On the instability of two entropic dynamical models[*]

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Abstract

In this paper we study two entropic dynamical models from the viewpoint of information geometry. We study the geometry structures of the associated statistical manifolds. In order to analyze the character of the instability of the systems, we obtain their geodesics and compute their Jacobi vector fields. The results of this work improve and extend a recent advance in this topics studied in [8].

1   Introduction

The aim of present article is to extend and study two entropic dynamical models introduced by Peng, Sun, Sun, and Yi in [8]. An entropic dynamical model of a system is a statistical manifold constructed using maximum entropy methods (see for instance [5]) where each macrostate of the system (a probability distribution) is represented by a point on it. The dynamics of the system's macro-states is determined by the geometry induced by the Fisher-information metric, i.e., the geodesics represent the evolution in time of the macrostates. There are several references related with the study of entropic dynamical models from the viewpoint of information geometry, see for instance [3], [4], [6] among others.

In [8], the authors studied the character of the instability of two entropic dynamical models:

- $\mathcal{M}_1$: with a statistical manifold induced by a family of a joint Gamma and Exponential distributions

- $\mathcal{M}_2$: with a statistical manifold induced by a family of a joint Gamma and Gaussian distributions.

From the study of the geometry of both models, they found out that M_1 have first order linear divergent instability and M_2 have exponential instability.

The first model that we consider is given by the statistical manifold induced by the $k-$joint one parametric exponential family (it model a system of uncorrelated k particles). Second, we study a system of two correlated particles modelled by the statistical manifold of the multivariate Gaussian probability family. Finally, we discuss how these models can be combined in order to generalize the obtained results to a large class of models.

The paper is organized as follows. In Section 2, we introduce a k-dimensional statistical manifold induced by densities of a one parameter exponential family and we study its geometrical structure. We analyze the character of the stability of this model when $k = 4$. In section 3, we study the geometric structure and the stability of a Gaussian statistical manifolds with correlations. Conclusions and some extensions are presented in Section 4.

2 Geometric structure and stability of $k-$dimensional statistical manifold

We refer the reader to [1] and [7] for definitions and standard results concerning to the geometry of statistical manifolds.

We consider a system of k particles in a one dimensional space named $\mathbf{x} = (x_1, \ldots, x_k)$. We assume that all information relevant to the dynamical model comes from the probability distribution which in this case is the joint distribution of k independent one parameter exponential family. More precisely, we consider the following joint density function

$$p(\mathbf{x}, \boldsymbol{\theta}) = h(\mathbf{x}) \exp \left(\sum_{s=1}^{k} (\eta_s(\theta_s) T_s(x_s) - \gamma_s(\theta_s)) \right)$$

with $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$, $\mathbf{x} = (x_1, \ldots, x_k)$, T_s is a continuous function and η_s and γ_s are twice-differentiable functions for $s = 1, \ldots, k$. Therefore, we can define the associated statistical manifold as follows

$$M_k := \left\{ p(\mathbf{x}, \boldsymbol{\theta}) = h(\mathbf{x}) \exp \left(\sum_{s=1}^{k} (\eta_s(\theta_s) T_s(x_s) - \gamma_s(\theta_s)) \right) \quad \theta_s \in \mathbb{R} \text{ for } s = 1, \ldots, k \right\}.$$

We are going to consider M_k endowed with the Fisher-information matrix. This metric is proportional to the amount of information that the distribution function contains about the parameter. Recall that the local expression of the Fisher-information metric with respect to the coordinate system $\boldsymbol{\theta}$ is:

$$g_{ij}(\boldsymbol{\theta}) = E \left(\partial_i l(\boldsymbol{\theta}) \partial_j l(\boldsymbol{\theta}) \right)$$

where $\partial_i l(\theta) = \frac{\partial}{\partial \theta_i} \log p(\mathbf{x}, \boldsymbol{\theta})$. It is easy to see that the Fisher-information metric on M_k can be computed as

$$g_{ij}(\boldsymbol{\theta}) = E \left((\eta_i'(\theta_i) T_i(x_i) - \gamma_i'(\theta_i)) (\eta_j'(\theta_j) T_j(x_j) - \gamma_j'(\theta_j)) \right).$$

Since the variables x_s $(s = 1, \ldots, k)$ have density function belonging to one parameter exponential family, the expected value and the variance of T_s can be computed easily in terms of η_s and γ_s. Indeed,

$$E(T_s) = \frac{\gamma_s'(\theta_s)}{\eta_s'(\theta_s)} \qquad Var(T_s) = \frac{\gamma_s''(\theta_s) \eta_s'(\theta_s) - \gamma_s'(\theta_s) \eta_s''(\theta_s)}{(\eta_s'(\theta_s))^3}.$$

From the independence of the variables x_s we have

$$g_{ij}(\boldsymbol{\theta}) = \delta_{ij} (\eta_i'(\theta_i))^2 \, Var(T_i) = \delta_{ij} \frac{\gamma_i''(\theta_i) \eta_i'(\theta_i) - \gamma_i'(\theta_i) \eta_i''(\theta_i)}{\eta_i'(\theta_i)},$$

where δ_{ij} is the Kronecker's delta. Note that we have assumed uncoupled constraints between the micro-variables. This assumptions leads to a metric tensor with trivial off diagonal elements.

