ON YAMABE CONSTANTS OF PRODUCTS WITH HYPERBOLIC SPACES

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Abstract. We study the $\mathbf{H}^n$-Yamabe constants of Riemannian products $(\mathbf{H}^n \times M^m, g_h^n + g)$, where (M, g) is a compact Riemannian manifold of constant scalar curvature and g_h^n is the hyperbolic metric on $\mathbf{H}^n$. Numerical calculations can be carried out due to the uniqueness of (positive, finite energy) solutions of the equation $\Delta u - \lambda u + u^q = 0$ on hyperbolic space $\mathbf{H}^n$ under appropriate bounds on the parameters λ, q, as shown by G. Mancini and K. Sandeep. We do explicit numerical estimates in the cases $(n, m) = (2, 2), (2, 3)$ and $(3, 2)$.

1. Introduction

For a closed Riemannian k-dimensional manifold (W^k, g) the Yamabe constant of its conformal class $[g]$ is defined as

$$Y(W, [g]) = \inf_{h \in [g]} \frac{\int_W s_h \, dv_h}{Vol(W, h)^{\frac{k-2}{k}}}$$

where s_h is the scalar curvature, dv_h the volume element and $Vol(W, h) = \int_W dv_h$ is the volume of (W, h).

We let $a_k = \frac{4(k-1)}{k-2}$ and $p = p_k = \frac{2k}{k-2}$. For $h \in [g]$ we write $h = f^{p-2} g$ for a function $f : W \to \mathbb{R}_{>0}$ and write the previous expression in terms of f and g: we have

$$Y(W, [g]) = \inf_{f \in C^\infty_+} Y_g(f),$$

where

$$Y_g(f) = \frac{\int_W a_k |\nabla f|^2 + s_g f^2 \, dv_g}{\|f\|^2_{p_k}}.$$

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We will call Y_g the Yamabe functional. Its critical points are solutions of the Yamabe equation:

$$-a_k \Delta_g f + s_g f = \mu f^{p-1},$$

where μ is a constant ($\mu = Y_g(f)\|f\|_p^{2-p}$): this means that the corresponding metric $h = f^{p-2} g$ has constant scalar curvature. The *Yamabe problem*, which consists in finding metrics of constant scalar curvature in a given conformal class, was solved for closed Riemannian manifolds by showing that the infimum in the definition of the Yamabe constant is always achieved.

There are different possible ways to try to generalize these ideas to non-compact manifolds. The non-compact case has attracted the atention of many authors (see for instance [1, 2, 7, 9, 10]) for the interest in the problem itself and also because non-compact examples play an important role when studying the Yamabe invariant of a closed manifold N, $Y(N)$ (the supremum of the Yamabe constants over the family of conformal classes of Riemannian metrics on N).

In this article we will study the case when the manifold is a Riemannian product $(\mathbf{H}^n \times M^m, g_h^n + g)$ where (M^m, g) is a closed Riemannian manifold of constant scalar curvature and $(\mathbf{H}^n, g_h^n)$ is the $n-$dimensional hyperbolic space of curvature -1. We denote by $\mathbf{s} = s_g - n(n-1)$ the scalar curvature of $g_h^n + g$. We define their Yamabe constant as:

$$Y(\mathbf{H}^n \times M^m, g_h^n + g) = \inf_{f \in L_1^2(\mathbf{H}^n \times M^m) - \{0\}} Y_{g_h^n + g}(f).$$

Note that this is well defined since the Sobolev embedding $L_1^2(\mathbf{H}^n \times M^m) \subset L^p(\mathbf{H}^n \times M^m)$ holds (see [6, Theorem 2.21]).

It is important the case when (M, g) is $(S^m, r g_0^m)$, where g_0^m is the round metric of constant curvature 1 and r is a positive constant, since it plays a fundamental role in understanding the behaviour of the Yamabe invariant under surgery. In [3, Theorem 1.3 and Section 3], it is proved that if M^k is obtained from N^k by performing surgery on a sphere of dimension $l \leq k-3$ then

$$Y(M) \geq \min\{Y(N), \Lambda_{k,l}\}$$

for positive constant $\Lambda_{k,l}$. If $l \leq k-4$ or $l \leq k-3$ and $k \leq 6$ this constant is actually the infimum for $r \in (0, 1]$ of the Yamabe constants of $(H^{l+1} \times S^{k-l-1}, g_h^{l+1} + r g_0^{k-l-1})$ (see [5], in the case $l = k-3$ and $k \geq 7$ one might need to deal with solutions of the Yamabe equation which are not in L^2). It can be seen via symmetrizations that the infimum in the definition is achieved by a function which is radial in both variables (it depends only on the distance to the origin in hyperbolic space and on the distance to a fixed point in the sphere). It seems reasonable to expect [2, 3] that for $r \in (0,1]$ there is a minimizer which depends only on the $\mathbf{H}^n$-variable (when $r=1$ this is known to be true, see [3, Proposition 3.1]; the limit when r approaches 0 is the case of the product of the round sphere with Euclidean space, considered in [2]). The main objective of this article is to show that if this were the case then one could compute the corresponding Yamabe constants numerically.

