Detecting Linear Groups

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(joint with Fred Cohen, Marston Conder, Jonathan Lopez, Vassilis Metaftsis)

Basic Question

Question. Given a group homomorphism $f : \Gamma \to G$, when is f 1-1?

Simple Answer. f is 1-1 $\Longleftrightarrow$ ker$(f) = \{1\}$.

The answer is not easy because it is related to the word problem.

We are interested in the case when $G = GL(n, \mathbb{R})$ for some n.

Topological Applications

Farrell – Jones. If a group Γ is discrete linear then it satisfies the Fiber Isomorphism Conjecture for the Pseudoisotopy Spectrum.

- A group is linear if it admits a faithful finite dimensional linear representation.
- The Fiber Isomorphism Conjecture is one of the most important conjectures in Geometric Topology. It implies almost all the classical rigidity conjectures (Borel Conjecture, Novikov Conjecture).

Examples

- Coxeter groups are linear (Tits).
- Braid groups B_n is linear (Bigelow, Krammer).
- Right-angled Artin groups are linear (Hsu–Wise).
- Aut(F_2) is linear (because B_4 is) (Dyer–Formanek–Grossman).
- Aut(F_n) is not linear for $n \geq 3$ (Formanek–Procesi). They showed that it contains a non-linear subgroup.
- Open for general mapping class groups.

Main Question

Consider the (split) short exact sequence of groups:

$$1 \longrightarrow \pi \xrightarrow{i} G \xrightarrow{p} \Gamma \longrightarrow 1$$

Assume that both π and Γ are linear.

Main Problem. Identify conditions for G that guarantee that G is linear.

Poison Group

Formanek – Procesi constructed a group H that fits into a split exact sequence:

$$1 \longrightarrow F_3 \xrightarrow{j} H \xrightarrow{p} F_2 \longrightarrow 1$$

The presentation of H is:

$$H = \langle a_1, a_2, a_3, \phi_1, \phi_2 : \ \phi_i a_j \phi_i^{-1} = a_j, \ \phi_i a_3 \phi_i^{-1} = a_3 a_1, i, j = 1, 2 \rangle$$

They proved that H is not linear.

Facts about H

- H is an HNN-extension of $F_2 \times F_2$:

$$H = \langle F_2 \times F_2, t : \ t(g, g) = (1, g) \rangle.$$

- $H < \mathrm{Aut}(F_n)$, $n \geq 3$ (Formanek–Procesi).
- $H < IA_n$, $n \geq 5$ (Pettet).

$$IA_n = \ker(\mathrm{Aut}(F_n) \to \mathrm{Aut}(F^{ab}) = GL(n, \mathbb{Z})).$$

- Brendle – Hamid-Tehrani showed that H is not a subgroup of the mapping class group of a surface of genus g with one fixed point.

The Basic Construction

Recall that a semi–direct product

$$1 \longrightarrow \pi \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

is determined by a homomorphism

$$\rho : \Gamma \longrightarrow \mathrm{Aut}(\pi)$$

The method that will be used is to consider filtrations of a group that fit together with the map ρ.

A filtration of π is a descending chain of normal subgroups

$$\cdots \subseteq L_j(\pi) \subseteq \cdots \subseteq L_1(\pi) \subseteq L_0(\pi) = \pi$$

such that

$$\bigcap_{j \geq 1} L_j(\pi) = \{1\}$$

A bounded p-congruence system is a filtration of π such that

- $\pi / L_1(\pi)$ is finite.
- $L_1(\pi)/L_j(\pi)$ is a finite p-group, for all $j \geq 0$.
- $d(L_i(\pi)/L_j(\pi))$ is uniformly bounded ($0 \leq i < j$), where $d(-)$ denotes the minimal number of generators.

Lubotzky's Theorem

Theorem. A finitely generated group H admits a faithful finite dimensional representation over a field of characteristic 0 if and only if there is a prime p together with a bounded p-congruence system for H.

Two Filtrations

Remember we start with a split exact sequence

$$1 \longrightarrow \pi \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

The main approach is to intertwine filtrations for both π and Γ to get a filtration of G. The ingredients are two filtrations:

1. $F_*(\Gamma): \quad \cdots \subseteq F_j(\Gamma) \subseteq \cdots \subseteq F_1(\Gamma) \subseteq F_0(\Gamma) = \Gamma$
2. $L_*(\pi): \quad \cdots \subseteq L_j(\pi) \subseteq \cdots \subseteq L_1(\pi) \subseteq L_0(\pi) = \pi$

Stable Filtrations

The group extension

$$1 \to \pi \to G \to \Gamma \to 1$$

together with the filtrations $F_*(\Gamma)$ and $L_*(\pi)$ is said to be stable provided that for every

$$(g, y) \in F_{r+s}(\Gamma) \times L_{r+s}(\pi), \quad (f, x) \in F_r(\Gamma) \times L_r(\pi),$$

the following properties are satisfied:

1. $f(y) \in L_{r+s}(\pi)$.
2. $x^{-1}g(x) \in L_{r+s}(\pi)$.

Main Theorem

Theorem. (F.Gohen – M. Conder – J. Lopez – P.) Assume that the extension

$$1 \to \pi \to G \to \Gamma \to 1$$

satisfies the following properties:

1. The groups π and Γ are filtered by bounded p-congruence systems.
2. The extension is stable with respect to these two filtrations.

Then G admits a faithful finite dimensional representation in $GL(N, \mathbb{R})$ for some N.