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The inverse matrix of g is

$$g^{-1} = [g^{ij}] = diag\left(\frac{\eta_1'(\theta_1)}{\gamma_1''(\theta_1)\eta_1'(\theta_1) - \gamma_1'(\theta_1)\eta_1''(\theta_1)}, \ldots, \frac{\eta_k'(\theta_k)}{\gamma_k''(\theta_k)\eta_k'(\theta_k) - \gamma_k'(\theta_k)\eta_k''(\theta_k)}\right).$$

The length element is given by

$$ds^2 = g_{ij}d\theta_i\theta_j = \sum_i \frac{\gamma_i''(\theta_i)\eta_i'(\theta_i) - \gamma_i'(\theta_i)\eta_i''(\theta_i)}{\eta_i'(\theta_i)} \, d\theta_i^2,$$

and the volume element is

$$dV_g = \sqrt{g} \, d\theta_1 \wedge \cdots \wedge d\theta_k = \left(\prod_i \frac{\gamma_i''(\theta_i)\eta_i'(\theta_i) - \gamma_i'(\theta_i)\eta_i''(\theta_i)}{\eta_i'(\theta_i)}\right)^{1/2} d\theta_1 \wedge \cdots \wedge d\theta_k. \tag{1}$$

Recall that the Christoffel symbols Γ_{ij}^l is defined by $\Gamma_{ij}^l = \Gamma_{ijs}g^{sl}$ $(i,j,l,s = 1,2,\ldots,k)$ where

$$\Gamma_{ijs} = \frac{1}{2}(\partial_i g_{js} + \partial_j g_{si} - \partial_s g_{ij}), \qquad i,j,s = 1,\ldots,k.$$

For this model the Christoffel symbols that are not zero are:

$$
\begin{aligned}
\Gamma_{ii}^i &= \frac{\gamma_i'''(\theta_i)(\eta_i'(\theta_i))^2 - \gamma_i'(\theta_i)\eta_i'''(\theta_i)\eta_i'(\theta_i) - \eta_i''(\theta_i)(\gamma_i''(\theta_i)\eta_i'(\theta_i) - \gamma_i'(\theta_i)\eta_i''(\theta_i))}{2\eta_i'(\theta_i)(\gamma_i''(\theta_i)\eta_i'(\theta_i) - \gamma_i'(\theta_i)\eta_i''(\theta_i))} \\
&= \frac{1}{2}\left(\frac{\gamma_i'''(\theta_i)\eta_i'(\theta_i) - A_i'(\theta_i)\eta_i''(\theta_i)}{A_i''(\theta_i)\eta_i'(\theta_i) - \gamma_i'(\theta_i)\eta_i''(\theta_i)} - \frac{\eta_i''(\theta_i)}{\eta_i'(\theta_i)}\right). \tag{2}
\end{aligned}
$$

The Ricci curvature R_{is} is defined by $R_{is} = R_{ijsl}g^{jl}$ $i,j,s,l = 1,\ldots,k$ where

$$R_{ijsl} = (\partial_j\Gamma_{is}^u - \partial_i\Gamma_{js}^u)g_{ul} + (\Gamma_{jtl}\Gamma_{is}^t - \Gamma_{itl}\Gamma_{js}^t).$$

Therefore, it is easy to see that the curvature tensor components are all zero and the scalar curvature $S_g = 0$.

Recall that the geodesic equations are given by the following non linear system of second order ordinary differential equations:

$$\frac{\partial^2\theta_l}{\partial\tau^2} + \Gamma_{ij}^l\frac{\partial\theta_i}{\partial\tau}\frac{\partial\theta_j}{\partial\tau} = 0 \qquad \text{for } i,j,l = 1\ldots,k. \tag{3}$$

From (2) we obtain that the geodesics are determined by the following k differential equations:

$$\frac{\partial^2\theta_i}{\partial^2\tau} + \Gamma_{ii}^i\left(\frac{\partial\theta_i}{\partial\tau}\right)^2 = 0 \qquad \text{for} \qquad i = 1,\ldots,k. \tag{4}$$

Remark 2.1: The entropic dynamical model $\mathcal{M}_\infty$ studied in [8] (Section 3) is a particular case of the model introduced in this section. More precisely, taken $k = 2$, $\eta_1(\theta_1) = \frac{-\rho}{\theta_1}$, $\eta_2(\theta_2) = \frac{-1}{\theta_2}$, $\gamma_1(\theta_1) = -\rho\ln(\theta_1)$, $\gamma_2(\theta_2) = -\ln(\theta_2)$ and $h(\mathbf{x}) = \frac{1}{\Gamma(\rho)}x_1^{\rho-1}$ we get $\mathcal{M}_\infty$. Therefore, the results given in this section extend those ones obtained in [8] .