We recall the following definition from [2]:

Definition 1.1. *For a Riemannian product $(N \times M, h+g)$ we define the N-Yamabe constant as*

$$Y_N(N \times M, h+g) = \inf_{f \in L_1^2(N)-\{0\}} Y_{h+g}(f).$$

We will study $Y_{\mathbf{H}^n}(\mathbf{H}^n \times M^m, g_h^n + g)$, where (M,g) is a closed Riemannian manifold of constant scalar curvature s_g and volume V_g. Note that

$$Y_{\mathbf{H}^n}(\mathbf{H}^n \times M^m, g_h^n + g) = V_g^{\frac{2}{m+n}} \inf_{f \in L_1^2(\mathbf{H}^n)} \frac{\int_{\mathbf{H}^n} a_{n+m}|\nabla f|^2 + \mathbf{s} f^2 \, dv_{g_h^n}}{\|f\|^2_{p_{n+m}}}$$

If $f \in L_1^2(\mathbf{H}^n)$ is a critical point of $Y_{g_h^n+g}$ restricted to $L_1^2(\mathbf{H}^n)$, then it satisfies the subcritical equation

$$-a_{n+m}\Delta_h f + \mathbf{s} f = \mu f^{p_{m+n}-1}$$

where μ is a constant (it is called a subcritical equation since $p_{m+n} < p_n$).

Let

$$\mathbf{c} = c_{m,n} = (n-1)(m-1)/(m+n-2).$$

In Section 2 we will prove:

Theorem 1.2. *If $s_g > \mathbf{c}$ then $Y_{\mathbf{H}^n}(\mathbf{H}^n \times M^m, g_h^n + g) > 0$ and the constant is achieved. If $s_g = \mathbf{c}$ then $Y_{\mathbf{H}^n}(\mathbf{H}^n \times M^m, g_h^n + g) > 0$ but the constant is not achieved. If $s_g < \mathbf{c}$ then $Y_{\mathbf{H}^n}(\mathbf{H}^n \times M^m, g_h^n + g) = -\infty$.*

In Section 3 and Section 4 we will consider the case $(M,g) = (S^m, r g_0^m)$, for $r \in [0,1]$, $m \geq 2$. The case when $r \in [0,1]$ is of interest because these are the values that appear in the surgery formula [3]. Note that for $r \in (0,1]$ we have $s_{rg_0^m} = (1/r)m(m-1) > \mathbf{c}$. Let $g(r) = g_h^n + r g_0^m$ and denote by g_e^n the Euclidean metric on $\mathbb{R}^n$. We define $Q_{n,m} : [0,1] \longrightarrow \mathbb{R}_{>0}$ by

$$Q_{n,m}(r) = \begin{cases} Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, g(r)) & if\ r > 0, \\ Y_{\mathbb{R}^n}(\mathbb{R}^n \times S^m, g_e^n + g_0^m) & if\ r = 0. \end{cases}$$

In section 2 we will also show the following:

Proposition 1.3. *$Q_{n,m}$ is a continuous function.*

Note that $Q_{n,m}(0)$ is computed in [2, Theorem 1.4] in terms of the best constants in the Gagliardo-Nirenberg inequalities (which can be computed numerically).

If (M^m, g) is a closed Riemannian manifold of constant scalar curvature $s_g > \mathbf{c}$ and f realizes $Y_{\mathbf{H}^n}(\mathbf{H}^n \times M^m, g_h^n + g)$, then f is a positive smooth solution of the subcritical Yamabe equation

$$-a_{m+n}\Delta_{g_h^n} f + s_{g_n^h+g} f = a_{m+n} f^{p_{m+n}-1}$$

(of course, f is a minimizer then for any positive constant α, αf is also a minimizer. One obtains a solution of the previous equation by picking α appropriately).

Due to the symmetries of hyperbolic space, using symmetrization, one can see that f is a radial function (with respect to some fixed point). Consider the following model for hyperbolic space:

$$\mathbf{H}^n = (\mathbb{R}^n, \sinh^2(r)\ g_0^{n-1} + dr^2).$$

For a radial function f write $f(x) = \varphi(\|x\|)$, where $\|x\|$ denotes the distance to the fixed point. Then f is a solution of the Yamabe equation if $\varphi : [0,\infty) \to \mathbb{R}_{>0}$ solves the ordinary differential equation:

$$EQ_{\lambda,n,q}: \quad \varphi'' + (n-1)\frac{e^{2t}+1}{e^{2t}-1}\varphi' = \lambda\varphi - \varphi^q$$

where $\lambda = s_{g_h^n+g}/a_{n+m}$ and $q = p_{n+m} - 1$.

