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Sobre caracterizaciones estructurales de clases de grafos relacionadas con los grafos perfectos y la propiedad de Kőnig

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias de la Computación

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Un grafo es *balanceado* si su matriz clique no contiene como submatriz ninguna matriz de incidencia arista-vértice de un ciclo impar. Se conoce una caracterización para estos grafos por subgrafos inducidos prohibidos, pero ninguna que sea por subgrafos inducidos prohibidos *minimales*. En esta tesis probamos caracterizaciones por subgrafos inducidos prohibidos minimales para los grafos balanceados restringidas a ciertas clases de grafos y mostramos que dentro de algunas de ellas conducen a algoritmos lineales para reconocer el balanceo.

Un grafo es *clique-perfecto* si en cada subgrafo inducido el mínimo número de vértices que intersecan todas las cliques coincide con el máximo número de cliques disjuntas dos a dos. Contrariamente a los grafos perfectos, para estos grafos no se conoce una caracterización por subgrafos inducidos prohibidos ni la complejidad del problema de reconocimiento. En esta tesis caracterizamos los grafos clique-perfectos por subgrafos inducidos prohibidos dentro de dos clases de grafos, lo que implica algoritmos de reconocimiento polinomiales para la clique-perfección dentro de dichas clases.

Un grafo tiene la *propiedad de Kőnig* si el mínimo número de vértices que intersecan todas las aristas iguala al máximo número de aristas que no comparten vértices. En esta tesis caracterizamos estos grafos por subgrafos prohibidos, lo que nos permite también caracterizar los grafos arista-perfectos por arista-subgrafos prohibidos.

Palabras clave. *algoritmos de reconocimiento, grafos arco-circulares, grafos arista-perfectos, grafos balanceados, grafos bipartitos, grafos clique-Helly hereditarios, grafos clique-perfectos, propiedad de Kőnig, grafos coordinados, grafos de línea, grafos* K*-perfectos hereditarios, grafos perfectos, subgrafos prohibidos*

On structural characterizations of graph classes related to perfect graphs and the Kőnig property

A graph is *balanced* if its clique-matrix contains no edge-vertex incidence matrix of an odd cycle as a submatrix. While a forbidden induced subgraph characterization of balanced graphs was given, no such characterization by *minimal* forbidden induced subgraphs is known. In this thesis, we prove minimal forbidden induced subgraph characterizations of balanced graphs, restricted to graphs that belong to certain graph classes. We also show that, within some of these classes, our characterizations lead to linear-time recognition algorithms for balancedness.

A graph is *clique-perfect* if, in each induced subgraph, the minimum size of a set of vertices meeting all the cliques equals the maximum number of vertex-disjoint cliques. Unlike perfect graphs, neither a forbidden induced subgraph characterization nor the complexity of the recognition problem are known for clique-perfect graphs. In this thesis, we characterize clique-perfect graphs by means of forbidden induced subgraphs within two different graph classes, which imply polynomial-time recognition algorithms for clique-perfectness within the same two graph classes.

A graph has the *Kőnig property* if the minimum number of vertices needed to meet every edge equals the maximum size of a set of vertex-disjoint edges. In this thesis, we characterize these graphs by forbidden subgraphs, which, in its turn, allows us to characterize edge-perfect graphs by forbidden edge-subgraphs.

Keywords. *balanced graphs, bipartite graphs, circular-arc graphs, clique-perfect graphs, coordinated graphs, edge-perfect graphs, forbidden subgraphs, Kőnig property, hereditary clique-Helly graphs, hereditary* K*-perfect graphs, line graphs, perfect graphs, recognition algorithms*

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On structural characterizations of graph classes related to perfect graphs and the Kőnig property

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Chapter 1

Introduction

In 1969, Berge defined a $\{0,1\}$-matrix to be *balanced* [7] if it contains no edge-vertex incidence matrix of any cycle of odd length as a submatrix. Balanced matrices have remarkable properties studied in polyhedral combinatorics. Most notably, if A is a balanced matrix, then A is *perfect* and *ideal*, meaning, respectively, that the *fractional set packing polytope* $P(A) = \{x \in \mathbb{R}^n : Ax \leqslant \mathbf{1},\ \mathbf{0} \leqslant x \leqslant \mathbf{1}\}$ and the *fractional set covering polytope* $Q(A) = \{x \in \mathbb{R}^n : Ax \geqslant \mathbf{1},\ \mathbf{0} \leqslant x \leqslant \mathbf{1}\}$ are *integral* (i.e., all their extreme points have integer coordinates) [58].

Perfect graphs were defined by Berge around 1960 [5] and are precisely those graphs whose clique-matrix is perfect [36], where by a *clique* we mean an inclusion-wise maximal set of pairwise adjacent vertices and by a *clique-matrix* we mean a clique-vertex incidence matrix. Some years ago, the minimal forbidden induced subgraphs for perfect graphs were identified [34], settling affirmatively a conjecture posed more than 40 years before by Berge [5, 6]. This result is now known as the *Strong Perfect Graph Theorem* and states that the minimal forbidden induced subgraphs for the class of perfect graphs are the chordless cycles of odd length having at least 5 vertices, called *odd holes*, and their complements, the *odd antiholes*.

Balanced graphs were defined to be those graphs whose clique-matrix is balanced. These graphs were already considered by Berge and Las Vergnas in 1970 [12] but the name 'balanced graphs' appears explicitly in [11]. It follows from [12] that balanced graphs form a subclass of the class of perfect graphs. Moreover, from [8] it follows that balanced graphs belong to another interesting graph class, the class of *hereditary clique-Helly graphs* [104]; i.e., the class of graphs whose induced subgraphs satisfy that the intersection of any nonempty family of pairwise intersecting cliques is nonempty. Prisner [104] characterized hereditary clique-Helly graphs as those graphs containing no induced 0-, 1-, 2-, or 3-pyramid (see Figure 1.1). Hence, no balanced graph contains an odd hole, an odd antihole, or any pyramid as an induced subgraph.

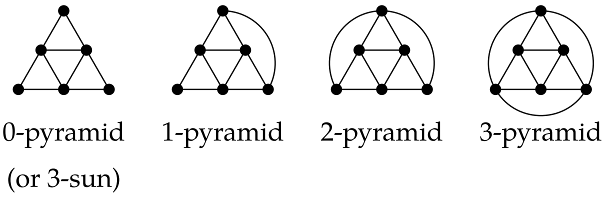


Figure 1.1: *The pyramids*

A graph is *bipartite* if it has no cycle of odd length. The *line graph* $L(G)$ of a graph G has the edges of G as vertices and two different edges of G are adjacent in $L(G)$ if and only if they share an endpoint. Bipartite graphs, complements of bipartite graphs, line graphs of bipartite graphs, and complements of line graphs of bipartite graphs are well-known classes of perfect graphs. Their perfectness follows already from the works of Kőnig [76, 77]. Moreover, these four graph classes constitute four of the five basic perfect graph classes in the decomposition of perfect graphs devised for the proof of the Strong Perfect Graph Theorem [34]. The validity of the Strong Perfect Graph Theorem within line graphs was first proved by Trotter [114]. Bipartite graphs and line graphs of bipartite graphs are balanced [10], but their complements are not always balanced. This is due to the fact that, contrary to perfect graphs, balanced graphs are not closed under graph complementation. For example, the graphs in Figure 1.1 are not balanced but have balanced complements.

The *intersection graph* of a finite family $\mathcal{F}$ is a graph whose vertices are the members of $\mathcal{F}$ and in which two different members of $\mathcal{F}$ are adjacent if and only if they have nonempty intersection. An *interval graph* [62] is the intersection graph of a finite number of intervals on a line. The class of interval graphs is properly contained in the class of *strongly chordal graphs* [54], which consists of all graphs whose clique-matrices are *totally balanced*; i.e., whose clique-matrices contain no edge-vertex incidence matrix of a cycle of length at least 3 as a submatrix [1]. As totally balanced matrices are balanced by definition, strongly chordal graphs, and consequently also interval graphs, are balanced. A *circular-arc graph* [79] is the intersection graph of a finite family of arcs on a circle. Contrary to the case of interval graphs, not all circular-arc graphs are balanced. Indeed, circular-arc graphs are neither perfect nor hereditary clique-Helly in general as odd holes, odd antiholes, and pyramids are easily seen to be circular-arc graphs. Perfectness of circular-arc graphs was addressed in [119], but the study of balancedness of circular-arc graphs is still in order.

Balanced graphs were characterized by a family of forbidden induced subgraphs known as *extended odd sun* [21]. Nevertheless, this characterization is not by *minimal* forbidden induced subgraphs because there are some extended odd suns that contain some other extended odd sun as a proper induced subgraph, as in the example given

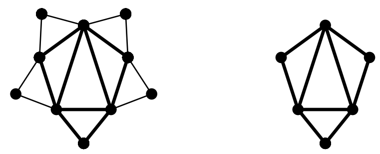


Figure 1.2: *On the left, an extended odd sun that is not minimal. Bold lines correspond to the edges of a proper induced extended odd sun, depicted on the right.*

in Figure 1.2.

In Chapter 3, we address the problem of characterizing balanced graphs by minimal forbidden induced subgraphs, giving several partial solutions by restricting ourselves to different graph classes. We prove structural characterization of balanced graphs, including characterizations by minimal forbidden induced subgraphs, restricted to complements of bipartite graphs, line graphs of multigraphs, and complements of line graphs of multigraphs. As a consequence of our structural characterizations, we show that the recognition problem of balanced graphs is linear-time solvable within each of these graph classes. This is in contrast, for instance, with the fact that $O(n^9)$ is the currently best time bound for algorithms deciding whether or not a given split graph having n vertices is balanced. In addition, we prove minimal forbidden induced subgraph characterizations of balanced graphs within three subclasses of circular-arc graphs: a superclass of the class of Helly circular-arc graphs and the classes of claw-free and gem-free circular-arc graphs.

Perfect graphs were originally defined by Berge in terms of a min-max type equality involving two important graph parameters: the *clique number* and the *chromatic number*. In many situations we are interested in knowing the minimum number of different colors needed to color all the vertices of a certain graph G in such a way that no two adjacent vertices receive the same color. This minimum number is called the *chromatic number* of G and is denoted by $\chi(G)$. The maximum size of a clique of a graph G is called the *clique number* of G and is denoted by $\omega(G)$. Clearly, $\omega(G)$ is a trivial lower bound for $\chi(G)$; i.e., the min-max type inequality

$$\omega(G) \leqslant \chi(G) \quad \text{holds for every graph } G.$$

Moreover, the difference between $\chi(G)$ and $\omega(G)$ can be arbitrarily large. Mycielski presented in [102] a family of graphs G_n such that $\omega(G_n) = 2$ and $\chi(G_n) = n$ for each $n \geqslant 2$. In this context, Berge defined a graph G to be *perfect* if and only if the equality $\omega(G') = \chi(G')$ holds for each induced subgraph G' of G.

An important property of perfect graphs is that the complement of a perfect graph is also perfect. This fact was conjectured by Berge. The first proof was given by Lovász [92] and there is an alternative proof due to Fulkerson based on the theory of antiblocking polyhedra [56]. The result is known as the *Perfect Graph Theorem* and implies that a graph is perfect if and only if the clique number and the chromatic number coincide in each induced subgraph of its complement. Let the *stability number* $\alpha(G)$ of a graph G be the clique number of its complement $\overline{G}$; i.e., $\alpha(G)$ is the maximum number of pairwise nonadjacent vertices. Similarly, let the *clique covering number* $\theta(G)$ be the chromatic number of $\overline{G}$; i.e., $\theta(G)$ is the minimum number of cliques covering all the vertices. So, the min-max type inequality

$$\alpha(G) \leqslant \theta(G) \quad \text{holds for every graph } G$$

and, by the Perfect Graph Theorem, a graph G is perfect if and only if the equality $\alpha(G') = \theta(G')$ holds for each induced subgraph G' of G.

There is an interesting connection between the equality $\alpha(G) = \theta(G)$ and a property of some families of sets known as the *Kőnig property*. The *transversal number* of a finite family $\mathcal{F}$ of nonempty sets with ground set X is the minimum number of elements of X needed to meet every member of $\mathcal{F}$ and the *matching number* of $\mathcal{F}$ is the maximum size of a collection of pairwise disjoint members of $\mathcal{F}$. If these two numbers coincide, the family $\mathcal{F}$ is said to have the *Kőnig property* (see [9, Chapter 2]). Given a $\{0, 1\}$-matrix A with no null columns, we may interpret its columns as the characteristic vectors of the members of some finite family $\mathcal{F}$ of nonempty sets. In this context, we say that two columns are *disjoint* if they do not have a 1 in the same row. Similarly, we say that a row *meets* a column if there is a 1 at the common entry of the row and the column. So, the columns of A have the Kőnig property if the maximum number of disjoint columns equals the minimum number of rows meeting every column. If we let G be a graph and A_G be a clique-matrix of G, then the maximum number of pairwise disjoint columns of A_G is $\alpha(G)$ and the minimum number of rows meeting every column of A_G is $\theta(G)$. Thus, the columns of A_G have the Kőnig property if and only if $\alpha(G) = \theta(G)$. Interestingly, Berge and Las Vergnas [12] proved that if a $\{0, 1\}$-matrix is balanced and has no null columns then its columns have the Kőnig property, from which they deduced that $\alpha(G) = \theta(G)$ holds for every balanced graph G. Moreover, as the class of balanced graphs is hereditary, they concluded that balanced graphs are perfect.

