Some relationships between the geometry of the tangent bundle and the geometry of the Riemannian base manifold

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Abstract. We compute the curvature tensor of the tangent bundle of a Riemannian manifold endowed with a natural metric and we get some relationships between the geometry of the base manifold and the geometry of the tangent bundle.

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1 Introduction

Let \((M, g)\) be a Riemannian manifold of dimension \(n \geq 2\). Let \(\pi : TM \rightarrow M\) and \(P : O(M) \rightarrow M\) be the tangent and the orthonormal bundle over \(M\) respectively. In this paper we deal with certain class of Riemannian metrics on \(TM\). A metric \(G\) belongs to this class if the canonical projection \(\pi : (TM, G) \rightarrow (M, g)\) is a Riemannian submersion, the horizontal distribution induced by the Levi-Civita connection of \((M, g)\) is orthogonal to the vertical distribution and \(G\) is the image by a natural operator of order two of the metric \(g\). The Sasaki metric and the Cheeger-Gromoll metric are well known examples of these class of metrics, and there were extensively studied by Kowalski [7], Aso [2], Sekizawa [11], Musso and Tricerri [9], Gudmundsson and Kappos [4] among others. The notion of natural tensor on the tangent bundle of a Riemannian manifold as a tensor that is the image by a natural operator of order two of the base manifold metric, was introduced and characterized by Kowalski and Sekizawa in [8]. In [3], Calvo and the second author showed that for a given Riemannian manifold \((M, g)\), any \((0, 2)\) tensor field on \(TM\) admits a global matrix representation. Using this one to one relationship, they defined and characterized, without making use of the theory of differential invariants, what they also called natural tensor. In the symmetric case this concept coincide with the one defined by Kowalski and Sekizawa. In [5], the first author gives a new approach of the concept of naturality, introducing the notion of s-space and \(\lambda\)-naturality. This approach avoids jets and natural operators theory and generalized the one given in [3] and [8].

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In section 2, we introduce natural metrics on $TM$ by means of [3]. For any $q \in M$, let $M_q$ be the tangent space of $M$ at $q$. Let $\psi : N := O(M) \times \mathbb{R}^n \rightarrow TM$ be the projection defined by

$$\psi(q, u, \xi) = \sum_{i=1}^{n} \xi^i u_i$$

where $u = (u_1, \ldots, u_n)$ is an orthonormal basis for $M_q$ and $\xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$. It is well known (see [9]), that for a fixed Riemannian metric $G$ on $TM$ a suitable Riemannian metric $G^\ast$ on $N$ can be defined such that $\psi : (N, G^\ast) \rightarrow (TM, G)$ is a Riemannian submersion. Based on this fact and the O’Neill formula, in Section 3, we compute the curvature tensor of $(TM, G)$, when $G$ is a natural metric. As an application, we get in Section 4 some relationships between the geometry of $TM$ and the geometry of $M$. In [1] Abbassi and Sarid studied some relationships between the geometry of $TM$ and the geometry of $M$, when $TM$ is endowed with a $g$ – natural metric. For example (Theorem 0.1) states that if $(TM, G)$ is flat, then $(M, g)$ is flat. Since in this paper we deal with a subclass of $g$ – natural metrics we get Corollary 4.2 as a converse of this theorem. Throughout, all geometric objects are assumed to be differentiable, i.e. $C^\infty$.

2 Preliminaries.

Let $\nabla$ be the Levi-Civita connection of $g$ and $K : TTM \rightarrow TM$ be the connection map induced by $\nabla$. For any $q \in M$ and $v \in M_q$, let $\pi_{sv} : (TM)_v \rightarrow M_q$ be the differential map of $\pi$ at $v$, and $K_v : (TM)_v \rightarrow M_q$ be the restriction of $K$ to $(TM)_v$.

Since the linear map $\pi_{sv} \times K_v : (TM)_v \rightarrow M_q \times M_q$ defined by $(\pi_{sv} \times K_v)(b) = (\pi_{sv}(b), K_v(b))$ is an isomorphism that maps the horizontal subspace $(TM)_v^h = \ker K_v$ onto $M_q \times \{0_q\}$ and the vertical subspace $(TM)_v^v = \ker \pi_{sv}$ onto $\{0_q\} \times M_q$, where $0_q$ denotes the zero vector, we define differentiable mappings $e_i, e_{n+i} : N = O(M) \times \mathbb{R}^n \rightarrow TTM$ for $i = 1, \ldots, n$ and $v = \psi(q, u, \xi)$ by

$$e_i(q, u, \xi) = (\pi_{sv} \times K_v)^{-1}(u_i, 0_q),$$

$$e_{n+i}(q, u, \xi) = (\pi_{sv} \times K_v)^{-1}(0_q, u_i).$$

The action of the orthonormal group $O(n)$ of $\mathbb{R}^{n \times n}$ on $N$ is given by the family of maps $R_a : N \rightarrow N$, $a \in O(n)$, $R_a(q, u, \xi) = (q, u.a, \xi.a)$ where $u.a = (\sum_{i=1}^{n} a_i^iu_i, \ldots, \sum_{i=1}^{n} a_i^nu_i)$ and $\xi.a = (\sum_{i=1}^{n} a_i^{i\xi^i}, \ldots, \sum_{i=1}^{n} a_i^{n\xi^i})$. Clearly, $\psi \circ R_a = \psi$. It follows from (2) that

$$e_j(R_a(p, u, \xi)) = \sum_{i=1}^{n} e_i(p, u, \xi)a_j^i$$

for $j = 1, \ldots, n$

and

$$e_{n+j}(R_a(p, u, \xi)) = \sum_{i=1}^{n} e_{n+i}(p, u, \xi)a_j^i$$

for $j = 1, \ldots, n$. 