2.1 Instability

In this section we consider a system of 4 particles in a one dimensional space. We assume that the particles $\mathbf{x} = (x_1, \ldots, x_4)$ (the micro-states) do not interact between them and are distributed according to Poisson, Pareto, Laplace, and Weibull distributions, respectively. More precisely, the joint probability density function is

$$p(\mathbf{x}, \boldsymbol{\theta}) = \frac{\theta_1^{x_1} e^{-\theta_1}}{x_1!} \, \theta_2 a^{\theta_2} x_2^{-(\theta_2+1)} \frac{1}{2\theta_3} e^{-\frac{|x_3|}{\theta_3}} \frac{b x_4^{b-1}}{\theta_4^b} e^{-\frac{x_4^b}{\theta_4^b}} \tag{5}$$

with $\theta_i > 0$ for $i = 1, \ldots, 4$. This is a particular case of the entropic dynamic model introduced in the previous section. Indeed, taking $k = 4$ and

$$\begin{aligned}
\eta_1(\theta_1) &= \ln(\theta_1) & \gamma_1(\theta_1) &= \theta_1 \\
\eta_2(\theta_2) &= -(\theta_2+1) & \gamma_2(\theta_2) &= -\ln(\theta_2) - \theta_2 \ln(a) \\
\eta_3(\theta_3) &= -\frac{1}{\theta_3} & \gamma_3(\theta_3) &= \ln(2\theta_3) \\
\eta_4(\theta_4) &= -\frac{1}{\theta_4^b} & \gamma_4(\theta_4) &= b \ln(\theta_4) - \ln(b)
\end{aligned}$$

(a and b are fixed and known) M_4 modelled the system described above. From (4) we get that the geodesic equations are

$$\frac{\partial^2 \theta_1}{\partial^2 \tau} = \frac{1}{2\theta_1} \left(\frac{\partial \theta_1}{\partial \tau} \right)^2 \tag{6}$$

$$\frac{\partial^2 \theta_i}{\partial^2 \tau} = \frac{1}{\theta_i} \left(\frac{\partial \theta_i}{\partial \tau} \right)^2 \qquad \text{for } i = 2, 3, 4, \tag{7}$$

whose solution is

$$\theta_1(\tau) = A_1 (t + B_1)^2 \qquad \theta_i(\tau) = A_i e^{B_i \tau} \quad \text{for} \quad i = 2, 3, 4 \quad A_i \in \mathbb{R} - \{0\}, B_i \in \mathbb{R}. \tag{8}$$

Let $\mathbf{A} = (A_1, \ldots, A_4)$ ($A_i \neq 0$) and $\mathbf{B} = (B_1, \ldots, B_4)$. We denote with $\alpha_{\mathbf{A}, \mathbf{B}}$ the geodesic obtained from replace $\mathbf{A}$ and $\mathbf{B}$ in (8). The arch length of $\alpha_{\mathbf{A}, \mathbf{B}}$ between $\alpha_{\mathbf{A}, \mathbf{B}}(0)$ and $\alpha_{\mathbf{A}, \mathbf{B}}(\tau)$ is

$$\begin{aligned}
\ell^\tau(\alpha_{\mathbf{A}, \mathbf{B}}) &= \int_0^\tau \left(\sum_i g_{ii} \left(\frac{\partial \theta_i}{\partial \tau} \right)^2 \right)^{1/2} ds = \int_0^\tau \left(4A_1 + B_2^2 + B_3^2 + b^2 B_4^2 \right)^{1/2} ds \\
&= \left(4A_1 + B_2^2 + B_3^2 + b^2 B_4^2 \right)^{1/2} \tau.
\end{aligned}$$

Note that the geodesic length is independent of A_2, A_3, A_4 and B_1. The difference of the length of two geodesics with close initial condition diverges. For instance,

$$D(\tau) = |\ell^\tau(\alpha_{(A_1+\delta,\ldots,A_4), \mathbf{B}}) - \ell^\tau(\alpha_{\mathbf{A}, \mathbf{B}})| \longrightarrow_{\tau \to \infty} +\infty.$$

From (1), we have that the volume element is

$$dV_g = \left(\frac{1}{\theta_1} \right)^{1/2} \frac{1}{b\theta_2\theta_3\theta_4} d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge d\theta_4.$$

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Thus the volume of an extended region of M_4 is

$$\Delta V_{M_4}(\tau) = \int_{\theta_1(0)}^{\theta_1(\tau)} \int_{\theta_2(0)}^{\theta_2(\tau)} \int_{\theta_3(0)}^{\theta_3(\tau)} \int_{\theta_4(0)}^{\theta_4(\tau)} dV_g = 2\sqrt{A_1}B_2B_3B_4(\tau + B_1)\tau^3$$

and the average volume is

$$\frac{1}{\tau}\int \Delta V_{M_4}(\tau) = 2\sqrt{A_1}B_2B_3B_4(\frac{\tau}{5} + \frac{B_1}{4})\tau^3.$$

This quantity encodes relevant information about the stability of neighbouring volume region. The asymptotic behaviour of the average volume has diffusive expansion that increase as a polynomial function.

