
\left[\left(\underset{\beta\in A_\alpha}{+} f^2_\beta\right) G_\alpha j \nu'\right] &= \left[\mathrm{inc}\left(\underset{\beta\in A_\alpha}{+} g^2_\beta F_{(d_3(1_\alpha))_\beta}\right)\nu'\right] = \mathrm{inc}_*\left(\sum_{\beta\in A_\alpha} [g^2_\beta F_{(d_3(1_\alpha))_\beta}]\right) = \\
&= \mathrm{inc}_*\left(\sum_{\beta\in A_\alpha} [g^2_\beta][F_{(d_3(1_\alpha))_\beta}]\right) = 0
\end{aligned}$$

in $\pi_1(X^2)$, since $d_2 d_3 = 0$.

Thus, the map $\left(\underset{\beta\in A_\alpha}{+} f^2_\beta\right) G_\alpha j \nu' : S^1 \to X^2$ can be extended to a map $h_\alpha : D^2 \to X^2$. Let $q : D^2 \to D^2/Z$ be the quotient map. By construction, it follows that there exists a map $\overline{h_\alpha} : D^2/Z \to X^2$ such that $\overline{h_\alpha} q = h_\alpha$.

Now, let g_α be the dotted arrow defined by the commutative diagram

$$\begin{tikzcd}
\bigvee_{\beta\in A_\alpha} S^1 \arrow[r, "\mathrm{inc}"] \arrow[d] \arrow[dr, phantom, "push"] & \bigvee_{\beta\in A_\alpha} D^2 \arrow[d, "i"] \arrow[ddr, bend left, "\left(\underset{\beta\in A_\alpha}{+} f^2_\beta\right) G_\alpha"] & \\
D^2/Z \arrow[r] \arrow[drr, bend right, "\overline{h_\alpha}"'] & S^2 \arrow[dr, dotted, "g_\alpha"] & \\
& & X^2.
\end{tikzcd}$$

Let X be obtained by attaching 3-cells to X^2 by the maps g_α, $\alpha \in J_3$. By construction, it follows that the cellular chain complex of X is (C_*, d_*). □

Lemma 7.1.4. *Let X be a topological space and let A be a Moore space of type $(\mathbb{Z}_d, 1)$ obtained by attaching a 2-cell to S^1 by a map α of degree d. Let $m \geq 2$ and suppose we attach to X an m-cell and an $(m+1)$-cell by maps g^m and g^{m+1} respectively*

$$\begin{tikzcd} S^{m-1} \arrow[r, "g^m"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "push"] & X \arrow[d, "i_1"] \\ D^m \arrow[r, "f^m"'] & X \cup e^m \end{tikzcd} \qquad \begin{tikzcd} S^{m} \arrow[r, "g^{m+1}"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "push"] & X \cup e^m \arrow[d, "i_2"] \\ D^m \arrow[r, "f^{m+1}"'] & X \cup e^m \cup e^{m+1} \end{tikzcd}$$

where g^{m+1} in the northern hemisphere $H_+ \subseteq S^m$ is the map $f^m \circ \mathrm{C}\Sigma^{m-2}\alpha : D^m \simeq H_+ \to X \cup e^n$ and for the southern hemisphere H_- we have $g^{m+1}(H_-) \subseteq X$.

Then $X \cup e^m \cup e^{m+1}$ is homeomorphic to a space $X \cup e^{A,m-1}$ obtained by attaching an A-$(m-1)$-cell to X.

Proof. Let $g_A : \Sigma^{m-2}A \to X$ be the map defined by

$$\begin{tikzcd} S^{m-1} \arrow[r, "\Sigma^{m-2}\alpha"] \arrow[d] \arrow[dr, phantom, "push"] & S^{m-1} \arrow[d] \arrow[ddr, bend left, "g^m"] & \\ D^m \arrow[r] \arrow[drr, bend right, "g^{m+1}|_{H_-}"'] & \Sigma^{m-2}A \arrow[dr, "g_A"] & \\ & & X \end{tikzcd}$$

and let $X \cup e^{A,m-1}$ be defined by the pushout

$$\begin{tikzcd} \Sigma^{m-2}A \arrow[r, "g_A"] \arrow[d] \arrow[dr, phantom, "push"] & X \arrow[d] \\ \mathrm{C}\Sigma^{m-2}A \arrow[r, "f_A"'] & X \cup e^{A,m-1} \end{tikzcd}$$

Using the universal properties of pushouts and colimits we will prove that $X \cup e^m \cup e^{m+1}$ and $X \cup e^{A,m-1}$ are colimits of the diagram

$$\begin{tikzcd} & S^{m-1} \arrow[r, "\Sigma^{m-2}\alpha"] \arrow[d] \arrow[ddl] & S^{m-1} \arrow[ddl] \arrow[ddr, bend left, "g^m"] & \\ & D^m \arrow[ddl] \arrow[drr, bend right, "g^{m+1}|_{H_-}"] & & \\ \mathrm{C}S^{m-1} \arrow[r, "\mathrm{C}\Sigma^{m-2}\alpha"] \arrow[d] & \mathrm{C}S^{m-1} & & X \\ \mathrm{C}D^m & & & \end{tikzcd}$$

Let us define first arrows ψ_1, ψ_2, ψ_3, ϕ_1, ϕ_2 and ϕ_3 such that the following diagrams commute

$$\begin{tikzcd}
 & S^{m-1} \arrow[rr, "\Sigma^{m-2}\alpha"] \arrow[d] \arrow[ddl] & & S^{m-1} \arrow[ddl] \arrow[ddrr, bend left, "g^m"] & & \\
 & D^m \arrow[ddl] \arrow[drrrr, bend right, "g^{m+1}|_{H_-}"] & & & & \\
\mathrm{C}S^{m-1} \arrow[rr, "\mathrm{C}\Sigma^{m-2}\alpha"] \arrow[d] & & \mathrm{C}S^{m-1} \arrow[drr, "\psi_2"'] & & & X \arrow[dl, "\psi_1"] \\
\mathrm{C}D^m \arrow[rrrr, "\psi_3"'] & & & & X \cup e^m \cup e^{m+1} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & S^{m-1} \arrow[rr, "\Sigma^{m-2}\alpha"] \arrow[d] \arrow[ddl] & & S^{m-1} \arrow[ddl] \arrow[ddrr, bend left, "g^m"] & & \\
 & D^m \arrow[ddl] \arrow[drrrr, bend right, "g^{m+1}|_{H_-}"] & & & & \\
\mathrm{C}S^{m-1} \arrow[rr, "\mathrm{C}\Sigma^{m-2}\alpha"] \arrow[d] & & \mathrm{C}S^{m-1} \arrow[drr, "\phi_2"'] & & & X \arrow[dl, "\phi_1"] \\
\mathrm{C}D^m \arrow[rrrr, "\phi_3"'] & & & & X \cup e^{A,m-1} &
\end{tikzcd}$$

We define the maps $\psi_1 : X \to X \cup e^m \cup e^{m+1}$ and $\phi_1 : X \to X \cup e^{A,m-1}$ to be the corresponding inclusions and $\psi_2 : \mathrm{C}S^{m-1} \simeq D^m \to X \cup e^m \cup e^{m+1}$ by $\psi_2 = i_2 f^m$, where $i_2 : X \cup e^m \to X \cup e^m \cup e^{m+1}$ is the inclusion. We also define $\psi_3 : \mathrm{C}D^m \simeq D^{m+1} \to X \cup e^m \cup e^{m+1}$ as the characteristic map of e^{m+1}, that is $\psi_3 = f^{m+1}$.

Now we define ϕ_2 and ϕ_3 in the following way. We consider the commutative cube

$$\begin{tikzcd}
 & S^{m-1} \arrow[rr, "\Sigma^{m-2}\alpha"] \arrow[d] \arrow[ddl] \arrow[drr, phantom, "push"] & & S^{m-1} \arrow[d, "\iota"] \arrow[ddl] \\
 & D^m \arrow[rr, "j"] \arrow[ddl] & & \Sigma^{m-2}A \arrow[ddl] \\
\mathrm{C}S^{m-1} \arrow[rr, "\mathrm{C}\Sigma^{m-2}\alpha"] \arrow[d] \arrow[drr, phantom, "push"] & & \mathrm{C}S^{m-1} \arrow[d, "\mathrm{C}\iota"] & \\
\mathrm{C}D^m \arrow[rr, "\mathrm{C}j"'] & & \mathrm{C}\Sigma^{m-2}A &
\end{tikzcd}$$

where the front and rear faces are pushout squares, and set $\phi_2 = f_A \mathrm{C}\iota$, $\phi_3 = f_A \mathrm{C}j$.

Now we will prove that both $X \cup e^m \cup e^{m+1}$ and $X \cup e^{A,m-1}$ satisfy the universal property of the colimit. Suppose that Y is a topological space and that there are continuous maps $\varphi_1 : X \to Y$, $\varphi_2 : \mathrm{C}S^{m-1} \to Y$ and $\varphi_3 : \mathrm{C}D^m \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
& S^{m-1} \arrow[r, "\Sigma^{m-2}\alpha"] \arrow[d] \arrow[ddl] & S^{m-1} \arrow[ddl] \arrow[ddr, bend left, "g^m"] & \\
& D^m \arrow[ddl] \arrow[drr, bend right, "g^{m+1}|_{H_-}"'] & & \\
\mathrm{C}S^{m-1} \arrow[r, "\mathrm{C}\Sigma^{m-2}\alpha"] \arrow[d] & \mathrm{C}S^{m-1} \arrow[dr, "\varphi_2"'] & & X \arrow[dl, "\varphi_1"] \\
\mathrm{C}D^m \arrow[rr, "\varphi_3"'] & & Y &
\end{tikzcd}$$

Then there exist maps $\beta_1 : X \cup e^m \to Y$, $\beta_2 : S^m \to Y$ $\beta : X \cup e^m \cup e^{m+1} \to Y$, $\gamma_1 : \mathrm{C}\Sigma^{m-2}A \to Y$ and $\gamma : X \cup e^{A,m-1}$ such that the following diagrams commute

$$\begin{tikzcd}
S^{m-1} \arrow[r, "g^m"] \arrow[d] \arrow[dr, phantom, "push"] & X \arrow[d] \arrow[ddr, bend left, "\varphi_1"] & \\
\mathrm{C}S^{m-1} \arrow[r, "f^m"] \arrow[drr, bend right, "\varphi_2"'] & X \cup e^m \arrow[dr, "\beta_1"] & \\
& & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{m-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "push"] & D^m \arrow[d] \arrow[ddr, bend left, "\beta_1\circ f^m\circ \mathrm{C}\Sigma^{m-2}\alpha=\beta_1\circ g^{m+1}|_{H_+}"] & \\
D^m \arrow[r] \arrow[drr, bend right, "\varphi_1\circ g^{m+1}|_{H_-}"'] & S^m \arrow[dr, "\beta_2"] & \\
& & Y
\end{tikzcd}$$

$$\begin{tikzcd}
S^m \arrow[r, "g^{m+1}"] \arrow[d] \arrow[dr, phantom, "push"] & X \cup e^n \arrow[d] \arrow[ddr, bend left, "\beta_1"] & \\
\mathrm{C}D^m \simeq D^{m+1} \arrow[r, "f^{m+1}"'] \arrow[drr, bend right, "\varphi_3"'] & X \cup e^m \cup e^{m+1} \arrow[dr, "\beta"] & \\
& & Y
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{C}S^{m-1} \arrow[r, "\mathrm{C}\Sigma^{m-2}\alpha"] \arrow[d] \arrow[dr, phantom, "push"] & \mathrm{C}S^{m-1} \arrow[d, "\mathrm{C}\iota"'] \arrow[ddr, bend left, "\varphi_2"] & \\
\mathrm{C}D^m \arrow[r, "\mathrm{C}j"'] \arrow[drr, bend right, "\varphi_3"'] & \mathrm{C}\Sigma^{m-2}A \arrow[dr, "\gamma_1"] & \\
& & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^{m-2}A \arrow[r, "g_A"] \arrow[d] \arrow[dr, phantom, "push"] & X \arrow[d, "\phi_1"'] \arrow[ddr, bend left, "\varphi_1"] & \\
\mathrm{C}\Sigma^{m-2}A \arrow[r, "f_A"'] \arrow[drr, bend right, "\gamma_1"'] & X \cup e^{A,m-1} \arrow[dr, "\gamma"] & \\
& & Y
\end{tikzcd}$$

It is routine to check that $\beta\psi_i = \varphi_i$ and $\gamma\phi_i = \varphi_i$ for $i = 1, 2, 3$. The uniqueness of the maps β and γ satisfying this equalities follows easily from the universal properties of the pushouts diagrams above.