The Kőnig property has its origins in the study of matchings and transversals in bipartite graphs. The *matching number* $\nu(G)$ of a graph G is the maximum size of a set of vertex-disjoint edges and the *transversal number* $\tau(G)$ is the minimum number of vertices necessary to meet every edge. Clearly, the min-max type inequality

$$\nu(G) \leqslant \tau(G) \quad \text{holds for every graph } G.$$

In 1931, Kőnig [77] and Egerváry [52] proved that every bipartite graph B satisfies $\nu(B) = \tau(B)$. This result is known as *Kőnig's matching theorem*. The theorem of Berge and Las Vergnas in [12] was originally conceived as a generalization of Kőnig's matching theorem in the following sense. As the transpose of a balanced matrix is also balanced, the result of Berge and Las Vergnas is equivalent to the fact that if A is a balanced $\{0,1\}$-matrix with no null rows, then the rows of A have the Kőnig property; i.e., the maximum number of disjoint rows equals the minimum number of columns meeting every row. Let G be a graph and let A_G be a clique-matrix of G. On the one hand, the maximum number of pairwise disjoint rows of A_G is the *clique-independence number* $\alpha_c(G)$, which is the maximum number of vertex-disjoint cliques of G. On the other hand, the minimum number of columns meeting every row of A_G is the *clique-transversal number* $\tau_c(G)$, which is the minimum number of vertices meeting every clique of G. Clearly, the min-max type inequality

$$\alpha_c(G) \leqslant \tau_c(G) \quad \text{holds for every graph } G.$$

What follows from the theorem of Berge and Las Vergnas is that $\alpha_c(G) = \tau_c(G)$ holds for every balanced graph G; i.e., the cliques of a balanced graph have the Kőnig property. In particular, if G is bipartite, as $\alpha_c(G) = \nu(G) + i(G)$ and $\tau_c(G) = \tau(G) + i(G)$ where $i(G)$ denotes the number of isolated vertices of G, $\alpha_c(G) = \tau_c(G)$ reduces to $\nu(G) = \tau(G)$, which is precisely the statement of Kőnig's matching theorem.

As the class of balanced graphs is hereditary, the equality $\alpha_c(G) = \tau_c(G)$ holds not only for every balanced graph G but also for each of its induced subgraphs. Graphs G such that $\alpha_c(G') = \tau_c(G')$ holds for each induced subgraph G' of G were named *clique-perfect* by Guruswami and Pandu Rangan [64] in 2000. It is important to mention that clique-perfect graphs are not perfect in general and that perfect graphs are not clique-perfect in general since, for instance, the antiholes that are clique-perfect are those having number of vertices multiple of 3 (Reed, 2001, see [50]). Notice that if the equality $\alpha_c(G) = \tau_c(G)$ holds for a graph G, the same equality may not hold for all its induced subgraphs. For instance, every graph G in the class of *dually chordal graphs* [29] satisfies the equality $\alpha_c(G) = \tau_c(G)$, dually chordal graphs are not clique-perfect in general; e.g., W_5 is dually chordal but it is not clique-perfect because it contains an induced C_5, for which $\alpha_c(C_5) = 2$ but $\tau_c(C_5) = 3$. A set of vertex-disjoint cliques of a graph is a *clique-independent set* and a set of vertices meeting all the cliques of a graph is called a *clique-transversal*. So, $\alpha_c(G)$ is the maximum size of a clique-independent set of a graph G and $\tau_c(G)$ is the minimum size of a clique-transversal of G. The difference between $\alpha_c(G)$ and $\tau_c(G)$ can be arbitrarily large. Durán, Lin, and Szwarcfiter presented in [50] a family of graphs G_n such that $\alpha_c(H_n) = 1$ and $\tau_c(H_n) = n$ for each $n \geqslant 2$, where the number of vertices of H_n grows exponentially on n. Later, Lakshmanan S. and Vijayakumar [84] found another family of graphs H'_n such that $\alpha_c(H'_n) = 2n + 1$ and $\tau_c(H'_n) = 3n + 1$ for each $n \geqslant 1$, where H'_n has only $5n + 2$ vertices.

Apart from balanced graphs, some other well-known graph classes are known to be clique-perfect: comparability graphs [2], complements of forests [15], and distance-hereditary graphs [87]. Unlike perfect graphs, the class of clique-perfect graphs is neither closed under graph complementation nor is a complete characterization of clique-perfect graphs by forbidden induced subgraphs known. Nevertheless, partial results in this direction were obtained; i.e., characterizations of clique-perfect graphs by a restricted list of forbidden induced subgraphs within graphs that belong to certain graph classes [16, 17, 25]. For instance, in [16], a characterization of those line graphs that are clique-perfect in terms of minimal forbidden induced subgraphs was given and, in [17], clique-perfect graphs were characterized within Helly circular-arc graphs also by minimal forbidden induced subgraphs. Another open question regarding clique-perfect graphs is the time complexity of the recognition problem.

In Chapter 4, we give structural characterizations of clique-perfect graphs restricted to two different graph classes. First, we characterize, by minimal forbidden induced subgraphs, which complements of line graphs are clique-perfect and show that this characterization leads to an $O(n^2)$-time algorithm that decides whether or not a given complement of line graph G having n vertices is clique-perfect and, if affirmative, computes a minimum clique-transversal. Finally, we show that, within gem-free circular-arc graphs, clique-perfect graphs coincide with perfect graphs and with two further superclass of balanced graphs: coordinated graphs and hereditary K-perfect graphs.

Graphs G satisfying the thesis of Kőnig's matching theorem, $\nu(G) = \tau(G)$, but not being necessarily bipartite, are called *Kőnig-Egerváry graphs* or simply said to have the *Kőnig property*. In 1979, Deming [44] and Sterboul [111] independently gave the first structural characterization of graphs having the Kőnig property. Moreover, in [44], also a polynomial-time recognition algorithm for graphs having the Kőnig property was devised. In 1983, Lovász [93] introduced the notion of *nice subgraphs* and characterized graphs having the Kőnig property by forbidden nice subgraphs within graphs with a perfect matching. We will show that it is not possible to extend his result to a characterization of all graphs having the Kőnig property by forbidden nice subgraphs. We introduce the notion of *strongly splitting subgraphs*, providing a suitable extension of Lovász's nice subgraphs, in the sense that all graphs having the Kőnig property can be characterized by forbidden strongly splitting subgraphs. Our result relies on a characterization by Korach, Nguyen, and Peis [82] of graphs having the Kőnig property by means of what we call *forbidden configurations* (certain arrangements of a subgraph and a maximum matching) which is itself an extension of Lovász's characterization.

Imposing the Kőnig property to each induced subgraph of a graph can be easily seen to coincide with requiring the graph to be bipartite. Instead, Escalante, Leoni, and Nasini defined a graph G to be *edge-perfect* [53] if each of its *edge-subgraphs* has the Kőnig property, where an edge-subgraph is any induced subgraph that arises by removing a (possibly empty) set of edges together with their endpoints. Edge-perfect graphs form a superclass of the class of bipartite graphs and a subclass of the class of graphs having the Kőnig property. The class of edge-perfect graphs cannot be characterized by forbidden induced subgraphs because it is not closed under taking induced subgraphs. Instead, our aim is to characterize them by forbidden edge-subgraphs.

In Chapter 5, we give a characterization of all graphs having the Kőnig property by forbidden strongly splitting subgraphs, which is a strengthened version of the characterization due to Korach et al. by forbidden configurations. Using our characterization of graphs having the Kőnig property, we state and prove a simple characterization of edge-perfect graphs by forbidden edge-subgraphs. Unfortunately, this result does not lead to a polynomial-time recognition algorithm for edge-perfect graphs. In fact, although the problem of recognizing edge-perfect graphs is known to be polynomial-time solvable when restricted to certain graph classes [47], it is NP-hard for the general class of graphs [48].

Chapter 2

Preliminaries

2.1 Basic definitions and notation

In this section, we give some general definitions; more specific definitions are introduced as needed. Graphs in this thesis are finite, undirected, without loops, and without multiple edges. We will also deal with multigraphs, introduced near the end of this section.

Let G be a graph. The vertex set of G is denoted by $V(G)$, the edge set by $E(G)$, and the complement of G by $\overline{G}$. A *edge-vertex incidence matrix* of G is a $\{0,1\}$-matrix having one row for each edge and one column for each vertex such that only two 1's of each row are in two columns corresponding to the endpoints of the edge the row represents. A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is *spanning* if $V(H) = V(G)$. If H_1 and H_2 are two subgraphs of G, we say that H_1 and H_2 *touch* if they share exactly one vertex of G. Moreover, if $V(H_1) \cap V(H_2) = \{v\}$, we say that H_1 and H_2 *touch at* v. If $W \subseteq V(G)$, the subgraph of G *induced* by W is the subgraph $G[W]$ whose vertex set is W and whose edge set is $\{vw \in E(G) : v, w \in W\}$. If $W \neq V(G)$, $G[W]$ is a *proper induced subgraph* of G. By $G - W$, we denote the subgraph of G induced by $V(G)\backslash W$. If $W = \{v\}$, we denote $G - W$ simply by $G - v$. If G is a graph and e is any edge of G, $G - e$ denotes the graph that arises from G by making the endpoints of e nonadjacent. If v and w are two nonadjacent vertices of G, then $G + vw$ denotes the graph that arises from G by making v and w adjacent. If $F \subseteq E(G)$, $G\backslash F$ denotes the graph that arises from G by removing the edges in F from the edge set of G. By *contracting* a subgraph H of G we mean replacing $V(H)$ with a single vertex h and making each vertex $v \in V(G)\backslash V(H)$ adjacent to h if and only if v was adjacent in G to some vertex of H. For any set S, $|S|$ denotes its cardinality. For any sets X and Y, $X \triangle Y$ denotes the *symmetric difference* $(X\backslash Y) \cup (Y\backslash X)$.

A vertex v of a graph G is *universal* if it is adjacent to every other vertex of G, *pendant* if it is adjacent to exactly one vertex of G, or *isolated* if it is adjacent to no vertex of G. An edge is *pendant* if it has at least one pendant endpoint. The *neighborhood* of v in G is the set consisting of all vertices of G adjacent to v and is denoted by $N_G(v)$, or simply $N(v)$ if G is clear from context. The *closed neighborhood* of v is $N_G[v] = N_G \cup \{v\}$. The *common neighborhood of an edge* $e = vw$ is $N_G(e) = N_G(v) \cap N_G(w)$ and, in general, the *common neighborhood of a nonempty set* $W \subseteq V(G)$ is $N_G(W) = \bigcap_{w \in W} N_G(w)$, whereas $N_G(\varnothing) = V(G)$. Two vertices v and w of G are *false twins* if $N_G(v) = N_G(w)$ and *true twins* if $N_G[v] = N_G[w]$. Two vertices are *twins* if they are either false or true twins. We denote by $E_G(v)$ the set of edges of G incident to v. The *degree* $d_G(v)$ of a vertex v of G is the number of different neighbors of v in G. The *maximum degree* of the vertices of G is denoted by $\Delta(G)$ and the *minimum degree* by $\delta(G)$.

A graph is *complete* if its vertices are pairwise adjacent and the complete graph on n vertices is denoted by K_n. A *complete* of a graph is a set of pairwise adjacent vertices and a *clique* is an inclusion-wise maximal complete set. A *clique-matrix* of a graph is a clique-vertex incidence matrix; i.e., a $\{0,1\}$-matrix having one row for each clique and one column for each vertex and such that there is a 1 in the intersection of a row and a column if and only if the clique corresponding to the row contains the vertex corresponding to the column. A complete on 3 vertices is called a *triangle*. A *stable set* of a graph is a set of pairwise nonadjacent vertices. A set $A \subseteq V(G)$ and a vertex v of $V(G)$ are *complete* to each other if $A \subseteq N_G(v)$, and *anticomplete* if $N_G[v] \cap A = \varnothing$. The set $A \subseteq V(G)$ is *complete* (resp. *anticomplete*) to the set $B \subseteq V(G)$ if A and b are complete (resp. anticomplete) for each $b \in B$.

Paths and cycles are simple; i.e., have no repeated vertices aside from the starting and ending vertices in the case of cycles. Trivial paths consisting of only one vertex (and no edges) will be allowed, but cycles must have at least three vertices. An n-*path* (resp. n-*cycle*) is a path (resp. cycle) on n vertices. The starting and ending vertices of a path are called the *endpoints* of the path. The cycles on three vertices are also called *triangles*. Let Z be a path or a cycle of a graph G. By the *edges* of Z we mean those edges of G joining two consecutive vertices of Z. We denote by $V(Z)$ the set of vertices of Z and by $E(Z)$ the set of edges of Z. The *length* of Z is $|E(Z)|$. The *distance* between two vertices in a graph is the minimum length of a path in the graph having them as endpoints. A *chord* of Z is an edge joining two nonconsecutive vertices of Z and Z is *chordless* if Z has no chords. The chordless n-path and the chordless n-cycle are denoted by P_n and C_n, respectively. For each $n \geqslant 4$, W_n denotes the graph that arises from C_n by adding a universal vertex. A *chord* ab of Z is *short* if there is some vertex c of Z which is consecutive to each of a and b in Z. If so, c is called a *midpoint* of the chord ab in Z. Three short chords of Z are *consecutive* if they admit three consecutive vertices of Z as their midpoints. A chord of Z that is not short is called *long*. Two chords ab and cd of a cycle C such that their endpoints are four different vertices of C that appear in the order a, c, b, d in C are called *crossing*. A cycle is *odd* if it has an odd number of vertices, and is *even* otherwise. A *hole* is a chordless cycle of length at least 4 and an *antihole* is the complement of a hole of length at least 5. A cycle of a graph is *Hamiltonian* if it visits every vertex of the graph. If $P = v_1v_2 \dots v_n$ and $P' = w_1w_2 \dots w_m$ are two paths (where the v_i's and the w_j's are vertices) and their only common vertex is $v_n = w_1$, then PP' denotes the concatenated path $v_1v_2 \dots v_n w_2 w_3 \dots w_m$. If v is a vertex outside $V(P)$ adjacent to v_1, vP denotes the path $vv_1v_2 \dots v_n$.