Finally, we study the temporal behaviour of the Jacobi field equation which is a natural tool to analyse dynamical chaos (analysing the geodesic spread). First, we recall that the Jacobi field equation (see [2]) is

$$\frac{D^2 J_i}{D\tau^2} = R^i_{kml}\frac{\partial \theta^k}{\partial \tau}J_m\frac{\partial \theta^l}{\partial \tau} \tag{9}$$

with $J = (J_1, J_2, J_3, J_4)$, $R^i_{kml} = \partial_m\Gamma^i_{kl} - \partial_k\Gamma^i_{ml} + \Gamma^j_{kl}\Gamma^i_{mk} - \Gamma^j_{ml}\Gamma^i_{kj}$ and the covariant derivative is defined as follows

$$\frac{D^2 J_i}{D\tau^2} = \frac{\partial^2 J_i}{\partial \tau^2} + 2\Gamma^i_{jk}\frac{\partial J_j}{\partial \tau}\frac{\partial \theta^k}{\partial \tau} + \Gamma^i_{jk}J_j\frac{\partial^2 \theta^k}{\partial^2 \tau} + \partial_h\Gamma^i_{jk}\frac{\partial \theta^h}{\partial \tau}\frac{\partial \theta^k}{\partial \tau}J_j + \Gamma^i_{jk}\Gamma^j_{ts}\frac{\partial \theta^s}{\partial \tau}\frac{\partial \theta^k}{\partial \tau}J_t.$$

In our case, the fact that the entropy dynamical model is uncoupled implies that $R^i_{kml} = 0$ for all i, k, m, l. Therefore the relative geodesic spread characterized by the Jacobi field equation is given by the following set of second order differential equations:

$$(\tau + B_1)^2\frac{\partial^2 J_1}{\partial \tau^2} - 2(\tau + B_1)\frac{\partial J_1}{\partial \tau} + 2J_1 = 0$$

$$\frac{\partial^2 J_i}{\partial \tau^2} - 2B_i\frac{\partial J_i}{\partial \tau} + B_i^2 J_i = 0 \quad \text{for} \quad i = 2, 3, 4.$$

Hence the coordinates of the Jacobi field are given by :

$$\begin{aligned} J_1(\tau) &= a_{1,1}(\tau + B_1) \\ J_i(\tau) &= (a_{1,i} + a_{2,i}\tau)e^{B_i\tau} \quad \text{for} \quad i = 2, 3, 4. \end{aligned}$$

where $a_{i,j}$ are integration constants. From this we can compute the square norm of the Jacobi field ($\|J\|^2 = \sqrt{g_{ij}J_iJ_j}$). We have that

$$\|J\|^2 = \frac{a_{1,1}^2}{A_1} + \left(\frac{a_{1,2} + a_{2,2}\tau}{A_2}\right)^2 + \left(\frac{a_{1,3} + a_{2,3}\tau}{A_3}\right)^2 + \left(b\frac{a_{1,4} + a_{2,4}\tau}{A_4}\right)^2.$$

Which shows that the Jacobi vector field intensity diverges polynomially.

3 Geometric structure and stability of Gaussian statistical manifold with correlations

In the previous sections, we analysed the geometry and the instability of a model of k particles with no interaction between them. In this section, we will consider two particles that interact between them with a certain correlation. More precisely, we will consider a Gaussian statistical manifold in presence of correlations. Recall that the density function of two random variables with joint Gaussian distribution is given by

$$p(\mathbf{x}, \boldsymbol{\theta}) = \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2\sigma^2(1-r^2)}\left[(x-\mu_x)^2 - 2r(x-\mu_x)(y-\mu_y) + (y-\mu_y)^2\right]\right\}, \quad (10)$$

where $\boldsymbol{\theta} = (\mu_x, \mu_y, \sigma)$, $x, y \in \mathbb{R}$ and $|r| < 1$ is a known parameter. Let us denote with M_G the statistical manifolds associated to p given by

$$M_G = \{p(\mathbf{x}, \boldsymbol{\theta}) : \text{ with } \mu_x, \mu_y \in \mathbb{R}, \quad \sigma > 0\}.$$

Note that the system is really different than the one we have analysed in the previous section. The coupled constraints would lead to a metric tensor with non-trivial off-diagonal elements given by the covariance terms. We compute the matrix of the Fisher-information metric $[g_{ij}]_{M_G}$ and its inverse $[g^{ij}]_{M_G}$

$$[g_{ij}] = \frac{1}{\sigma^2}\begin{pmatrix} \frac{1}{1-r^2} & \frac{-r}{1-r^2} & 0 \\ \frac{-r}{1-r^2} & \frac{1}{1-r^2} & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad [g^{ij}] = \sigma^2\begin{pmatrix} 1 & r & 0 \\ r & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Then the non zero coefficients Γ_{ijk} of the Levi-Civita connection are

$$\Gamma_{113} = \Gamma_{223} = \frac{1}{(1-r^2)\sigma^3} \qquad \Gamma_{131} = \Gamma_{232} = \Gamma_{311} = \Gamma_{322} = \frac{-1}{(1-r^2)\sigma^3} \qquad \Gamma_{333} = \frac{-4}{\sigma^3}$$
$$\Gamma_{213} = \Gamma_{123} = \frac{-r}{(1-r^2)\sigma^3} \qquad \Gamma_{132} = \Gamma_{231} = \Gamma_{312} = \Gamma_{321} = \frac{r}{(1-r^2)\sigma^3}$$
.

Therefore, the non zero Christoffel symbols are:

$$\Gamma^3_{11} = \Gamma^3_{22} = \frac{1}{4(1-r^2)\sigma} \qquad \Gamma^3_{33} = \Gamma^1_{13} = \Gamma^2_{23} = \Gamma^1_{31} = \Gamma^2_{32} = \frac{-1}{\sigma}$$
$$\Gamma^3_{21} = \Gamma^3_{12} = \frac{-r}{4(1-r^2)\sigma}$$
.