Note that

$$\int_{\mathbf{H}^n} f^k dv_{g_h^n} = V_{g_0^{n-1}} \int_0^\infty \varphi^k(t)\sinh^{n-1}(t)dt$$

(for any $k > 0$) and

$$\int_{\mathbf{H}^n} |\nabla f|^2 dv_{g_h^n} = V_{g_0^{n-1}} \int_0^\infty \varphi'^2 \sinh^{n-1}(t)dt.$$

Uniqueness of (positive, finite energy) solutions of the subcritical Yamabe equation (or equivalently $EQ_{\lambda,n,q}$) was proved by G. Mancini and K. Sandeep in [8, Theorem 1.3, Theorem 1.4]. We will describe the solutions of the ODE in Section 3 to see that one can numerically compute $Q_{n,m}(r)$ for any fixed $r \in (0,1]$ and use this in Section 4 to prove:

Theorem 1.4. *For* $(n,m) = (2,2),(2,3),(3,2)$ *and* $r \in [0,1]$ $Q_{n,m}(r) \geq 0.99\ Q_{n,m}(0)$.

Following the discussion above on applications to estimating the Yamabe invariants of closed manifolds, assuming that $Y(\mathbf{H}^{l+1} \times S^{k-l-1}, g_h^{l+1} + r g_0^{k-l-1}) = Y_{\mathbf{H}^{l+1}}(\mathbf{H}^{l+1} \times S^{k-l-1}, g_h^{l+1} + r g_0^{k-l-1})$ the previous results would show that the *surgery constants* $\Lambda_{k,l}$ could be proved to be very close to $Y_{\mathbb{R}^{l+1}}(\mathbb{R}^{l+1} \times S^{k-l-1}, g_e^{l+1} + g_0^{k-l-1})$. These constants are computed in [2] and would give a great improvement over the estimates for $\Lambda_{k,l}$ known at present.

It should be true that $Q_{n,m}(r) > Q_{n,m}(0)$ for $r > 0$, but we have not been able to prove it (the problem is to prove the inequality for r close to 0). But for any given $0 < \mu < 1$ and a given pair (n,m) one could prove that $Q_{n,m}(r) > \mu\ Q_{n,m}(0)$.

2. $\mathbf{H}^n$-Yamabe constants

In this section we will prove Theorem 1.2 and Proposition 1.3.

Proof of Theorem 1.2: This is a *subcritical* result and most of it actually appears in [8] with a different notation. We will sketch the proof for the convenience of the reader.

Recall that

$$\inf_{f \in L_1^2(\mathbf{H}^n) - \{0\}} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} = \frac{(n-1)^2}{4}.$$

Let

$$Y_{n,m}^s(f) = \frac{\int_{\mathbf{H}^n} a_{n+m} |\nabla f|^2 + (s - n(n-1)) f^2 dv_{g_h^n}}{\|f\|_{p_{n+m}}^2},$$

so that, if (M,g) is a closed Riemannian manifold of volume V and constant scalar curvature s then

$$Y_{\mathbf{H}^n}(\mathbf{H}^n \times M^m, g_h^n + g) = V^{\frac{2}{m+n}} \inf_{f \in L_1^2(\mathbf{H}^n) - \{0\}} Y_{n,m}^s(f).$$

We can rewrite the expression of $Y_{n,m}^s(f)$ as:

$$Y_{n,m}^s(f) = \frac{a_{m+n}}{\|f\|_{p_{n+m}}^2} \int_{\mathbf{H}^n} |\nabla f|^2 + \left(\frac{s - \mathbf{c}}{a_{m+n}} - \frac{(n-1)^2}{4} \right) f^2 dv_{g_h^n}.$$

It follows that if $s - \mathbf{c} < 0$ then there exists $f \in C_0^\infty(\mathbf{H}^n)$ such that $Y_{n,m}^s(f) < 0$. For each integer k we can consider $f_k \in C_0^\infty(\mathbf{H}^n)$ which consists of k disjoint copies of f. Then $Y_{n,m}^s(f_k) = k^{1-(2/p)} Y_{n,m}^s(f)$ and so

$$\lim_{k\to\infty} Y_{n,m}^s(f_k) = -\infty,$$

proving the last statement of Theorem 1.2.

If $s - \mathbf{c} \geq 0$ then $Y_{n,m}^s(f) > 0$ for all $f \in L_1^2(\mathbf{H}^n)$. To prove that the constant is strictly positive it is enough to consider the case when $s = \mathbf{c}$. But

$$\inf_{f \in L_1^2(\mathbf{H}^n)-\{0\}} Y_{n,m}^{\mathbf{c}}(f) = a_{m+n} S_{n,p_{m+n}-1},$$

where $S_{n,p_{m+n}-1}$ is the best constant in the Poincaré-Sobolev inequality proved in [8, (1.2)]. If the infimum were achieved in this case then the minimizing function would be a positive smooth solution in $L_1^2(\mathbf{H}^n)$ of $\Delta f + (n-1)^2/4 + f^{p-1} = 0$; but such a solution does not exist by [8, Theorem 1.1]. In case $s > \mathbf{c}$ then bounds on $Y_{n,m}^s(f)$ give bounds on the L_1^2-norm of f, so minimizing sequences are bounded in $L_1^2(\mathbf{H}^n)$. Then by the usual techniques one can show convergence to a smooth positive function in $L_1^2(\mathbf{H}^n)$. This is explicitly done in [8, Theorem 5.1].

This concludes the proof of Theorem 1.2.

