Operads and the chain rule for Goodwillie calculus

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Introduction

This is a talk about Goodwillie's calculus of homotopy functors:

- chain rules: derivatives of FG in terms of derivatives of F and derivatives of G
  - stable case (e.g. Spec): "simple"
  - unstable case (e.g. Top_*): more difficult
- application to algebraic K-theory of ring spectra

Plan

Review of Calculus of Functors
- Taylor Tower of a Functor
- Derivatives of a Functor

Chain Rules
- Stable Case
- Unstable Case

Proof and Application
- Key Step in Proof
- Application to Algebraic K-Theory

Functors

Study functors $F : \mathcal{C} \to \mathcal{D}$ where:

- $\mathcal{C}$ and $\mathcal{D}$ are ‘appropriate’ categories with a notion of (weak) homotopy equivalence: e.g.
  - Top_* (based spaces)
  - Spec (spectra)
  - A_∞-/E_∞-ring spectra
  - chain complexes of R-modules
- F preserves equivalences (F is a homotopy functor)

$$X \xrightarrow{\sim} Y \quad \Longrightarrow \quad FX \xrightarrow{\sim} FY$$

- F preserves filtered homotopy colimits

Taylor Tower of a Homotopy Functor

Theorem (Goodwillie)

$F : \mathcal{C} \to \mathcal{D}$: homotopy functor
$X \in \mathcal{C}$
For each map $Y \to X$ in $\mathcal{C}$ there is a sequence:

$$F(Y) \to \cdots \to P_n^X F(Y) \to P_{n-1}^X F(Y) \to \cdots \to P_0^X F(Y) = F(X)$$

such that:

- *the functor $P_n^X F : \mathcal{C}_X \to \mathcal{D}$ is n-excisive*
- *the map $F \to P_n^X F$ is universal*

This is the Taylor tower of F expanded at X.

Convergence of the Taylor Tower

$$F(Y) \to \cdots \to P_n^X F(Y) \to P_{n-1}^X F(Y) \to \cdots \to P_0^X F(Y) = F(X)$$

Definition

The Taylor tower for F expanded at X converges at Y if

$$F(Y) \simeq \operatorname{holim}_n P_n^X F(Y)$$

Typically, the tower converges when $Y \to X$ is sufficiently highly connected (if $\mathcal{C} = \mathrm{Top}_*$ or Spec).

Layers of the Taylor Tower

The layers of the Taylor tower of F:

$$D_n^X F = \text{hofib}(P_n^X F \to P_{n-1}^X F)$$

- $D_n^X F$ represents the n$^{\text{th}}$ term in the Taylor tower for F expanded at X
- $D_n^X F$ is a homogeneous degree n functor

To simplify things, we consider only Taylor towers expanded at $X = *$ and write:

$$P_n F := P_n^* F$$

$$D_n F := D_n^* F$$

Derivatives of a Homotopy Functor

Theorem (Goodwillie)

- $F : \mathrm{Spec} \to \mathrm{Spec}$

$$D_n F(X) \simeq (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$$

- $F : \mathrm{Top}_* \to \mathrm{Top}_*$

$$D_n F(X) \simeq \Omega^\infty(\partial_n F \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$$

$\partial_n F$ is a spectrum with Σ_n-action, the n^{th} derivative of F

The Chain Rule Problem

Questions Given $\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{F} \mathcal{E}$:

- how does $\partial_*(FG)$ depend on $\partial_*(F)$ and $\partial_*(G)$?
- how does $\{P_n(FG)\}$ depend on $\{P_nF\}$ and $\{P_nG\}$?

Our Answers:

- explicit formula for $\partial_n(FG)$ based on operads and modules
- approach to finding $P_n(FG)$

Previous Work on the Chain Rule

Theorem (Klein-Rognes, 2002)

$F, G : \mathrm{Top}_* \to \mathrm{Top}_*$, $F(*) = G(*) = *$

$$\partial_1(FG) \simeq \partial_1(F) \wedge \partial_1(G)$$

(They also do the case $G(*) \neq *$, etc...)

Chain Rule for Ordinary Calculus

Given $f, g : \mathbb{R} \to \mathbb{R}$, what is $(fg)_n$ (the n$^{\text{th}}$ Taylor coefficient of fg)?

$$f(gx) = \sum_{k \geq 1} \frac{f_k(\sum_{j \geq 1} g_j x^j / j!)^k}{k!}$$

Theorem (Faà di Bruno's Formula)

$$(fg)_n = \sum_{n_1 + \cdots + n_k = n} f_k g_{n_1} \ldots g_{n_k}$$

Chain rule if Middle Category is Stable

Theorem (C. 2007)

$F, G : \mathrm{Spec} \to \mathrm{Spec}$, $F(*) = G(*) = *$

$$\partial_n(FG) \simeq \bigvee_{n_1+\cdots+n_k=n} \partial_k F \wedge \partial_{n_1} G \wedge \ldots \wedge \partial_{n_k} G$$

or

$$\partial_*(FG) \simeq \partial_* F \circ \partial_* G$$

This is the composition product of symmetric sequences used to define operads.