The non zero components of the curvature tensor are:

$$R_{1212} = R_{2121} = \frac{-1}{4(1-r^2)\sigma^4} \qquad R_{1221} = R_{2112} = \frac{1}{4(1-r^2)\sigma^4}$$
$$R_{1323} = R_{2313} = R_{3132} = \frac{r}{(1-r^2)\sigma^4} \qquad R_{1313} = R_{2323} = R_{3131} = \frac{-1}{(1-r^2)\sigma^4}.$$

The components of the Ricci curvature are:

$$R_{11} = R_{22} = \frac{-1}{2(1-r^2)\sigma^2}, \quad R_{12} = R_{21} = \frac{r}{2(1-r^2)\sigma^2} \quad \text{and} \quad R_{33} = \frac{-2}{\sigma^2}.$$

From this we conclude that M_G is a manifold of constant negative scalar curvature. More precisely, $S_{M_G} = -\frac{3}{2}$. The sign of the scalar curvature is an expression of chaos. Negative scalar curvature is a sufficient condition for the presence of local instability.

The geodesic equations for this model are:

$$\begin{cases} \frac{\partial^2 \mu_x}{\partial^2 \tau} = \frac{2}{\sigma} \frac{\partial \mu_x}{\partial \tau} \frac{\partial \sigma}{\partial \tau} \\ \frac{\partial^2 \mu_y}{\partial^2 \tau} = \frac{2}{\sigma} \frac{\partial \mu_y}{\partial \tau} \frac{\partial \sigma}{\partial \tau} \\ \frac{\partial^2 \sigma}{\partial^2 \tau} = \frac{1}{\sigma} \left(\frac{\partial \sigma}{\partial \tau} \right)^2 - \frac{1}{4(1-r^2)\sigma} \left(\left(\frac{\partial \mu_x}{\partial \tau} \right)^2 + \left(\frac{\partial \mu_y}{\partial \tau} \right)^2 \right) + \frac{2r}{4(1-r^2)\sigma} \frac{\partial \mu_x}{\partial \tau} \frac{\partial \mu_y}{\partial \tau}. \end{cases} \qquad (11)$$

A set of solutions of the system (11) is given by

$$\mu_x(\tau) = C_x \frac{1}{1 + e^{2C\tau}}, \qquad \mu_y(\tau) = C_y \frac{1}{1 + e^{2C\tau}} \quad \text{and} \quad \sigma(\tau) = \frac{e^{C\tau}}{1 + e^{2C\tau}} \qquad (12)$$

where $C = \frac{1}{4} \sqrt{\frac{C_x^2 + C_x^2 - 2rC_xC_y}{1 - r^2}}$.

Let $\alpha_{C_x, C_y, r}$ be the geodesic obtained from replace C_x, C_y and r in (12). The arch length of $\alpha_{C_x, C_y, r}$ between $\alpha_{C_x, C_y, r}(0)$ and $\alpha_{C_x, C_y, r}(\tau)$ is given by

$$\begin{aligned} \ell^\tau(\alpha_{C_x, C_y, r}) &= \int_0^\tau \frac{1}{\sigma} \left(\frac{1}{1-r^2} \left(\left(\frac{\partial \mu_x}{\partial \tau} \right)^2 + \left(\frac{\partial \mu_y}{\partial \tau} \right)^2 \right) - \frac{2r}{1-r^2} \left(\frac{\partial \mu_x}{\partial \tau} \right) \left(\frac{\partial \mu_y}{\partial \tau} \right) + 4 \left(\frac{\partial \sigma}{\partial \tau} \right)^2 \right)^{1/2} ds \\ &= \int_0^\tau \left(\frac{1}{1-r^2} ((C_x)^2 + (C_y)^2) \sigma^2 - \frac{2r}{1-r^2} C_x C_y \sigma^2 + \frac{4}{\sigma^2} \left(\frac{\partial \sigma}{\partial \tau} \right)^2 \right)^{1/2} ds \\ &= \int_0^\tau \left(16 C^2 \sigma^2 + 4C^2 \frac{(1 - e^{2C\tau})^2}{(1 + e^{C\tau})^2} \right)^{1/2} ds \\ &= \int_0^\tau \left(16 C^2 \frac{e^{2C\tau}}{(1 + e^{C\tau})^2} + 4C^2 \frac{(1 - e^{2C\tau})^2}{(1 + e^{C\tau})^2} \right)^{1/2} ds \\ &= 2C \int_0^\tau \frac{\left(4 e^{2C\tau} + (1 - e^{2C\tau})^2 \right)^{1/2}}{(1 + e^{C\tau})} ds \\ &= 2C\tau. \end{aligned}$$

Therefore, if we consider the difference of arch length between $\alpha_{C_x + \delta, C_y, r}$ and $\alpha_{C_x, C_y, r}$, it diverges when $\tau \to \infty$. Therefore, as in the example of the previous section, two nearby geodesics could differ significantly in time.