Hence, $X \cup e^m \cup e^{m+1}$ and $X \cup e^{A,m-1}$ are homeomorphic. $\square$

Proposition 7.1.5. *Let p be a prime number and let A be a Moore space of type $(\mathbb{Z}_p, r)$. Then for every $m, n \in \mathbb{N}$ with $n \geq r + 1$ there exists a generalized CW(A)-complex X which is a Moore space of type $(\mathbb{Z}_{p^m}, n)$. Moreover, X has a finite number of A-cells.*

Proof. We suppose first that $r = 1$. Note that it suffices to prove the case $n = 2$ since suspensions of generalized CW(A)-complexes are also generalized CW(A)-complexes and the suspension of a Moore space of type (G, r) $(r \geq 2)$ is a Moore space of type $(G, r+1)$.

We will suppose that A is constructed by attaching a 2-cell to S^1 by a map of degree p and we will consider the general case later.

Consider the chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^{m+1} \xrightarrow{d_2} \mathbb{Z}^{m+2} \xrightarrow{d_1} \mathbb{Z}$$

where d_1 and d_2 are the morphisms of $\mathbb{Z}$-modules defined by left multiplication by the matrices

$$M_1 = \begin{pmatrix} p & p^m & -p^{m-1} & \dots & (-1)^{m-1}p & (-1)^m \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} -p^m & 0 & 0 & \dots & 0 & 0 \\ p & 1 & 0 & \dots & 0 & 0 \\ 0 & p & 1 & \dots & 0 & 0 \\ 0 & 0 & p & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & p & 1 \\ 0 & 0 & 0 & \dots & 0 & p \end{pmatrix}$$

respectively. Schematically, the chain complex above is represented by the diagram

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[r, "p"] & \mathbb{Z} \\
 & \oplus & \\
\mathbb{Z} \arrow[uur, "-p^m"] \arrow[r, "p"] & \mathbb{Z} \arrow[uur, "p^m"] & \\
\oplus & \oplus & \\
\mathbb{Z} \arrow[uur, "1"] \arrow[r, "p"] & \mathbb{Z} \arrow[uuuur, "-p^{m-1}"] & \\
\oplus & \oplus & \\
\dots & \dots & \\
\oplus & \oplus & \\
\mathbb{Z} \arrow[uur, "1"] \arrow[r, "p"] & \mathbb{Z} \arrow[uuuuuuuur, "(-1)^m"'] &
\end{tikzcd}$$

where an arrow labeled with $k \in \mathbb{Z}$ indicates multiplication by k.

It follows that the homology of this chain complex is 0 in degrees $i \neq 2$ and $\mathbb{Z}_{p^m}$ in degree 2 with generator the class of $(1, 0, 0, \dots, 0, (-1)^{m-1}p)$.

By theorem 7.1.3, there exists a CW-complex Z such that its cellular complex is the chain complex above. Applying the previous lemma, we obtain that Z is homeomorphic to a generalized CW(A)-complex X with a finite number of A-cells.

Now, if A' is any Moore space of type $(\mathbb{Z}_p, 1)$ and A is as above then there is a homotopy equivalence $A \to A'$. Hence, by a similar argument as the one in the proof of 3.3.1 there exists a generalized CW(A')-complex X' homotopy equivalent to X. Thus X' is also a Moore space of type $(\mathbb{Z}_{p^m}, 2)$. Note that X' has a finite number of A'-cells since X has finite number of A-cells.

It remains to prove the general case $r \in \mathbb{N}$. By the same reasoning as above, we may suppose that $A = \Sigma^{r-1}B$ where B is a Moore space of type $(\mathbb{Z}_p, 1)$. Hence, there exists a generalized CW(B)-complex X which is a Moore space of type $(\mathbb{Z}_{p^m}, n-r+1)$. Since the suspension functor commutes with colimits, it follows that $\Sigma^{r-1}X$ is a CW(A)-complex which is a Moore space of type $(\mathbb{Z}_{p^m}, n)$. $\square$

Corollary 7.1.6. *Let p be a prime number and let A be a Moore space of type $(\mathbb{Z}_p, r)$. Let G be a finitely generated p-torsion abelian group and let $n \in \mathbb{N}$, $n \geq r+1$. Then there exists a generalized CW(A)-complex X which is a Moore space of type (G, n). Moreover, X has a finite number of A-cells.*

Proof. Since G is a finitely generated p-torsion abelian group, we know that there exist an index set I and natural numbers m_i, $i \in I$, such that $G = \bigoplus_{i\in I} \mathbb{Z}_{p^{m_i}}$. We take X to be a wedge sum of Moore spaces $M(\mathbb{Z}_{p^{m_i}}, n)$, $i \in I$, which might be taken to be CW(A)-complexes by the previous proposition. The result follows applying the wedge axiom for (reduced) homology. $\square$

As an example of an application, we will show now which Eilenberg-MacLane spaces can be obtained as generalized CW(A)-complexes. We give a constructive proof at this point. However, it will be also deduced later by a homotopy classification theorem for generalized CW(A)-complexes.

We need the following lemma.

Lemma 7.1.7. *Let $m \in \mathbb{N}$ and let A be the Moore space of type $(\mathbb{Z}_m, 1)$ obtained by attaching a 2-cell to S^1 by a map g of degree m. Let $f : S^1 \to X$ be a continuous map. Then $[f]^m = 0$ in $\pi_1(X)$ if and only if f can be extended to A.*

Proof. Note that $[f]^m = [fg]$ in $\pi_1(X)$. Hence if $[f]^m = 0$ then $fg \simeq 0$ and thus it can be extended to D^2. The extension of f to A is obtained then by the universal property of pushouts.

To prove the converse, let $\overline{g} : D^2 \to A$ be the characteristic map of the 2-cell of A and let inc : $S^1 \to D^2$ be the inclusion map. If f can be extended to $\overline{f} : A \to X$ then $fg = \overline{f}\,\overline{g}\mathrm{inc}$ is nullhomotopic. Hence $[f]^m = [fg] = 0$. $\square$

Note that this result can also be deduced from the cofibration sequence $S^1 \hookrightarrow A \to S^2$.

Proposition 7.1.8. *Let p be a prime number and let A be a Moore space of type $(\mathbb{Z}_p, r)$. Let G be a finitely generated p-torsion abelian group and let $n \in \mathbb{N}$ with $n \geq r+1$. Then there exists a generalized CW(A)-complex X which is an Eilenberg-MacLane space of type (G, n).*

Proof. Applying the same argument as in the proof of 7.1.5, we may suppose that A is obtained by attaching a 2-cell to S^1 by a map of degree p.

We will prove only the case $r = 1$, since the general case is completely analogous to it.

By the previous corollary we may build a finite generalized CW(A)-complex X_0 which is also a Moore space of type (G, n). By the Hurewicz theorem, $\pi_r(X_0) = 0$ for $r \leq n-1$ and $\pi_n(X_0) = G$. Moreover, by the generalized Hurewicz theorem (2.3.13) we know that the groups $\pi_r(X_0)$ must be finite and of p-torsion for all $r \in \mathbb{N}$. Then, there exists $l \in \mathbb{N}$ such that $\exp(\pi_{n+1}(X_0)) = p^l$.

We attach now A-$(n+1)$-cells to X_0 to kill π_{n+1}. We proceed inductively in l. If $l \geq 1$, let J be a set of generators of the elements of order p in $\pi_{n+1}(X_0)$. For each $\alpha \in J$ we will attach an A-$(n+1)$-cell in the following way. By lemma 7.1.7, α can be extended to some $\overline{\alpha} : \Sigma^n A \to X_0$, which will be the attaching map of the A-$(n+1)$-cell.

Let Y be the space obtained in this way. It follows that $\exp(\pi_{n+1}(Y)) \leq p^{l-1}$. Thus, by induction, we may construct a finite generalized CW(A)-complex X_{n+1} such that $\pi_n(X_{n+1}) = \mathbb{Z}_p$ and $\pi_r(X_{n+1}) = 0$ for $r \leq n+1$, $r \neq n$. By 6.2.3, $\pi_r(X_{n+1})$ must be finite and of p-torsion for all $r \in \mathbb{N}$, so the previous argument may be applied again and the result follows. □

Theorem 7.1.9. *Let p be a prime number and let A be a Moore space of type $(\mathbb{Z}_p, r)$. Let X be an r-connected CW-complex such that $H_n(X)$ is a finitely generated p-torsion abelian group for all $n \geq r+1$. Then X has the homotopy type of a generalized CW(A)-complex.*

Proof. As in the proofs above, we will only analyse the case $r = 1$ since the general case is very similar to that.

By 1.4.39 we know that X admits a homology decomposition. Then, there exist a CW-complex Z, a homotopy equivalence $f : Z \to X$, and a sequence $(Z_n)_{n\in\mathbb{N}}$ of subcomplexes of Z such that

(a) $Z_n \subseteq Z_{n+1}$ for all $n \in \mathbb{N}$.

(b) $Z = \bigcup_{n\in\mathbb{N}} Z_n$.

(c) Z_1 is a Moore space of type $(H_1(X), 1)$.

(d) For all $n \in \mathbb{N}$, Z_{n+1} is the mapping cone of a cellular map $g_n : M_n \to Z_n$, where M_n is a Moore space of type $(H_{n+1}(X), n)$, and g_n is such that the induced map $(g_n)_* : H_n(M_n) \to H_n(Z_n)$ is trivial.

Now, by 7.1.6 we may suppose that M_n is a finite generalized CW(A)-complex for all $n \in \mathbb{N}$. By a similar argument as that of the proof of 3.1.26 it follows that Z is a generalized CW(A)-complex. □

Remark 7.1.10. Note that if A is a Moore space of type $(\mathbb{Z}_p, r)$ and X is an r-connected CW-complex which has the homotopy type of a generalized CW(A)-complex, then by 6.2.6, $H_n(X)$ is a p-torsion abelian group for all $n \geq r+1$.

Hence, from this fact and the previous theorem we obtain a homotopy classification theorem for r-connected generalized CW(A)-complexes which have finitely generated homology groups.

Note that from 7.1.9 and the generalized Hurewicz theorem (2.3.13) we may deduce proposition 7.1.8.