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For any \((0,2)\) tensor field \(T\) on \(TM\) we define the differentiable function \(gT : N \rightarrow \mathbb{R}^{2n \times 2n}\) as follows: If \((q,u,\xi) \in N\) and \(v = \psi(q,u,\xi)\), let \(gT(q,u,\xi)\) be the matrix of the bilinear form \(T_v : (TM)_v \times (TM)_v \rightarrow \mathbb{R}\) induced by \(T\) on \((TM)_v\) with respect to the basis \(\{e_1(q,u,\xi), \ldots, e_{2n}(q,u,\xi)\}\). One sees easily that \(gT\) satisfies the following invariance property:

\[
gT \circ R_a = (L(a))^t gT L(a)
\]

where \(L : O(n) \rightarrow \mathbb{R}^{2n \times 2n}\) is the map defined by

\[
L(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.
\]

Moreover, there is a one to one correspondence between the \((0,2)\) tensor fields on \(TM\) and differentiable maps \(gT\) satisfying (3).

A tensor field \(T\) on \(TM\) will be called natural with respect to \(g\) if \(gT\) depends only on the parameter \(\xi\), (see [3]). In the sense of [5], the collection \(\lambda = (N, \psi, O(n), \tilde{R}, \{e_i\})\) is a s-space over \(TM\), with base change morphism \(L\); and the natural tensors with respect to \(g\) are the \(\lambda\)-natural tensors with respect to \(TM\).

Writing \(gT\) in the block form \(gT = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}\), where \(A_i : N \rightarrow \mathbb{R}^{n \times n}\); it follows from Lemma 3.1 of [3], that \(T\) is natural if there exist differentiable functions \(\alpha_i, \beta_i : [0, +\infty) \rightarrow \mathbb{R}\) \((i = 1, 2, 3, 4)\), such that

\[
A_i(p, u, \xi) = \alpha_i(\|\xi\|^2) \text{Id}_{n \times n} + \beta_i(\|\xi\|^2)\xi^t \xi
\]

where \(\|\xi\|\) denotes the norm of \(\xi\) induced by the canonical inner product of \(\mathbb{R}^n\). In that case \(T\) is said to be a \(g\)–natural metric if in addition \(T\) is a Riemannian metric.

It is easy to check that a \((0,2)\)-tensor field \(T\) on \(TM\) is a \(g\)-natural metric if and only if \(T\) is natural, \(A_2 = A_4, \alpha_3(t) > 0, \alpha_1(t) \alpha_3(t) - \alpha_2^2(t) > 0, \phi_3(t) > 0\) and \(\phi_1(t) \phi_3(t) - \phi_2^2(t) > 0\) for all \(t \geq 0\); where \(\phi_i(t) = \alpha_i(t) + t \beta_i(t)\) for \(i = 1, 2, 3\).

In this paper we will call \(G\) a natural metric on \(TM\) if:

1. \(G\) is a Riemannian metric such that \(\pi : (TM, G) \rightarrow (M, g)\) is a Riemannian submersion.

2. For \(v \in TM\), the subspaces \((TM)_v^u\) and \((TM)_v^h\) are orthogonals.

3. \(G\) is natural with respect to \(g\).

It follows that \(G\) is a natural metric on \(TM\) if

\[
gG(p, u, \xi) = \begin{pmatrix} \text{Id}_{n \times n} & 0 \\ 0 & \alpha(\|\xi\|^2).\text{Id}_{n \times n} + \beta(\|\xi\|^2)(\xi^t \xi) \end{pmatrix}
\]

where \(\alpha, \beta : [0, +\infty) \rightarrow \mathbb{R}\) are differentiable functions satisfying \(\alpha(t) > 0\), and \(\alpha(t) + t \beta(t) > 0\) for all \(t \geq 0\).
Remark 2.1  The Sasaki metric $G_s$ corresponds to the case $\alpha = 1$, $\beta = 0$; and the Cheeger-Gromoll metric $G_{ch}$ to the case $\alpha(t) = \beta(t) = \frac{1}{1+t}$.

3 Curvature equations.

In this section we compute the curvature tensor of $TM$ endowed with a natural metric. Since this computation involves well known objects defined on $N$, we shall begin to describe them briefly using the connection map.

3.1 Canonical constructions on $N$.

Let $\theta^i$, $\omega^i_j$ be the canonical 1-forms on $O(M)$, which in terms of the connection map are defined as follows:

$$\theta^i(q, u)(b) = g_q\left(P_{(q,u)}(b), u_i\right)$$

and

$$\omega^i_j(q, u)(b) = g_q\left(K((\pi_j)_*(q,u))(b), u_i\right)$$

where $\pi_j : O(M) \to TM$ is the $j^{th}$ projection, i.e. $\pi_j(q, u) = u_j$ and $1 \leq i, j \leq n$.

From now on, let $\theta^i$, $\omega^i_j$, $d\xi^i$ be the pull backs of the canonical 1-forms on $O(M)$ and the usual 1-forms on $\mathbb{R}^n$ by the projections $P_1 : N \to O(M)$ and $P_2 : N \to \mathbb{R}^n$ respectively.

For any $z \in N$ let us denote by $V_z = \ker \psi^*_z$ and $H_z := \{b \in N_z : \omega^i_j(z)(b) = 0, 1 \leq i < j \leq n\}$ the vertical and the horizontal subspace of $N_z$ respectively. By letting (see [9])

$$\theta^{n+i} = d\xi^i + \sum_{j=1}^{n} \xi^j \omega^i_j$$

we get that for any $z \in N$, $\{\theta^1(z), \ldots, \theta^{2n}(z), \{\omega^i_j(z)\}\}$ is a basis for $N_z^*$ and $V_z := \{b \in N_z : \theta^l(z)(b) = 0 \text{ for } 1 \leq l \leq 2n\}$.

