Invertibility in Bicategories

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Abstract

These slides are slightly modified from a talk presented at the 2008 Algebraic Topology Conference in Buenos Aires. We give a bicategorical perspective on invertibility beginning with Morita theory and duality, and then describing generalized Brauer groups and Azumaya objects. We develop the theory of invertibility in triangulated bicategories and give a characterization of Azumaya objects therein.

Brauer Group of a Field

Let R be a field, and A an R-algebra.

- If A is simple with center R, then $A = M_n(D)$ for some division ring D (Wedderburn).
- The Brauer group is defined by introducing an equivalence relation: $M_n(D) \sim M_m(D)$ for all m, n.
- If A is a central, simple R-algebra, then $A \otimes A^{op} \sim R$; the set of similarity classes of central simple R-algebras is a group under $\otimes_R$.
- This is the Brauer group, $Br(R)$.

Brauer Group of a Commutative Ring

Let R be a commutative ring, and A an R-algebra.

- A is *central* if the center of A is equal to R.
- A is *separable* if A is projective as a module over $A^e = A \otimes_R A^{op}$.
- A is *faithfully-projective* if A is finitely-generated and projective as an R-module, and if,
  for any R-module M, $A \otimes_R M = 0 \Rightarrow M = 0$.

Brauer Group of a Commutative Ring

Theorem

The following are equivalent for an R-algebra A:

1. *A is central and separable over R.*
2. *A is faithfully-projective over R and* $\mu : A^e \xrightarrow{\cong} Hom_R(A, A)$ *is an isomorphism.*
3. A^e *is Morita equivalent to R.*
4. *There is an R-algebra B such that* $A \otimes_R B$ *is Morita equivalent to R.*

These conditions define an *Azumaya algebra* over R.
The Brauer group of R is the group of Azumaya R-algebras: $Br(R)$.

Goals

Brauer, Azumaya in derived and topological settings

- Understand invertibility in practice
- Azumaya objects in triangulated bicategories
- Topologically motivated development of localization

Outline

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A Bicategory is a Weak 2-Category

A bicategorical context provides:

- organizational framework
- conceptual advantage

Definition by example; Modules over a commutative ring, R: $\mathscr{M}_R$

- 0-cells: R-algebras
- 1-cells: bimodules $\mathscr{M}_R(A, B)$ is the category of (B, A)-bimodules
- 2-cells: bimodule morphisms

The horizontal composite of 1-cells is given by the tensor product

$$N \otimes_B M : A \overset{M}{\nrightarrow} B \overset{N}{\nrightarrow} C$$

Additional Structure

Work in a *closed autonomous monoidal* bicategory

Right-adjoints to $M \otimes_A -$ and $- \otimes_B M$
given by Source-Hom and Target-Hom

$$M : A \nrightarrow B$$

Notation:

$$\mathscr{B}(M \otimes_A X, Z) \cong \mathscr{B}(X, \mathrm{Hom}_B(M, Z))$$

$$\mathscr{B}(Y \otimes_B M, W) \cong \mathscr{B}(Y, \mathrm{Hom}_A(M, Y))$$

Additional Structure

Work in a *closed autonomous monoidal* bicategory

$\mathscr{M}_R$ has

- $\otimes_R$ on 0-, 1-, and 2-cells; a symmetric monoidal product
- An involution $(-)^{op}$

For $M \in \mathscr{M}_R(A, B)$, this gives

- $M^{op} \in \mathscr{M}_R(B^{op}, A^{op})$
- $M_r \in \mathscr{M}_R(A \otimes_R B^{op}, R)$
- $M_\ell \in \mathscr{M}_R(R, A^{op} \otimes B)$
- ...compatibility axioms

Further Examples

Let R be a commutative DG-algebra or ring spectrum.

Other examples of interest

- Ch_R
  $Ch_R(A, B)$ is the category of DG-(B, A)-bimodules
- $\mathscr{D}_R$
  $\mathscr{D}_R(A, B)$ is the homotopy category of (B, A)-bimodules
  Note: Use $\otimes$ and Hom to denote the derived tensor and hom.

Duality and Invertibility

Duality in a closed bicategory generalizes duality in a closed monoidal category.

A pair of 1-cells (X, Y) is a *dual pair* if

$X : A \nrightarrow B$ and $Y : B \nrightarrow A$

- unit: $B \to X \otimes_A Y$
  counit: $Y \otimes_B X \to A$.
  satisfying the triangle identities

equivalently:

- (X, Y) defines an adjunction $(- \otimes_B X) \dashv (- \otimes_A Y)$.
- (X, Y) defines an adjunction $(Y \otimes_B -) \dashv (X \otimes_A -)$.

A dual pair, (X, Y) is *invertible* if the induced functors are an equivalence;
if and only if the unit/counit are isomorphisms

Remark: (X, Y) is an invertible pair if and only if (Y, X) is invertible.