7.2 General case: A is a $M(\mathbb{Z}_m, r)$

To study the case where A is a Moore space of type $(\mathbb{Z}_m, r)$, with m not necessarily prime, we will make use of the following theorem, due to J.H.C. Whitehead [23].

Theorem 7.2.1. *Let (C_*, d_*) be a chain complex such that $C_0 = \mathbb{Z}$, $C_1 = 0$ and $C_n = \bigoplus_{J_n} \mathbb{Z}$ for each $n \geq 2$, where J_n, $n \geq 2$, are index sets. Then there exists a simply-connected CW-complex X such that its cellular chain complex is (C_*, d_*).*

Proof. For each $n \in \mathbb{N}$, we will construct a CW-complex X_n of dimension n with cellular chain complex $(C_*^{(n)}, d_*^{(n)})$ such that

- $(X_n)^{n-1} = X_{n-1}$
- $C_i^{(n)} = C_i$ if $0 \leq i \leq n$
- $C_i^{(n)} = 0$ if $i > n$
- $d_i^{(n)} = d_i$ if $1 \leq i \leq n$
- $d_i^{(n)} = 0$ if $i > n$
- If, for $n \in \mathbb{N}$, $j_n : \pi_n(X_n) \to \pi_n(X_n, X_{n-1})$ and $\partial_n : \pi_n(X_n, X_{n-1}) \to \pi_{n-1}(X_{n-1})$ are morphisms coming from the long exact sequence of homotopy groups associated to the pair (X_n, X_{n-1}), then $\operatorname{Im} j_n = \ker(j_{n-1}\partial_n)$.

We proceed by induction on n. For $n = 1$ we take $X_1 = *$ and for $n = 2$ we take $X_2 = \bigvee_{J_2} S^2$. We also define $X_0 = *$. From the long exact sequence of homotopy groups associated to the pair (X_2, X_1) it follows that $\operatorname{Im} j_2 = \pi_2(X_2, X_1)$ which coincides with $\ker(j_1\partial_2)$, since $j_1\partial_2$ is the trivial map.

Suppose now that $n \geq 3$ and that X^{n-1} is constructed. Since (X_{n-1}, X_{n-2}) is $(n-2)$-connected, $n-2 \geq 1$ and X_{n-2} is simply-connected, by the relative version of the Hurewicz theorem we obtain that $\pi_{n-1}(X_{n-1}, X_{n-2}) \simeq H_{n-1}(X_{n-1}, X_{n-2}) \simeq C_{n-1}$. Similarly, $\pi_{n-2}(X_{n-2}, X_{n-3}) \simeq H_{n-2}(X_{n-2}, X_{n-3}) \simeq C_{n-2}$ (note that this holds trivially if $n = 3$).

Let $\phi_{n-1} : C_{n-1} \to \pi_{n-1}(X_{n-1}, X_{n-2})$ and $\phi_{n-2} : C_{n-2} \to \pi_{n-2}(X_{n-2}, X_{n-3})$ be the inverse maps of the respective Hurewicz isomorphisms. Hence, from the naturality of the Hurewicz morphisms it follows that there is a commutative diagram

$$\begin{tikzcd}
C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\phi_{n-1}"', "\simeq"] & C_{n-2} \arrow[d, "\simeq"', "\phi_{n-2}"] \\
 & \pi_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "j_{n-2}\partial_{n-1}"] \arrow[d, "\partial_{n-1}"'] & \pi_{n-2}(X_{n-2}, X_{n-3}) \\
\pi_{n-1}(X_{n-1}) \arrow[ur, "j_{n-1}"'] & \pi_{n-2}(X_{n-2}) \arrow[ur, "j_{n-2}"'] &
\end{tikzcd}$$

Let $\{z_i : i \in J'\}$ be a basis of $\operatorname{Im} d_n$. For each $i \in J'$ let $c_i \in C_n$ be such that $d_n(c_i) = z_i$ and let $G_n \subseteq C_n$ be the subgroup generated by $\{c_i : i \in J'\}$. Note that a relation between the c_i's will imply the corresponding relation between the z_i's. Hence, $\{c_i : i \in J'\}$ is a basis of G_n. Therefore, $C_n \simeq \ker d_n \oplus G_n$ and $d_n|_{G_n} : G_n \to \operatorname{Im} d_n$ is an isomorphism. Let $\{x_i : i \in I\}$ be a basis of $\ker d_n$.

Let B be the basis of C_n defined by $B = \{x_i : i \in I\} \cup \{c_i : i \in J'\}$. Note that there exists a bijection $B \simeq J_n$ and that $j_{n-2}\partial_{n-1}\phi_{n-1}d_n(a) = \phi_{n-2}d_{n-1}d_n(a) = 0$ for all $a \in B$. Hence, $\phi_{n-1}d_n(a) \in \ker(j_{n-2}\partial_{n-1}) = \operatorname{Im} j_{n-1}$. Thus, there exists $b_a \in \pi_{n-1}(X_{n-1})$ such that $j_{n-1}(b_a) = \phi_{n-1}d_n(a)$. We take $b_a = 0$ if $a \in \ker d_n$.

For $a \in B$, let $g_a : S^{n-1} \to X_{n-1}$ be a continuous map such that $[g_a] = b_a$. If $b_a = 0$ we take g_a to be the constant map. Let X_n be obtained from X_{n-1} by attaching n-cells by the maps g_a, $a \in B \simeq J_n$. Note that $\pi_n(X_n, X_{n-1}) \simeq \bigoplus\limits_{J_n} \mathbb{Z} \simeq C_n$. As above, let $\phi_n : C_n \to \pi_n(X_n, X_{n-1})$ be induced by the inverse map of the Hurewicz isomorphism. By construction, it is not hard to check that the following diagram commutes

$$\begin{tikzcd}
C_n \arrow[r, "d_n"] \arrow[d, "\phi_n"', "\simeq"] & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\phi_{n-1}"', "\simeq"] & C_{n-2} \arrow[d, "\simeq"', "\phi_{n-2}"] \\
\pi_n(X_n, X_{n-1}) \arrow[r, "j_{n-1}\partial_n"] \arrow[d, "\partial_n"'] & \pi_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "j_{n-2}\partial_{n-1}"] \arrow[d, "\partial_{n-1}"'] & \pi_{n-2}(X_{n-2}, X_{n-3}) \\
\pi_{n-1}(X_{n-1}) \arrow[ur, "j_{n-1}"'] & \pi_{n-2}(X_{n-2}) \arrow[ur, "j_{n-2}"'] &
\end{tikzcd}$$

Thus, the first five conditions above are satisfied. It remains to prove that $\operatorname{Im} j_n = \ker(j_{n-1}\partial_n)$.

Note that $\operatorname{Im} j_n = \ker(\partial_n) \subseteq \ker(j_{n-1}\partial_n)$. On the other hand, let $a \in \ker d_n \cap B$ and let e^n_a be the cell with attaching map g_a. Let $f_a : (D^n, S^{n-1}) \to (X_n, X_{n-1})$ be the characteristic map of e^n_a. Since by construction g_a is the constant map, f_a induces a continuous map $\overline{f_a} : D^n/S^{n-1} \simeq S^n \to X_n$. Note that, $j_n(\overline{f_a}) = f_a = \phi_n(a)$. Hence, $\ker(j_{n-1}\partial_n) = \phi_n(\ker d_n) \subseteq \operatorname{Im} j_n$. It follows that $\operatorname{Im} j_n = \ker(j_{n-1}\partial_n)$ as desired.

Finally, we take $X = \operatorname*{colim}\limits_{n \in \mathbb{N}} X_n$. □

The previous theorem allows us to obtain the following results.

Proposition 7.2.2. *Let $m, d \in \mathbb{N}$ such that $d|m$ and let A be a Moore space of type $(\mathbb{Z}_m, r)$. Then, for all $n \geq \max\{r, 2\}$, there exists a CW(A)-complex which is a $M(\mathbb{Z}_d, n)$.*

Proof. As in the proofs above, we will only analyse the case $r = 1$, $n = 2$, since the general case is completely analogous to that.

Let (C_*, d_*) be the chain complex defined in the following way. The groups C_n, $n \in \mathbb{N}_0$ are defined by

$$C_n = \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ or } n = 2 \\ 0 & \text{if } n = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \geq 3 \end{cases}$$

and the morphism $d_n : C_n \to C_{n-1}$ is trivial if $n = 1$ or $n = 2$ and is defined by left multiplication by the matrix D_n if $n \geq 3$, where

$$D_n = \begin{cases} \begin{pmatrix} m & d \end{pmatrix} & \text{if } n = 3 \\ \begin{pmatrix} -d & -1 \\ m & \frac{m}{d} \end{pmatrix} & \text{if } n \geq 4 \text{ and } n \text{ is even} \\ \begin{pmatrix} -\frac{m}{d} & -1 \\ m & d \end{pmatrix} & \text{if } n \geq 4 \text{ and } n \text{ is odd} \end{cases}$$

It follows that $H_2(C_*, d_*) = \mathbb{Z}_d$ and $H_n(C_*, d_*) = 0$ if $n \neq 2$ and $n \neq 0$.

By the previous theorem, there exists a CW-complex X such that its cellular chain complex is (C_*, d_*). Hence, X is a Moore space of type $(Z_d, 2)$.

Finally, with a similar argument to the one in the proof of 7.1.5 we conclude that X is homotopy equivalent to a CW(A)-complex. $\square$

Now, proceeding as in the proof of 7.1.9, we obtain the following result. We mention that we will generalize this result later on.

Proposition 7.2.3. *Let $m \in \mathbb{N}$ and let A be a Moore space of type $(\mathbb{Z}_m, r)$. Let X be an r-connected CW-complex such that for all $n \geq r+1$, $H_n(X) = \bigoplus_{j \in J_n} \mathbb{Z}_{m_j}$ with $m_j | m$ for all $j \in J_n$. Then X has the homotopy type of a generalized CW(A)-complex.*

Remark 7.2.4. The converse of the previous proposition does not hold, since the space of example 5.2.8 is a generalized CW($\mathbb{D}_4^2$)-complex but its homology in degree 3 is $\mathbb{Z}_8$.

Proposition 7.2.5. *Let $m \in \mathbb{N}$ and let A be a Moore space of type $(\mathbb{Z}_m, r)$. Let $k \in \mathbb{N}$, $k \geq \max\{r+1, 3\}$, and let G be a finite abelian group such that the prime divisors of the orders of its elements also divide m. Then there exists a generalized CW(A)-complex which is a Moore space of type (G, k).*

Proof. As in the proof of 7.1.5, it suffices to prove the case $k = \max\{r+1, 3\}$. We know that G is the direct sum of cyclic groups of order a power of a prime that divides m. Since reduced singular homology groups satisfy the wedge axiom, it suffices to analyse the case $G = \mathbb{Z}_{p^l}$, where $p|m$ and $l \in \mathbb{N}$.

By 7.2.2, there exists a CW(A)-complex B which is a Moore space of type $(\mathbb{Z}_p, k-1)$. By 7.1.5, there exists a generalized CW(B)-complex X which is a Moore space of type $(\mathbb{Z}_{p^l}, k)$. Finally, X is a CW(A)-complex by 3.1.29. $\square$

Now we obtain another homotopy classification theorem.