A graph is *connected* if every two of its vertices are the endpoints of some path. A *component* of a graph is a containment-wise maximal connected subgraph. A component is *nontrivial* if it has at least two vertices, and is *trivial* otherwise. A connected graph without cycles is a *tree*. A graph is a *forest* if all its components are trees. A *cutpoint* is a vertex whose removal increases the number of components. A graph is *nonseparable* if it is connected, has at least two vertices, and has no cutpoints. A *block* of a graph is a containment-wise maximal nonseparable subgraph. An edge e of a graph G is a *bridge* if $G - e$ has more components than G.

A *dominating set* of a graph G is a set $A \subseteq V(G)$ such that each $v \in V(G) \setminus A$ is adjacent to at least one element of A. We say that a subset W of the vertex set of a graph H is *edge-dominating* if each edge of H has at least one endpoint in W. A path or cycle Z is *dominating* (resp. *edge-dominating*) if $V(Z)$ is dominating (resp. edge-dominating).

Let G and H be two graphs. We say that G *contains* H if H is isomorphic to a subgraph (induced or not) of G and that G *contains an induced* H if H is isomorphic to an induced subgraph of G. A class $\mathcal{G}$ of graphs is called *hereditary* if, for every graph G of $\mathcal{G}$, each induced subgraph of G belongs to $\mathcal{G}$. We say that G is H-*free* to mean that G contains no induced H. If $\mathcal{H}$ is a collection of graphs, we say that G *is* $\mathcal{H}$-*free* to mean that G contains no induced H for any $H \in \mathcal{H}$. A graph H is a *forbidden induced subgraph for a graph class* $\mathcal{G}$ if no graph of $\mathcal{G}$ contains an induced H. Moreover, if $\mathcal{G}$ is a hereditary class, H is said a *minimal forbidden induced subgraph for the class* $\mathcal{G}$ or a *minimally not* $\mathcal{G}$ *graph* if H does not belong to $\mathcal{G}$ but each proper induced subgraph of H belongs to $\mathcal{G}$.

Let G_1 and G_2 be two graphs and assume that $V(G_1) \cap V(G_2) = \emptyset$. The *join* of G_1 and G_2 is the graph $G_1 + G_2$ having vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{vw : v \in V(G_1), w \in V(G_2)\}$.

A graph H is *bipartite* if its vertex set can be partitioned into two stable sets X and Y. If so, $\{X, Y\}$ is called a *bipartition* of H. If, in addition, every vertex of X is adjacent to every vertex of Y, the graph is called *complete bipartite*.

A *matching* of a graph G is a set of vertex-disjoint edges of G. Let M be a matching of G. The endpoints of the edges belonging to M are called M-*saturated* and the

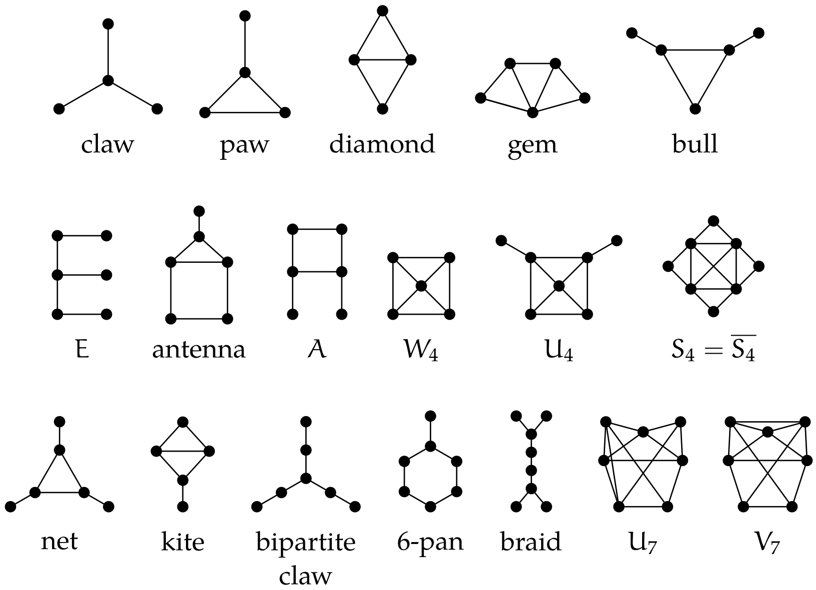


Figure 2.1: *Some small graphs*

remaining vertices of G are called M-*unsaturated*. M is *maximal* if it is inclusion-wise maximal and *maximum* if it is of maximum size; i.e., if $|M| = \nu(G)$ (where $\nu(G)$ denotes the matching number defined in the Introduction). M is *perfect* if it saturates every vertex of G and *near-perfect* if it saturates all but one vertex of G. Clearly, graphs with a perfect matching have an even number of vertices, while graphs with a near-perfect matching have an odd number of vertices. Perfect and near-perfect matchings are trivially maximum. A path is M-*alternating* if, for each two consecutive edges of the path, exactly one of them belongs to M. An M-*augmenting path* is an M-alternating path starting and ending in M-unsaturated vertices. Notice that if P is an M-augmenting path then $M' = M \triangle E(P)$ is also a matching and $|M'| = |M| + 1$. Indeed, a matching M is maximum if and only if it has no M-augmenting paths [4]. The following is a well-known result about matchings in bipartite graphs.

Theorem 2.1 (Hall's theorem [66])**.** *Let* H *be a bipartite graph with bipartition* $\{X, Y\}$*. Then, there is a matching* M *of* H *that saturates each vertex of* X *if and only if*

$$\left|\bigcup_{a \in A} N_H(a)\right| \geqslant |A| \quad \textit{for each } A \subseteq X.$$

Some small graphs to be referred in what follows are depicted in Figure 2.1. We will call any of the graphs in Figure 1.1 a *pyramid*. The *center* of a bipartite-claw is its vertex of degree 3.

Multigraphs are an extension of graphs obtained by allowing different edges to have the same pair of endpoints. Multigraphs are still finite, undirected, and without loops. Two edges joining the same pair of vertices are called *parallel*. We denote the vertex

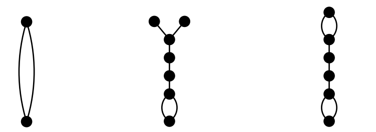


Figure 2.2: *Some special multigraphs*

set of a multigraph H by $V(H)$ and its edge set by $E(H)$. If H is a multigraph, the *underlying graph* of H is the graph $\widehat{H}$ having the same vertices as H and two vertices of $\widehat{H}$ are adjacent if there is at least one edge in H joining them. If v is a vertex of a multigraph H, we denote by $\hat{d}_H(v)$ the degree of v in the underlying graph $\widehat{H}$. A vertex of a multigraph is *pendant* if it has exactly one neighbor; i.e., if it is a pendant vertex of the underlying graph. Notice that there may be many edges joining a pendant vertex to its only neighbor.

Let H' and H be two multigraphs. We say that H' is a *submultigraph* of H if $V(H') \subseteq V(H)$ and, for each pair of adjacent vertices v and w of H', there are at least as many edges in H joining them as there are in H'. We say that H' *is contained in* H or that H *contains* H' if and only if H' is isomorphic to a submultigraph of H. Two submultigraphs *touch at vertex* v if v is their only common vertex. A multigraph is *connected* if its underlying graph is connected and a *component* of a multigraph is a containment-wise maximal connected submultigraph.

The *paths* and *cycles* of a multigraph are the paths and cycles of its underlying graph. A *multitree* is a connected multigraph without cycles; i.e., a multigraph whose underlying graph is a tree. Some multigraphs needed in what follows are displayed in Figure 2.2. Notice that we denote the multigraph consisting of two vertices and two parallel edges joining them by C_2, despite not being a cycle under our definition.

Two edges are *incident* if they share at least one endpoint, so that parallel edges are considered incident. If R is a graph or multigraph, the *line graph* $L(R)$ of R has the edges of R as vertices and two different edges e_1, e_2 of R are adjacent in $L(R)$ if and only if e_1 and e_2 are incident. A graph G is a *line graph of a multigraph* if there exists some multigraph R such that $G = L(R)$. If R can be chosen to be a graph, G is simply said to be a *line graph* and R is called a *root graph* of G. A *matching* of a multigraph H is any set M of pairwise non-incident edges of H and M is *maximal* if it is inclusionwise-maximal.

Let H_1 and H_2 be two vertex-disjoint graphs or multigraphs. The *disjoint union* $H_1 \cup H_2$ of H_1 and H_2 has vertex set $V(H_1) \cup V(H_2)$, two vertices v and w are adjacent in H if and only if they are adjacent in H_i for some $i \in \{1,2\}$, and there are exactly as many edges joining u and v in H as there are in H_i. If t is a nonnegative integer and H is a graph or multigraph, tH denotes the disjoint union of t copies of H.

2.2 Some special graph classes and the modular decomposition

In this section, we give some background about some special graph classes and the modular decomposition. Some results already stated in the Introduction are formally restated for future reference.

2.2.1 Perfect graphs

In the 1960's, Berge posed two conjectures regarding the structure of perfect graphs, the weaker of which is now known as the the Perfect Graph Theorem and states that the class of perfect graphs is closed by graph complementation.

Theorem 2.2 (Perfect Graph Theorem [92]). *A graph is perfect if and only if its complement is perfect.*

The stronger conjecture posed by Berge, concerning the minimal forbidden induced subgraph characterization for the class of perfect graphs, was proved only some years ago.

Theorem 2.3 (Strong Perfect Graph Theorem [34]). *A graph is perfect if and only if it has no odd holes and no odd antiholes.*

In addition, an $O(n^9)$-time algorithm was devised in [33] that decides whether or not a given graph G having n vertices has an odd hole or an odd antihole.

The following result characterizes perfect graphs by means of the integrality of their fractional set packing polytopes.

Theorem 2.4 ([36]). *A graph is perfect if and only if its clique-matrix is perfect.*

2.2.2 Helly property and hereditary clique-Helly graphs

A family $\mathcal{F}$ of sets has the *Helly property* if every nonempty subfamily of $\mathcal{F}$ of pairwise intersecting members has nonempty intersection. A graph is *clique-Helly* if the family of its cliques has the Helly property. So, a hereditary clique-Helly graph is a graph such that each of its induced subgraphs is clique-Helly. Prisner characterized hereditary clique-Helly graphs both by forbidden submatrices of their clique-matrices and by minimal forbidden induced subgraphs, as follows.

Theorem 2.5 ([104]). *A graph is hereditary clique-Helly if and only if its clique-matrices contain no edge-vertex incidence matrix of C_3 as a submatrix or, equivalently, if and only if it does not contain any of the graphs in Figure 1.1 as an induced subgraph.*

Prisner also gave a recognition algorithm for hereditary clique-Helly graphs.

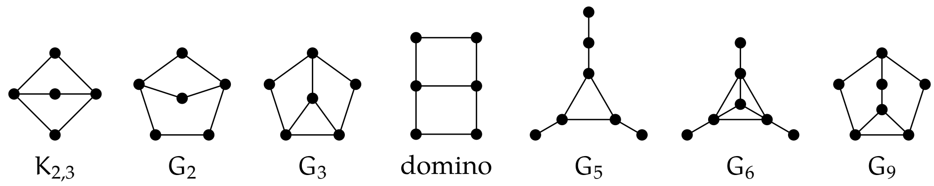


Figure 2.3: *Some forbidden induced subgraphs for the class of circular-arc graphs*

Theorem 2.6 ([104])**.** *It can be decided in* $O(n^2m)$ *time whether or not a given graph having* n *vertices and* m *edges is hereditary clique-Helly.*

Moreover, he proved that if G is a connected hereditary clique-Helly graph, then G has at most m cliques and concluded that all the cliques of G can be found in $O(m^2n)$ time by means of the algorithm devised in [116] that enumerates the cliques of G, one after the other, in $O(mn)$ time per clique. Therefore, the following holds.

Theorem 2.7 ([104, 116])**.** *In* $O(m^2n)$ *time, it can be decided whether or not a given a connected graph* G *having* n *vertices and* m *edges is hereditary clique-Helly and, if affirmative, output a clique-matrix of* G*, which has at most* m *rows.*

2.2.3 Circular-arc graphs and Helly circular-arc graphs

A *circular-arc* graph is the intersection graph of a finite family of arcs on a circle. Such a family of arcs is called a *circular-arc model* of the graph. The structure of circular-arc graphs was first studied by Tucker [117, 118, 119, 120] and these graphs can be recognized in linear time [100]. Some minimal forbidden induced subgraphs for the class of circular-arc graphs are $K_{2,3}$, G_2, G_3, domino, G_5, G_6, $\overline{C_6}$, net $\cup K_1$, $C_n \cup K_1$ for each $n \geqslant 4$, and G_9 [115] (see Figure 2.3).

Since $C_n \cup K_1$ is not a circular-arc graph for any $n \geqslant 4$, if G is a circular-arc graph and H is a hole of G, then $V(H)$ is dominating in G. We state the following slightly more general result for future reference (see [18]).