Let $H_1, \ldots, H_{2n}, \{V^{l}_{m}\}_{1 \leq l < m \leq n}$ be the dual frame of $\{\theta^1, \ldots, \theta^{2n}, \{\omega^i_j\}\}$. These vector fields were constructed as follow: If $z = (q, u, \xi)$, let $c_i$ be the geodesic that satisfies $c_i(0) = q$ and $\dot{c}_i(0) = u_i$. Let $E^i_1, \ldots, E^i_n$ be the parallel vector fields along $c_i$ such that $E^i_1(0) = u_i$. If we define $\gamma_i(t) = (c_i(t), E^i_1(t), \ldots, E^i_n(t), \xi)$, then

$$H_i(z) = \dot{\gamma}_i(z)$$

and

$$H_{n+i}(z) = (i_{(q,u)}\xi)(\frac{\partial}{\partial \xi})$$

for $1 \leq i \leq n$, where $i_{(q,u)} : \mathbb{R}^n \to N$ is the inclusion map given by $i_{(q,u)}(\xi) = (q, u, \xi)$.
Let \( \sigma_z : O(n) \rightarrow N \) be the map defined by \( \sigma_z(a) = R_a(z) = z.a. \) Since \( V_z = \ker(\psi_z) = (\sigma_z)_{*id}(\mathfrak{o}(n)) \), where \( \mathfrak{o} \) is the space of skew symmetric matrices of \( \mathbb{R}^{n \times n} \), let

\[
V_{m+i}^l(z) = (\sigma_z)_{*id}(A_{m+i}^l)
\]

where \([A_{m+i}^l]_j = 1, [A_{m+i}^l]_j = -1\) and \([A_{m+i}^l]_j = 0\) otherwise. Hence,

\[
\psi_z(V_{m+i}^l(z)) = 0.
\]

An easy check shows that

\[
\psi_z(H_i(z)) = e_i(z)
\]

and

\[
\psi_z(H_{n+i}(z)) = e_{n+i}(z).
\]

Let \( \omega = \sum_{1 \leq i < j \leq n} \omega^i_j \otimes \omega^j_i \), if \( G \) is a Riemannian metric on \( TM \) then

\[
G^* = \psi^*(G) + \omega
\]

is also a Riemannian metric on \( N \). It follows easily that \( V_z \perp G^* H_z \) and \( \psi_z : H_z \rightarrow (TM)_{\psi(z)} \) is an isometry, therefore \( \psi : (N,G^*) \rightarrow (TM,G) \) is a Riemannian submersion. We shall use this fact to compute the curvature tensor of \((TM,G)\) when \( G \) is a natural metric.

**Remark 3.1** Let \( X \) be a vector field on \( TM \), the horizontal lift of \( X \) is a vector field \( X^h \) on \( N \) such that \( X^h(z) \in H_z \) and \( \psi_z(X^h(z)) = X(\psi(z)) \). If \( X(\psi(z)) = \sum_{i=1}^{2n} x^i(z)e_i(z) \), from (11), (12) and (13) it follows that \( X^h(z) = \sum_{i=1}^{2n} a^i(z)H_i(z) \).

**Proposition 3.2** For \( 1 \leq i,j,l,m \leq n \) let \( R_{ijlm} : N \rightarrow \mathbb{R} \) be the maps defined by \( R_{ijlm}(q,u,\xi) = g(R(u_i,u_j)u_l,u_m) \), where \( R \) is the curvature tensor of \((M,g)\). The Lie bracket on vertical and horizontal vector fields on \( N \) satisfies:

a) \([H_i,H_j] = \sum_{l,m=1}^{n} R_{ijlm}\xi^m H_{n+i} + \frac{1}{2} \sum_{l,m=1}^{n} R_{ijlm}V_{m+i}^l \).

b) \([H_i,H_{n+j}] = 0 \).

c) \([H_i,V_{m+i}^l] = \delta_{il}H_m - \delta_{im}H_i \).

d) \([H_{n+i},H_{n+j}] = 0 \).

e) \([H_{n+i},V_{m+i}^l] = \delta_{il}H_{n+m} - \delta_{im}H_{n+i} \).

f) \([V_{j}^l,V_{m+i}^l] = \delta_{jl}V_{m+j} + \delta_{jl}V_{m-1} + \delta_{jl}V_{j+i} \).

g) If \( f : N \rightarrow \mathbb{R} \) is a function that depends only on the parameter \( \xi \), then \( H_i(f) = 0 \) and \( V_{j}^l(f) = \xi^i H_{n+j}(f) - \xi^j H_{n+i}(f) \).

h) If \( X \) and \( Y \) are tangent vector fields on \( TM \) and \( v = \psi(q,u,\xi) \) then \( [X^h,Y^h] \mathfrak{v}_{(q,u,\xi)} = \sum_{1 \leq i < m \leq n} g_{ij}(\pi\psi(X(v)),\pi\psi(Y(v)))u_l,u_m)V_{m+i}^l(q,u,\xi) \).

The proof is straightforward and follows by taking local coordinates in \( M \) and the induced one in \( TM \) and evaluating the forms \( \theta^i, \theta^{m+i}, \omega^i_j \) on the fields \([H_r,H_s], [H_r,V_{m+i}^l] \) and \([V_{m+i}^l,V_{m+i}^{l'}] \) for \( 1 \leq r,s \leq 2n, 1 \leq l < m \leq n \) and \( 1 \leq l' < m' \leq n \).
3.2 The main result.