Another useful indicator of dynamical chaoticity is given by the average volume elements on M_G. The volume element on M_G is given by

$$dV_g = \left(\frac{4}{\sigma^6 (1 - r^2)} \right)^{1/2} d\mu_x \wedge d\mu_y \wedge d\sigma.$$

Then the volume of an extended region of M_G is

$$\Delta V_{M_G}(\tau) = \int_{\mu_x(0)}^{\mu_x(\tau)} \int_{\mu_y(0)}^{\mu_y(\tau)} \int_{\theta(0)}^{\theta(\tau)} dV_g = \frac{-C_x C_y}{4\sqrt{1 - r^2}} \frac{(1 - e^{2C\tau})^4}{e^{2C\tau}(1 + e^{2C\tau})^2},$$

and the average volume is

$$\frac{1}{\tau} \int_0^\tau \Delta V_{M_G}(t)\, dt = \frac{-C_x C_y}{8C\sqrt{1 - r^2}} \left(\frac{e^{2C\tau} - e^{-2C\tau}}{\tau} - \frac{16}{\tau(1 + e^{2C\tau})} - 12C + \frac{8}{\tau} \right).$$

Note, that the asymptotic behaviour of the average volume has a regime of diffusive evolution that increase exponentially when $\tau \to \infty$. This behaviour is similar to the one obtained in [8] (Section 6) for the model with one Gaussian variable. Also, it is interesting to note that the diffusive behaviour depends on the correlation r (through the constant C) but the asymptotic behaviour does not change even when the two particles do not interact between them (i.e., $r = 0$).

Finally, we consider the parameter family of neighbouring geodesics

$$\alpha_{C_x, C_y, r} = \{\mu_x(\tau; C_x, C_y, r), \ \mu_y(\tau; C_x, C_y, r), \ \sigma(\tau; C_x, C_y, r)\}$$

where $\mu_x(\tau; C_x, C_y, r), \mu_y(\tau; C_x, C_y, r)$ and $\sigma(\tau; C_x, C_y, r)$ are given in (12). The Jacobi field equations are:

$$\frac{\partial^2 J_x}{\partial^2 \tau} - \frac{2}{\sigma}\frac{\partial \sigma}{\partial \tau}\frac{\partial J_x}{\partial \tau} - 2C_x\sigma\frac{\partial J_\sigma}{\partial \tau} + \left(C\sigma^2 + (C_x + C_y)\frac{\partial \sigma}{\partial \tau}\right)J_x + C_x\left(\frac{rC_y - C_x}{4(1 - r^2)}\sigma^2 + \frac{\partial \sigma}{\partial \tau}\right)J_\sigma = 0 \qquad (13)$$

$$\frac{\partial^2 J_y}{\partial^2 \tau} - \frac{2}{\sigma}\frac{\partial \sigma}{\partial \tau}\frac{\partial J_y}{\partial \tau} - 2C_y\sigma\frac{\partial J_\sigma}{\partial \tau} + \left(C\sigma^2 + (C_x + C_y)\frac{\partial \sigma}{\partial \tau}\right)J_y + C_y\left(\frac{rC_x - C_y}{4(1 - r^2)}\sigma^2 + \frac{\partial \sigma}{\partial \tau}\right)J_\sigma = 0 \qquad (14)$$

$$\frac{\partial^2 J_\sigma}{\partial^2 \tau} + \frac{\sigma}{2(1 + r)}\left(C_x\frac{\partial J_x}{\partial \tau} + C_y\frac{\partial J_y}{\partial \tau}\right) - \frac{2}{\sigma}\frac{\partial \sigma}{\partial \tau}\frac{\partial J_\sigma}{\partial \tau} + \left(\frac{(\frac{J_\sigma}{\partial \tau})^2}{\sigma^2} + (C_x + C_y)\frac{\partial \sigma}{\partial \tau}\right)J_\sigma$$

$$+ \frac{1}{4(1 - r^2)}\left(\frac{C_x^2 - rC_y^2 + (r^2 - 2)C_xC_y}{4}\sigma^2 + (C_x - rC_y)\frac{\partial \sigma}{\partial \tau}\right)J_x$$

$$+ \frac{1}{4(1 - r^2)}\left(\frac{C_y^2 - rC_x^2 + (r^2 - 2)C_xC_y}{4}\sigma^2 + (C_y - rC_x)\frac{\partial \sigma}{\partial \tau}\right)J_y = 0 \qquad (15)$$

The technical details of this computation can be found in the Appendix. Note that $\lim_{\tau \to \infty} \sigma(\tau) = \lim_{\tau \to \infty} \frac{\partial \sigma(\tau)}{\partial \tau} = 0$ and $\lim_{\tau \to \infty} \frac{\frac{\partial \sigma(\tau)}{\partial \tau}}{\sigma(\tau)} = -C$. Therefore, if we assume as in [8] that

$$\lim_{\tau \to \infty} \sigma(\tau)\frac{\partial J_\sigma}{\partial \tau} = \lim_{\tau \to \infty} \sigma(\tau)\frac{\partial J_x}{\partial \tau} = \lim_{\tau \to \infty} \sigma(\tau)\frac{\partial J_y}{\partial \tau} = 0$$

$$\lim_{\tau \to \infty} \frac{\partial \sigma(\tau)}{\partial \tau}J_\sigma = \lim_{\tau \to \infty} \frac{\partial \sigma(\tau)}{\partial \tau}J_x = \lim_{\tau \to \infty} \frac{\partial \sigma(\tau)}{\partial \tau}J_y = 0,$$

and we consider the asymptotic limit as $\tau \to \infty$, the Jacobi field equations become,

$$\frac{\partial^2 J_x}{\partial^2 \tau} + 2C\frac{\partial J_x}{\partial \tau} = 0,$$

$$\frac{\partial^2 J_y}{\partial^2 \tau} + 2C\frac{\partial J_y}{\partial \tau} = 0, \qquad (16)$$

$$\frac{\partial^2 J_\sigma}{\partial^2 \tau} + 2C\frac{\partial J_\sigma}{\partial \tau} + C^2 J_\sigma = 0.$$