Lemma 2.8. *Let* G *be a circular-arc graph and* H *be a hole of* G*. If* $v \in V(G)\setminus V(H)$*, then either* v *is adjacent to every vertex of* H *or* $N_G(v) \cap V(H)$ *induces a path in* G*.*

A *Helly circular-arc* graph [61] is a circular-arc graph admitting a circular-arc model having the Helly property. We call any circular-arc model $\mathcal{A}$ having the Helly property a *Helly circular-arc model* of the graph. The class of Helly circular-arc graphs contains all interval graphs because every set of intervals of a line has the Helly property [72]. Let G be a Helly circular-arc graph and let us denote by A_v the arc of $\mathcal{A}$ that corresponds to vertex $v \in V(G)$. For a clique Q of G, we call any point $p \in \bigcap_{v \in Q} A_v$ an *anchor* of Q. Since Q is an inclusion-wise maximal complete, for each anchor p of Q and each

$v \in V(G)$, it holds that $p \in A_v$ if and only if $v \in Q$. In [75], a linear-time recognition algorithm for Helly circular-arc graphs was given, as well as a characterization by forbidden induced subgraphs of Helly circular-arc graphs within circular-arc graphs (see Theorem 3.46 on page 64).

2.2.4 Cographs and modular decomposition

Let G be a graph. A set M of vertices of G is a *module* if every vertex outside M is either adjacent to all vertices of M or to none of them. The empty set, the singletons $\{v\}$ for each $v \in V(G)$, and $V(G)$ are the *trivial modules* of G. A nonempty module M of G is *strong* if, for every other module M' of G, either $M \cap M' = \emptyset$, M, or M'. The *modular decomposition tree* $T(G)$ of a graph G is a rooted tree having one node for each strong module of G and such that a node h representing a strong module M has as its children the nodes representing the inclusion-wise maximal strong modules of G properly contained in M. Therefore, the root of $T(G)$ is $V(G)$ and the leaves of $T(G)$ are the singletons $\{v\}$ for each $v \in V(G)$. We will identify the module $\{v\}$ with the vertex v and say that the leaves of $T(G)$ are the vertices of G. For each node h of $T(G)$, we denote by $M(h)$ the strong module of G represented by h. By definition, $M(h)$ is the set of vertices of G having h as their ancestor in $T(G)$. For each node h of $T(G)$, we denote $G[M(h)]$ by $G[h]$. Each internal node of $T(G)$ is labeled P, S, or N, according to whether $G[h]$ is disconnected, $\overline{G[h]}$ is disconnected, or both $G[h]$ and $\overline{G[h]}$ are connected, respectively. Nodes labeled P, S, or N are called *parallel*, *series*, or *neighborhood*, respectively. Therefore, if h is an internal node of $T(G)$ and $h_1, \ldots, h_k$ are the children of h in $T(G)$, the following conditions holds:

(i) If $G[h]$ is disconnected, then h is labeled P and $G[h_1], \ldots, G[h_k]$ are the components of G.

(ii) If $\overline{G[h]}$ is disconnected, then h is labeled S and $\overline{G[h_1]}, \ldots, \overline{G[h_k]}$ are the components of $\overline{G}$.

(iii) If $G[h]$ and $\overline{G[h]}$ are both connected, then h is labeled N and $G[h_1], \ldots, G[h_k]$ is the set of inclusion-wise maximal proper submodules of $G[h]$.

There are linear-time algorithms for computing the modular decomposition tree of any given graph [40, 41, 101, 113].

A *cograph* if a P_4-free graph. The following result implies that a graph is a cograph precisely when each internal node of its modular decomposition tree is either a parallel or a series node.

Theorem 2.9 ([108])**.** *If G is a cograph having at least two vertices, then either G or $\overline{G}$ is disconnected.*

Seinsche [108] used this fact to prove that cographs are perfect since K_1 is perfect and the disjoint union and the join of two perfect graphs are perfect.

Chapter 3

Balanced graphs

In this chapter, we address the problem of characterizing balanced graphs by *minimal* forbidden induced subgraphs within different graph classes. The chapter is organized as follows:

- In Section 3.1, we give some background about balanced graphs.
- In Section 3.2, we prove basic properties about minimally not balanced graphs.
- In Section 3.3, we show that there is a strong tie between the time complexities of the problem of recognizing balanced graphs and that of recognizing balanced matrices.
- In Sections 3.4 to 3.6, we give structural characterizations of balanced graph, including minimal forbidden induced subgraphs characterizations, within each of the following graph classes: complements of bipartite graphs, line graphs of multigraphs, and complements of line graphs of multigraphs. These characterizations lead to linear-time algorithms for recognizing balancedness within each of these graph classes. This is in contrast with the fact that the currently best bound on the running time of an algorithm that recognizes balanced graphs within split graphs is $O(n^9)$, where n denotes the number of vertices of the input graph.
- In Section 3.7, we present a minimal forbidden induced subgraph characterization of balanced graphs within a superclass of the class of Helly circular-arc graphs. In Sections 3.8 and 3.9, we prove analogous characterizations within the classes of claw-free circular-arc graphs and gem-free circular-arc graphs, respectively.

The main results of this chapter appeared in [22] and [23].

3.1 Background

Recall that a $\{0,1\}$-matrix A is balanced if and only if it contains no edge-vertex incidence matrix of an odd cycle as a submatrix. Notice that if A contains the edge-vertex incidence matrix of an odd cycle, then A contains the edge-vertex incidence matrix of an odd *chordless* cycle. Equivalently, A is balanced if and only if it contains no odd square submatrix with exactly two 1's per row and per column. Notice that any matrix that arises by permuting the rows and/or columns of a balanced matrix is balanced and that the transpose of a balanced matrix is also balanced.

In [12], Berge and Las Vergnas reported to have found a new class of perfect graphs in an attempt to prove a conjecture about perfect graphs. In fact, they concluded the following.

Theorem 3.1 ([12])**.** *A graph G has a balanced clique-matrix if and only if every odd cycle in G contains at least one edge with the property that every maximal clique containing this edge contains a third vertex of the cycle. Moreover, any such graph G is perfect.*

In [8], Berge gave a more detailed characterization of these graphs, which we reproduce below. For each graph G, each $W \subseteq V(G)$, and each subfamily $\mathcal{D}$ of the family of cliques of G, let $G_{W,\mathcal{D}}$ be the graph that arises from G by deleting the vertices of $V(G)\backslash W$ and the edges that do not belong to a clique in $\mathcal{D}$.

Theorem 3.2 ([8])**.** *Let G be a graph. Then, the following assertions are equivalent:*

(i) *The clique-matrix of G is balanced.*

(ii) $\omega(G_{W,\mathcal{D}}) = \chi(G_{W,\mathcal{D}})$ *for each W and each $\mathcal{D}$.*

(iii) $\alpha(G_{W,\mathcal{D}}) = \theta(G_{W,\mathcal{D}})$ *for each W and each $\mathcal{D}$.*

(iv) *Every odd cycle in G contains at least one edge with the property that every maximal clique containing this edge contains a third vertex of the cycle.*

So, a balanced graph is any graph satisfying all of the above assertions. The name 'balanced graphs' for these graphs appears in [11]. As Berge [8] also proved that the rows (resp. columns) of a balanced matrix have the Helly property, we have the following.

Theorem 3.3 ([8])**.** *Balanced graphs are hereditary clique-Helly.*

Theorem 3.1 characterizes balanced graphs by means of the absence of *unbalanced cycles*; i.e., the absence of odd cycles C such that, for each edge $e \in E(C)$, there exists a (possibly empty) complete subgraph W_e of G such that $W_e \subseteq N(e)\backslash V(C)$ and

$N(W_e) \cap N(e) \cap V(C) = \varnothing$. More recently, balanced graphs were characterized by forbidden induced subgraphs, called extended odd suns. An *extended odd sun* [21] is a graph G with an unbalanced cycle C such that $V(G) = V(C) \cup \bigcup_{e \in E(C)} W_e$ and $|W_e| \leqslant |N(e) \cap V(C)|$ for each edge $e \in E(C)$. The extended odd suns with the smallest number of vertices are C_5 and the pyramids in Figure 1.1. The characterization of balancedness by forbidden induced subgraphs is as follows.

Theorem 3.4 ([21])**.** *A graph is balanced if and only if it contains no induced extended odd sun.*

As already noted in [21], extended odd suns are not necessarily minimal forbidden induced subgraphs because some extended odd suns may contain some others as proper induced subgraphs.

A graph is *chordal* [65] if every cycle of length at least 4 has some chord. For each $t \geqslant 3$, a t-*sun*, or simply *sun*, is a chordal graph G on $2t$ vertices whose vertex set can be partitioned into two sets, $W = \{w_1, \ldots, w_t\}$ and $U = \{u_1, \ldots, u_t\}$, such that W is a stable set and, for each $i = 1, 2, \ldots, t$, $N_G(w_i) = \{u_i, u_{i+1}\}$ (where u_{t+1} stands for u_1). Such a sun is *odd* if t is odd and *complete* if U is a complete. We denote the complete t-sun by S_t. For instance, S_3 coincides with the graph 3-sun of Figure 1.1. The graph S_4 is depicted in Figure 2.1. Clearly, extended odd suns contain odd suns as a special case.

Strongly chordal graphs, which we mentioned in the Introduction as one example of balanced graphs, are precisely the sun-free chordal graphs [54]. More generally, the following characterization of those chordal graphs that are balanced was proved in [88].

Theorem 3.5 ([88])**.** *Let G be a chordal graph. Then, G is balanced if and only if it contains no induced odd sun.*

Notice that the extended odd suns in Figure 1.2 are also odd suns and, consequently, not all odd suns are minimal forbidden induced subgraphs for balancedness. Indeed, characterizing balanced graphs by minimal forbidden induced subgraphs is unresolved even when the problem is restricted to chordal graphs.

Notice, however, that the problem is easily settled within the class of split graphs, which is a subclass of the class of chordal graphs. A graph is *split* [55] if its vertex set can be partitioned into a complete and a stable set. In [55], it was shown that split graphs are precisely those graphs that are chordal and whose complement is also chordal, and also that they coincide with the $\{2K_2, C_4, C_5\}$-free graphs. A *pseudo-split* graph [13, 98] is a $\{2K_2, C_4\}$-free graph. So, the class of pseudo-split graphs is a superclass of the class of split-graphs, but not of the class of chordal graphs. The following corollary of Theorem 3.5 gives the characterization of balanced graphs within pseudo-split graphs by *minimal* forbidden induced subgraphs.

Corollary 3.6. *Let G be a pseudo-split graph. Then, G is balanced if and only if it contains no induced C_5 and no induced odd complete sun.*

Proof. Let H be a pseudo-split graph that is minimally not balanced. We must show that H is either C_5 or a complete odd sun. If H contains an induced C_5, then the minimality of H implies that H is C_5. Therefore, assume, without loss of generality, that H is C_5-free. So, as H is pseudo-split, H is a split graph. Then, by Theorem 3.5, H is an odd sun and let $\{U, V\}$ be a partition of the vertex set of H as in the definition of odd sun. If there were two nonadjacent vertices in U, say u_i and u_j, then $\{u_i, w_i, u_j, w_j\}$ would induce $2K_2$ in H, a contradiction with the fact that H is split. So, U is a complete and H is an odd complete sun. □

3.2 Some properties of minimally not balanced graphs

The aim of this section is to prove some basic properties of minimally not balanced graphs; i.e., those graphs that are not balanced but such that each of their induced subgraphs are balanced.

Lemma 3.7. *If H is a minimally not balanced graph, then each of the following holds:*

(i) H *is connected.*

(ii) H *has no pendant vertices.*

(iii) H *has no true twins.*

(iv) H *has no universal vertices.*

(v) H *has no cutpoints.*

Proof. (i) Suppose, by the way of contradiction, that H is not connected. Let $H = H_1 \cup H_2$ for some graphs H_1 and H_2 having at least one vertex each and let A_1 and A_2 be clique-matrices of H_1 and H_2, respectively. Then, $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ is a clique-matrix of H. As H is not balanced, there is some submatrix A' of A which is the edge-vertex incidence matrix of an odd chordless cycle. As A' cannot intersect both A_1 and A_2 in A, A' is a submatrix of either A_1 or A_2. But then, H_1 or H_2 is not balanced, contradicting the minimality of H. This contradiction proves that H is connected.

(ii) Suppose, by the way of contradiction, that H has some pendant vertex v. Then, a clique-matrix of $H - v$ arises from the clique-matrix of H by first removing a column with exactly one 1 (which is the column corresponding to vertex v) and then removing a row with exactly one 1 (which is the row corresponding to the clique $N_H[v]$). Since an edge-vertex incidence matrix of a chordless cycle has two 1's per row and per column, H is balanced if and only if $H - v$ is balanced. This contradicts the minimality of H and proves that H has no pendant vertices.

(iii) Suppose, by the way of contradiction, that there are two true twins v and w in H. Then, a clique-matrix of $H - v$ arises from a clique-matrix of H by removing the column corresponding to vertex v, which is identical to the column corresponding to vertex w. Since an edge-vertex incidence matrix of a chordless cycle contains no two identical columns, H is balanced if and only if $H - v$ is balanced. This contradicts the minimality of H and proves that H has no true twins.

(iv) Suppose, by the way of contradiction, that there is some universal vertex v in H. Then, a clique-matrix of $H - v$ arises from a clique-matrix of H by removing a column with all its entries equal to 1. Since an edge-vertex incidence matrix of a chordless cycle contains no columns with all entries equal to 1, H is balanced if and only if $H - v$ is balanced. This contradicts the minimality of H and proves that H has no universal vertices.