From now on, let $\bar{R}$ and $R^*$ be the curvature tensors of $(TM, G)$ and $(N, G^*)$ respectively. For simplicity we denote by $< , >$ the metrics $G$ and $G^*$. Since $\psi : (N, G^*) \longrightarrow (TM, G)$ is a Riemannian submersion, by the O’Neill formula (see [10]) we have that

$$< \bar{R}(X, Y)Z, W > \circ \psi = < R^*(X^h, Y^h)Z^h, W^h > + \frac{1}{4} < [Y^h, Z^h]^v, [X^h, W^h]^v >$$

$$- \frac{1}{4} < [X^h, Z^h]^v, [Y^h, W^h]^v > - \frac{1}{2} < [Z^h, W^h]^v, [X^h, Y^h]^v > .$$

(15)

If $Y^h(z) = \sum_{i=1}^{2n} y^i(z)H_i(z)$, $Z^h(z) = \sum_{i=1}^{2n} z^k(z)H_i(z)$ and $W^h(z) = \sum_{i=1}^{2n} w^l(z)H_i(z)$, then the first term of the right side of equality (15) is

$$< R^*(X^h, Y^h)Z^h, W^h > = \sum_{ijkl=1}^{2n} x^i y^j z^k w^l < R^*(H_i, H_j)H_k, H_l > .$$

On the other hand, if $v = \psi(q, u, \xi)$, it follows from Proposition 3.2 (part h) that

$$< [X^h, Y^h]^v, [Z^h, W^h]^v > |_{(q, u, \xi)} =$$

$$= \frac{1}{2} \sum_{r, s=1}^{n} < R(\pi_s(X(v)), \pi_s(Y(v)))u_r, u_s > . < R(\pi_s(Z(v)), \pi_s(W(v)))u_r, u_s > .$$

(16)

**Remark 3.3** In order to compute $< \bar{R}(X(v), Y(\psi))Z(v), W(v) >$ it is sufficient to evaluate the right side of (15) on points of $N$ of the form $z = (q, u, t, 0, \ldots, 0)$ such that $v = \psi(z)$, where $t = |v|$, and where $|v|$ is the norm induced by the metric $g$.

Let $f : [0, +\infty) \longrightarrow \mathbb{R}$ be a differentiable map, from now on, let us denote by $\dot{f}(t)$ the derivate of $f$ at $t$.

**Theorem 3.4** Let $G$ be a natural metric on $TM$. Let $\alpha$ and $\beta$ be the functions that characterizes $G$. If $1 \leq i, j, k, l \leq n$ and $z = (q, u, t, 0, \ldots, 0)$ we have that

a) $< R^*(H_i(z), H_j(z))H_k(z), H_l(z) >=$

$$t^2 \alpha(t^2) \sum_{r=1}^{n} \left\{ \frac{1}{2} R_{ijr1}(z)R_{kldr1}(z) + \frac{1}{4} R_{i1r1}(z)R_{kjlr1}(z) + \frac{1}{4} R_{j1r1}(z)R_{iklr1}(z) \right\}$$

$$+ \sum_{1 \leq r < s \leq n} \left\{ \frac{1}{2} R_{ijr1}(z)R_{klrs}(z) + \frac{1}{4} R_{i1r1}(z)R_{kjrs}(z) + \frac{1}{4} R_{j1r1}(z)R_{ikrs}(z) \right\} + R_{ijkl}(z).$$

b) Let $\epsilon_{ijkl} = \delta_{il}\delta_{jk} - \delta_{ij}\delta_{lk}$, then
b.1) If no index is equal to one, then
\[ < R^*(H_{n+i}(z), H_{n+j}(z))H_{n+k}(z), H_{n+l}(z) > = \epsilon_{ijkl}F(t^2) \]
where \( F : [0, +\infty) \rightarrow \mathbb{R} \) is defined by
\[ F(t) = \frac{\alpha(t)\beta(t) - t(\dot{\alpha}(t))^2 - 2\alpha(t)\dot{\beta}(t)}{\alpha(t) + t\beta(t)}. \] (17)

b.2) If some index equals one, for example \( l = 1 \), then
\[ < R^*(H_{n+i}(z), H_{n+j}(z))H_{n+k}(z), H_{n+l}(z) > = \epsilon_{ijkl}H(t^2) \]
where \( H : [0, +\infty) \rightarrow \mathbb{R} \) is defined by
\[ H(t) = \phi(t)\frac{\partial}{\partial t}\ln(\alpha\Delta)|_t - 2\phi(t) \] (18)

and \( \phi(t) = \alpha(t) + t\dot{\alpha}(t), \Delta(t) = \alpha(t) + t\beta(t) \).

c) \( < R^*(H_i(z), H_{n+j}(z))H_{n+k}(z), H_{n+l}(z) > = 0 \).
d) \( < R^*(H_{n+i}(z), H_{n+j}(z))H_k(z), H_l(z) > = \)
\[ \frac{1}{2}(2\alpha(t^2) + (\delta_{i1} + \delta_{j1})\beta(t^2)t^2)R_{ijkl}(z) + \frac{1}{2}\delta_{i1}(\beta(t^2) - 2\dot{\beta}(t^2))t^2R_{klj1}(z) \]
\[ + \frac{1}{2}\delta_{j1}(2\dot{\alpha}(t^2) - \beta(t^2))t^2R_{kl1}(z) + \frac{(\alpha(t^2))^2t^2}{4}\sum_{r=1}^{n}\{R_{krj1}(z)R_{rli1}(z) - R_{kr1}(z)R_{rtl1}(z)\}. \]
e) \( < R^*(H_i(z), H_{n+j}(z))H_k(z), H_{n+l}(z) > = \)
\[ \frac{1}{2}\alpha(t^2)R_{klij}(z) + \frac{(\alpha(t^2))^2t^2}{4}\sum_{r=1}^{n}R_{krj1}(z)R_{rli1}(z) + \frac{t^2}{2}(\delta_{j1} + \delta_{i1})\dot{\alpha}(t^2)(R_{kil1}(z) - R_{kij1}(z)). \]
f) \( < R^*(H_i(z), H_j(z))H_{n+k}(z), H_l(z) > = \)
\[ \frac{\alpha(t^2)t}{2}\{ < \nabla_D R(E^i_j(s), E^j_j(s))E^k_k(s)|_{s=0}, u_1 > - < \nabla_D R(E^j_i(s), E^i_i(s))E^k_k(s)|_{s=0}, u_1 > \}. \]

The proof follows from the Koszul formula and Proposition 3.2 and it involves a lot of calculation. For more details we refer the reader to [6] pages 132-151.