Theorem 7.2.6. *Let $m \in \mathbb{N}$ and let A be a Moore space of type $(\mathbb{Z}_m, r)$, with $r \geq 2$. Let X be an $(r-1)$-connected CW-complex satisfying the following conditions*

(a) $H_r(X) = \bigoplus_{j \in J} \mathbb{Z}_{m_j}$ *with $m_j | m$ for all $j \in J$*

(b) For all $n \geq r+1$, $H_n(X)$ is a finite abelian group such that the prime divisors of the orders of its elements also divide m.

Then X has the homotopy type of a generalized CW(A)-complex.

Its proof is analogous to that of 7.1.9. Clearly, there is a similar result for $r = 1$.

Note that by 6.2.6 if a topological space X has the homotopy type of a generalized CW(A)-complex, where A is a Moore space of type $(\mathbb{Z}_m, r)$, then X is $(r-1)$-connected and for all $n \geq r$, $H_n(X)$ is a torsion abelian group such that the prime divisors of the orders of its elements also divide m. Thus, the previous theorem is a weak converse to this statement.

As a corollary of 7.2.6 we obtain a sufficient condition for the existence of a homotopical approximation by a generalized CW(A)-complex.

Theorem 7.2.7. *Let $m \in \mathbb{N}$ and let A be a Moore space of type $(\mathbb{Z}_m, r)$ with $r \geq 2$. Let X be an $(r-1)$-connected topological space satisfying conditions (1) and (2) of the previous theorem. Then there exists a generalized CW(A)-complex Z and a weak homotopy equivalence $f : Z \to X$.*

Chapter 8

Obstruction theory

In this chapter we start developing an obstruction theory for CW(A)-complexes. However, the A-cellular chain complex introduced in chapter 5 is not suitable for this purpose. Thus, we define a new A-cellular chain complex which fulfils our requirements. It is interesting to point out that although this new chain complex is completely different from the previous one, it coincides with the classical cellular chain complex if $A = S^0$.

We also define an obstruction cocycle and a difference cochain, which give the exact obstructions to extension problems for maps, as we shall see. These are generalizations of the classical ones. We emphasize that the crucial part is the definition of an adequate A-cellular chain complex so that classical obstruction theory can be taken to our more general setting.

This chapter is only an introduction to obstruction theory for CW(A)-complexes. Much more work can be done.

8.1 A new A-cellular chain complex

Let A be an l-connected and compact CW-complex of dimension k with $k \leq 2l$ and $l \geq 1$. By theorem 11 of [19] p. 458 the map $\Sigma : [\Sigma^n A, \Sigma^n A] = \pi_n^A(A) \to [\Sigma^{n+1} A, \Sigma^{n+1} A] = \pi_{n+1}^A(A)$ is a bijection for $n \geq 0$ and hence an isomorphism of groups for $n \geq 1$.

Let $R = \pi_0^{A,\mathrm{st}}(X) = \operatorname{colim}_n \pi_n^A(\Sigma^n X)$. Then R is isomorphic to $\pi_r^A(\Sigma^r A)$ for $r \geq 2$. We will denote by $+$ the usual abelian group operation in $\pi_r^A(\Sigma^r A)$. We will also define a product in $\pi_r^A(\Sigma^r A)$ as follows: $[f][g] = [g \circ f]$. It is clear that this operation is well defined and associative and has an identity element $[\mathrm{Id}]$.

Also, $([f]+[g])[h] = h_*([f]+[g]) = h_*([f]) + h_*([g]) = [f][h]+[g][h]$. We will prove now that $[h]([f]+[g]) = [h][f]+[h][g]$. If $[h] \in \pi_r^A(\Sigma^r A)$ by the isomorphism $\pi_{r-1}^A(\Sigma^{r-1} A) \simeq \pi_r^A(\Sigma^r A)$ we know that there exists $[h'] \in \pi_{r-1}^A(\Sigma^{r-1} A)$ such that $[h] = [\Sigma h']$. We denote by $q : \Sigma^r A \to \Sigma^r A \vee \Sigma^r A$ the quotient map which induces the sum $+$. Then

$$\begin{aligned} [h]([f]+[g]) &= [h][(f+g)\circ q] = [(f+g)\circ q \circ h] = [(f+g)\circ q \circ \Sigma h'] = \\ &= [(f+g)\circ (\Sigma h' \vee \Sigma h')\circ q] = [(f\circ \Sigma h' + g \circ \Sigma h')\circ q] = \\ &= [f \circ \Sigma h'] + [g \circ \Sigma h'] = [f\circ h] + [g \circ h] = [h][f] + [h][g]. \end{aligned}$$

Hence $\pi_r^A(\Sigma^r A)$ is a unital ring.

Note that $\Sigma : [\Sigma^n A, \Sigma^n A] = \pi_n^A(A) \to [\Sigma^{n+1}A, \Sigma^{n+1}A] = \pi_{n+1}^A(A)$ is an isomorphism of rings for $n \geq 2$. Hence R is also a unital ring with the inherited structure and the induced maps $\pi_r^A(\Sigma^r A) \to \operatorname{colim}_r \pi_r^A(\Sigma^r X) = \pi_0^{A,\mathrm{st}}(X) = R$ are ring isomorphisms.

Note also that since $\Sigma : [\Sigma A, \Sigma A] = \pi_1^A(A) \to [\Sigma^2 A, \Sigma^2 A] = \pi_2^A(A)$ is an isomorphism, we get that $[\Sigma A, \Sigma A] = \pi_1^A(A)$ is an abelian group.

We will define now another A-cellular complex which will be used to develop the obstruction theory for CW(A)-complexes. For a CW(A)-complex X, let C_n be the free R-module generated by the A-n-cells of X. We define the boundary map $d : C_n \to C_{n-1}$ in the following way. Let e_α^n be an A-n-cell of X, let g_α be its attaching map and let J_{n-1} be an index set for the A-$(n-1)$-cells. As usual, for $\beta \in J_{n-1}$, let $q_\beta : X^{n-1} \to X^{n-1}/(X^{n-1} - \overset{\circ}{e}{}_\beta^{n-1}) = \Sigma^{n-1}A$ be the quotient map. We define $d(e_\alpha^n) = \displaystyle\sum_{\beta \in J_{n-1}} [q_\beta g_\alpha] e_\beta^{n-1}$. Recall that this sum is finite since A is compact.

Now we wish to prove that (C_*, d) is a chain complex. We proceed as in [11] p. 95-98. Note that applying 4.1.14 we get

$$C_n = R^{(J_n)} = \bigoplus_{J_n} R \simeq \bigoplus_{J_n} \pi_n^A(\Sigma^n A) \simeq \pi_n^A(\bigvee_{J_n} \Sigma^n A) \simeq \pi_n^A(X^n/X^{n-1})$$

Thus, up to isomorphisms, we may think $d : \pi_n^A(X^n/X^{n-1}) \to \pi_{n-1}^A(X^{n-1}/X^{n-2})$.

We will give another description of the boundary map d. We define $\partial_n : X^n/X^{n-1} \to \Sigma X^{n-1}/X^{n-2}$ as the composition

$$X^n/X^{n-1} \xrightarrow{\psi^{-1}} X^n \underset{X^{n-1}}{\cup} \mathrm{C}X^{n-1} \xrightarrow{q_1} \Sigma X^{n-1} \xrightarrow{\Sigma q} \Sigma(X^{n-1}/X^{n-2})$$

where q_1 and q are quotient maps and ψ^{-1} is a homotopy inverse of the quotient map $\psi : X^n \underset{X^{n-1}}{\cup} \mathrm{C}X^{n-1} \to X^n/X^{n-1}$.

We will prove that the triangle

$$\begin{tikzcd}
\pi_n^A(X^n/X^{n-1}) \arrow[rr, "d"] \arrow[dr, "(\partial_n)_*"'] & & \pi_{n-1}^A(X^{n-1}/X^{n-2}) \\
& \pi_n^A(\Sigma(X^{n-1}/X^{n-2})) \arrow[ur, "\Sigma^{-1}"'] &
\end{tikzcd}$$

commutes.

As usual, for each A-n-cell e_α^n, $\alpha \in J_n$, let g_α be its attaching map and f_α be its characteristic map and consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{C}\Sigma^{n-1}A/\Sigma^{n-1}A \arrow[d, "\overline{qf_\alpha}"'] & \mathrm{C}\Sigma^{n-1}A \underset{\Sigma^{n-1}}{\cup} \mathrm{C}\Sigma^{n-1}A \arrow[l, "\psi_0"'] \arrow[r, "q_0"] \arrow[d, "f_\alpha \cup \mathrm{C}g_\alpha"] & \Sigma(\Sigma^{n-1}A) \arrow[d, "\Sigma g_\alpha"] \\
X^n/X^{n-1} & X^n \underset{X^{n-1}}{\cup} \mathrm{C}X^{n-1} \arrow[l, "\psi"] \arrow[r, "q_1"'] & \Sigma X^{n-1}
\end{tikzcd}$$

where ψ_0 and q_0 are quotient maps and $\overline{qf_\alpha}$ is the map induced by $f_\alpha \cup \mathrm{C}g_\alpha$ in the quotient spaces. Let $(\psi_0)^{-1}$ be a homotopy inverse of ψ_0. It follows that $q_1\psi^{-1}\overline{qf_\alpha} \simeq \Sigma g_\alpha q_0(\psi_0)^{-1}$.

It is easy to prove that, up to the standard homeomorphisms $\mathrm{C}\Sigma^{n-1}A/\Sigma^{n-1}A = \Sigma^{n-1}A \underset{\Sigma^{n-1}}{\cup} \mathrm{C}\Sigma^{n-1}A = \Sigma(\Sigma^{n-1}A) = \Sigma^n A$, the maps q_0 and ψ_0 are both homotopic to the identity map. Note also that the isomorphism $C_n \simeq \pi_n^A(X^n/X^{n-1})$ takes the basis $\{e_\alpha^n : \alpha \in J_n\}$ to $\{\overline{qf_\alpha} : \alpha \in J_n\}$. Hence, up to this isomorphism,

$$\begin{aligned}\Sigma^{-1}(\partial_n)_*(e_\alpha^n) &= \Sigma^{-1}[\partial_n\overline{qf_\alpha}] = \Sigma^{-1}[\Sigma q q_1 \psi^{-1}\overline{qf_\alpha}] \simeq \Sigma^{-1}[\Sigma q \Sigma g_\alpha q_0(\psi_0)^{-1}] \simeq \\ &\simeq \Sigma^{-1}[\Sigma q \Sigma g_\alpha] = \Sigma^{-1}[\Sigma(qg_\alpha)] = [qg_\alpha].\end{aligned}$$

Thus, $\Sigma^{-1}(\partial_n)_*(e_\alpha^n) = [qg_\alpha] \in \pi_{n-1}^A(X^{n-1}/X^{n-2}) \simeq C_{n-1}$.

We need now the following lemma. Its proof is formally identical to that of lemma 2.5.3.

Lemma 8.1.1. *Let A be an l-connected and compact CW-complex of dimension k with $k \leq 2l+2$ and let X be a CW(A)-complex. Let $r \geq 2$ and let $g : \Sigma^r A \to X^r$ be a continuous map. Let $q : X^r \to X^r/X^{r-1} = \bigvee_{r\text{-cells}} \Sigma^r A$ be the quotient map. Let J_r be an index set for the A-r-cells of X and for each $\beta \in J_r$ let $q_\beta : X^r \to \Sigma^r A$ be the quotient map which collapses $X^r - \overset{\circ}{e}{}^r_\beta$ to a point. Then*

$$[qg] = \sum_{\beta \in J_r} [i_\beta q_\beta g]$$

in $\pi_r(\bigvee_{J_r} \Sigma^r A)$.