Proof of Proposition 1.3:

Note that for $r > 0$, $Q_{n,m}(r) = Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, (1/r)g_h^n + g_0^m)$ and continuity at 0 means that

$$\lim_{T\to\infty} Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, Tg_h^n + g_0^m) = Y_{\mathbb{R}^n}(\mathbb{R}^n \times S^m, g_e^n + g_0^m).$$

If we had a closed Riemannian manifold instead of hyperbolic space, then we would be in the situation of [2, Theorem 1.1]. As in [2] one has to prove

$$\limsup_{T\to\infty} Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, Tg_h^n + g_0^m) \leq Y_{\mathbb{R}^n}(\mathbb{R}^n \times S^m, g_e^n + g_0^m),$$

and

$$\liminf_{T\to\infty} Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, Tg_h^n + g_0^m) \geq Y_{\mathbb{R}^n}(\mathbb{R}^n \times S^m, g_e^n + g_0^m).$$

The proof of the first inequality given in [2] does not use compactness and works in our situation. The second inequality is actually very simple in our case. It follows for instance from the Proposition 4.2 in Section 4 of this article.

Now consider $r \in (0,1]$.

$$Q_{n,m}(r) = r^{\frac{m}{n+m}} V_{g_0^m}^{\frac{2}{n+m}} \inf_{f \in L_1^2(\mathbf{H}^n)-\{0\}} \frac{a_{n+m}\|\nabla f\|_2^2 + s_{g(r)}\|f\|_2^2}{\|f\|_{p_{n+m}}^2}.$$

Let

$$F(r) = \inf_{f \in L_1^2(\mathbf{H}^n)-\{0\}} \frac{a_{n+m}\|\nabla f\|_2^2 + s_{g(r)}\|f\|_2^2}{\|f\|_{p_{n+m}}^2}.$$

It is clear that $F(r)$ is uniformly bounded in any interval $[r_0, 1]$, for $r_0 > 0$. Let f_r be a minimizer for $F(r)$ (i.e. f_r is a minimizer for $Q_{n,m}(r)$). We can normalize it to have $\|f_r\|_{p_{n+m}} = 1$. Then

$$F(r) = a_{n+m}\|\nabla f_r\|_2^2 + s_{g(r}\|f_r\|_2^2 \geq \left(\frac{a_{n+m}(n-1)^2}{4} + s_{g(r)} \right) \|f_r\|_2^2$$

Since

$$\frac{a_{n+m}(n-1)^2}{4} + s_{g(r)} \geq \frac{a_{n+m}(n-1)^2}{4} + m(m-1) - n(n-1)$$

$$= m(m-1) - \mathbf{c} > 0,$$

it follows that $\|f_r\|_2^2$ is uniformly bounded. Then F is clearly continuous at any $r > 0$ and so $Q_{n,m}$ is continuous.

3. Computing $Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, g(r))$ for $r \in (0,1]$

In this section $r \in (0,1]$ will be fixed. It follows from Theorem 1.2 that there is a function f_r which acieves $Q_{n,m}(r) = Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, g(r))$, where we call $g(r) = g_h^n + r g_0^m$. Then (after normalizing it appropriately) $f_r(x) = \varphi_r(\|x\|)$ where φ_r is a solution of $EQ_{\lambda,n,q}$ with $\lambda = \lambda(r) = \frac{-n(n-1)+r^{-1}m(m-1)}{a_{n+m}}$ and $q = p_{n+m} - 1$.

Then

$$Q_{n,m}(r) = Y_{g(r)}(f_r) = r^{\frac{m}{n+m}} V_{g_0^m}^{\frac{2}{n+m}} \frac{a_{n+m}\|\nabla f_r\|_2^2 + s_{g(r)}\|f_r\|_2^2}{\|f_r\|_{p_{n+m}}^2}$$

$$= a_{n+m} r^{\frac{m}{n+m}} V_{g_0^m}^{\frac{2}{n+m}} \|f_r\|_{p_{n+m}}^{\frac{4}{n+m-2}}$$

(where all the norms are taken considering f_r as a function on $\mathbf{H}^n$).

In this section r (and λ) will be fixed and we want to show that we can effectively numerically compute $Q_{n,m}(r)$, which means that we can compute numerically $\|f_r\|_{p_{n+m}}$.

Let φ be the solution of $EQ_{\lambda,n,q}$ with $\varphi(0)=\alpha>0$ and $\varphi'(0)=0$. Of course φ depends only on α and we will use the notation $\varphi=\varphi_\alpha$ when we want to make explicit this dependence. We will use the notation $f_\alpha(x)=\varphi_\alpha(\|x\|)$.

We are interested in the cases $\lambda \in [a_{m+n}^{-1}(m(m-1)-n(n-1)),\infty)$. The cases when $\lambda>0$ have some qualitative differences to the cases when $\lambda \leq 0$.

Consider the energy function associated with φ:

$$E=E(\varphi):=(1/2)(\varphi')^2-\lambda\varphi^2/2+\varphi^{q+1}/(q+1).$$

Then

$$E'(t)=-(n-1)\frac{e^{2t}+1}{e^{2t}-1}(\varphi'(t))^2 \leq 0.$$

If a solution φ intersects the t axis, let $b(\varphi)$ be the first point such that $\varphi(b(\varphi))=0$. If φ does not cross the t axis, we define $b(\varphi)=\infty$. Note that in the first case $\varphi'(b_\varphi)<0$ and therefore $E(b(\varphi))>0$. We are going to consider the function φ defined in $[0,b(\varphi)]$.