**Theorem 3.5** The curvature tensor \( \tilde{R} \) evaluated on \( e_i(z), e_{n+i}(z) \) satisfies:

a) \( < \tilde{R}(e_i(z), e_j(z))e_k(z), e_l(z) > = \)
\[ t^2\alpha(t^2)\sum_{r=1}^{n}\left\{ \frac{1}{2}R_{ijr1}(z)R_{klr1}(z) + \frac{1}{4}R_{dir1}(z)R_{kjr1}(z) + \frac{1}{4}R_{jir1}(z)R_{kjr1}(z) \right\} + R_{ijkl}(z). \]
b) b.1) If no index is equal to one, then
\[
< \bar{R}(e_{n+1}(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z) > = \epsilon_{ijkl}F(t^2). \tag{19}
\]

b.2) If some index equals one, for example \( l = 1 \), then
\[
< \bar{R}(e_{n+1}(z), e_{n+j}(z))e_{n+k}(z), e_{n+1}(z) > = \epsilon_{ijk1}H(t^2). \tag{20}
\]

c) \( < \bar{R}(e_{1}(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z) > = 0. \)
d) \( < \bar{R}(e_{n+1}(z), e_{n+j}(z))e_{k}(z), e_{l}(z) > = \)
\[
\frac{1}{2}\left(2\alpha(t^2) + (\delta_{11} + \delta_{j1})\beta(t^2)t^2\right)R_{ijkl}(z) + \frac{1}{2}\delta_{j1}\left(\beta(t^2) - 2\dot{\alpha}(t^2)\right)t^2R_{kij1}(z)
\]
\[
+ \frac{1}{2}\delta_{j1}\left(2\dot{\alpha}(t^2) - \beta(t^2)\right)t^2R_{klj1}(z) + \frac{(\alpha(t^2))^2t^2}{4}\sum_{r=1}^{n}\{R_{krj1}(z)R_{rl1}(z) - R_{krl1}(z)R_{rj1}(z)\}.
\]
e) \( < \bar{R}(e_{1}(z), e_{n+j}(z))e_{k}(z), e_{n+l}(z) > = \)
\[
\frac{1}{2}\alpha(t^2)R_{klj}(z) + \frac{(\alpha(t^2))^2t^2}{4}\sum_{r=1}^{n}R_{krj1}(z)R_{rl1}(z) + \frac{t^2}{2}(\delta_{j1} + \delta_{j1})\dot{\alpha}(t^2)(R_{klj1}(z) - R_{klj1}(z)).
\]
f) \( < \bar{R}(e_{1}(z), e_{j}(z))e_{n+k}(z), e_{l}(z) > = \)
\[
\frac{\alpha(t^2)t}{2}\{ < \nabla_D R(E_i^j(s), E_i^j(s))E_j^k(s)|_{s=0, u_1} > - < \nabla_D R(E_i^j(s), E_i^j(s))E_j^k(s)|_{s=0, u_1} >\}.
\]

Proof. The proof is straightforward and follows form Theorem 3.4 and equality (15).

The functions \( F \) and \( H \) satisfy the following proposition:

**Proposition 3.6** Let \( \alpha, \beta : [0, +\infty) \rightarrow \mathbb{R} \) be differentiable functions such that \( \alpha(t) > 0 \) and \( \alpha(t) + t\beta(t) > 0 \) for all \( t \geq 0. \) If \( F \) is the zero function, then:

i) \( \beta(t) = \frac{t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)}{\alpha(t)}. \)

ii) \( \alpha(t)\alpha(t) + t\beta(t) = (t\ddot{\alpha}(t) + \alpha(t))^2. \)

iii) \( \alpha(t) + t\ddot{\alpha}(t) > 0. \)

iv) \( H(t) = 0 \) for all \( t \geq 0. \)

Proof. Assertion i) follows from equality (17) and ii) is a consequence of i). Equality ii) shows that \( \alpha(t) + t\ddot{\alpha}(t) \neq 0 \) for all \( t \geq 0, \) and since \( \alpha(0) + 0\dot{\alpha}(0) = \alpha(0) > 0, \) then we get iii). Equality ii) says that \( \alpha \Delta = \phi^2, \) and assertion iii) says that \( \phi > 0. \) Therefore, from equality (18) we get that \( H = 0. \)
Corollary 3.7 Let $\alpha, \beta : [0, +\infty) \rightarrow \mathbb{R}$ be differentiable functions such that $\alpha(t) > 0$, $\alpha(t) + t\dot{\alpha}(t) > 0$ and $\alpha(t) + t\beta(t) > 0$ if $t \geq 0$. If $H$ is the zero function, then it is also $F$.

Proof. Since $\phi > 0$ and $H = 0$, the equality (18) implies that $\ln(\alpha \Delta) = \ln(\phi^2) + C$ for some constant $C$. In particular $2 \ln(\alpha(0)) = 2 \ln(\alpha(0)) + C$, hence $C = 0$. Since $\alpha \Delta = \phi^2$, we obtain that $F = 0$.

4 Geometric consequences of curvature equations.

In this section the Riemannian metric $G$ on $TM$ is assumed natural. As throughout all the paper, $G$ is characterized by the functions $\alpha$ and $\beta$. As in Remark 3.3, if $v \in TM$, let $z = (q, u, t, 0, \ldots, 0) \in N$ such that $\psi(z) = v$ and $t = |v|$. From Theorem 3.5 and Proposition 3.6 we get inmediatly

Corollary 4.1 (Theorem 0.1, [1]) If $(TM, G)$ is flat then $(M, g)$ is flat.

Proof. It follows from part a) of Theorem 3.5 by setting $t = 0$.

Corollary 4.2 If $\dim M \geq 3$, $(TM, G)$ is flat if and only if $(M, g)$ is flat and 

$$\beta(t) = \frac{t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)}{\alpha(t)}$$

Proof. Assume that $(TM, G)$ is flat. From Theorem 3.5 part b.1) and $1 < i < j \leq n$ we have that

$$< \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+i}(z), e_{n+j}(z) >= -F(t^2)$$

Therefore $F = 0$, and the desired equality on $\beta$ follows from Proposition 3.6 part i).