Applying the previous lemma we obtain that $[qg_\alpha] = \sum_{\beta \in J_{n-1}} [i_\beta q_\beta g_\alpha] = \sum_{\beta \in J_{n-1}} [q_\beta g_\alpha][i_\beta]$ which corresponds to $\sum_{\beta \in J_{n-1}} [q_\beta g_\alpha] e_\beta^{n-1}$ under the isomorphism $\pi_{n-1}^A(\bigvee_{J_{n-1}} \Sigma^{n-1}A) \simeq C_{n-1}$. Hence, $d = \Sigma^{-1}(\partial_n)_*$.

Using this description of the boundary map d we will prove that $d^2 = 0$. We consider the following commutative diagram

$$\begin{tikzcd} X^n \underset{X^{n-1}}{\cup} CX^{n-1} \arrow[r, "q_1"] \arrow[d, "\psi"] & \Sigma X^{n-1} \arrow[r, "\Sigma i"] \arrow[d, "\Sigma q"] & \Sigma(X^{n-1} \underset{X^{n-2}}{\cup} CX^{n-2}) \arrow[r, "q_1'"] \arrow[d, "\Sigma\psi'"] & \Sigma^2 X^{n-2} \arrow[d, "\Sigma^2 q'"] \\ X^n/X^{n-1} \arrow[r, "\partial_n"'] & \Sigma(X^{n-1}/X^{n-2}) \arrow[r, "\mathrm{Id}"'] & \Sigma(X^{n-1}/X^{n-2}) \arrow[r, "\Sigma\partial_{n-1}"'] & \Sigma^2(X^{n-2}/X^{n-3}) \end{tikzcd}$$

where q_1' and q' are quotient maps and i is an inclusion.

Since $\Sigma q_1' \circ \Sigma i$ is the constant map and ψ is a homotopy equivalence we conclude that

$\Sigma\partial_{n-1} \circ \partial_n \simeq *$. Thus, from the commutative diagram

$$\begin{tikzcd}[column sep=small]
\pi_n^A(X^n/X^{n-1}) \arrow[rr, "d"] \arrow[dr, "(\partial_n)_*"'] & & \pi_{n-1}^A(X^{n-1}/X^{n-2}) \arrow[rr, "d"] \arrow[dr, "(\partial_{n-1})_*"'] & & \pi_{n-2}^A(X^{n-2}/X^{n-3}) \\
& \pi_n^A(\Sigma(X^{n-1}/X^{n-2})) \arrow[ur, "\Sigma^{-1}"'] \arrow[dr, "(\Sigma\partial_{n-1})_*"'] & & \pi_{n-1}^A(\Sigma(X^{n-2}/X^{n-3})) \arrow[ur, "\Sigma^{-1}"'] & \\
& & \pi_n^A(\Sigma^2(X^{n-2}/X^{n-3})) \arrow[ur, "\Sigma^{-1}"'] & &
\end{tikzcd}$$

we deduce that $d^2 = 0$.

Remark 8.1.2. Although S^0 does not satisfy the required relation between dimension and degree of connectedness (see p. 168), the suspension functor induces isomorphisms of groups $[\Sigma^n S^0, \Sigma^n S^0] \simeq [\Sigma^{n+1} S^0, \Sigma^{n+1} S^0]$ for $n \geq 1$. Hence, the previous construction also works for S^0 if the CW-complex X has only one 0-cell (recall that this is not a homotopical restriction). In this case, by [11] p. 95-98, the S^0-cellular chain complex coincides with the classical cellular chain complex.

The homology of this chain complex is not a topological invariant as the following example shows.

Remark 8.1.3. The homology of this A-cellular chain complex is not invariant by homeomorphisms. Indeed, take $A = S^3 \vee S^4$. Let $X = D^4 \vee D^5 \vee D^6 \vee D^7$. We will give two CW($A$)-complex structures to X. The first one is defined by the following pushouts

$$\begin{tikzcd}
A \arrow[r, "\mathrm{Id}"] \arrow[d] \arrow[dr, phantom, "push"] & X^0 = A \arrow[d] \\
\mathrm{C}A \arrow[r] & X^1 = D^4 \vee D^5
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma A \arrow[r, "*"] \arrow[d] \arrow[dr, phantom, "push"] & X^1 \arrow[d] \\
\mathrm{C}\Sigma A \arrow[r] & X^2 = D^4 \vee D^5 \vee S^5 \vee S^6
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^2 A = S^5 \vee S^6 \arrow[r, "\mathrm{inc}"] \arrow[d] \arrow[dr, phantom, "push"] & X^2 \arrow[d] \\
\mathrm{C}\Sigma^2 A \arrow[r] & X
\end{tikzcd}$$

while the second one is defined by the pushouts

$$\begin{tikzcd}
A = S^3 \vee S^4 \arrow[r, "\mathrm{in}_1 + *"] \arrow[d] \arrow[dr, phantom, "push"] & A = S^3 \vee S^4 \arrow[d] \\
\mathrm{C}A \arrow[r] & X^1 = D^4 \vee S^4 \vee S^5
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma A = S^4 \vee S^5 \arrow[r, "*"] \arrow[d] \arrow[dr, phantom, "push"] & X^1 = D^4 \vee S^4 \vee S^5 \arrow[d] \\
\mathrm{C}\Sigma A \arrow[r] & X^2 = D^4 \vee D^5 \vee S^5 \vee S^6
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^2 A = S^5 \vee S^6 \arrow[r, "\mathrm{inc}"] \arrow[d] \arrow[dr, phantom, "push"] & X^2 \arrow[d] \\
\mathrm{C}\Sigma^2 A \arrow[r] & X
\end{tikzcd}$$

Since A is 2-connected and has dimension 4, by theorem 11 of [19] p. 458 we get that $R \simeq \pi_0^A(A)$. Applying proposition 6.36 of [20] we obtain that

$$\begin{aligned} R &\simeq \pi_0^A(A) = [S^3 \vee S^4, S^3 \vee S^4] = \pi_3(S^3 \vee S^4) \oplus \pi_4(S^3 \vee S^4) = \\ &= \pi_3(S^3) \oplus \pi_3(S^4) \oplus \pi_4(S^3) \oplus \pi_4(S^4) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}. \end{aligned}$$

Before analyzing the A-cellular chain complexes we need to study the ring structure in R. Recall that we have defined the product in R as a composition in $\pi_0^A(A) \simeq R$. More precisely, if $f, g : A \to A$ are continuous maps then $[f].[g] = [g \circ f]$. We wish to obtain a concise description of this product under the isomorphism $\pi_0^A(A) \simeq \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$.

If $\alpha : A = S^3 \vee S^4 \to A = S^3 \vee S^4$ is a continuous map, we may decompose $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 : S^3 \to S^3 \vee S^4$ and $\alpha_2 : S^4 \to S^3 \vee S^4$. Let $i_1 : S^3 \to S^3 \vee S^4$ and $i_2 : S^4 \to S^3 \vee S^4$ be the inclusions and let $\rho_1 : S^3 \vee S^4 \to S^3$ and $\rho_2 : S^3 \vee S^4 \to S^4$ be the standard quotient maps. Note that, up to the isomorphism given by proposition 6.36 of [20] we have that $[\alpha_1] = ([q_1\alpha_1], [q_2\alpha_1])$ and $[\alpha_2] = ([q_1\alpha_2], [q_2\alpha_2])$. Hence, the isomorphism $\pi_0^A(A) \simeq \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$ takes $[\alpha]$ to $([q_1\alpha_1], [q_1\alpha_2], [q_2\alpha_2]) = ([q_1\alpha i_1], [q_1\alpha i_2], [q_2\alpha i_2])$.

Taking this into consideration we get that the product $[f].[g] = [g \circ f]$ can be translated into

$$([q_1 f i_1], [q_1 f i_2], [q_2 f i_2]).([q_1 g i_1], [q_1 g i_2], [q_2 g i_2]) = ([q_1(g \circ f)i_1], [q_1(g \circ f)i_2], [q_2(g \circ f)i_2])$$

As mentioned above, $[f_2] = ([q_1 f_2], [q_2 f_2])$ under the isomorphism $((q_1)_*, (q_2)_*)$ which is the inverse of $((i_1)_*, (i_2)_*) : \pi^4(S^3) \oplus \pi^4(S^4) \to \pi^4(S^3 \vee S^4)$. Hence,

$$[q_2(g \circ f)i_2] = [q_2 g i_1 q_1 f i_2] + [q_2 g i_2 q_2 f i_2] = [q_2 g i_2 q_2 f i_2]$$

since $q_2 g i_1 : S^3 \to S^4$ is nullhomotopic.

By a similar argument we get that

$$[q_1(g \circ f)i_1] = [q_1 g i_1 q_1 f i_1]$$

and

$$[q_1(g \circ f)i_2] = [q_1 g i_1 q_1 f i_2] + [q_1 g i_2 q_2 f i_2].$$

From these equations it is easy to conclude that the product in $\pi_0^A(A) \simeq \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$ is defined by $(a_1, a_2, a_3)(b_1, b_2, b_3) = (b_1 a_1, b_1 a_2 + b_2 a_3, b_3 a_3)$. Indeed, we only have to note that if $\gamma : S^n \to S^n$ and $\beta : S^n \to S^m$ are continuous maps and $[\gamma] \in \pi_n(S^n)$ is represented by $m \in \mathbb{Z}$ then $[\beta \circ \gamma] = m[\beta]$, which follows from the identities $[\beta \circ \gamma] = \beta_*([\gamma]) = \beta_*(m[\mathrm{Id}]) = m\beta_*([\mathrm{Id}]) = m[\beta]$.

Now it is easy to prove that the A-cellular chain complex corresponding to the first CW(A)-complex structure of X is

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \xrightarrow{\mathrm{Id}} R \xrightarrow{0} R \xrightarrow{\mathrm{Id}} R$$

whose homology is 0 in every degree.

On the other hand, we will prove now that the A-cellular chain complex corresponding to the second CW(A)-complex structure of X is

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \xrightarrow{d_3} R \xrightarrow{0} R \xrightarrow{d_1} R$$

where d_3 and d_1 are defined by $d_3(r_1, r_2, r_3) = (0, 0, r_3)$ and $d_1(r_1, r_2, r_3) = (r_1, r_2, 0)$. Hence its homology is $\mathbb{Z}$ in degrees 0 and 1 and $\mathbb{Z} \oplus \mathbb{Z}_2$ in degrees 2 and 3.

The map d_1 is defined by $d_1(e^2) = [q^{(2)} g^{(2)}]e^1$, where $q^{(2)} : X^1 \to X^1/(X^1 - \overset{\circ}{e^1}) = \Sigma A$ is the quotient map and $g^{(2)} = \text{in}_1 + *$ is the attaching map of the A-cell e^2. Note that $q^{(2)}$ is the identity map and that $g^{(2)} = \text{in}_1 + *$ corresponds to $[(1, 0, 0)] \in \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$ under the isomorphism mentioned before. Hence $d_1(r_1, r_2, r_3) = (r_1, r_2, r_3).d_1(e^2) = (r_1, r_2, r_3).(1, 0, 0) = (r_1, r_2, 0)$.