We divide the solutions φ into these families:

- $N=\{$Solutions for which $b(\varphi)<\infty\}$.
- $P=\{$Solutions which stay positive but are not in $L^{q+1}\}$.
- $G=\{$Solutions for which $b(\varphi)=\infty$ and are in $L^{q+1}\}$.

The minimizing solution belongs to G. It is proved in [8, Theorem 1.2] that there exists exactly one such solution. If the initial value of this solution is $\varphi(0)=\alpha_\lambda$, then they also show [8, Corollary 4.6] that if $\alpha<\alpha_\lambda$ then $\varphi_\alpha \in P$ and if $\alpha>\alpha_\lambda$ then $\varphi_\alpha\in N$. Note that we are using the notation $f_r=f_{\alpha_{\lambda(r)}}$.

To see that one can compute $Q_{n,m}(r)$ numerically we will argue that we can numerically approximate the value of $\alpha_{\lambda(r)}$ and that for any given $\epsilon>0$ we can explicitly find $t>0$ such that

$$\|f_r|_{\{\|x\|>t\}}\|_p<\varepsilon.$$

Claim 1 For fixed λ the value α_λ can be approximated to any given precision.

If for some α the solution φ_α hits 0, then $\alpha > \alpha_\lambda$. To analyse the case when φ_α stays positive up to some large value T we consider the next lemma.

Lemma 3.1. *Fix λ, let $\alpha > \alpha_\lambda$ and let f_α, f_{α_λ} be the corresponding functions in hyperbolic space. Then $\|f_\alpha|_{B(0,b_\alpha)}\|_p \geq \|f_{\alpha_\lambda}\|_p$. Moreover, if α_i, $i = 1,2$ are such that $\infty > b_{\alpha_1} > b_{\alpha_2}$ then $\|f_{\alpha_2}|_{B(0,b_{\alpha_2})}\|_p \geq \|f_{\alpha_1}|_{B(0,b_{\alpha_1})}\|_p$.*

Proof. Restrict the Yamabe functional $Y_{g(r)}$ to smooth functions with support in the Riemannian ball $B(0,T)$. It can be seen that the infimum of this functional is achieved by a smooth solution of $EQ_{\lambda,n,q}$ which is positive in $[0,T)$ and vanishes at T. But there is exactly one such solution by [8, Proposition 4.4]. It follows that if $b_\alpha = T$ then f_α is the minimizer and then the infimum is $r^{\frac{m}{n+m}} V_{g_0^m}^{\frac{2}{n+m}} \|f_\alpha\|_p^{\frac{4}{n+m-2}}$. If $b_{\alpha_1} > b_{\alpha_2}$ then we restrict φ_{α_2} to $[0, b_{\alpha_2}]$ and extend it by 0 to $[b_{\alpha_2}, b_{\alpha_1}]$. The corresponding function on hyperbolic space has support in $B(0, b_{\alpha_1})$ and it then follows from the previous comments that $\|f_{\alpha_2}|_{B(0,b_{\alpha_2})}\|_p \geq \|f_{\alpha_1}|_{B(0,b_{\alpha_1})}\|_p$.

□

Then for some given value of α one can numerically compute the corresponding solution φ_α and decide if $\alpha > \alpha_\lambda$ (in case it hits 0 at some point) or $\alpha < \alpha_\lambda$ (in case for some $T > 0$, $\|f_\alpha|_{B(0,T)}\|_p$ is bigger than the L^p-norm of a solution in N).

In the case when $\lambda > 0$ one can do it a little easier since solutions of the equation which are in P will have positive local minimums.

Claim 2: For a given $\varepsilon > 0$ one can find t such that $\|f_r|_{\{\|x\|>t\}}\|_p < \varepsilon$.

Note first that there are known explicit positive lower bounds for $\|f_r\|_p$: this is of course equivalent to have lower bounds for $Q_{n,m}(r)$ and in [4, Theorem 4.1, Corollary 4.2] the authors give lower bounds for $Y(\mathbf{H}^n \times S^m, g(r))$ (and of course $Y(\mathbf{H}^n \times S^m, g(r)) \leq Q_{n,m}(r)$).

Now

$$\frac{\|f_r\|_p^2 Q_{n,m}(r)}{r^{\frac{m}{m+n}} V_{g_0^m}^{2/(m+n)}} = \int_{\mathbf{H}^n} a_{n+m} |\nabla f_r|^2 + \left(\frac{m(m-1)}{r} - n(n-1) \right) f_r^2 dv_{g_h^n}$$

$$\geq \left(a_{m+n} \frac{(n-1)^2}{4} + m(m-1) - n(n-1) \right) \|f_r\|_2^2 = D \|f_r\|_2^2.$$

Note that $D > 0$.