(v) As H is minimally not balanced, Theorem 3.4 implies that H is an extended odd sun. Let C and $\{W_e\}_{e \in E(C)}$ be as in the definition of extended odd sun. It is clear that neither the vertices of C nor the vertices of the W_e's are cutpoints of H. Since $H = V(C) \cup \bigcup_{e \in E(C)} W_e$, H has no cutpoints. □

We will now establish necessary and sufficient conditions for the join of two graphs to be balanced. They involve the notion of trivially perfect graphs, introduced by Golumbic [63]. A graph is *trivially perfect* if each induced subgraph H has a stable set meeting all the cliques of H. Trivially perfect graphs coincide with $\{P_4, C_4\}$-free graphs [63] and also arise as the *comparability graphs of trees*, which means that trivially perfect graphs are those in which the closed neighborhoods of any two adjacent vertices are nested (see [124, 125] or [126]). The latter characterization can also be phrased in terms of clique-distinguishability: we say that two vertices u and v of a graph are *clique-distinguishable* if there is a clique containing u and not containing v and vice versa. As two vertices are clique-distinguishable if and only if their closed neighborhoods are not nested, we have the following.

Theorem 3.8 ([124]). *A graph is trivially perfect if and only if every two clique-distinguishable vertices are nonadjacent.*

This also means that a graph is trivially perfect if and only if a clique-matrix of it contains no submatrix that arises by permuting the rows of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$. This immediately means that trivially perfect graphs are balanced. Moreover, we have the following.

Lemma 3.9. *Let G be a graph that is the join of two graphs G_1 and G_2. Then, G is balanced if and only if at least one of the following assertions holds:*

(i) *One of G_1 and G_2 is complete and the other one is balanced.*

(ii) *Both G_1 and G_2 are trivially perfect.*

Proof. Suppose that G is balanced. Since G_1 and G_2 are induced subgraphs of G, they are balanced too. Therefore, if at least one of G_1 and G_2 were complete, then (i) would hold. Suppose, on the contrary, that none of G_1 and G_2 is complete. Then, G_1 is trivially perfect; otherwise G_1 would contain an induced P_4 or C_4 and, since G_2 is not complete, $G = G_1 + G_2$ would contain an induced $P_4 + 2K_1 =$ 2-pyramid or $C_4 + 2K_1 =$ 3-pyramid, respectively, contradicting the fact that G is balanced. Symmetrically, G_2 is also trivially perfect. Thus, (ii) holds.

Now suppose that G is not balanced. If G_1 were complete, then the clique-matrix of G_2 would arise from the clique-matrix of G by removing some columns all whose entries are 1's and, as G is not balanced, necessarily G_2 would not be balanced. Symmetrically, if G_2 were complete, then G_1 would not be balanced. We conclude that (i) does not hold. Assume now that none of G_1 and G_2 is complete. Since G is not balanced, there exist some cliques $Q_1, \ldots, Q_{2t+1}$ of G and some pairwise different vertices $v_1, \ldots, v_{2t+1}$ of G for some $t \geqslant 1$ such that $Q_i \cap \{v_1, v_2, \ldots, v_{2t+1}\} = \{v_i, v_{i+1}\}$ for each $i = 1, \ldots, 2t+1$ (where v_{2t+2} stands for v_1). In particular, $C = v_1 v_2 \ldots v_{2t+1} v_1$ is an odd cycle of G. Since C is odd, there are two consecutive vertices of C that belong both to G_1 or both to G_2. Without loss of generality, assume that v_1 and v_2 both belong to G_1. As Q_i is a clique of G, $Q'_i = Q_i \cap V(G_1)$ is a clique of G_1 for each $i = 1, 2, \ldots, 2t+1$. By construction, $Q'_{2t+1} \cap \{x_1, x_2\} = \{x_1\}$, $Q'_1 \cap \{x_1, x_2\} = \{x_1, x_2\}$, and $Q'_2 \cap \{x_1, x_2\} = \{x_2\}$. Therefore, x_1 and x_2 are two adjacent clique-distinguishable vertices and, by Theorem 3.8, G_1 is not trivially perfect and (ii) does not hold. □

The above lemma implies the following fact about minimally not balanced graphs.

Corollary 3.10. *The only minimally not balanced graphs whose complements are disconnected are the 2-pyramid and the 3-pyramid.*

Proof. Let H be a minimally not balanced graph whose complement $\overline{H}$ is disconnected. Since $\overline{H}$ is disconnected, H is the join of two graphs H_1 and H_2 with at least one vertex each. Therefore, as H is minimally not balanced, H_1 and H_2 are balanced. Nevertheless, as H is not balanced, Lemma 3.9 implies that H_1 or H_2 is not trivially perfect.

Without loss of generality, assume that H_1 is not trivially perfect; i.e., H_1 contains an induced P_4 or an induced C_4. Since $H = H_1 + H_2$ is not balanced and H_1 is balanced, Lemma 3.9 implies that H_2 is not complete. Thus, H_2 contains an induced $2K_1$. Finally, $H = H_1 + H_2$ contains an induced $P_4 + 2K_1 =$ 2-pyramid or an induced $C_4 + 2K_1 =$ 3-pyramid. By minimality, H is the 2-pyramid or the 3-pyramid. □

By Lemma 3.9, the join of two trivially perfect graphs is balanced. Below, we state a generalization of this fact for future reference. Notice that, in the result below, $G[X]$ and $G[Y]$ are trivially perfect by Theorem 3.8.

Lemma 3.11. *If the vertex set of a graph G can be partitioned into two sets X and Y such that every two clique-distinguishable vertices in G that belong both to X or both to Y are nonadjacent, then G is balanced.*

Proof. Suppose, by the way of contradiction, that G is not balanced. Then, there is some submatrix A of a clique-matrix A_G of G such that A is an edge-vertex incidence matrix of an odd chordless cycle. Notice that no row of A has two 1's in columns corresponding to vertices of X; otherwise, these two columns would correspond to adjacent vertices of X which, by hypothesis, are not clique-distinguishable in G, meaning that one of these columns would dominate the other in A_G, contradicting the fact that A has no dominated columns. Similarly, no row of A contains two 1's in columns corresponding to vertices of Y. So, as each row of A has exactly two 1's, each row of A has exactly one 1 in a column corresponding to a column corresponding to a vertex of X and exactly one 1 in a column corresponding to a vertex of Y, which contradicts the fact that A is an edge-vertex incidence matrix of an odd chordless cycle. This contradiction arose from assuming that G was not balanced. □

We close this section with the following reformulation of Lemma 3.9, also for future reference.

Lemma 3.12. *A graph G is balanced if and only if exactly one of the following assertions holds:*

(i) $\overline{G}$ *has only trivial components.*

(ii) $\overline{G}$ *has only one nontrivial component and the complement of this component is balanced.*

(iii) $\overline{G}$ *has exactly two nontrivial components and the complements of these two components are trivially perfect.*

Proof. If $\overline{G}$ has only trivial components, then G is a complete graph and, in particular, balanced. If $\overline{G}$ has only one nontrivial component H, then G is the join of a (possibly empty) complete and $\overline{H}$ and, by Lemma 3.9, G is balanced if and only if $\overline{H}$ is balanced. Suppose now that $\overline{G}$ has two nontrivial components H_1 and H_2. Then, G is the join of a (possibly empty) complete with the join of $\overline{H}_1$ and $\overline{H}_2$, where none of $\overline{H}_1$ and $\overline{H}_2$ is a complete graph. Therefore, by Lemma 3.9, G is balanced if and only if $\overline{H}_1$ and $\overline{H}_2$ are trivially perfect. Finally, notice that if $\overline{G}$ has 3 or more nontrivial components, then G is not balanced because it contains an induced $\overline{3K_2}$ = 3-pyramid. □

3.3 Recognition of balanced graphs and balanced matrices

As noted in [42], a polynomial-time algorithm for recognizing balanced graphs follows from Theorem 2.7 and the fact, first proved in [37], that balanced matrices can be recognized in polynomial time. The purpose of this section is to show that there is a stronger tie between the recognition of balanced graphs and of balanced matrices.

In [128], Zambelli devised a recognition algorithm for balanced $\{0, 1\}$-matrices, which has the currently best time bound.

Theorem 3.13 ([128])**.** *There is a* $O((r + c)^9)$*-time algorithm that decides whether or not a given* $r \times c$ $\{0, 1\}$*-matrix is balanced.*

It is easy to see that the above result immediately implies that whether or not a given graph G having n vertices and m edges is balanced can be decided in $O(m^9 + n)$ time. Indeed, as it takes only $O(m+n)$ time to compute the components of G, it suffices to show that if G is connected then it can be decided in $O(m^9)$ time whether or not G is balanced. Indeed, if G is connected, then Theorem 2.7 ensures that in $O(m^2 n)$ time it can either be detected that G is not hereditary clique-Helly (and, consequently, not balanced) or a clique-matrix of G be computed. In the latter case, such a clique-matrix of G has at most m rows and at most m columns and Zambelli's algorithm is able to determine whether or not the clique-matrix of G is balanced in $O(m^9)$ time.

We observe that for graphs having the number of cliques bounded from above by a linear function on the number of vertices, like chordal graph [57], pseudo-split graphs [13], planar graphs [105], and Helly circular-arc graphs [61], the same analysis shows that deciding their balancedness can be completed in $O(n^9)$ time. One might be tempted to consider the $O(n^9)$ time bound too loose, for instance, for chordal graphs, given that, in order to decide the balancedness of chordal graphs, there is no need to test for balancedness of arbitrary $\{0, 1\}$-matrices, but just those that are clique-matrices of chordal graphs. The lemma below shows that it is not the case, as any improvement on the $O(n^9)$-time bound for the recognition of balanced graphs within split graphs is tied to the existence of recognition algorithms for balanced matrices asymptotically faster than that of Zambelli, and vice versa. The reduction we apply here was used in [32] to prove the NP-completeness of determining α_c and τ_c for split graphs.

Lemma 3.14. *Let $p \geqslant 2$. Then, there exists an $O(n^p)$-time algorithm for deciding the balancedness of any given split graph having n vertices if and only if there exists an $O((r+c)^p)$-time algorithm for deciding the balancedness of any given $r \times c$ $\{0,1\}$-matrix.*

Proof. Suppose that there is an $O(n^p)$-time algorithm for deciding the balancedness of split graphs having n vertices and let $A = (a_{ij})$ be a given $r \times c$ $\{0,1\}$-matrix. Without loss of generality, assume that no row of A is full of 1's, as such rows can be ignored when deciding the balancedness of A. Consider the graph $G(A)$ with vertex set $\{s_1, \ldots, s_r, k_1, \ldots, k_c\}$, where $\{s_1, \ldots, s_r\}$ is a stable set, $\{k_1, \ldots, k_c\}$ is a complete, and such that s_i is adjacent to k_j if and only if $a_{ij} = 1$. Clearly, $G(A)$ can be constructed in $O((r+c)^2)$ time and a clique-matrix of $G(A)$ is $A' = \left(\begin{smallmatrix} I_r & A \\ 0 & 1 \end{smallmatrix}\right)$ where I_r denotes the identity matrix of order r, 0 a row of r entries equal to 0's, and 1 denotes a row of c entries equal to 1's. Clearly, A' is balanced if and only if A is balanced. So, A is balanced if and only if $G(A)$ is balanced, which, by hypothesis, can be decided in $O((r+c)^p) = O(n^p)$ time.

Conversely, suppose that there is an $O((r+c)^p)$-time algorithm that decides the balancedness of $r \times c$ $\{0,1\}$-matrices and let G be a split graph. Let $\{S, K\}$ be a partition of $V(G)$ such that $S = \{s_1, \ldots, s_x\}$ is a stable set and $K = \{k_1, \ldots, k_y\}$ is complete of G, and let $A = (a_{ij})$ be the $x \times y$ $\{0,1\}$-matrix such that $a_{ij} = 1$ if and only if s_i is adjacent to k_j. Reasoning as in the preceding paragraph, G is balanced if and only if A is balanced, which, by hypothesis, can be decided in $O(n^p)$ time once the matrix A is constructed in $O(n^2)$ time, where $n = x + y$ is the number of vertices of G. □

Notice that if $p \geqslant 2$ and there were an $O((r+c)^p)$-time recognition algorithm for balanced matrices, then, by reasoning as we did with Zambelli's algorithm, one concludes that there would be an $O(m^p + m^2 n)$-time algorithm for deciding the balancedness of any given graph having n vertices and m edges. Notice also that above proof of the lemma leads to an alternative derivation of Corollary 3.6.

3.4 Balancedness of complements of bipartite graphs

Recall from the Introduction that bipartite graphs are balanced, but also that the class of balanced graphs is not self-complementary. In particular, it turns out that the complements of bipartite graphs are not necessarily balanced. In this section, we characterize those complements of bipartite graphs that are balanced by minimal forbidden induced subgraphs. In fact, we show that the complement of a bipartite graph is balanced if and only if it is hereditary clique-Helly.

Theorem 3.15. *Let G be the complement of a bipartite graph. Then, the following statements are equivalent:*

(i) G *is balanced.*

(ii) *A clique-matrix of* G *has no edge-vertex incidence matrix of* C_3 *as a submatrix.*

(iii) G *is hereditary clique-Helly.*

(iv) G *contains no induced 1-pyramid, 2-pyramid, or 3-pyramid.*

Proof. The implication (i) $\Rightarrow$ (ii) follows by definition and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follows from Theorem 2.5. In order to prove that (iv) $\Rightarrow$ (i), assume that G contains no induced 1-pyramid, 2-pyramid, or 3-pyramid, and we will prove that G is balanced. Since G is the complement of a bipartite graph, its vertex set can be partitioned into red and blue vertices such that any two vertices of the same color are adjacent. Suppose, by the way of contradiction, that G is not balanced. Let $C = v_1v_2\dots v_{2t+1}v_1$ be an unbalanced cycle in G and let the W_e's for each $e \in E(C)$ be as in the corresponding definition. Since the 3-sun is not the complement of a bipartite graph, G is pyramid-free and, by Theorem 2.5, $t > 1$.