Assuming that $(M, g)$ is flat and $\beta(t) = \left(t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)\right)/\alpha(t)$, we only need to show that

$$< \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z) >= 0$$

for $1 \leq i, j, k, l \leq 2n$. The other cases also satisfies (21) because $R = 0$. Equality on $\beta$ implies that $F = 0$, therefore by Proposition 3.6 part iv) we have that $H = 0$, and equality (21) is satisfied.

We get also the following result:

Corollary 4.3 If $\dim M = 2$, $(TM, G)$ is flat if and only if $(M, g)$ is flat and $H = 0$.

Remark 4.4 Let $\alpha(t) > 0$ be a differentiable function that satisfies $t\dot{\alpha}(t) + \alpha(t) > 0$ for all $t \geq 0$ and define $\beta(t) = \left(t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)/\alpha(t)\right)$. If we consider the natural metric $G$ induced by $\alpha$ and $\beta$, then $(TM, G)$ is flat if $(M, g)$ is flat.
Remark 4.5 The above Corollaries generalizes the well known fact that \((TM, G_s)\) is flat if and only if \((M, g)\) is flat (Kowalski [7], Aso [2]). This fact follows from the Corollaries taking \(\alpha = 1\) and \(\beta = 0\).

We will denote by \(K\) and \(\bar{K}\) the sectional curvatures of \((M, g)\) and \((TM, G)\) respectively.

Theorem 4.6 Let \(v \in TM\) and \(z = (q, u, t, 0, \ldots, 0) \in N\) such that \(\psi(z) = v\) (\(t = |v|\)). We have the following expression for the sectional curvature of \((TM, G)\):

a) For \(1 \leq i, j \leq n\):

\[
\bar{K}(e_i(z), e_j(z)) = K(u_i, u_j) - \frac{3}{4}\alpha(t^2)|R(u_i, u_j)v|^2.
\]

b) b.1) If \(2 \leq i, j \leq n\) and \(i \neq j\)

\[
\bar{K}(e_{n+1}(z), e_{n+j}(z)) = \frac{F(t^2)}{(\alpha(t^2))^2}.
\]

b.2) If \(2 \leq i \leq n\)

\[
\bar{K}(e_{n+1}(z), e_{n+j}(z)) = \frac{H(t^2)}{\alpha(t^2)(\alpha(t^2)+t^2\beta(t^2))}.
\]

c) For \(1 \leq i, j \leq n\):

\[
\bar{K}(e_i(z), e_{n+j}(z)) = \frac{\alpha(t^2)}{4}|R(u_j, v)u_i|^2.
\]

In particular \(\bar{K}(e_i, e_{n+1}) = 0\) if \(1 \leq i \leq n\) because \(v = tu_1\).

Proof. From equality (4) we get that \(\{e_1(z), \ldots, e_{2n}(z)\}\) is an orthogonal basis for \((TM)_v\) such that \(<e_i(z), e_j(z)> = \delta_{ij}\) if \(1 \leq i, j \leq n\), \(<e_{n+1}(z), e_{n+1}(z)> = \alpha(t^2) + t^2\beta(t^2)\) and \(<e_{n+1}(z), e_{n+1}(z)> = \alpha(t^2)\) if \(2 \leq i \leq n\). Let \(1 \leq i, j \leq n\), \(i \neq j\). By setting \(k = j\) and \(l = i\) in equation a) of Theorem 3.5 we have that

\[
\bar{K}(e_i(z), e_j(z)) = -<\bar{R}(e_i(z), e_j(z)e_j(z), e_i(z)> = R_{ijji}(z) - \frac{3}{4}t^2\alpha(t^2)\sum_{r=1}^{n} R_{rjj1}(z).
\]

Since \(K(u_i, u_j) = R_{ijji}(z)\) and \(v = tu_1\), we can write

\[
\bar{K}(e_i(z), e_j(z)) = K(u_i, u_j) - \frac{3}{4}\alpha(t^2)|R(u_i, u_j)v|^2.
\]

Part b) follows directly from equations b.1) and b.2) of Theorem 3.5.

Since \(|e_i(z)| = 1\) and \(<e_i(z), e_{n+j}(z)> = 0\) for \(1 \leq i, j \leq n\), from Theorem 3.5 equation e), we see that

\[
\bar{K}(e_i(z), e_{n+j}(z)) = -\frac{(\alpha(|v|^2))^2|v|^2}{4(\alpha(|v|^2) + \delta_{ij}\beta(|v|^2)|v|^2)}\sum_{r=1}^{n} R_{rijj1}(z)R_{rijj1}(z)
\]

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Corollary 4.7

i) \((TM, G)\) is never a manifold with negative sectional curvature.

ii) If \(\bar{K}\) is constant, then \((TM, G)\) and \((M, g)\) are flat.

iii) If \(\bar{K}\) is bounded and \(\lim_{t \to +\infty} t\alpha(t) = +\infty\), then \((M, g)\) is flat.

iv) If \(c \leq \bar{K} \leq C\) (possibly \(c = -\infty\) and \(C = +\infty\)), then \(c \leq K \leq C\).

Proof. Assertions i), ii) and ii) follow from Theorem 4.6 part c). Let \(q \in M\) and \(u = (u_1, \ldots, u_n)\) be an orthonormal basis for \(M_q\). Then, if we consider \(z = (q, u, t, 0, \ldots, 0)\) and \(v = 0, \ldots, 0\), from Theorem 4.6 part a) we have that \(\bar{K}(e_i(z), e_j(z)) = K(u, u_j)\) and part iv) holds. Also ii) follows from Theorem 3.5 part a) taking \(t = 0\).