In this case (16) can be easily solved. Thus, the asymptotic solutions are given by

$$J_x(\tau) = a_{x,1} + a_{x,2}e^{-2C\tau} \qquad J_y(\tau) = a_{y,1} + a_{y,2}e^{-2C\tau} \qquad J_\sigma(\tau) = (a_{\sigma,1} + a_{\sigma,2\tau})e^{-C\tau}, \qquad (17)$$

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where $a_{x,j}, a_{y,j}$ and $a_{\sigma,j1}$ for $j = 1, 2$ are real integration constants. Hence, we have that the square norm of the Jacobi field is

$$\|J\|^2 = \frac{1}{\sigma^2(1-r^2)}\Big(\big(a_{x,1} + a_{x,2}e^{-2C\tau}\big)^2 + \big(a_{y,1} + a_{y,2}e^{-2C\tau}\big)^2 - 2r\big(a_{x,1} + a_{x,2}e^{-2C\tau}\big)\big(a_{y,1} + a_{y,2}e^{-2C\tau}\big)\Big)$$
$$+ \frac{4}{\sigma^2}(a_{\sigma,1} + a_{\sigma,2\tau})^2 e^{-2C\tau}.$$

Finally, it is easy to see that the main term of the asymptotic expansion has exponential behaviour equivalent to $2\left(\frac{a_{1,x}^2 + a_{1,y}^2 - 2ra_{1x}a_{1,y}}{1-r^2}\right)e^{2C\tau}$. Note that when the particles do not interact between them, the Jacobi field has the same asymptotic behavior. On the other hand, if the particles are strongly related (i.e., $r \to 1$) the norm of the Jacobi field goes to infinity.

4   Final remarks and conclusions

In this paper, we investigate two entropic dynamical models corresponding to statistical manifolds with different characteristics. The first one corresponds to a system of four uncorrelated particles in a one dimensional space modelled by a statistical manifold of $4-$joint one parameter exponential density. The second one describes the behaviour of two particle interacting between them according to a multivariate Gaussian distribution. For both models, we study their geometric structure from the viewpoint of information geometry. In order to analyse the character of the stability for both models, we obtain explicit parametrizations of the geodesics and we study their behaviour. Also, we compute the volume of an extended region of each manifold and the Jacobi field associated with the geodesic deviation equations on the manifolds. We concluded that both models show clear signs of instability.

Finally, we want to note that if we combine the studied models, we obtain a large class of statistical manifolds that can be analysed easily using the results obtained here. More precisely, assume that we have the following statistical manifold

$$M = \{p_1((x_1, x_2, x_3, x_4), (\theta_1, \theta_2, \theta_3, \theta_4))\, p_2((x_5, x_6), (\mu_{x_6}, \mu_{x_7}, \sigma))\quad \theta_i > 0 \text{ for } i = 1, \ldots, 4 \text{ and } \sigma > 0\}$$

where p_1 and p_2 are defined as in (5) and (10), respectively. M modelled a system of six particle $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$ in a one dimensional space with no interaction between them, except (x_5, x_6). We will not stop here in the details, however it is easy to see that the scalar curvature of M is $\frac{-3}{2}$, the geodesic equations correspond to a system of seven equations given by (6) and (11) and the square norm of the Jacobi field is $\|J\|^2 = \|J_{M_4}\|^2 + \|J_{M_G}\|^2$, where J_{M_4} and J_{M_G} are the Jacobi fields of M_4 and M_G, respectively. The negative sign of the scalar curvature and the exponential grow of $\|J\|^2$ show local instability of this system.

The progress presented in this work constitute an advance for characterize the chaos of the entropic dynamical models and extend the important results obtained in [8].

Appendix

In this section we sketch the steps to get the equations (16). First step, we compute R^i_{klm}

$$R^1_{111} = R^1_{122} = R^2_{222} = R^2_{211} = \frac{1}{4\sigma^2(1-r^2)} = -R^1_{131} = -R^2_{232}$$

$$R^1_{121} = R^1_{112} = R^2_{212} = R^2_{221} = \frac{-r}{4\sigma^2(1-r^2)} = -R^1_{132} = -R^2_{231}$$

$$R^1_{311} = R^1_{133} = R^1_{312} = R^2_{322} = R^2_{233} = R^2_{321} = R^3_{331} = R^3_{332} = \frac{1}{\sigma^2} = -R^1_{331} = -R^1_{313} = -R^2_{332} = -R^2_{323}$$