Since C is odd, there exist consecutive vertices v_k and v_{k+1} in C having the same color (here, and all along the proof, subindices should be understood modulo $2t + 1$). Either, there is another vertex v_ℓ in $V(C)\setminus\{v_k, v_{k+1}\}$ of this color, or all vertices in $V(C)\setminus\{v_k, v_{k+1}\}$ have the other color. In any case, as $t > 1$, C has three pairwise different vertices v_i, v_{i+1}, and v_j of the same color, say red. Thus, v_i, v_{i+1}, and v_j induce a triangle and $v_j \in N_G(v_iv_{i+1}) \cap V(C)$ follows.

Next, we shall construct a blue triangle u_1, u_2, and u_3 in G. By the definition of an unbalanced cycle, $N(W_{v_iv_{i+1}}) \cap N(v_iv_{i+1}) \cap V(C) = \varnothing$ and there exists some $u_1 \in W_{v_iv_{i+1}}$ such that u_1 is nonadjacent to v_j. Since v_j is red, u_1 is blue. If v_{i-1} is nonadjacent to v_{i+1}, we let $u_2 = v_{i-1}$; otherwise, $v_{i+1} \in N(v_{i-1}v_i) \cap V(C)$ and we let u_2 be any vertex of $W_{v_{i-1}v_i}$ nonadjacent to v_{i+1}. In both cases, u_2 is blue because it is nonadjacent to the red vertex v_{i+1}. Similarly, if v_{i+2} is nonadjacent to v_i, we define $u_3 = v_{i+2}$; otherwise, we let u_3 be any vertex of $W_{v_{i+1}v_{i+2}}$ nonadjacent to v_i. In both cases, u_3 is blue because it is nonadjacent to v_i. By construction, u_1, u_2, and u_3 are pairwise different because $N_G(u_1) \cap \{v_i, v_{i+1}\} = \{v_i, v_{i+1}\}$, $N_G(u_2) \cap \{v_i, v_{i+1}\} = \{v_i\}$, and $N_G(u_3) \cap \{v_i, v_{i+1}\} = \{v_{i+1}\}$. Since u_1, u_2, and u_3 are blue, they induce a triangle in G. Therefore, $\{u_1, v_i, v_{i+1}, v_j, u_2, u_3\}$ induces a 1-pyramid, 2-pyramid, or 3-pyramid in G, a contradiction. This contradiction arose from assuming that G was not balanced. Hence, G is balanced, which concludes the proof of (iv) $\Rightarrow$ (i) and of the theorem. □

As a consequence of the equivalence between (i) and (iii) of the above theorem, deciding if the complement of a bipartite graph is balanced is equivalent to determining whether it is hereditary clique-Helly. The currently best known time bound for recognizing hereditary clique-Helly graphs is $O(m^2 + n)$ where m is the number of edges of the input graph [91]. Notice that if the input graph is the complement of a bipartite graph with n vertices and m edges, then $m^2 = \Theta(n^4)$, which means that $O(m^2 + n)$ is not a linear-time bound. In fact, the algorithm in [91] 'as is' takes $\Omega(n^3)$ time when applied to the complement of a bipartite graph with n vertices because its main loop runs over all the triangles of the input graph. We will show that there is a simple linear-time recognition algorithm for hereditary clique-Helly graphs (or, equivalently, balanced graphs) when the input graph is known to be the complement of a bipartite graph.

As a consequence of Theorem 3.15, Lemma 3.12 becomes the following when specialized to complements of bipartite graphs.

Corollary 3.16. *Let G be the complement of a bipartite graph. Then, G is balanced if and only if one of the following assertions holds:*

(i) $\overline{G}$ *has only trivial components.*

(ii) $\overline{G}$ *has exactly one nontrivial component and this component is $\{E, P_4 \cup P_2, 3K_2\}$-free.*

(iii) $\overline{G}$ *has exactly two nontrivial components and these two components are complete bipartite graphs.*

Proof. The results follows from Lemma 3.9 by noticing that if H is a connected bipartite graph then: (1) $\overline{H}$ is balanced if and only if H is $\{E, P_4 \cup P_2, 3K_2\}$-free, and (2) $\overline{H}$ is trivially perfect if and only if H is a complete bipartite graph. Assertion (1) follows immediately from Theorem 3.15. When considering (2), it is clear that, if H is a complete bipartite graph, then $\overline{H}$ is trivially perfect because the endpoints of any pair of non-incident edges in H induce C_4 in H. Conversely, suppose that $\overline{H}$ is trivially perfect. In particular, H is P_4-free, which means that any two nonadjacent vertices u and v belonging to a same component of H are at distance 2 in H. So, since we are assuming that H is a connected bipartite graph, any two nonadjacent vertices of H are on the same set of the bipartition of H. This proves that H is complete bipartite, which completes the proof of (2) and of the corollary. □

Let G be the complement of a bipartite graph H and let n and m be the number of vertices and edges of G. We will show that there is a simple $O(n^2)$-time algorithm that decides whether or not G is balanced. Notice that, in this case, $O(n^2)$ is a linear-time bound because, being G the complement of a bipartite graph, $m = \Theta(n^2)$. Since conditions (i) and (iii) of Corollary 3.16 can be clearly verified in $O(n^2)$ time, it suffices to show that it is easy to decide in $O(n^2)$ time whether or not a connected bipartite graph having n vertices is $\{E, P_4 \cup P_2, 3K_2\}$-free.

If H is any bipartite graph, we write $H = (X, Y; F)$ to mean that $\{X, Y\}$ is a bipartition of H and $F = E(H)$. The *bipartite complement* of a connected bipartite graph $H =$

$(X, Y; F)$ is the bipartite graph $\overline{H}^{bip} = (X, Y; (X \times Y) \setminus F)$. For instance, $\overline{P_5}^{bip} = 2K_2 \cup K_1$. The recognition algorithm for $\{E, P_4 \cup P_2, 3K_2\}$-free bipartite graphs follows from the study of E-free bipartite graphs in [95]. In particular, we make use of the following result.

Theorem 3.17 ([95])**.** *Let H be a connected bipartite graph. Then, the following assertions are equivalent:*

(i) H *is* $\{E, P_7\}$*-free.*

(ii) H *is* $\overline{P_5}^{bip}$*-free.*

(iii) *Each component of* $\overline{H}^{bip}$ *is* $2K_2$*-free.*

We have the following immediate consequence.

Corollary 3.18. *Let H be a connected bipartite graph. Then, H is $\{E, P_4 \cup P_2, 3K_2\}$-free if and only if each component of $\overline{H}^{bip}$ is $2K_2$-free.*

Proof. In fact, if H is $\{E, P_4 \cup P_2, 3K_2\}$-free, then, in particular, H is $\{E, P_7\}$-free (because P_7 contains an induced $P_4 \cup P_2$) and, by Theorem 3.17, each component of $\overline{H}^{bip}$ is $2K_2$-free.

Conversely, suppose that each component of $\overline{H}^{bip}$ is $2K_2$-free. Then, by Theorem 3.17, H is $\overline{P_5}^{bip}$-free. Since each of E, $P_4 \cup P_2$, and $3K_2$ contains an induced $\overline{P_5}^{bip}$, H is $\{E, P_4 \cup P_2, 3K_2\}$-free. □

Bipartite $2K_2$-free graphs are known as *chain graphs* [127] or *difference graphs* [67]. It is well-known that a linear-time recognition for these graphs follows from the fact that, in any bipartite chain graph $H = (X, Y; F)$, the neighborhoods of the vertices of X (resp. Y) are nested. (For a detailed account, the reader may consult [71].) Therefore, as a consequence of Corollary 3.18, given a connected bipartite graph H with n vertices, it can be decided whether or not H is $\{E, P_4 \cup K_2, 3K_2\}$-free in $O(n^2)$ time, as follows: $\overline{H}^{bip}$ can be clearly computed in $O(n^2)$ time and, since bipartite chain graphs can be recognized in linear time, we can decide whether each of the components of $\overline{H}^{bip}$ is $2K_2$-free also in $O(n^2)$ time.

Altogether, we have a simple $O(n^2)$-time algorithm to decide whether or not a given complement of bipartite graph with n vertices is balanced. Recalling that an $O(n^2)$-time algorithm is linear-time if its input is the complement of a bipartite graph, we conclude the following.

Corollary 3.19. *It can be decided in linear time whether or not the complement of a bipartite graph is balanced (or, equivalently, hereditary clique-Helly).*

3.5 Balancedness of line graphs of multigraphs

The first characterization of perfect line graphs appeared in [114] and an alternative algorithmic proof was given in [43]. This characterization was later extended in [97]. It is known that line graphs of bipartite graphs are balanced [10]. In this subsection, we prove structural characterizations of those line graphs that are balanced, including a characterization by minimal forbidden induced subgraphs. Near the end of this subsection, we show how these structural results naturally extend to line graphs of multigraphs.

In order to state our results we need to introduce some definitions. First, we note that the cliques in the line graph $L(R)$ of a given graph R correspond to the inclusion-wise maximal sets of pairwise incident edges in R, called by us the L-*cliques* of R, which are the edge sets of the triangles of R, called *triads*, and the *stars* $E_R(v)$ $(v \in V(R))$ that are not contained in another star or triad.

A t-*bloom* $\{v; v_1, \dots, v_t\}$ in a graph is a set of $t > 0$ different pendant vertices $v_1, \dots, v_t$ all being adjacent to vertex v. By *identifying* two nonadjacent vertices u and v, we mean replacing them by a new vertex w with $N(w) = N(u) \cup N(v)$. If G_1 and G_2 are two vertex-disjoint graphs, $A = \{a; a_1, \dots, a_t\}$ is a t-bloom in G_1, and $B = \{b; b_1, \dots, b_t\}$ is a t-bloom in G_2, then $G_1 \triangle_{AB} G_2$ denotes the graph that arises from $G_1 \cup G_2$ by adding the edge ab and identifying a_i with b_i for each $i = 1, \dots, t$.

The following result characterizes which line graphs are balanced, including a characterization by minimal forbidden induced subgraphs.

Theorem 3.20. *Let G be a line graph and let R be a graph such that $G = L(R)$. Then, the following assertions are equivalent:*

(i) G *is balanced.*

(ii) G *is perfect and hereditary clique-Helly.*

(iii) G *has no odd holes and contains no induced 3-sun, 1-pyramid, or 3-pyramid.*

(iv) R *has no odd cycles of length at least* 5 *and contains no net, kite, or* K_4*.*

(v) *Each component of* R *belongs to the graph class* $\mathcal{S}$ *which is the minimal graph class satisfying the following two conditions:*

   (a) All connected bipartite graphs belong to $\mathcal{S}$*.*

   (b) If $G_1, G_2 \in \mathcal{S}$ *and the sets* A *and* B *are* t*-blooms of* G_1 *and* G_2*, respectively, then* $G_1 \triangle_{AB} G_2$ *belongs to* $\mathcal{S}$*.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Theorems 3.1 and 3.3 and (ii) $\Rightarrow$ (iii) from Theorem 2.5. That (iii) $\Rightarrow$ (iv) follows from the definition of line graph.

We prove that (iv) $\Rightarrow$ (v) by induction on the number n of edges of R. Assume that R has no odd cycles of length at least 5 and contains no kite, net, or K_4. If $n = 1$, (v) holds trivially. Let $n > 1$ and assume that (v) holds for graphs with less than n edges. Let S be any component of R and assume that S is not bipartite. In order to prove that S belongs to $\mathcal{S}$, we need to show that $S = S_1 \triangle_{AB} S_2$ for some $S_1, S_2 \in \mathcal{S}$ and some blooms A and B. Since S is not bipartite and has no odd cycles of length at least 5, there is some triangle T in S. Since S contains no net, kite, or K_4, there is some vertex of T of degree 2 in S. Let $T = \{a, b, c_1\}$ where $d_S(c_1) = 2$. Let $c_1, c_2, \ldots, c_t$ be all the vertices of S with $\{a, b\} \subseteq N_S(c_i)$. Since S contains no K_4, $\{c_1, \ldots, c_t\}$ is a stable set of S. Moreover, we have $\{a, b\} = N_S(c_i)$, for each $i = 2, \ldots, t$, because S contains no kite. Let S' be the graph that arises from S by removing the edge ab and the vertices $c_1, \ldots, c_t$; i.e., $S' = (S - ab) - \{c_1, \ldots, c_t\}$. Since S has no odd cycles of length at least 5, there is no path joining a and b in S'. Nevertheless, $S' + ab = S - \{c_1, \ldots, c_t\}$ is connected because S is connected. Consequently, S' consists of two components S'_1 and S'_2 such that a belongs to S'_1 and b belongs to S'_2. Let S_1 be the graph that arises from S'_1 by adding t pendant vertices $a_1, \ldots, a_t$ adjacent to a. Analogously, let S_2 be the graph that arises from S'_2 by adding t pendant vertices $b_1, \ldots, b_t$ adjacent to b. Then, $A = \{a; a_1, \ldots, a_t\}$ and $B = \{b; b_1, \ldots, b_t\}$ are t-blooms of S_1 and S_2, respectively, and $S = S_1 \triangle_{AB} S_2$. Moreover, S_1 and S_2 satisfy (iv) because they are subgraphs of S. Therefore, as S_1 and S_2 are connected and have less edges than S, by induction hypothesis, $S_1, S_2 \in \mathcal{S}$. This completes the proof of (iv) $\Rightarrow$ (v).