Corollary 4.8 Let \((M, g)\) be a manifold of constant sectional curvature \(K_0\) and \(TM\) endowed with a natural metric \(G\), then we have for \(z = (q, u, t, 0, \ldots, 0)\) and \(\psi(z) = v\) that

a) \(\bar{K}(e_i(z), e_j(z)) = K_0 - \frac{3}{4}(K_0)^2 \alpha(||v||^2)(\delta_{ij})||v||^2\) with \(i \neq j\).

b) \(\bar{K}(e_i(z), e_{n+j}(z)) = \frac{\alpha(||v||^2)}{4} K_0 ||v||^2 (\delta_{ij} + \delta_{i1})\).

The vertical case \(\bar{K}(e_{n+i}, e_{n+j})\) is as Theorem 4.6 part b).

From Theorem 4.6 we get the following result

Corollary 4.9 Let \(G_1\) and \(G_2\) be two natural metrics on \(TM\) such that are characterized by the functions \(\{\alpha_i\}_{i=1,2}\) and \(\{\beta_i\}_{i=1,2}\) respectively. If \(K_1(u)(V, W) = K_2(u)(V, W)\) for all \(u \in TM\) and \(V, W \in (TM)_u\) and \((M, g)\) is not flat, then \(\alpha_1 = \alpha_2\).

Remark 4.10 Let \(G_{+\exp}\) and \(G_{-\exp}\) be the natural metrics on \(TM\) defined by

\[
\begin{pmatrix}
\alpha G_{+\exp}(q, u, \xi) &= \left( I_{d \times n} \quad 0 \quad A^+(\xi) \right)
\end{pmatrix}
\begin{pmatrix}
\alpha G_{-\exp}(q, u, \xi) &= \left( I_{d \times n} \quad 0 \quad A^-(\xi) \right)
\end{pmatrix}
\]

where \(A^+(\xi) = e^{\xi}(I_{d \times n} + \xi^t \xi)\) and \(A^-(\xi) = e^{-\xi}(I_{d \times n} + \xi^t \xi)\). We call \(G_{+\exp}\) and \(G_{-\exp}\) the positive and negative exponential metric.

It is known ([11]) that \(TM\) endowed with the Cheeger-Gromoll metric is never a manifold of constant sectional curvature. Theorem 4.6 applied to \(G_{+\exp}\) and \(G_{-\exp}\) shows that these metrics satisfy the same property.
4.1 Ricci tensor and scalar curvature.

Let $Ric$ and $\bar{Ric}$ be the Ricci tensor of $(M, g)$ and $(TM, G)$ respectively. We will denote by $S$ and $\bar{S}$ the scalar curvature of $(M, g)$ and $(TM, G)$.

**Theorem 4.11** For $1 \leq i, j \leq n$ and $z = (q, u, t, 0 \ldots, 0)$ we have the following expressions for $\bar{Ric}$:

a) $\bar{Ric}(e_i(z), e_j(z)) = -\frac{\alpha(t^2)t^2}{2} \sum_{1 \leq r, l \leq n} R_{irl1}(z)R_{jr1l}(z) + Ricc(u_i, u_j)$.

b) $\bar{Ric}(e_i(z), e_{n+j}(z)) = -\frac{\alpha(t^2)t^2}{2} \sum_{1 \leq r \leq n} \left\{ <\nabla D R(E^i_r, E^r_j)|_{s=0}, u_1 > - <\nabla D R(E^i_r, E^r_i)|_{s=0}, u_1 > \right\}$.

c.1) If $2 \leq i \leq n$, then

$$\bar{Ric}(e_{n+i}(z), e_{n+i}(z)) = \frac{t^2\alpha(t^2)}{4} \sum_{1 \leq r, l \leq n} R_{rl1l}^2(z) + \frac{(n-2)}{\alpha(t^2)} F(t^2) \frac{1}{\alpha(t^2) + t^2\beta(t^2)} H(t^2).$$

c.2) If $2 \leq i, j \leq n$ and $i \neq j$, then

$$\bar{Ric}(e_{n+i}(z), e_{n+j}(z)) = \frac{t^2\alpha(t^2)}{4} \sum_{1 \leq r, l \leq n} R_{rl1l}(z)R_{rlj1}(z).$$

c.3) If $1 \leq j \leq n$, then

$$\bar{Ric}(e_{n+1}(z), e_{n+j}(z)) = \frac{(n-1)}{\alpha(t^2)} H(t^2)\delta_{j1}.$$ 

**Proof.** Let $\bar{e}_1(z), \ldots, \bar{e}_{2n}(z)$ be the orthonormal basis for $(TM)_v$ induced by the orthogonal basis $e_1(z), \ldots, e_{2n}(z)$, where $\psi(z) = v$. For $X, Y \in (TM)_v$ we have that

$$\bar{Ricc}(X, Y) = \sum_{l=1}^{2n} < \bar{R}(X, \bar{e}_l(z))\bar{e}_l(z), Y > .$$

Equalities a), b) and c) follow directly from Theorem 3.5 and the fact that $< e_{n+1}(z), e_{n+1}(z) > = \alpha(t^2) + t^2\beta(t^2)$ and $< e_{n+i}(z), e_{n+i}(z) > = \alpha(t^2)$ if $2 \leq i \leq n$. □

In [1], it is shown in the general $g$-Riemannian natural case that if $(TM, G)$ is Einstein then $(M, g)$ is Einstein. In our situation we have
Corollary 4.12 If \((TM, G)\) is Einstein, then \((M, g)\) and \((TM, G)\) are flats.