In a similar way, the map d_3 is defined by $d_3(e^4) = [q^{(4)} g^{(4)}]e^3$, where $q^{(4)} : X^3 \to X^3/(X^3 - \overset{\circ}{e^3}) = \Sigma^3 A$ is the quotient map and $g^{(2)} = \text{inc}$ is the attaching map of the A-cell e^2. Note that $q^{(2)} g^{(2)}$ corresponds to $[(0, 0, 1)] \in \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. Hence $d_1(r_1, r_2, r_3) = (r_1, r_2, r_3).d_1(e^2) = (r_1, r_2, r_3).(0, 0, 1) = (0, 0, r_3)$.

Note that, as it occurs with the A-cellular chain complex of chapter 5, this new A-cellular chain complex does not take into consideration the way that A-n-cells are attached to A-k-cells for $k \leq n - 2$. Hence, different CW(A)-decompositions of the same space may lead to different results, as the example above shows. Again, the key fact here is including in one of the structures an A-2-cell and an A-3-cell such that their attaching maps involve in a non-trivial way the A-0-cell and the A-1-cell respectively.

Remark 8.1.4. Suppose that A is a finite l-connected CW-complex of dimension k with $k \leq 2l$ and X is a finite CW(A)-complex. Let $C = (C_*, d)$ be the A-cellular complex defined above. Then $\chi(C) = \chi_A(X).\text{rg}(R)$. But since $\chi(H_*(C)) = \chi(C)$ and $\chi_A(X).\chi(A) = \chi(X)$ we obtain

$$\chi(H_*(C)).\chi(A) = \chi(X).\text{rg}(R).$$

Hence for fixed X and A with $\chi(A) \neq 0$, $\chi(H_*(C))$ is uniquely determined although $H_*(C)$ is not a topological invariant.

On the other hand, if A is fixed and we know $H_*(C)$, not every CW(A)-complex X is possible, since if $\text{rg}(R) \neq 0$ then $\chi(X) = \dfrac{\chi(H_*(C)).\chi(A)}{\text{rg}(R)}$.

In a similar way, we consider now another case. Let A be finite CW-complex with finite homology groups and let X be a finite CW(A)-complex. We have proved that $\chi_m(X) = \chi_m(A)^{\chi_A(X)}$. By 6.2.2 and 6.1.10 we obtain that the groups $\pi_n^A(\Sigma^n A)$ are finite for all $n \geq 2$. Hence $R \simeq \pi_0^{A,\text{st}}(A)$ is a finite group. Then, $(\#R)^{\chi_A(X)} = \chi_m(C) = \chi_m(H_*(C))$.

8.2 Obstruction cocycle

Let $f : X^{n-1} \to Y$ be a continuous map. If $n \geq 2$, the abelian group $\pi_n^A(Y)$ has a right $\pi_n^A(\Sigma^n A)$-module structure defined by $[\alpha][g] = [\alpha \circ g]$ for $\alpha \in \pi_n^A(\Sigma^n A)$, $g \in \pi_n^A(Y)$. This structure induces a right R-module structure in $\pi_n^A(Y)$. We call $C^n(X, \pi_{n-1}^A(Y)) = \text{Hom}_R(C_n, \pi_{n-1}^A(Y))$ and define $c(f) \in C^n(X, \pi_{n-1}^A(Y))$ by $c(f)(e_\alpha^n) = [f \circ g_\alpha]$ and extend it linearly.

Note also that there is a bijection $\text{Hom}_R(C_n, \pi_{n-1}^A(Y)) \leftrightarrow \text{Hom}_{\text{Sets}}(A\text{-}n\text{-cells}, \pi_{n-1}^A(Y))$. It is clear that $c(f) = 0$ if and only if f can be extended to X^n.

Theorem 8.2.1. *$c(f)$ is a cocycle.*

Proof. Recall the following commutative diagrams from earlier (pages 169 and 170), where we have relabeled some of the arrows.

$$\begin{tikzcd}
X^n/X^{n-1} \arrow[r, "(\psi_n^X)^{-1}"] \arrow[rrr, bend right, "\partial_n"'] & X^n \underset{X^{n-1}}{\cup} \mathrm{C}X^{n-1} \arrow[r, "{}_nq_1^X"] & \Sigma X^{n-1} \arrow[r, "\Sigma q_{n-1}^X"] & \Sigma(X^{n-1}/X^{n-2})
\end{tikzcd}$$

$$\begin{tikzcd}
C_n \arrow[rr, "d"] & & C_{n-1} \\
\pi_n^A(X^n/X^{n-1}) \arrow[u, "\phi_n", "\simeq"'] \arrow[rr, "d"] \arrow[dr, "(\partial_n)_*"'] & & \pi_{n-1}^A(X^{n-1}/X^{n-2}) \arrow[u, "\simeq", "\phi_{n-1}"'] \\
& \pi_n^A(\Sigma(X^{n-1}/X^{n-2})) \arrow[ur, "\Sigma^{-1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{C}\Sigma^{n-1}A/\Sigma^{n-1}A \arrow[d, "\overline{qf_\alpha}"'] & \mathrm{C}\Sigma^{n-1}A \underset{\Sigma^{n-1}}{\cup} \mathrm{C}\Sigma^{n-1}A \arrow[l, "\psi_n^A"'] \arrow[r, "{}_nq_1^A"] \arrow[d, "f_\alpha \cup \mathrm{C}g_\alpha"] & \Sigma(\Sigma^{n-1}A) \arrow[d, "\Sigma g_\alpha"] \\
X^n/X^{n-1} & X^n \underset{X^{n-1}}{\cup} \mathrm{C}X^{n-1} \arrow[l, "\psi_n^X"] \arrow[r, "{}_nq_1^X"'] & \Sigma X^{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
X^n \underset{X^{n-1}}{\cup} \mathrm{C}X^{n-1} \arrow[r, "{}_nq_1^X"] \arrow[d, "\psi_n^X"'] & \Sigma X^{n-1} \arrow[r, "\Sigma i_{n-1}"] \arrow[d, "\Sigma q_{n-1}^X"'] & \Sigma(X^{n-1} \underset{X^{n-2}}{\cup} \mathrm{C}X^{n-2}) \arrow[r, "\Sigma({}_{n-1}q_1^X)"] \arrow[d, "\Sigma\psi_{n-1}^X"'] & \Sigma^2 X^{n-2} \arrow[d, "\Sigma^2 q_{n-2}^X"'] \\
X^n/X^{n-1} \arrow[r, "\partial_n"'] & \Sigma(X^{n-1}/X^{n-2}) \arrow[r, "\mathrm{Id}"'] & \Sigma(X^{n-1}/X^{n-2}) \arrow[r, "\Sigma\partial_{n-1}"'] & \Sigma^2(X^{n-2}/X^{n-3})
\end{tikzcd}$$

Since the composition $\Sigma({}_{n-1}q_1^X) \circ \Sigma i_{n-1}$ is the constant map and ψ_n^X is a homotopy equivalence we get that $\Sigma({}_{n-1}q_1^X)\Sigma(\psi_{n-1}^X)^{-1}\partial_n \simeq *$.

It easy to see that the isomorphism $\pi_n^A(\bigvee_{J_n} \Sigma^n A) \simeq \pi_n^A(X^n/X^{n-1})$ is an isomorphism of $\pi_n^A(\Sigma^n A)$-modules since the distributive properties hold by a similar argument to that in page 168. Hence it is also an isomorphism of R-modules.

We will prove now that the isomorphisms ϕ_n are morphisms of R-modules for $n \geq 2$. It suffices to prove that $\phi_n : \pi_n^A(\bigvee_{J_n} \Sigma^n A) \to \bigoplus_{J_n} \pi_n^A(\Sigma^n A)$ is a morphism of $\pi_n^A(\Sigma^n A)$-modules.

But $(\phi_n)^{-1} = \bigoplus_{\alpha \in J_n} (i_\alpha)_*$ is easily seen to be a morphism of $\pi_n^A(\Sigma^n A)$-modules as for $[g] \in \pi_n^A(\Sigma^n A)$ and $\{f_\alpha\}_{\alpha \in J_n} \in \bigoplus_{J_n} \pi_n^A(\Sigma^n A)$ we have that

$$\begin{aligned}
\bigoplus_{\alpha \in J_n} (i_\alpha)_*([g].\{f_\alpha\}_{\alpha \in J_n}) &= \bigoplus_{\alpha \in J_n} (i_\alpha)_*(\{f_\alpha \circ g\}_{\alpha \in J_n}) = \sum_{\alpha \in J_n} [i_\alpha f_\alpha g] = \sum_{\alpha \in J_n} [g][i_\alpha f_\alpha] = \\
&= [g] \sum_{\alpha \in J_n} [i_\alpha f_\alpha] = [g] \bigoplus_{\alpha \in J_n} (i_\alpha)_*(\{f_\alpha\}_{\alpha \in J_n}).
\end{aligned}$$

Hence (ϕ_n) is an isomorphism of R-modules for $n \geq 2$.

Now,

$$\begin{aligned}\partial_n\overline{qf_\alpha} &= (\partial_n)_*(\phi_n^{-1}(e^n_\alpha)) = \Sigma(d(\phi_n^{-1}(e^n_\alpha))) = \Sigma(\phi_n^{-1}(de^n_\alpha)) = \\ &= \Sigma(\phi_n^{-1}(\sum_{\beta\in J_{n-1}}[q_\beta g_\alpha]e_\beta^{n-1})) = \Sigma(\sum_{\beta\in J_{n-1}}[q_\beta g_\alpha]i_\beta) = \Sigma(\sum_{\beta\in J_{n-1}}[i_\beta q_\beta g_\alpha])\end{aligned}$$

Thus, $\partial_n\overline{qf_\alpha} = \Sigma(\sum_{\beta\in J_{n-1}}[i_\beta q_\beta g_\alpha])$.

Since we have proved that $\Sigma({}_{n-1}q_1^X)\Sigma(\psi_{n-1}^X)^{-1}\partial_n \simeq *$, we get

$$\begin{aligned} * &\simeq \Sigma({}_{n-1}q_1^X)\Sigma(\psi_{n-1}^X)^{-1}\partial_n\overline{qf_\alpha} = \Sigma({}_{n-1}q_1^X(\psi_{n-1}^X)^{-1}(\sum_{\beta\in J_{n-1}}[i_\beta q_\beta g_\alpha])) = \\ &= \Sigma(\sum_{\beta\in J_{n-1}}[{}_{n-1}q_1^X(\psi_{n-1}^X)^{-1}i_\beta q_\beta g_\alpha])) = \Sigma(\sum_{\beta\in J_{n-1}}[{}_{n-1}q_1^X(\psi_{n-1}^X)^{-1}\overline{qf_\alpha}q_\beta g_\alpha])) = \\ &= \Sigma(\sum_{\beta\in J_{n-1}}[\Sigma g_\alpha({}_nq_1^A)(\psi_{n-1}^A)^{-1}q_\beta g_\alpha])) = \Sigma(\sum_{\beta\in J_{n-1}}[\Sigma g_\alpha q_\beta g_\alpha])\end{aligned}$$

because $({}_nq_1^A)(\psi_{n-1}^A)^{-1} \simeq \mathrm{Id}$. But Σ is an isomorphism, hence $\sum_{\beta\in J_{n-1}}[\Sigma g_\alpha q_\beta g_\alpha] = 0$. Then

$$\begin{aligned} d^*(c(f))(e_\beta^{n+2}) &= c(f)(de_\beta^{n+2}) = c(f)(\sum_{\alpha\in J_{n+1}}[q_\alpha g_\beta]e_\alpha^{n+1}) = \sum_{\alpha\in J_{n+1}}[q_\alpha g_\beta][fg_\alpha] = \\ &= \sum_{\alpha\in J_{n+1}}[\Sigma g_\alpha q_\alpha g_\beta][f] = 0.\end{aligned}$$

Hence $c(f)$ is a cocycle. $\square$

8.3 Difference cochain

Definition 8.3.1. Let A be a CW-complex and let X be a CW(A)-complex. Let $f, g : X^n \to Y$ be continuous maps such that $f|_{X^{n-1}} \simeq g|_{X^{n-1}}$ and let $H : IX^{n-1} \to Y$ be a homotopy between $f|_{X^{n-1}}$ and $g|_{X^{n-1}}$. We define the *difference cochain of f and g with respect to H* as the cochain $d(f, H, g) \in \mathrm{Hom}_R(C_n, \pi_n^A(Y))$ defined by

$$d(f, H, g)(e^n_\alpha) = [(f \circ f_\alpha) \cup (H \circ Ig_\alpha) \cup (g \circ f_\alpha)]$$

with

$$(f \circ f_\alpha) \cup (H \circ Ig_\alpha) \cup (g \circ f_\alpha) : \mathrm{C}\Sigma^{n-1}A \underset{\Sigma^{n-1}A}{\cup} I(\Sigma^{n-1}A) \underset{\Sigma^{n-1}A}{\cup} \mathrm{C}\Sigma^{n-1}A = \Sigma^n A \to Y$$

where g_α the attaching map of the cell e^n_α and f_α is its characteristic map.