Let us now turn to the proof of (v) $\Rightarrow$ (i). Assume that every component of R belongs to $\mathcal{S}$. We will prove that $G = L(R)$ is balanced by induction on the number n of edges of R. Without loss of generality we can assume that R has no isolated vertices. If $n = 1$, then $G = K_1$ is balanced. Let $n > 1$ and assume that (i) holds when R has less than n edges. If R is disconnected, each component S of R has less than n edges and, by induction hypothesis, each $L(S)$ is balanced, which implies that $G = L(R)$ is balanced, as desired. So, without loss of generality, we assume that R is connected. Suppose, by the way of contradiction, that G is not balanced; i.e, there exist some L-cliques $E_1, \ldots, E_r$ and some pairwise different edges $e_1, \ldots, e_r$ of R such that $E_i \cap \{e_1, \ldots, e_r\} = \{e_i, e_{i+1}\}$ (from this point on, all subindices should be understood modulo r) for some odd $r \geqslant 3$.

Recall from the Introduction that line graphs of bipartite graphs are balanced. Hence, R is not bipartite and, since $R \in \mathcal{S}$ by hypothesis, $R = R_1 \triangle_{AB} R_2$ where $R_1, R_2 \in \mathcal{S}$, $A = \{a; a_1, \ldots, a_t\}$ is a t-bloom of R_1, and $B = \{b; b_1, \ldots, b_t\}$ is a t-bloom of R_2. Since R_1 and R_2 have less edges than R, the induction hypothesis implies that

$L(R_1)$ and $L(R_2)$ are both balanced. If $E_R(a)$ is an L-clique of R, we will identify $E_R(a)$ with $E_{R_1}(a)$ and say that $E_R(a)$ *is an* L*-clique of* R_1. Similarly, if $E_R(b)$ is an L-clique of R, we will identify $E_R(b)$ with $E_{R_2}(b)$ and say that $E_R(b)$ *is an* L*-clique of* R_2. With this conventions, the L-cliques of R are the L-cliques of R_1 and R_2, plus the triads $T_k = \{ab, ac_k, bc_k\}$ for each $k = 1, \ldots, t$, where c_k is the vertex that results from identifying a_k with b_k. If $r = 3$, Theorem 2.5 implies that G contains an induced pyramid, which means that R contains net, kite, or K_4; and consequently, by definition of $\triangle$, either R_1 or R_2 contain net, kite or K_4, a contradiction with $L(R_1)$ and $L(R_2)$ balanced. Hence, we have $r \geqslant 5$ and suppose that at least one of $E_1, \ldots, E_r$ is an L-clique of R_1. Since $L(R_1)$ is balanced, not all of $E_1, \ldots, E_r$ are L-cliques of R_1. Therefore, there exists some $i \in \{1, \ldots, r\}$ such that E_i is an L-clique of R_1, but E_{i+1} is not. Since $E_i \cap E_{i+1} \neq \emptyset$, necessarily, $E_i = E_R(a)$. Similarly, there is some $j \in \{1, \ldots, r\}$ such that E_j is an L-clique of R_1 and E_{j-1} is not, and necessarily $E_j = E_R(a)$. Hence, every block of consecutive L-cliques of R_1 in the circular ordering $E_1E_2 \ldots E_rE_1$ starts and ends with $E_R(a)$. Since $E_1, \ldots, E_r$ are r pairwise different L-cliques of R, $E_R(a)$ is the only L-clique of R_1 that may belong to $E_1, \ldots, E_r$. Similarly, $E_R(b)$ is the only L-clique of R_2 that may belong to $E_1, \ldots, E_r$.

Since $r \geqslant 5$ and among $E_1, \ldots, E_r$ there are at most one L-clique of R_1 and at most one L-clique of R_2, there are two consecutive elements in the circular ordering $E_1E_2 \ldots E_kE_1$ that are triads T_k for some values of k. Without loss of generality, $E_1 = T_1$ and $E_2 = T_2$. Therefore, $e_2 \in E_1 \cap E_2 = \{ab\}$. But then, $e = ab$ belongs to each of $E_1, \ldots, E_r$, a contradiction. This contradiction arose from assuming that G was not balanced. So, G satisfies (i), as desired. □

As a corollary of the above theorem, we now prove another characterization of those line graphs that are balanced which leads to a linear-time recognition algorithm for balanced graphs within line graphs.

Corollary 3.21. *Let* G *be a line graph and let* R *be a graph such that* $G = L(R)$. *Let* U *be the set of vertices of* R *of degree* 2 *that belong to some triangle of* R *and let* E' *be the set of edges of* R *whose both endpoints are the two neighbors of some vertex of* U. *Then,* G *is balanced if and only if* $R - U$ *is a bipartite graph and every edge of* $R - U$ *that belongs to* E' *is a bridge of* $R - U$.

Proof. Suppose that G is balanced. By assertion (iv) of Theorem 3.20, R contains no kite, net, or K_4. Thus, every triangle of R has at least one vertex of degree 2 and, therefore, $R - U$ has no triangles. Since, in addition, R has no odd cycles of length at least 5, $R - U$ is bipartite. Let ab be an edge of $R - U$ that belongs to E' and suppose, by the way of contradiction, that ab is not a bridge of $R - U$. Thus, ab is an edge of some cycle C of $R - U$. Since $R - U$ is bipartite, $C = abv_1 \ldots v_{2k}a$ for some $k \geqslant 1$.

Since $ab \in E'$, there exists some vertex $c \in R$ such that $N_R(c) = \{a, b\}$. But then, $C' = acbv_1 \dots v_{2k}a$ is a cycle of R of length $2k + 3$ with $k \geqslant 1$, a contradiction since R has no odd cycles of length at least 5.

Conversely, assume that $R - U$ is bipartite and every edge of $R - U$ that belongs to E' is a bridge of $R - U$. We will prove that assertion (iv) of Theorem 3.20 holds. R contains no kite, net, or K_4 (otherwise, $R - U$ would contain a triangle, in contradiction with $R - U$ bipartite). It only remains to prove that R has no odd cycles of length at least 5. Suppose, by the way of contradiction, that R has a cycle $C = v_1v_2 \dots v_rv_1$ of odd length at least 5. Let $w_1, w_2, \dots, w_s$ be the sequence of vertices that arises from the sequence $v_1, v_2, \dots, v_r$ by removing all the vertices that belong to U. Notice that, if $v_i \in U$, then each of v_{i-1} and v_{i+1} has degree at least 3 in R and, therefore, none of v_{i-1} and v_{i+1} belongs to U and $v_{i-1}v_{i+1}$ is an edge of $R - U$. Therefore, $C' = w_1w_2 \dots w_sw_1$ is a cycle of $R - U$. Since C is an odd cycle and $R - U$ is bipartite, $C' \neq C$. So, necessarily, there is at least one vertex of C that belongs to U. Without loss of generality assume that $v_2 \in U$. By construction, $w_1 = v_1$, $w_2 = v_3$, $v_1v_3 \in E'$, and v_1v_3 is an edge of the cycle C' in $R - U$. Therefore, v_1v_3 is an edge of $R - U$ that belongs to E' but is not a bridge of $R - U$, a contradiction. This contradiction proves that R has no odd cycles of length at least 5. Hence statement (iv) of Theorem 3.20 holds and, consequently, G is balanced. □

From Corollary 3.21, we deduce the following.

Corollary 3.22. *It can be decided in linear time whether a given line graph* G *is balanced.*

Proof. Let n and m be the number of vertices and edges of G. A graph R without isolated vertices such that $L(R) = G$ can be computed in $O(m + n)$ time [89, 107]. Additionally, the neighborhoods of the vertices of R can be easily sorted, consistently with some fixed total ordering of $V(R)$, in $O(n)$ time (see, e.g., [80, p. 115]). Notice that $O(n)$ time means linear time of R because R has n edges and no isolated vertices. We now show that U and E' defined as in Corollary 3.21 can also be computed in $O(n)$ time. Let H be an auxiliary multigraph whose vertex set is $V(R)$ and having each of its edges labeled with a vertex of R defined as follows: two vertices v and w of H are joined by one (and exactly one) edge labeled with x if and only if $N_R(x) = \{v, w\}$. Clearly, H can be computed in $O(n)$ time and, as we did with R, we can sort the neighborhoods of H (ignoring the edge labels), consistently with the total ordering of $V(R)$ used for the neighborhoods of R, also in $O(n)$ time. Now, as both $N_R(v)$ and $N_H(v)$ are sorted consistently for each $v \in V(R)$, we can find, in overall $O(n)$ time, the set D of all triples (v, w, x) that satisfy both that $w \in N_R(v) \cap N_H(v)$ and that there is an edge joining v and w labeled with x. Then, U consists of all vertices x such that there is some triple $(v, w, x) \in D$ and E' consists of all edges vw such that there is some triple $(v, w, x) \in D$.

This shows that indeed U and E' can be computed in $O(n)$ time. Finally, we can also decide in $O(n)$ time whether $R - U$ is bipartite and whether the edges of $R - U$ that belong to E' are bridges of $R - U$, because the bridges of a graph can be determined in linear time by depth-first search [112]. □

In the above proof, the sets U and E' can also be computed in $O(m + n)$ time by enumerating all triangles of R using the approach sketched in [80, p. 115], which leads to an alternative linear-time algorithm to decide the balancedness of G. Nevertheless, our procedure has the advantage that it takes only linear time of R to decide the balancedness of $L(R)$ if R is given as input.

We will now briefly comment on how the above results for line graphs naturally extend to line graphs of multigraphs. Since two edges of a multigraph H are adjacent in $L(H)$ if and only if they have at least one endpoint in common, every two parallel edges of H are true twins in $L(H)$. This means that the line graph of the multigraph H arises from the line graph of its underlying graph $\widehat{H}$ by adding true twins. As adding a true twin to a graph only duplicates one column of its clique-matrix, its balancedness is not affected. Therefore, $L(H)$ is balanced if and only if $L(\widehat{H})$ is balanced. Moreover, adding true twins affects neither perfectness nor the fact of being hereditary clique-Helly (as follows, for instance, from Theorems 2.3 and 2.5 because no odd hole, no odd antihole, and no pyramid has true twins). Therefore, $L(H)$ is perfect and hereditary clique-Helly if and only if $L(\widehat{H})$ is so. As a consequence, Theorem 3.20 extends to line graphs of multigraphs as follows.

Theorem 3.23. *Let G be the line graph of a multigraph H. Then, the following assertions are equivalent:*

(i) G *is balanced.*

(ii) G *is perfect and hereditary clique-Helly.*

(iii) G *has no odd holes and contains no induced 3-sun, 1-pyramid, or 3-pyramid.*

(iv) H *has no odd cycles of length at least 5, and contains no net, kite, or K_4.*

(v) *Each component of the underlying graph of H belongs to the class $\mathcal{S}$ (as defined in the statement of Theorem 3.20).*

Finally, also the linear-time recognition algorithm for balanced graphs within line graphs can be extended to line graphs of multigraphs.

Corollary 3.24. *Given the line graph G of a multigraph, it can be decided in linear time whether or not G is balanced.*

Proof. In [80], an algorithm is proposed that, given a graph G, computes in linear time the *representative graph* $\mathcal{R}(G)$ of G, which is the graph that arises from G by successively removing one vertex of some pair of true twins, as long as this is possible. It is easy to see that $\mathcal{R}(G)$ is unique up to isomorphisms. Indeed, a representative graph of G is any induced subgraph of G induced by a set of representatives of the equivalence classes of the relation "is a true twin of" on the vertices of G. As $G = L(H)$ arises from $L(\widehat{H})$ by adding true twins, $\mathcal{R}(G)$ is also the representative graph of $L(\widehat{H})$. Thus, $\mathcal{R}(G)$ is an induced subgraph of $L(\widehat{H})$ and, in particular, $\mathcal{R}(G)$ is a line graph. In addition, as adding true twins does not affect balancedness, G is balanced if and only if $\mathcal{R}(G)$ is balanced. We conclude that the algorithm for computing the representative graph in [80] reduces the problem of deciding the balancedness of the line graphs of multigraphs G to that of deciding the balancedness of the line graphs $\mathcal{R}(G)$, which, as we have seen, is linear-time solvable. □

3.6 Balancedness of complements of line graphs of multigraphs

We say that a multigraph H is $\overline{L}$-*balanced* if the complement of its line graph is balanced. In this subsection, we will characterize those complements of line graphs of multigraphs that are balanced by determining which multigraphs are $\overline{L}$-balanced. As completes in $\overline{L(H)}$ correspond to matchings in H, the clique-matrices of $\overline{L(H)}$ are the maximal matching vs. edge incidence matrices of H, which we call the *matching-matrices* of H. Consequently, H is $\overline{L}$-balanced if and only if its matching-matrix is balanced.

3.6.1 Families of $\overline{L}$-balanced multigraphs

The main result of this subsection is Theorem 3.28 which establishes that certain multigraph families are $\overline{L}$-balanced. The proof of this theorem splits into two parts. The first part will follow from a sufficient condition for $\overline{L}$-balancedness given in Lemma 3.27, near the end of this subsection. The second part is postponed to Subsection 3.6.4. In order to prove the aforesaid sufficient condition, we introduce three multigraph families: $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$. In Figure 3.1, a generic member of each of these families is shown, where light lines represent single edges, bold lines one or more parallel edges, p is any positive integer, and $a_1, \dots, a_p$ are pairwise false twins.

Our next lemma shows that the multigraph families $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ arise naturally when characterizing those multigraphs H such that $\overline{L(H)}$ is trivially perfect.