Proof. Let \(c\) be a constant such that \(\bar{\text{Ricc}} = cG\). In order to prove that \(R = 0\), it is enough to show that for any \(q \in M\) and any orthonormal basis \(u = \{u_1, \ldots, u_n\}\) for \(M_q\) the following equalities are satisfied

\[
< R(u_i, u_r)u_l, u_1 > = 0
\]

for \(1 \leq i, r, l \leq n\). Let \(v \in M_q\), \(v \neq 0\) and \(z = (g, u, t, 0, \ldots, 0) \in N\) such that \(\psi(z) = tu_1 = v\). Since \(G(e_i(z), e_j(z)) = \delta_{ij}\) if \(1 \leq i, j \leq n\), from Theorem 4.11 part a) we have that

\[
e_{ij} = -\frac{\alpha(t^2)l^2}{2} \sum_{1 \leq r, l \leq n} R_{ir1l}(z)R_{jr1l}(z) + \text{Ricc}(u_i, u_j).
\]

Taking \(t = 0\), we get that \(\text{Ricc}(u_i, u_j) = c\delta_{ij}\). Replacing these values for \(i = j\) in (23) we obtain that

\[
0 = -\frac{\alpha(t^2)l^2}{2} \sum_{1 \leq r, l \leq n} (< R(u_i, u_r)u_l, u_1 >)^2
\]

for \(t \geq 0\) and equality (22) is satisfied. Since \(\text{Ricc} = c\) and \(R = 0\), it follows that \(\bar{\text{Ricc}} = 0\). Using that \((TM, G)\) is Ricci flat and \(R = 0\), from Theorem 4.11 parts c.1) and c.3) one gets that \(H = F = 0\). From Theorem 3.5 we have that \(\bar{R} = 0\). \(\Box\)

Remark 4.13 It is easy to see from Theorem 4.11 that if \((M, g)\) is not flat or if not exists a constant \(k\) such that \(H(t) = k\alpha(t)\) and \((n - 2)[\alpha(t) + t\beta(t)]F(t) = \alpha(t)[(n - 2)\alpha(t) + (n - 1)t\beta(t)]\), then \(\text{Ricc} \) is not a \(\lambda\) - natural tensor (see [5]).

Corollary 4.14 Let \(v \in TM\) and \(z = (\pi(v), u_1, \ldots, u_n, t, 0, \ldots, 0) \in N\) such that \(v = u_1t\). The scalar curvature of \((TM, G)\) at \(v\) is given by

\[
S(v) = S(\pi(v)) - \frac{t^2\alpha(t)}{4} \sum_{i=1}^{n} R_{ir1l}(z) + \frac{2(n - 1)}{\alpha(t^2)(\alpha(t^2) + \beta(t^2))^2} H(t^2)
\]

\[
+ \frac{(n - 1)(n - 2)}{(\alpha(t^2))^2} F(t^2).
\]

Proof. Since \(\{\bar{e}_1(z), \ldots, \bar{e}_{2n}(z)\}\) is an orthonormal basis for \((TM)_v\) and the scalar curvature \(\bar{S}(v) = \sum_{i=1}^{2n} \text{Ricc}(\bar{e}_i(z), \bar{e}_i(z))\), the expression for \(S\) follows straightforward from Theorem 4.11. \(\Box\)

Remark 4.15 Corollary 4.14 applied to \(G_{+, \exp}\) and \(G_{-, \exp}\) reads:

\[
S_{+, \exp}(v) = S(\pi(v)) - (n - 1)e^{-|v|^2}\frac{2 + (n - 2)(1 + |v|^2)}{(1 + |v|^2)}
\]

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\[- \frac{e^{\|v\|^2}}{4} \sum_{i,j=1}^{n} |R(u_i, u_j)v|^2 \]

and

\[
S_{-\exp}(v) = S(\pi(v)) + \frac{(n-1)e^{\|v\|^2}}{1 + \|v\|^2} \left[ (n-2)(3 - \|v\|^2) + \frac{6 + 2\|v\|^2}{1 + \|v\|^2} \right] - \frac{e^{-\|v\|^2}}{4} \sum_{i,j=1}^{n} |R(u_i, u_j)v|^2.
\]

**Proposition 4.16** If \((M, g)\) is a manifold of constant sectional curvature \(K_0\), then

\[
S_{+\exp}(v) = (n-1) \left\{ K_0 \left( n - \frac{K_0}{2} \|v\|^2 e^{\|v\|^2} \right) - e^{\|v\|^2} \left[ \frac{2 + (n-2)(1 + \|v\|^2)}{1 + \|v\|^2} \right] \right\}.
\]

and

\[
S_{-\exp}(v) = (n-1) \left\{ K_0 \left( n - \frac{K_0}{2} \|v\|^2 e^{-\|v\|^2} \right) + \frac{e^{\|v\|^2}}{1 + \|v\|^2} \left[ (n-2)(3 - \|v\|^2) + \frac{6 + 2\|v\|^2}{1 + \|v\|^2} \right] \right\}.
\]

**Corollary 4.17** Let \((M, g)\) be a flat manifold, then we have that:

a) \(S_{+\exp} < 0\).

b) If \(\dim M = 2\), then \(S_{-\exp} > 0\).

c) If \(\dim \geq 3\), \(S_{-\exp}(v) > 0\) if and only if \(0 \leq \|v\|^2 < \frac{(n-1)+\sqrt{4(n-2)n+1}}{n-2}\).

d) If \(\dim \geq 3\), \(S_{-\exp}(v) = 0\) if and only if \(\|v\|^2 = \frac{(n-1)+\sqrt{4(n-2)n+1}}{n-2}\).

**Proof.** It follows from Proposition 4.16.

\[\square\]

**Remark 4.18** In [1], it is shown (Theorem 0.3) that if \(G\) is a \(g\)-natural metric on \(TM\) and \((TM, G)\) has constant scalar curvature, then \((M, g)\) has constant scalar curvature. In our case, this property follows immediately from Corollary 4.14, taking \(t = 0\). We can see that if \((TM, G)\) has constant scalar curvature \(\bar{S}\) and \(F = 0\), then \((TM, G)\) is flat. If \(F = 0\) by Proposition 3.6, \(H = 0\), and by Corollary 4.14 it follows that \(R = 0\). Finally, from Theorem 3.5 we get that \((TM, G)\) is flat.
References


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