If $\varphi_r(t) < \varepsilon$ then $f_r^p(x) < \varepsilon^{p-2} f_r^2(x)$ for all x such that $\|x\| > t$. Therefore

$$\int_{\{\|x\|>t\}} f_r^p \leq \varepsilon^{p-2} \|f_r\|_2^2 < \varepsilon^{p-2} \frac{\|f_r\|_p^2 Q_{n,m}(r)}{r^{\frac{m}{m+n}} V_{g_0^m}^{2/(m+n)} D} \leq K(\varepsilon),$$

where $K(\varepsilon)$ is some explicit function of ε that goes to 0 with ε.

Upper bounds for $\|f_r\|_p$ are easy to obtain (for instance using Lemma 3.1) and this implies that given any positive ε, since φ_r is decreasing, one can explicitly find t such that $\varphi_r(t) < \varepsilon$.

Then for any given $\varepsilon > 0$ one can explicitly find t such that such that $\|f_r|_{\{\|x\|>t\}}\|_p < \varepsilon$.

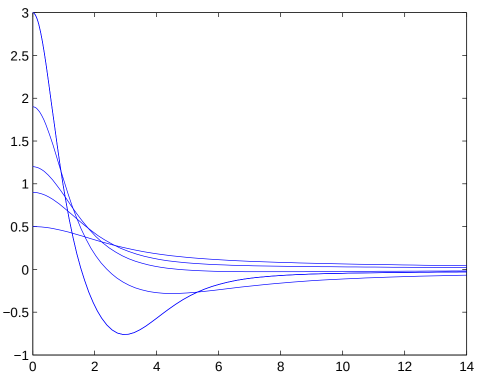
It follows from Claims 1 and 2 that $\|f_r\|_p$ can be effectively computed numerically.

To finish our description we show examples in each case $\lambda \leq 0$ and $\lambda > 0$.

3.1. **ODE for $\lambda \leq 0$.** In this case if t_0 is a local minimum of a solution φ then $\varphi(t_0) < 0$ and in case t_0 is a local maximum then $\varphi(t_0) > 0$.

If for some initial value the solution hits 0 we know that it belongs to N. Solutions in P are always decreasing and to decide if a solution belongs to P one has to apply Lemma 3.1.

The following graphic shows the solutions of the equation EQ_λ with parameters $\lambda = -3/32$ and $q = 7/3$ (which correspond to $m + n = 5$ and $\mathbf{s} = -1/2$) with initial condition $\varphi(0) = 0.5$, $\varphi(0) = 0.9$, $\varphi(0) = 1.2$, $\varphi(0) = 1.9$ and $\varphi(0) = 3$ respectively.



3.2. **ODE for $\lambda > 0$.** It is equivalent to solve

$$EQ_\lambda : \qquad \varphi'' + (n-1)\frac{e^{2t}+1}{e^{2t}-1}\varphi' = \lambda(\varphi - \varphi^q).$$

We normalize it in this way so we always have the constant solutions 0 and 1.

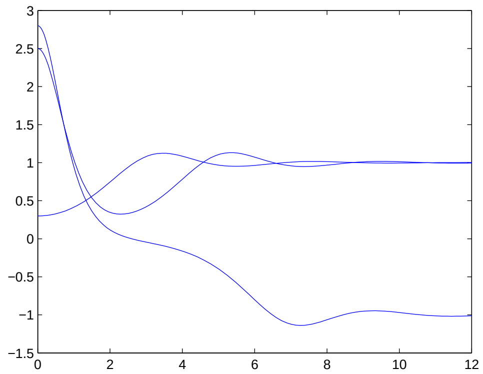
Note that if $\varphi \in N$ then $E(b(\varphi)) > 0$.

Since E is a decreasing function, the solutions φ are bounded.

Suppose that t_0 is a critical point of φ. Then $\varphi(t_0) < 1$ if t_0 is a local minimum and $\varphi(t_0) > 1$ if it is a local maximum (we are only considering φ defined where it stays positive). If t_0 is a local minimum of φ then $E(t_0) < 0$ and $\varphi \in P \cup G$.

Now suppose that φ is always positive and 0 is a limit point of φ. Then it follows that $E(\infty) = \lim_{t\to\infty} E(t) = 0$. Therefore φ cannot have any local minimum and φ must be monotone decreasing (to 0). So if φ has a local minimum them $\varphi \in P$

The graphic below shows the solutions of the equation EQ_λ with parameters $\lambda = 15/8$, $q = 7/3$ (which correspond to $m + n = 5$ and $\mathbf{s} = 10$) and with initial condition $\varphi(0) = 0.3$, $\varphi(0) = 2.5$ and $\varphi(0) = 2.8$ respectively.



4. Numerical computations: proof of Theorem 1.4

We saw in the previous section that for r fixed we can compute $Q_{n,m}(r)$. We want to estimate the infimum of $Q_{n,m}(r)$ for $r \in [0,1]$. Recall that we denote by $g(r)$ the metric $g_h^n + r g_0^m$. Note that the product $(\mathbf{H}^n \times S^m, g(r))$ is conformal (by a constant, $1/r$) to $(\mathbf{H}_r^n \times S^m, g_{hr}^n + g_0^m)$ where g_{hr}^n is the hyperbolic metric of constant curvature $-r$. Therefore $Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, g(r)) = Y_{\mathbf{H}_r^n}(\mathbf{H}_r^n \times S^m, g_{hr}^n + g_0^m)$. We proved that $Q_{n,m}(r)$ is continuous.