In the particular case that $f|_{X^{n-1}} = g|_{X^{n-1}}$ and that H is stationary we will write $d(f, H, g) = d(f, g)$.

Remark 8.3.2. If $A = \Sigma A'$ with A' a CW-complex and let X, f, g and H be as above. By 3.1.24 we know that IX is a CW(A)-complex if X is. Moreover, the CW(A)-complex structure of IX is induced by that of X as in the standard case. It follows that $d(f, H, g)(e^n_\alpha) = c(H)(Ie^n_\alpha)$.

Remark 8.3.3. If $f|_{X^{n-1}} = g|_{X^{n-1}}$ then $d(f,g)(e^n_\alpha) = [(f \circ f_\alpha) \cup (g \circ f_\alpha)]$ with

$$(f \circ f_\alpha) \cup (g \circ f_\alpha) : \mathrm{C}\Sigma^{n-1}A \underset{\Sigma^{n-1}A}{\cup} \mathrm{C}\Sigma^{n-1}A = \Sigma^n A \to Y.$$

Theorem 8.3.4. *Let $A = \Sigma A'$ with A' a CW-complex and let $\delta = d^*$. Let f, g be as above. Then $\delta(d(f,H,g)) = c(f) - c(g)$.*

Proof. By the remark above $\delta(d(f,H,g))(e^n_\alpha) = d(f,H,g)(de^n_\alpha) = c(H)(I(de^n_\alpha))$, where $I(de^n_\alpha) = \displaystyle\sum_{\beta \in J_{n-1}} a_\beta I e^{n-1}_\beta$ if $d(e^n_\alpha) = \displaystyle\sum_{\beta \in J_{n-1}} a_\beta e^{n-1}_\beta$.

On the other hand, by theorem 8.2.1, $c(H)$ is a cocycle. Hence,

$$\begin{aligned} 0 &= \delta(c(H))(Ie^n_\alpha) = c(H)(d(Ie^n_\alpha)) = c(H)(e^n_\alpha \times \{1\} - I(de^n_\alpha) - e^n_\alpha \times \{0\}) = \\ &= c(H)(e^n_\alpha \times \{1\}) - c(H)(I(de^n_\alpha)) - c(H)(e^n_\alpha \times \{0\}) \end{aligned}$$

But clearly $c(H)(e^n_\alpha \times \{0\}) = c(f)(e^n_\alpha)$ and $c(H)(e^n_\alpha \times \{1\}) = c(g)(e^n_\alpha)$ and the result follows. $\square$

Theorem 8.3.5. *Let A, X and f be as above and let $d \in \mathrm{Hom}_R(C_n, \pi^A_n(Y))$. Then there exists a continuous map $g : X^n \to Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$ and $d(f,g) = d$.*

Proof. We will define g in each A-n-cell of X extending $f|_{X^{n-1}}$. For each A-n-cell e^n_α, let g_α be its attaching map and f_α its characteristic map. Let $\gamma = d(e^n_\alpha) \in \pi^A_n(Y)$ and let $H : \mathrm{C}\Sigma^{n-1}A \times I \to Y$ be a homotopy from $\gamma|_{\mathrm{C}\Sigma^{n-1}A}$ to $f \circ f_\alpha$.

Let $j_1, j_2 : \mathrm{C}\Sigma^{n-1}A \to \Sigma^n A$ be the inclusion maps defined by the following pushout diagram

$$\begin{tikzcd} \Sigma^{n-1}A \arrow[r, "\mathrm{inc}"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "push"] & \mathrm{C}\Sigma^{n-1}A \arrow[d, "j_1"] \\ \mathrm{C}\Sigma^{n-1}A \arrow[r, "j_2"'] & \Sigma^n A \end{tikzcd}$$

Note that j_1 and j_2 are analogous to the inclusions of the northern and southern hemispheres in the sphere.

We consider the following commutative diagram of solid arrows

$$\begin{tikzcd} \mathrm{C}\Sigma^{n-1}A \arrow[r, "i_0"] \arrow[d, "j_1"'] & I(\mathrm{C}\Sigma^{n-1}A) \arrow[d, "Ij_1"] \arrow[ddr, bend left, "H"] & \\ \Sigma^n A \arrow[r, "i_0"] \arrow[drr, bend right, "\gamma"'] & I(\Sigma^n A) \arrow[dr, dotted, "\overline{H}"] & \\ & & Y \end{tikzcd}$$

Since j_1 is a cofibration, the dotted arrow $\overline{H}$ exists. We extend g to e^n_α in the following way

$$\begin{tikzcd}
\Sigma^{n-1}A \arrow[r, "g_\alpha"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "push"] & X^{n-1} \arrow[d] \arrow[ddr, bend left, "g|_{X^{n-1}}=f|_{X^{n-1}}"] & \\
C\Sigma^{n-1}A \arrow[r, "f_\alpha"'] \arrow[drr, bend right, "\overline{H}i_1j_2"'] & X^n \arrow[dr, dotted, "g"] & \\
& & Y
\end{tikzcd}$$

Note that $\overline{H}i_1j_2\mathrm{inc} = \overline{H}i_1j_1\mathrm{inc} = \overline{H}Ij_1i_1\mathrm{inc} = Hi_1\mathrm{inc} = ff_\alpha\mathrm{inc} = fg_\alpha$. Hence, the extension exists.

Finally, $\overline{H}$ is a homotopy from γ to $(f \circ f_\alpha) \cup (g \circ f_\alpha)$. Thus, $d(f,g)(e^n_\alpha) = [(f \circ f_\alpha) \cup (g \circ f_\alpha)] = [\gamma] = d(e^n_\alpha)$. □

Theorem 8.3.6. *Let X be a CW(A)-complex and let $f : X^n \to Y$ be a continuous map. Then there exists a continuous map $g : X^{n+1} \to Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$ if and only if $c(f)$ is a coboundary.*

Proof. If such a map g exists then $\delta(d(f,g)) = c(f) - c(g)$ but since $g : X^n \to Y$ may be extended to X^{n+1} then $c(g) = 0$. Hence $c(f)$ is a coboundary.

Conversely, if $c(f) = \delta d$ we take $g : X^n \to Y$ such that $g|_{X^{n-1}} = f|_{X^{n-1}}$ and $d(f,g) = d$. Then $c(f) = \delta d = \delta d(f,g) = c(f) - c(g)$. Hence $c(g) = 0$ and g can be extended to X^{n+1}. □

Theorem 8.3.7. *Let A be the suspension of a CW-complex and let X be a CW(A)-complex. Let $f, g : X^n \to Y$ be continuous maps. Then*

(a) $f \simeq g$ rel X^{n-1} if and only if $d(f,g) = 0$.

(b) $f \simeq g$ rel X^{n-2} if and only if $\overline{d(f,g)} = 0$ in $H^n(C^*, \delta)$.

Proof.
(a) We define $H : IX^{n-1} \cup (X^n \times \{0,1\}) \to Y$ by

$$H(x,t) = \begin{cases} f(x) & \text{if } t = 0 \text{ or } x \in X^{n-1} \\ g(x) & \text{if } t = 1 \end{cases}$$

It is clear that $f \simeq g$ rel X^{n-1} if and only if H can be extended to IX^n, which is equivalent to $c(H) = 0$. By remark 8.3.2 this holds if and only if $d(f,g) = 0$.

(b) Define H as above. By (a relative version of) the previous theorem the map $H|_{IX^{n-2} \cup (X^n \times \{0,1\})}$ can be extended to IX^n if and only if $c(H)$ is a coboundary, or equivalently, if and only if $\overline{d(f,g)} = 0$ in $H^n(C^*, \delta)$. □

8.4 Stable A-homotopy

Theorem 8.4.1. *Let A be a CW-complex. Then $\pi_*^{A,st}(\text{–})$ defines a reduced homology theory on the category of CW(A)-complexes.*

Moreover, $\pi_*^{A,st}(\text{–})$ satisfies the wedge axiom if A is compact.

Proof. The graded functor $\pi_*^{A,\mathrm{st}}(-)$ is clearly homotopy invariant and the suspension axiom holds trivially. The exact sequence axiom follows from 4.1.18.

Suppose now that A is compact. Let J be an index set and let X_α, $\alpha \in J$ be CW(A)-complexes. We wish to prove that $\pi_n^{A,\mathrm{st}}(\bigvee_{\alpha\in J} X_\alpha) = \bigoplus_{\alpha\in J} \pi_n^{A,\mathrm{st}}(X_\alpha)$.

Firstly, we prove the case $\#J = 2$. From 4.1.12 we deduce that for fixed $r \geq 2$ $\pi_{r+n}^A(\Sigma^n X \vee \Sigma^n Y) = \pi_{r+n}^A(\Sigma^n X) \oplus \pi_{r+n}^A(\Sigma^n Y)$ for $n \geq r + \dim(A)$. Hence, taking colimits we get $\pi_r^{A,\mathrm{st}}(X \vee Y) = \pi_r^{A,\mathrm{st}}(X) \oplus \pi_r^{A,\mathrm{st}}(Y)$.

If J is a finite set, the result follows from the case $\#J = 2$ using an inductive argument.

For the general case, note that if $K \subseteq \bigvee_{\alpha\in J} X_\alpha$ is a compact set there exists a finite subset $J' \subseteq J$ such that $K \subseteq \bigvee_{\alpha\in J'} X_\alpha$. Here we are using that X_α is a $T1$ space for all $\alpha \in J$. The result follows by a similar argument as the one used in the proof of 4.1.14. $\square$

Remark 8.4.2. Let X be a CW(A)-complex and let $B \subseteq X$ be an A-subcomplex. There are long exact sequences

$$\cdots \longrightarrow \pi_n^{A,\mathrm{st}}(B) \longrightarrow \pi_n^{A,\mathrm{st}}(X) \longrightarrow \pi_n^{A,\mathrm{st}}(X/B) \longrightarrow \pi_{n-1}^{A,\mathrm{st}}(B) \longrightarrow \cdots$$

where the index n runs through the integers.