$$R^3_{121} = R^3_{112} = R^3_{212} = R^3_{221} = \frac{-r}{16\sigma^2(1-r^2)}$$

$$R^3_{111} = R^3_{222} = \frac{1}{16\sigma^2(1-r^2)}$$

$$R^3_{211} = R^3_{122} = \frac{r^2}{16\sigma^2(1-r^2)}$$

$$R^3_{331} = R^3_{332} = \frac{-r}{2\sigma^2(1-r^2)}$$

$$R^3_{321} = R^3_{312} = \frac{-r}{2\sigma^2(1-r^2)}$$

Therefore using the fact that $\frac{\partial \mu_x(\tau)}{\partial \tau} = C_x \sigma^2$ and $\frac{\partial \mu_y(\tau)}{\partial \tau} = C_y \sigma^2$, the Jacobi field equations reduce to

$$\frac{D^2 J_x}{D\tau^2} + J_x \left(\sigma^2 \frac{C_x^2 - rC_xC_y}{4(1-r^2)} + \frac{\partial \sigma}{\partial \tau}(C_x + C_y) - \left(\frac{\frac{\partial \sigma}{\partial \tau}}{\sigma} \right)^2 \right)$$

$$+ J_y \left(\sigma^2 \frac{C_xC_y - rC_x^2}{4(1-r^2)} \right) + J_\sigma \left(\sigma^2 \frac{-C_x^2 + rC_xC_y}{4(1-r^2)} \right) = 0$$

$$\frac{D^2 J_y}{D\tau^2} + J_y \left(\sigma^2 \frac{C_y^2 - rC_xC_y}{4(1-r^2)} + \frac{\partial \sigma}{\partial \tau}(C_x + C_y) - \left(\frac{\frac{\partial \sigma}{\partial \tau}}{\sigma} \right)^2 \right) \tag{18}$$

$$+ J_x \left(\sigma^2 \frac{C_xC_y - rC_y^2}{4(1-r^2)} \right) + J_\sigma \left(\sigma^2 \frac{-C_y^2 + rC_xC_y}{4(1-r^2)} \right) = 0$$

$$\frac{D^2 J_\sigma}{D\tau^2} + J_x \left(\sigma^2 \frac{C_x^2 + (r^2-r)C_xC_y - rC_y^2}{16(1-r^2)^2} + \frac{\partial \sigma}{\partial \tau} \frac{C_x - rC_y}{2(1-r^2)} \right)$$

$$+ J_y \left(\sigma^2 \frac{C_y^2 + (r^2-r)C_xC_y - rC_x^2}{16(1-r^2)^2} + \frac{\partial \sigma}{\partial \tau} \frac{C_y - rC_x}{2(1-r^2)} \right)$$

$$+ J_\sigma \frac{\partial \sigma}{\partial \tau}(C_x + C_y) = 0.$$

The expression of the second derivative of J_x, J_y and J_σ are

$$
\begin{aligned}
\frac{D^2 J_x}{D\tau^2} =\ & -\ 2\frac{\frac{\partial \sigma}{\partial \tau}}{\sigma}\frac{\partial J_x}{\partial \tau} - 2C_x\sigma\frac{\partial J_\sigma}{\partial \tau} + J_x\left(-\frac{\frac{\partial^2 \sigma}{\partial \tau^2}}{\sigma} + 2\left(\frac{\frac{\partial \sigma}{\partial \tau}}{\sigma}\right)^2 - \sigma^2\frac{C_x^2 - rC_xC_y}{4(1-r^2)}\right) \\
& +\ J_y\left(\sigma^2\frac{rC_x^2 - C_xC_y}{4(1-r^2)}\right) + J_\sigma\frac{\partial \sigma}{\partial \tau}Cx \\
\frac{D^2 J_y}{D\tau^2} =\ & -\ 2\frac{\frac{\partial \sigma}{\partial \tau}}{\sigma}\frac{\partial J_y}{\partial \tau} - 2C_y\sigma\frac{\partial J_\sigma}{\partial \tau} + J_y\left(-\frac{\frac{\partial^2 \sigma}{\partial \tau^2}}{\sigma} + 2\left(\frac{\frac{\partial \sigma}{\partial \tau}}{\sigma}\right)^2 - \sigma^2\frac{C_y^2 - rC_xC_y}{4(1-r^2)}\right) \\
& +\ J_x\left(\sigma^2\frac{rC_y^2 - C_xC_y}{4(1-r^2)}\right) + J_\sigma\frac{\partial \sigma}{\partial \tau}Cy \\
\frac{D^2 J_\sigma}{D\tau^2} =\ & -\ 2\frac{\frac{\partial \sigma}{\partial \tau}}{\sigma}\frac{\partial J_\sigma}{\partial \tau} + \frac{\sigma(1-r)C_x}{2(1-r^2)}\frac{\partial J_x}{\partial \tau} + \frac{\sigma(1-r)C_y}{2(1-r^2)}\frac{\partial J_y}{\partial \tau} \\
& +\ J_\sigma\left(\frac{\frac{\partial \sigma}{\partial \tau}}{\sigma}\right)^2 + \frac{1}{4(1-r^2)}\frac{\partial \sigma}{\partial \tau}J_x + \frac{1}{4(1-r^2)}\frac{\partial \sigma}{\partial \tau}J_y
\end{aligned}
\tag{19}
$$

Finally the equations (9) follow from (18) and (19).

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