Recall also that $Q_{n,m}(1) = Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, g(1)) = Y(\mathbf{H}^n \times S^m, g(1)) = Y(S^{n+m})$, as was noted in [3, Proposition 3.1]. $Q_{n,m}(0) = Y_{\mathbb{R}^n}(\mathbb{R}^n \times S^m, g_e^n + g_0^m)$ was computed in [2] and

$$Q_{n,m}(0) < Q_{n,m}(1).$$

To prove Theorem 1.4 we will use the fact that we can compute $Q_{m,n}(r)$ for r fixed and two simple results. The first one is a simpler case of [3, Lemma 3.7].

Lemma 4.1. *Let $0 < r_0 \leq r_1$, then*

$$Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, g(r_1)) \leq (\frac{r_1}{r_0})^{\frac{m}{n+m}} Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, g(r_0)).$$

Proof. We have that $s_{g(r_1)} \leq s_{g(r_0)}$, $dv_{g(r_0)} = (\frac{r_0}{r_1})^{\frac{m}{2}} dv_{g(r_1)}$ and $\|\nabla f\|^2_{g(r_1)} = \|\nabla f\|^2_{g(r_0)}$ for any $f \in L^2_1(\mathbf{H}^n)$. Then $Y_{g_h^n + r_1 g_0^m}(f) \leq (\frac{r_1}{r_0})^{\frac{m}{n+m}} Y_{g_h^n + r_0 g_0^m}(f)$ for any $f \in L^2_1(\mathbf{H}^n)$ and the Lemma follows. □

The other simple result we will use is the following proposition. It is proved in a more general situation in [4, Corollary 3.3]: we give a short proof of this simpler case.

Proposition 4.2. *For any small $r > 0$, $Q_{n,m}(r) \geq \frac{m(m-1) - rn(n-1)}{m(m-1)} Q_{n,m}(0)$.*

Proof. For $r > 0$, $Q_{n,m}(r) = Y_{\mathbf{H}^n}(\mathbf{H}^n \times S^m, (1/r)g_h^n + g_0^m)$. Given any non-negative function $f \in C_0^\infty(\mathbf{H}^n)$, considered as a function in $(\mathbf{H}^n, (1/r)g_h^n)$, we consider its Euclidean radial symmetrizations: this is the radial, non-increasing, non-negative function $f_* \in C_0^\infty(\mathbb{R}^n)$ such that for each $t > 0$ $Vol(\{f > t\}) = Vol(\{f_* > t\})$. It is elementary that for any $q > 0$, $\|f_*\|_q = \|f\|_q$. On the other hand since the isoperimetric profile of $(\mathbf{H}^n, (1/r)g_h^n)$ is greater than that of Euclidean space it follows from the coarea formula that $\|\nabla f\|_2 \geq \|\nabla f_*\|_2$.

Then, if we let $\mathbf{s} = -rn(n-1) + m(m-1)$, we have

$$Y_{(1/r)g_h^n + g_0^n}(f) = V_{g_0^m}^{\frac{2}{m+n}} \frac{\int_{\mathbf{H}^n} a_{n+m} |\nabla f|^2 + \mathbf{s} f^2 dv_{(1/r)g_h^n}}{\|f\|^2_{p_{n+m}}} \geq$$

$$\frac{-rn(n-1) + m(m-1)}{m(m-1)} V_{g_0^m}^{\frac{2}{m+n}} \frac{\int_{\mathbb{R}^n} a_{n+m} |\nabla f_*|^2 + m(m-1) f_*^2 dv_{g_e^n}}{\|f_*\|^2_{p_{n+m}}}$$

$$= \frac{-rn(n-1) + m(m-1)}{m(m-1)} Y_{g_e^m + g_0^n}(f_*)$$

And the proposition follows.

□

4.1. **$\mathbf{H}^2 \times S^2$.** It follows from Proposition 4.2 that if $r \in [0, 0.01]$ then $Q_{2,2}(r) \geq 0.99 Q_{2,2}(0) = 58.81076$. It is known that $Q_{2,2}(1) = Y(\mathbf{H}^2 \times S^2, g_h^2 + g_0^2) = Y(S^4) = 61.56239 > Q_{2,2}(0) = Y_{\mathbb{R}^2}(S^2 \times \mathbb{R}^2) = 59.40481$.