Duality and Invertibility

Lemmas

- X is right-dualizable if and only if the coevaluation $X \otimes_A \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_A(X, X)$ is an isomorphism.
  Any right dual of X is isomorphic to $\mathrm{Hom}_A(X, A)$.
- X is left-dualizable if and only if the coevaluation $\mathrm{Hom}_B(X, B) \otimes X \to \mathrm{Hom}_B(X, X)$ is an isomorphism.
  Any left dual of X is isomorphic to $\mathrm{Hom}_B(X, B)$.

- If X is right-dualizable and the action map $B \to \mathrm{Hom}_A(X, X)$ is an isomorphism, then the evaluation $X \otimes_A \mathrm{Hom}_B(X, B) \to B$ is an isomorphism.
- If X is left-dualizable and the action map $A \to \mathrm{Hom}_B(X, X)$ is an isomorphism, then the evaluation $\mathrm{Hom}_A(X, A) \otimes_B X \to A$ is an isomorphism.

Outline

Let A be a 0-cell of a closed autonomous monoidal bicategory with unit R. Recall $A_r : A^e \nrightarrow R$.

The following are equivalent

0. A_r is invertible $(A_r, \mathrm{Hom}_{A^e}(A_r, A^e))$ $(\mathrm{Hom}_R(A_r, R), A_r)$
1. eval: $\mathrm{Hom}_{A^e}(A_r, A^e) \otimes_R A_r \xrightarrow{\cong} A^e$ e.g. $\mathscr{M}_R$
   coeval: $A_r \otimes_{A^e} \mathrm{Hom}_{A^e}(A_r, A^e) \xrightarrow{\cong} \mathrm{Hom}_{A^e}(A_r, A_r)$ (separable)
   action: $R \xrightarrow{\cong} \mathrm{Hom}_{A^e}(A_r, A_r)$ (central)
2. eval: $A_r \otimes_{A^e} \mathrm{Hom}_R(A_r, R) \xrightarrow{\cong} R$ (faithfully-projective)
   coeval: $\mathrm{Hom}_R(A_r, R) \otimes_R A_r \xrightarrow{\cong} \mathrm{Hom}_R(A_r, A_r)$
   action: $A^e \xrightarrow{\cong} \mathrm{Hom}_R(A_r, A_r)$

These conditions define an *Azumaya object*, A.

Note: in the case of $\mathscr{M}_R$, the ''action'' and ''coeval'' isomorphisms of part 1 imply that ''eval'' is also an isomorphism. We do not yet know how generally this holds.

Let A be a 0-cell of a closed autonomous monoidal bicategory $\mathscr{D}$ with unit R.

Theorem

A is an Azumaya object if and only if there is a 0-cell B such that B_r is left-dualizable and $A \otimes_R B \simeq_{Morita} R$.

***Proof*: Diagram chase**

$$\begin{tikzcd}
\mathscr{D}(R,-) \arrow[r, shift left, "-\otimes_R A_r"] \arrow[dr, shift left, "\simeq"'] & \mathscr{D}(A^e,-) \arrow[l, shift left, "\mathrm{Hom}_{A^e}(A_r,-)"] \arrow[d, shift left, "-\otimes_{A^e}(A^e\otimes_R B_r)"] \\
& \mathscr{D}(A^e \otimes_R B^e,-) \arrow[u, shift left, "\mathrm{Hom}"] \arrow[ul, shift left]
\end{tikzcd}$$

If B_r is left-dualizable, the vertical counit is an isomorphism. (Lemma)

The Brauer Group

Let R be the unit of a closed autonomous monoidal bicategory $\mathscr{D}$.

- $[A]$ is the equivalence class of A under Morita equivalence.
- $[A] \in Br(R)$ if there is a 0-cell B with $A \otimes_R B \simeq_{Morita} R$.
- This is a group, the *Brauer group* of R.

The Azumaya objects of $\mathscr{D}$ are those A for which $[A] \in Br(R)$ and A_r is left-dualizable.

Question: Is there an example for which $A_r \otimes_R B_r$ is invertible, but A_r and B_r are not?

`Note: In` $\mathscr{M}_r$`, this cannot happen; we do not yet know if it can happen more generally. This issue is the same as that raised following the definition of Azumaya objects.`

Outline

Triangulated Bicategories

Let $\mathscr{D}$ be a closed autonomous monoidal bicategory, and each $\mathscr{D}(A, B)$ is a triangulated category such that:

- The functors $X \otimes_A -$, $- \otimes_B X$, $\mathrm{Hom}_A(X, -)$, $Hom_B(X, -)$ are exact.
- axioms relating Σ and units, autonomous structure ...

For the remainder of the talk, we suppose $\mathscr{D}$ has such a triangulated structure.

$\mathscr{D}[Z, W]_*$ denotes the graded Abelian group of 2-cells $Z \to W$.

Let $T : A \nrightarrow B$ be a 1-cell in $\mathscr{D}$.

Definition (T-acyclic)

M is *$T_{\otimes}$-acyclic* if $T \otimes_A M = 0$. (push-forward)

M' is *$T^{\otimes}$-acyclic* if $M' \otimes_B T = 0$. (pull-back)

Definition (T-local)

N is *$T_{\otimes}$-local* if $\mathscr{D}[M, N]_* = 0$ for all $T_{\otimes}$-acyclic M.