Thus, if A is a finite CW-complex and X is a CW(A)-complex there is a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots \arrow[r] & \pi_{n+1}^{A,\mathrm{st}}(X_A^p) \arrow[r] \arrow[d] & \pi_{n+1}^{A,\mathrm{st}}(X_A^p/X_A^{p-1}) \arrow[r] & \pi_n^{A,\mathrm{st}}(X_A^{p-1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_{n+1}^{A,\mathrm{st}}(X_A^{p+1}) \arrow[r] \arrow[d] & \pi_{n+1}^{A,\mathrm{st}}(X_A^{p+1}/X_A^p) \arrow[r] & \pi_n^{A,\mathrm{st}}(X_A^p) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_{n+1}^{A,\mathrm{st}}(X_A^{p+2}) \arrow[r] \arrow[d] & \pi_{n+1}^{A,\mathrm{st}}(X_A^{p+2}/X_A^{p+1}) \arrow[r] & \pi_n^{A,\mathrm{st}}(X_A^{p+1}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & & \vdots &
\end{tikzcd}$$

where $X_A^p = *$ for $p < 0$. So, $\pi_n^{A,\mathrm{st}}(X_A^p) = 0$ for $p < 0$.

On the other hand, $\pi_n^A(X_A^p) = \pi_n^A(X)$ for $p \geq \dim(A) + n + 1$. Hence $\pi_n^{A,\mathrm{st}}(X_A^p) = \pi_n^{A,\mathrm{st}}(X)$ for $p \geq \dim(A) + n + 1$.

Then there is an spectral sequence $\{E_{p,q}^a\}$ with $E_{p,q}^1 = \pi_{p+q}^{A,\mathrm{st}}(X^p/X^{p-1})$ which converges to $\pi_n^{A,\mathrm{st}}(X)$. Note that, by the wedge axiom, we get

$$E_{p,q}^1 = \pi_{p+q}^{A,\mathrm{st}}(X^p/X^{p-1}) = \pi_{p+q}^{A,\mathrm{st}}(\bigvee_{A\text{-}p\text{-cells}} \Sigma^p A) = \bigoplus_{A\text{-}p\text{-cells}} \pi_{p+q}^{A,\mathrm{st}}(\Sigma^p A) = \bigoplus_{A\text{-}p\text{-cells}} \pi_q^{A,\mathrm{st}}(A).$$

So, we have proved the following theorem

Theorem 8.4.3. *Let A be a finite CW-complex and let X be a CW(A)-complex. Then there exists a spectral sequence $\{E^a_{p,q}\}$ with $E^1_{p,q} = \bigoplus_{A-p-cells} \pi_q^{A,st}(A)$ which converges to $\pi_*^{A,st}(X)$.*

Note also that (for $n \geq 1$) the map $\pi_n^A(X) \to \operatorname{colim}_j \pi^A_{n+j}(\Sigma^j X) = \pi_n^{A,\mathrm{st}}(X)$ is a kind of Hurewicz morphism. Also, by theorem 11 of [19] p. 458 we get that if X is m-connected and $n + \dim(A) \leq 2m$ then the previous map is an isomorphism.

Appendix A

Universal coefficient theorems and Künneth formula

In this appendix we recall some useful formulas for computation of homology and cohomology groups: the universal coefficient theorems and the Künneth formula. The first ones relate homology and cohomology groups with coefficients with homology groups with integral coefficients. The latter helps to compute the homology groups of the product of spaces from the homology groups of each one.

We begin by recalling some basic facts about Tor and Ext functors.

Proposition A.1. *Let G, H and G_i, $i \in I$, be abelian groups.*

(a) $\mathrm{Tor}(G, H) \simeq \mathrm{Tor}(H, G)$.

(b) $\mathrm{Tor}(\bigoplus_{i\in I} G_i, H) \simeq \bigoplus_{i\in I} \mathrm{Tor}(G_i, H)$.

(c) $\mathrm{Tor}(G, H) = 0$ *if G or H is torsionfree.*

(d) If $T(G)$ is the torsion subgroup of G then $\mathrm{Tor}(G, H) \simeq \mathrm{Tor}(T(G), H)$.

(e) If $n \in \mathbb{N}$ and $\mu_n : G \to G$ is defined by $\mu_n(g) = ng$, then $\mathrm{Tor}(\mathbb{Z}_n, G) \simeq \ker(\mu_g)$.

(f) If $0 \to A \to B \to C \to 0$ is a short exact sequence, then there is an exact sequence

$$0 \longrightarrow \mathrm{Tor}(G, A) \longrightarrow \mathrm{Tor}(G, B) \longrightarrow \mathrm{Tor}(G, C) \longrightarrow G \otimes A \longrightarrow G \otimes B \longrightarrow G \otimes C \longrightarrow 0$$

Proposition A.2. *Let G, H and H' be abelian groups. Then*

(a) $\mathrm{Ext}(H \oplus H', G) \simeq \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.

(b) $\mathrm{Ext}(H, G) = 0$ *if H is free.*

(c) $\mathrm{Ext}(\mathbb{Z}_n, G) \simeq G/nG$.

Theorem A.3 (Universal coefficient theorem for homology)**.** *Let (C_*, d_*) be a chain complex of free abelian groups and let G be an abelian group. Then for each $n \in \mathbb{N}$ there are natural short exact sequences*

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C; G) \longrightarrow \mathrm{Tor}(H_{n-1}(C), G) \longrightarrow 0$$

Moreover, these exact sequences split (but not naturally).

Corollary A.4. *Let X be a topological space and let $A \subseteq X$ be a subspace. Then*

(a) There exist natural short exact sequences

$$0 \longrightarrow H_n(X; \mathbb{Z}) \otimes G \longrightarrow H_n(X; G) \longrightarrow \mathrm{Tor}(H_{n-1}(X; \mathbb{Z}), G) \longrightarrow 0$$

for all $n \in \mathbb{N}$, and these exact sequences split, but not naturally.

(b) There exist natural short exact sequences

$$0 \longrightarrow H_n(X, A; \mathbb{Z}) \otimes G \longrightarrow H_n(X, A; G) \longrightarrow \mathrm{Tor}(H_{n-1}(X, A; \mathbb{Z}), G) \longrightarrow 0$$

for all $n \in \mathbb{N}$, and these exact sequences split, but not naturally.

Corollary A.5. *Let X be a topological space and let $n \in \mathbb{N}$.*

(a) $H_n(X; \mathbb{Q}) \simeq H_n(X; \mathbb{Z}) \otimes \mathbb{Q}$.

(b) If $H_n(X; \mathbb{Z})$ and $H_{n-1}(X; \mathbb{Z})$ are finitely generated and p is a prime number then $H_n(X; \mathbb{Z}_p)$ consists of

- *a $\mathbb{Z}_p$ summand for each $\mathbb{Z}$ summand of $H_n(X; \mathbb{Z})$.*
- *a $\mathbb{Z}_p$ summand for each $\mathbb{Z}_{p^k}$ summand of $H_n(X; \mathbb{Z})$ with $k \geq 1$.*
- *a $\mathbb{Z}_p$ summand for each $\mathbb{Z}_{p^k}$ summand of $H_{n-1}(X; \mathbb{Z})$ with $k \geq 1$.*

Corollary A.6. *(a) Let X be a topological space. Then $\widetilde{H}_n(X; \mathbb{Z}) = 0$ for all $n \geq 0$ if and only if $\widetilde{H}_n(X; \mathbb{Q}) = 0$ for all $n \geq 0$ and $\widetilde{H}_n(X; \mathbb{Z}_p) = 0$ for all $n \geq 0$ and for all prime numbers p.*

(b) A continuous map $f : X \to Y$ induces isomorphisms on integral homology groups if and only if it induces isomorphisms on homology groups with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients for all prime numbers p.

Theorem A.7 (Universal coefficient theorem for cohomology)**.** *Let (C_*, d_*) be a chain complex of free abelian groups and let G be an abelian group. Then for each $n \in \mathbb{N}$ there are split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \longrightarrow \mathrm{Hom}(H_n(C), G) \longrightarrow 0$$

Corollary A.8. *Let X be a topological space and let G be an abelian group. Then for each $n \in \mathbb{N}$ there are split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X; \mathbb{Z}), G) \longrightarrow H^n(X; G) \longrightarrow \mathrm{Hom}(H_n(X; \mathbb{Z}), G) \longrightarrow 0$$

Corollary A.9. *Let (C_*, d_*) be a chain complex of free abelian groups and let G be an abelian group. For $n \in \mathbb{N}$ let T_n be the torsion subgroup of $H_n(C)$ and let $k \in \mathbb{N}$. If $H_k(C)$ and $H_{k-1}(C)$ are finitely generated then $H^n(C;\mathbb{Z}) \simeq (H_n(C)/T_n) \oplus T_{n-1}$.*

Corollary A.10. *Let G be an abelian group and let $f : X \to Y$ be a continuous map such that f induces isomorphisms in homology groups. Then $f_* : H^n(X;G) \to H^n(Y;G)$ is an isomorphism for all $n \geq 0$.*

We give now the topological Künneth formula which derives directly from its algebraic version. However, we will not develop this one here.

Theorem A.11 (Topological Künneth formula)**.** *Let X and Y be CW-complexes and let R be a principal ideal domain. Then, there are natural short exact sequences*

$$0 \longrightarrow \bigoplus_{0\leq i\leq n} (H_i(X) \underset{R}{\otimes} H_{n-i}(Y)) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{0\leq i\leq n-1} \mathrm{Tor}_R(H_i(X), H_{n-1-i}(Y)) \longrightarrow 0$$

for all $n \in \mathbb{N}$, where homology means homology with coefficients in R. Moreover, these short exact sequences split (but not naturally).

There exists also a relative version of this formula which can be found in [8]. From this relative version we can deduce the reduced Künneth formula:

Theorem A.12 (Reduced Künneth formula)**.** *Let X and Y be CW-complexes with base points x_0 and y_0 respectively and let R be a principal ideal domain. Then, there are natural short exact sequences*

$$0 \longrightarrow \bigoplus_{0\leq i\leq n} (\widetilde{H}_i(X) \underset{R}{\otimes} \widetilde{H}_{n-i}(Y)) \longrightarrow \widetilde{H}_n(X \wedge Y) \longrightarrow \bigoplus_{0\leq i\leq n-1} \mathrm{Tor}_R(\widetilde{H}_i(X), \widetilde{H}_{n-1-i}(Y)) \longrightarrow 0$$

for all $n \in \mathbb{N}$, where (reduced) homology means (reduced) homology with coefficients in R and where $X \wedge Y$ denotes the smash product of X and Y, i.e. $X \wedge Y = X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$. Moreover, these short exact sequences split (but not naturally).

In particular, if $Y = S^r$, we obtain isomorphisms $\widetilde{H}_n(X) \simeq \widetilde{H}_{n+r}(\Sigma^r X)$.
Also, if Y is a Moore space we obtain the following corollary.

Corollary A.13. *Let G be an abelian group, let $r \in \mathbb{N}$ and let Y be a Moore space of type (G, r). Let X be a CW-complex and let $n \in \mathbb{N}$. Then there are natural isomorphisms $\widetilde{H}_n(X;G) \simeq \widetilde{H}_{n+r}(X \wedge Y;\mathbb{Z})$.*

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