Let

$$s_2 = \Big(\frac{0.99 Q_{2,2}(0)}{Q_{2,2}(1)}\Big)^2.$$

Then $0.99 Q_{2,2}(0) = s_2^{1/2} Q_{2,2}(1)$. By Lemma 4.1 it follows that $Q_{2,2}(s) \geq 0.99 Q_{2,2}(0)$ for any $s \in [s_2, 1]$. On the other hand, as explained in Section 3, we can numerically compute $Q_{2,2}(s_2) = 61.55039 > 0.99 Q_{2,2}(0) = 58.81076$. Let

$$s_3 = \Big(\frac{0.99 Q_{2,2}(0)}{Q_{2,2}(s_2)}\Big)^2 \ s_2 = 0.83317 \ .$$

Since $0.99 Q_{22}(0) = (s_3/s_2)^{1/2} Q_{22}(s_2)$, by Lemma 4.1 and the inequality above $Q_{2,2}(s) \geq 0.99 Q_{2,2}(0)$ if $s \in [s_3, 1]$. Following this procedure we found a finite succession s_i with $i = 1, \ldots, 126$ such that

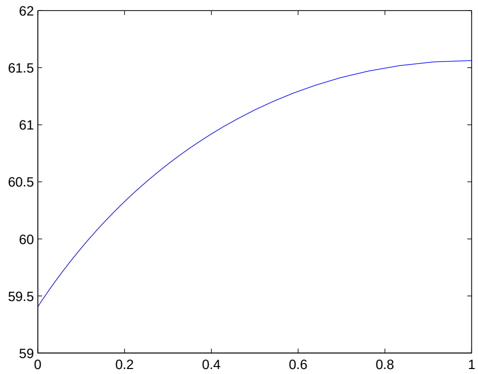
$$s_{i+1} = \Big(\frac{0.99 Q_{2,2}(0)}{Q_{2,2}(s_i)}\Big)^2 \ s_i,$$

$Q_{2,2}(s_i) > 0.99 Q_{2,2}(0)$ and $s_{126} < 0.01$. Then by Lemma 4.1 and Proposition 4.2 $Q_{2,2}(r) \geq 0.99 Q_{2,2}(0)$ for all $r \in [0, 1]$.

In the following table we exhibit some values of the succession s_i:

i	21	42	63	84	105	126
s_i	0.22732	0.09051	0.04630	0.02641	0.01593	0.00992
$Q_{2,2}(s_i)$	60.42277	59.87433	59.65783	59.55268	59.49515	59.46143

The graph of $Q_{2,2}$ in $[0, 1]$ is :



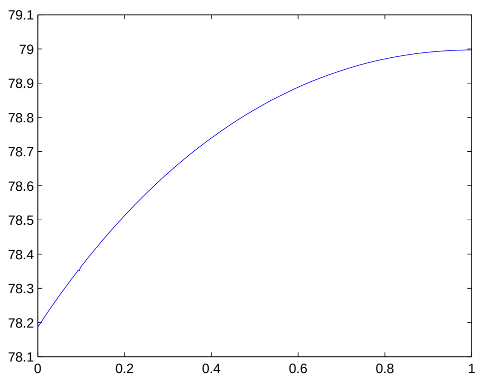
4.2. $\mathbf{H}^2 \times S^3$**.** By Proposition 4.2 $Q_{2,3}(r) \geq 0.99 Q_{2,3}(0)$ for $r \in [0, 0.03]$. It is known that $Q_{2,3}(0) = 78.18644$ and $Q_{2,3}(1) = 78.99686$. As in the case $\mathbf{H^2} \times S^2$ we can numerically compute a finite succession s_i with $i = 1, \ldots, 152$, such that $s_1 = 1$,

$$s_{i+1} = \left(\frac{0.99 Q_{2,3}(0)}{Q_{2,3}(s_i)}\right)^{\frac{5}{3}} s_i,$$

$Q_{2,3}(s_i) > 0.99 Q_{2,3}(0)$ and $s_{152} < 0.03$. Therefore, Theorem 1.4 for $(n, m) = (2, 3)$ follows from Lemma 4.1 and Proposition 4.2. The following table includes some values of the succession s_i:

i	25	51	76	101	126	152
s_i	0.46075	0.22854	0.12886	0.07706	0.04774	0.02968
$Q_{2,3}(s_i)$	78.79217	78.55030	78.40924	78.32559	78.27483	78.24226

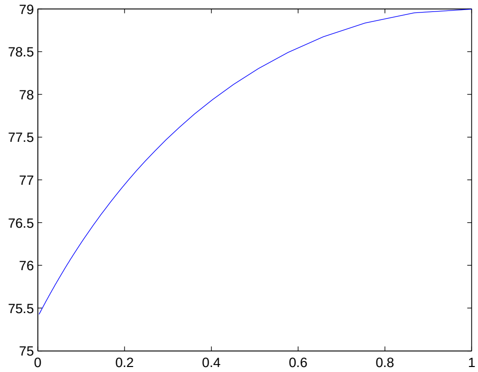
The graph of $Q_{2,3}$ in $[0, 1]$ is :



4.3. $\mathbf{H}^3 \times S^2$**.** Recall that $s_{g(r)} \leq 0$ for $r \geq 1/3$. As in the cases $\mathbf{H^2} \times S^2$ and $\mathbf{H^2} \times S^3$ we found a succession s_i with the properties described above, which proves the Theorem 1.3 in this situation. In this case $Q_{3,2}(1) = 78.99686$, $Q_{3,2}(0) = 75.39687$ and the last term $s_{132} < 1/300$.

i	9	33	57	81	105	132
s_i	0.36158	0.07155	0.02794	0.01315	0.00668	0.00325
$Q_{3,2}(s_i)$	77.77070	76.03779	75.66151	75.52397	75.46201	75.42872

The graph of $Q_{3,2}$ in $[0, 1]$ is :



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