N' is *$T^{\otimes}$-local* if $\mathscr{D}[M', N']_* = 0$ for all $T^{\otimes}$-acyclic M'.

Notation:

The subcategory of $T_{\otimes}$-local 1-cells $C \nrightarrow A$ is $\mathscr{D}(C, A)_{\langle T_{\otimes} \rangle}$

The subcategory of $T^{\otimes}$-local 1-cells $B \nrightarrow C$ is $\mathscr{D}(B, C)_{\langle T^{\otimes} \rangle}$

Baker-Lazarev Factorization

The adjunctions induced by $T : A \nrightarrow B$ factor through the T-local categories [Baker-Lazarev 2004]

$$\begin{tikzcd}
\mathscr{D}(B,-) \arrow[rr, shift left, "-\otimes_B T"] \arrow[dr, shift left, "\text{localization}"] & & \mathscr{D}(A,-) \arrow[ll, shift left, "\mathrm{Hom}_A(T,-)"] \arrow[dl, shift left, "\mathrm{Hom}_A(T,-)"] \\
& \mathscr{D}(B,-)_{\langle T\otimes \rangle} \arrow[ul, shift left] \arrow[ur, shift left, "-\otimes_B T"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{D}(-,A) \arrow[rr, shift left, "T\otimes_A -"] \arrow[dr, shift left, "\text{localization}"] & & \mathscr{D}(-,B) \arrow[ll, shift left, "\mathrm{Hom}_B(T,-)"] \arrow[dl, shift left, "\mathrm{Hom}_B(T,-)"] \\
& \mathscr{D}(-,A)_{\langle T_\otimes \rangle} \arrow[ul, shift left] \arrow[ur, shift left, "T\otimes_A -"] &
\end{tikzcd}$$

Proposition (Baker-Lazarev 2004)

If T is right-dualizable and the action $B \xrightarrow{\cong} \mathrm{Hom}_A(T,T)$ is an isomorphism, then $\mathscr{D}(-,A)_{\langle T_\otimes \rangle} \simeq \mathscr{D}(-,B)$. (Lemma)
Likewise for left-dualizability.

Invertibility in Triangulated Bicategories

Invertibility and Localization

The following are equivalent for a 1-cell $T : A \nrightarrow B$ in $\mathscr{D}$:

0. T is invertible
1. T is right-dualizable
   action: $B \xrightarrow{\cong} \mathrm{Hom}_A(T, T)$
   A is $T_{\otimes}$-local
2. T is left-dualizable
   action: $A \xrightarrow{\cong} \mathrm{Hom}_B(T, T)$
   B is $T^{\otimes}$-local

Let A be a 0-cell of $\mathscr{D}$ and take $T = A_r : A^e \twoheadrightarrow R$.

Baker-Lazarev: $\mathscr{D}$ = spectra

Corollary

The following are equivalent:

0. A_r is Azumaya (as defined previously)
1. A_r is right-dualizable
   action: $R \xrightarrow{\cong} \mathrm{Hom}_{A^e}(A_r, A_r) = THH_R(A, A)$
   A^e is $A_{r\otimes}$-local
2. A_r is left-dualizable
   action: $A^e \xrightarrow{\cong} \mathrm{Hom}_R(A_r, A_r)$
   R is $A_r^{\otimes}$-local

Example: Morava $K(1)$ is Azumaya over $\widehat{KU_2}$ (Baker-Lazarev).

Let R be a commutative differential graded algebra (Rickard)
or a commutative ring spectrum (Schwede-Shipley).
Recall: $\mathscr{D}_R(A, B)$ is the homotopy category of (B, A)-bimodules.

Corollary

Let $T : A \nrightarrow R$ be a 1-cell of $\mathscr{D}_R$, and let $E = \mathrm{Hom}_A(T, T)$.
Let $\widetilde{T}$ be the induced 1-cell $A \nrightarrow E$.
If T has the following two properties, then $\widetilde{T}$ provides an equivalence $\mathscr{D}_R(A) \simeq \mathscr{D}_R(E)$.

- T is right-dualizable
- T generates the triangulated category $\mathscr{D}_R(A)$.

$\mathscr{D}_R(A) = \mathscr{D}_R(A, R)$; $\mathscr{D}_R(E) = \mathscr{D}_R(E, R)$

T right-dualizable $\Rightarrow$ $\widetilde{T}$ is right-dualizable
T generates $\mathscr{D}_R(A)$ $\Rightarrow$ A is $T_{\otimes}$-local.

For general $\mathscr{D}$, need to know E and $\widetilde{T} : A \nrightarrow E$ exist.

Conclusion

Ideas for Future Work

- Compare with Picard group calculations for $K(n)$-local and $E(n)$-local spheres (Hopkins-Mahowald-Sadofsky, Hovey-Sadofsky)
- Study relative Brauer group, $Br(S, R)$, for ring map $S \to R$. (Vitale)
- Categorical description of 3-stage spectra

- Morita theory: Study of equivalence and invertibility in bicategory
- General bicategories: Description of invertibility generalizes classical work with Azumaya algebras
- Triangulated bicategories: Factorization relates localization and invertibility
- Conceptual unification of algebraic and topological theory