A Classical Example

For each $q \in \mathbb{N}$, let

$$\Gamma_q = \text{ker}(SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/q\mathbb{Z}))$$

For $p \neq 2$ a prime, consider the “tower” of groups

$$\cdots \subseteq \Gamma_{p^r} \subseteq \ldots \Gamma_{p^2} \subseteq \Gamma_p$$

It is a bounded p-congruence system for the free group on $1 + p(p^2-1)/12$ letters with

$$\Gamma_{p^r}/\Gamma_{p^{r+1}} = \bigoplus_3 \mathbb{Z}/p\mathbb{Z}$$

An Application

Theorem. Let

$$\cdots \subseteq \pi_{p^3} \subseteq \pi_{p^2} \subseteq \pi_p = \pi$$

be a p-congruence system or the group $\pi = \pi_p$. If the split extension

$$1 \to \pi \to G \to \Gamma$$

1. is classified by a map

   $$\rho : \Gamma \to \text{Aut}(\pi^*) = \{f \in \text{Aut}(\pi) : \ f \text{ preserves the filtration}\}$$

2. Γ is linear with a filtration that satisfies $x^{-1}g(x) \in \pi_{p^{r+s}}$, for $g \in F_{r+s}(\Gamma)$ and $x \in \pi_{p^r}$.

Then G is linear.

A Question

Let F_n be the free group on n letetrs. Let

$$IA_n = \text{ker}(\text{Aut}(F_n) \to GL(n, \mathbb{Z}))$$

Question. Is it the case that all the elements of IA_n can be realized by the automorphisms of a tower for a p-congruence system for the group F_n?

An Easier Question

Fix a basis $\{x_1, \ldots, x_n\}$ of F_n. Let $\chi_{k,i} \in \mathrm{Aut}(F_n)$ is defined by:

$$\chi_{k,i}(x_j) = \begin{cases} x_j, & \text{if } k \neq j \\ x_i^{-1} x_k x_i, & \text{if } k = j \end{cases}$$

The group M_n generated by $\chi_{k,i}$ is called the McCool Group or the group of basis conjugating automorphisms.

Question. Is it the case that all the elements of M_n can be realized by the automorphisms of a tower for a p-congruence system for the group F_n? Guess: Yes

Examples

Example 1. It is not clear that the methods imply Bigelow's and Krammer's result for the Artin's Braid group. It is unclear whether the required monodromy can be realized as an automorphism of a tower.

Example 2. Some examples arise by automorphims of towers given by the principal congruence subgroups. They do not suffice to show that Artin's braid groups are linear. They do apply to show that some choices of subgroups of McCool's group are linear.

Example 3. Consider the extension:

$$1 \longrightarrow F[a_1, a_2, \ldots, a_n, b] \longrightarrow G_n \longrightarrow F[x, y] \longrightarrow 1$$

for which the action of $F[x, y]$ is given as follows:

- $x(a_q) = a_{q+1}$, if $1 \leq q < n$ with $x(a_n) = b{\cdot}a_1{\cdot}b^{-1}$.
- $x(b) = b$.
- $y(a_q) = a_1{\cdot}a_q{\cdot}a_1^{-1}$.
- $y(b) = a_1{\cdot}b{\cdot}a_1^{-1}$.

The group G_n is linear. After knowing that, it is direct to construct a concrete faithful representation in $GL(8, \mathbb{R})$.

Example 4. Let

$$1 \to F_n \to G \to \Gamma \to 1$$

be a split extension such that

1. $\Gamma < GL(2, \mathbb{Z})$ (and thus Γ has a normal subgroup of finite index that is free).
2. F_n is isomorphic to a principal congruence subgroup of level p^r in $PSL(2, \mathbb{Z})$.
3. Γ acts on F_n by conjugation.

Then G is linear.

Question. Let

$$1 \longrightarrow F_n \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

be a split exact sequence with F_n a free group and Γ linear. Assume that Γ acts trivially on $H_*(F_n)$. Is G linear?

- The condition on the action of Γ on $H_*(F_n)$ is necessary because of the Poison Group.
- F. Cohen – P. and D. Cohen – F. Cohen – P. have worked out conditions for detecting monomorphisms on certain poly-free groups. Those include the pure braid group, the orientable McCool group, and some fundamental groups of complements of hyperplane arrangements.The conditions use the Lie algebra of the descending central series of the group and thus they homological in nature.

Further Results

Theorem 1. (F. Cohen – V. Metaftsis – P.) The holomorph of F_2 is linear.

The homolomorph of G is the universal split extansion with kernel G:

$$1 \to G \to \mathrm{Hol}(G) \to \mathrm{Aut}(G) \to 1.$$

Using the results of F. Cohen–Wu, then $\mathrm{Hol}(F_2)$ is a large linear subgroup of $\mathrm{Aut}(F_3)$.

<u>Conjecture</u>. Any semidirect product $F_n \rtimes \mathbb{Z}$ is linear.

F. Cohen – V. Metaftsis – P.: True for $n = 2$.

<u>Conjecture</u>. (V. Metaftsis – P.) The HNN-extension

$$G = \langle F_2 \times F_2, t : \ t(g, g)t^{-1} = (g, g) \rangle$$

is linear.

<u>Remember</u>: The Formanek–Procesi group is given by:

$$G = \langle F_2 \times F_2, t : \ t(g, g)t^{-1} = (1, g) \rangle.$$