Chain Rule if Middle Category is Unstable

Key Fact about Calculus for Topological Spaces:

- The derivatives of the identity functor $I : \mathrm{Top}_* \to \mathrm{Top}_*$ are non-trivial:

$$\partial_n I \simeq \bigvee_{(n-1)!} S^{1-n}$$

This means:

$$\partial_*(FI) = \partial_* F \neq \partial_* F \circ \partial_* I$$

Instead we want:

$$\partial_*(FG) = \partial_* F \circ_{\partial_* I} \partial_* G \qquad \text{(compare } M \otimes_R N\text{)}$$

Chain Rule if Middle Category is Unstable

Theorem (Arone-C. 2008)

1. *There is an operad structure on $\partial_* I$ (the derivatives of the identity functor on based spaces):*

$$\partial_* I \circ \partial_* I \to \partial_* I$$

2. *Given $F : \mathrm{Top}_* \to \mathrm{Top}_*$, the derivatives of F have a $\partial_* I$-bimodule structure:*

$$\partial_* F \circ \partial_* I \to \partial_* F, \quad \partial_* I \circ \partial_* F \to \partial_* F$$

3. $F, G : \mathrm{Top}_* \to \mathrm{Top}_*$, $F(*) = G(*) = *$:

$$\partial_*(FG) \simeq \partial_* F \circ_{\partial_* I} \partial_* G$$

Cosimplicial Cobar Construction

- $F, G : \mathrm{Top}_* \to \mathrm{Top}_*$
- $(\Sigma^\infty, \Omega^\infty)$ adjunction between Top_* and Spec

Define a cosimplicial object:

$$F\Omega^\infty\Sigma^\infty G \quad \substack{\rightarrow \\ \leftarrow \\ \rightarrow} \quad F\Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty G \cdots$$

using the unit and counit of the adjunction

$$1 \to \Omega^\infty\Sigma^\infty \qquad\qquad \Sigma^\infty\Omega^\infty \to 1$$

Key Proposition

$$P_n(FG) \to \operatorname{Tot}\left(\begin{array}{c} \vdots \\ P_n(F\Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty G) \\ \uparrow\downarrow\uparrow \\ P_n(F\Omega^\infty\Sigma^\infty G) \end{array} \right)$$

Proposition

This map is an equivalence for all n.

Proof.

Induction on Taylor tower of F reduces to homogeneous case.
Then use formula for D_nF. □

Cobar Construction for Derivatives

We see that $\partial_*(FG)$ is given by a cobar construction:

$$\partial_*(FG) \simeq \mathrm{Tot}\left(\begin{array}{c} \vdots \\ \partial_*(F\Omega^\infty) \circ \partial_*(\Sigma^\infty\Omega^\infty) \circ \partial_*(\Sigma^\infty G) \\ \uparrow\downarrow\uparrow \\ \partial_*(F\Omega^\infty) \circ \partial_*(\Sigma^\infty G) \end{array} \right)$$

Example

- $F = G = I$: $\implies$ operad structure on $\partial_* I$.
- $G = I$: $\implies$ right $\partial_* I$-module structure on $\partial_* F$.
- $F = I$: $\implies$ left $\partial_* I$-module structure on $\partial_* G$.

Algebraic K-Theory of Ring Spectra

- R-alg: augmented associative R-algebras (= A_∞-ring spectra over/under R)
- $K : R\text{-alg} \to \text{Spec}$,
  $K(A) =$ algebraic K-theory of finite cell A-modules
- (Basterra-Mandell): the $(\Sigma^\infty, \Omega^\infty)$ adjunction between R-alg and Spec(R-alg) $=$ R-bimod is given by
  - $\Sigma^\infty(A) = TAQ_R(A)$
  - $\Omega^\infty(M) = R \vee M$

Taylor Tower of K-Theory

Apply Key Proposition with $F = K$ and $G = I_{R\text{-alg}}$:

$$P_n(K) \to \mathrm{Tot}\left(\begin{array}{c} \vdots \\ P_n(K\Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty) \\ \uparrow\downarrow\uparrow \\ P_n(K\Omega^\infty\Sigma^\infty) \end{array} \right)$$

We need:

- Taylor tower of $K\Omega^\infty$ (calculated by Lindenstrauss-McCarthy)
- Taylor tower of $\Sigma^\infty\Omega^\infty$ (easy)
- how these interact (hard)

$P_2(K)$

- For $P_1(K)$ recover Dundas-McCarthy result:

$$P_1(K)(A) \simeq THH(R, \Sigma TAQ_R(A))$$

- $$P_2(K)(A) \simeq \operatorname{holim} \begin{pmatrix} W_2(R, \Sigma TAQ_R(A)) \\ \downarrow\ \downarrow \\ W_2(R, \Sigma^2 TAQ_R(A)^{\wedge 2}) \end{pmatrix}$$

  where
  - W_2 comes from Taylor tower of $K\Omega^\infty$ (Lindenstrauss-McCarthy)
  - the vertical maps are induced by the coface maps in the cosimplicial object