Lemma 3.25. *Let G be the line graph of a multigraph H. Then, the following assertions are equivalent:*

(i) G *is trivially perfect.*

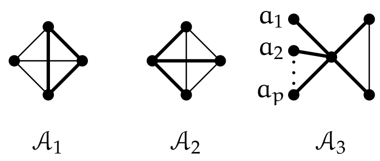


Figure 3.1: *Multigraphs families $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$. Light lines represent single edges, whereas bold lines represent one or more parallel edges. Parameter p varies over all positive integers and $a_1, a_2, \dots, a_p$ are pairwise false twins*

(ii) H *contains no* P_5, $2P_3$, $P_3 \cup C_2$, *or* $2C_2$.

(iii) *Some component of* H *is contained in some member of* $\mathcal{A}_1$, $\mathcal{A}_2$, *or* $\mathcal{A}_3$, *and each of the remaining components of* H *has at most one edge.*

Proof. The equivalence between (i) and (ii) follows immediately from the definitions of trivially perfect graphs and line graphs of multigraphs. It is also clear, by simple inspection, that each of the members of the families $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ contains no P_5, $2P_3$, $P_3 \cup C_2$, or $2C_2$. Therefore, the same holds also for any submultigraph of them, which proves that (iii) implies (ii). To complete the proof, we prove that (ii) implies (iii). Recall that $\hat{d}_H(v)$ denotes the degree of v in the underlying graph $\widehat{H}$ and that a vertex v of H is pendant if and only if $\hat{d}_H(v) = 1$.

Suppose that H satisfies (ii) and let S be any component of H. First assume that S is a multitree and let $P = v_1 v_2 \dots v_t$ be a longest path in S. Since S contains no P_5 and P is maximal, necessarily $t \leqslant 4$, v_1 and v_t are pendant vertices, and each neighbor of $v_2, \dots, v_{t-1}$ outside P is a pendant vertex. If $t \leqslant 3$, S is contained in some member of $\mathcal{A}_3$, as desired. So, let $t = 4$. Since S contains no $P_3 \cup C_2$ or $2C_2$, we can assume, by symmetry, that there is a single edge joining v_1 to v_2 and $\hat{d}_S(v_2) = 2$. We conclude that S is contained in some member of $\mathcal{A}_3$, as desired. So, from now on, we assume, without loss of generality, that S is not a multitree and let ℓ be the length of the longest cycle of S. Since S contains no P_5, $\ell = 3$ or $\ell = 4$.

Suppose that $\ell = 3$ and let $T = v_1 v_2 v_3 v_1$ be some triangle of S. Since S contains no P_5 or bipartite claw and $\ell = 3$, at most one vertex of T has some neighbor $v \in V(S) \setminus V(T)$ and each of these neighbors v is a pendant vertex. Without loss of generality, we assume that $\hat{d}_S(v_1) = \hat{d}_S(v_2) = 2$. If $\hat{d}_S(v_3) > 3$ or v_3 is joined to some pendant vertex through two or more parallel edges, then there is a single edge joining v_1 to v_2 (because S contains no $P_3 \cup C_2$ or $2C_2$) and S is contained in some member of $\mathcal{A}_3$. If $\hat{d}_S(v_3) \leqslant 3$ and there are no two parallel edges joining v_3 to a pendant neighbor, then S is contained in some member of $\mathcal{A}_1$.

Finally, suppose that $\ell = 4$ and let C be a 4-cycle of S. Since C contains no P_5 or $2C_2$, $V(S) = V(C)$ and S has no two non-incident pairs of parallel edges. Therefore, S is some member of $\mathcal{A}_1$ or $\mathcal{A}_2$.

We conclude that H satisfies (iii), which completes the proof. □

We say that two edges e_1 and e_2 of a multigraph H are *matching-distinguishable* if there is some maximal matching of H that contains e_1 but not e_2 and vice versa. Equivalently, e_1 and e_2 are matching-distinguishable in H if and only if they are clique-distinguishable as vertices of $\overline{L(H)}$. Notice that every two parallel edges are always matching-distinguishable. It is easy to see that, for each member of $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$, any two matching-distinguishable edges are incident. Indeed, in each of the multigraphs represented in Figure 3.1, the edges in bold are pairwise incident and each light edge is not matching-distinguishable from any edge that is non-incident to it. (Alternatively, the result follows by applying Theorem 3.8 to $\overline{L(H)}$ for each multigraph H in Figure 3.1, as we know that $\overline{L(H)}$ is trivially perfect.)

Let F be a submultigraph of a multigraph H. We say that F is a *fragment* of H if there is an embedding of F in some of the multigraphs represented in Figure 3.1 such that the edges of F corresponding, under the embedding, to light edges in Figure 3.1 are incident in H to edges of F only. We observe the following.

Lemma 3.26. *If F is a fragment of H, then any pair of edges of F matching-distinguishable in H are incident.*

Proof. Indeed, the edges of F corresponding under the embedding to bold edges are pairwise incident and, if M is a maximal matching of H that does not contain some edge e of F corresponding to a light edge, then M must contain some edge e' of F that is incident to e and it follows that M cannot contain any edge e'' of F that is non-incident to e (because the edges e'' of F that are non-incident to e turn out to be necessarily incident to e'). □

In Figure 3.2, we introduce multigraph families $\mathcal{B}_1$, $\mathcal{B}_2$, …, $\mathcal{B}_{16}$ by presenting a generic member of each family: light lines represent single edges, bold lines represent one or more parallel edges, p is any positive integer, and $a_1, \dots, a_p$ are pairwise false twins. Notice, for instance, that for each member of $\mathcal{B}_2$, $\mathcal{B}_3$, and $\mathcal{B}_4$, its edge set can be partitioned into the edge sets of two fragments. Our next result shows that this condition is sufficient for $\overline{L}$-balancedness.

Lemma 3.27. *If the edge set of a multigraph H can be partitioned into the edge sets of two fragments of H, then H is $\overline{L}$-balanced.*

Proof. Let F_1 and F_2 be two fragments of H such that $\{E(F_1), E(F_2)\}$ is a partition of $E(H)$. Let $G = \overline{L(H)}$ and let $X = E(F_1)$ and $Y = E(F_2)$. Then, $\{X, Y\}$ is a partition of the vertex set of G and, by Lemma 3.26, any two vertices clique-distinguishable in G that belong both to X or both to Y are nonadjacent. So, by Lemma 3.11, G is balanced, as desired. □

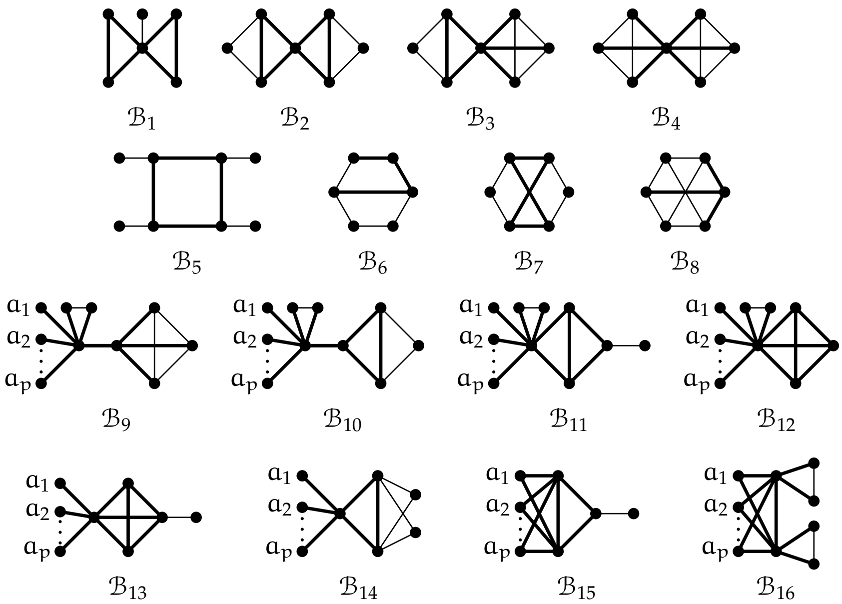


Figure 3.2: *Multigraph families $\mathcal{B}_1$, $\mathcal{B}_2$, …, $\mathcal{B}_{16}$. Light lines represent single edges, whereas bold lines represent one or more parallel edges. Parameter p varies over the positive integers, and $a_1, a_2, \ldots, a_p$ are pairwise false twins*

In Figure 3.2, we introduce the multigraph families $\mathcal{B}_1$, $\mathcal{B}_2$, …, $\mathcal{B}_{16}$ by presenting a generic member of each family. If follows, by direct application of the above lemma, that the families $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{B}_9$, $\mathcal{B}_{10}$, $\mathcal{B}_{11}$, $\mathcal{B}_{12}$, and $\mathcal{B}_{16}$ are $\overline{L}$-*balanced*; i.e., each of their members are $\overline{L}$-balanced. In Subsection 3.6.4, we provide separate proofs of the $\overline{L}$-balancedness of each of the remaining families displayed in Figure 3.2. As a result, we conclude the following.

Theorem 3.28. *The families $\mathcal{B}_1$, $\mathcal{B}_2$, …, $\mathcal{B}_{16}$ are $\overline{L}$-balanced.*

3.6.2 Characterizing balanced complements of line graphs of multigraphs

In this subsection, we characterize those complements of line graphs of multigraphs that are balanced, including a characterization by minimal forbidden induced subgraphs.

Theorem 3.29. *Let G be the complement of the line graph of a multigraph H. Then, the following assertions are equivalent:*

(i) G *is balanced.*

(ii) *A clique-matrix of* G *has no edge-vertex incidence matrix of* C_3, C_5, *or* C_7 *as a submatrix.*

(iii) G *contains no induced 3-sun, 2-pyramid, 3-pyramid,* C_5, $\overline{C_7}$, U_7, *or* V_7.

(iv) H *contains no bipartite claw,* $P_5 \cup P_3$, $P_5 \cup C_2$, $3P_3$, $2P_3 \cup C_2$, $P_3 \cup 2C_2$, $3C_2$, C_5, C_7, *6-pan, braid, 1-braid, or 2-braid.*

(v) *One of the following conditions holds:*

(a) *Each component of* H *has at most one edge.*

(b) H *has exactly one component with more than one edge, which is contained in a member of* $\mathcal{B}_1$, $\mathcal{B}_2$, …, *or* $\mathcal{B}_{16}$

(c) H *has exactly two components with more than one edge each, each of which is contained in a member of* $\mathcal{A}_1$, $\mathcal{A}_2$, *or* $\mathcal{A}_3$.

Proof. The implication (i) $\Rightarrow$ (ii) follows by definition. The implication (ii) $\Rightarrow$ (iii) follows from the fact that a clique-matrix of each of 3-sun, 2-pyramid, 3-pyramid, C_5, $\overline{C_7}$, U_7, and V_7 contains an edge-vertex incidence matrix of C_3, C_5, or C_7 as a submatrix. The implication (iii) $\Rightarrow$ (iv) follows by definition of the line graph of a multigraph.

The implication (v) $\Rightarrow$ (i) can be proved as follows. If (a) holds, then $G = \overline{L(H)}$ is a clique and, in particular, G is balanced. So, assume that (b) or (c) holds. Without loss of generality, H has no isolated vertices. Moreover, we can also assume that H has no component with only one edge because removing these components from H amounts to removing the universal vertices from $\overline{L(H)}$, which does not affect the balancedness of $\overline{L(H)}$ (because each universal vertex corresponds to a column full of 1's in the clique-matrix). Therefore, we can assume that H is contained in a member of $\mathcal{B}_1$, $\mathcal{B}_2$, …, or $\mathcal{B}_{16}$ or H has two components, each of which is contained in a member of $\mathcal{A}_1$, $\mathcal{A}_2$, or $\mathcal{A}_3$. If the former holds, $\overline{L(H)}$ is balanced by Theorem 3.28, if the latter holds, $\overline{L(H)}$ is balanced by Lemma 3.27. This concludes the proof of (v) $\Rightarrow$ (i).

The rest of the proof is devoted to showing that (iv) $\Rightarrow$ (v). In order to do so, assume that H satisfies (iv). Suppose first that H has two or more components with two or mores edges each. Since H contains no $3P_3$, $2P_3 \cup C_2$, $P_3 \cup 2C_2$, or $3C_2$, H has exactly two components S_1 and S_2 with at least two edges each. In particular, S_2 contains P_3 or C_2, which means that S_1 contains no P_5, $2P_3$, $P_3 \cup C_2$, or $2C_2$ and, by Lemma 3.25, S_1 is contained in some member of $\mathcal{A}_1$, $\mathcal{A}_2$, or $\mathcal{A}_3$. By symmetry, S_2 is also contained in some member of $\mathcal{A}_1$, $\mathcal{A}_2$, or $\mathcal{A}_3$. This proves that if H has at least two components with two or more edges each, (c) holds. If each component of H has at most one edge, (a) holds. Therefore, we assume that H has exactly one component S having at least two edges. We will prove that S is contained in some member of $\mathcal{B}_1$, $\mathcal{B}_2$, …, $\mathcal{B}_{16}$ and, consequently, (b) holds, concluding the proof of the theorem.

We split the proof into four main cases. In the first case S is a multitree. In the other cases, we assume that S is not a multitree and we let ℓ be the length of the longest cycle in S. Since S contains no C_5, C_7, or $P_5 \cup P_3$, necessarily $\ell = 3, 4$, or 6.

Along this proof, we adopt the following convention: Given any two adjacent vertices u and v of S, we will say that uv *is a simple edge* if there is exactly one edge joining u to v; otherwise, we say that uv *is a multiple edge*. Recall that we say that a vertex v of S is pendant if and only if $\hat{d}(v) = 1$ (where $\hat{d}_S(v)$ denotes the degree of v in the underlying graph $\widehat{S}$).

Case 1. *S is a multitree.*

Let $P = v_1v_2 \ldots v_t$ be a path of S of maximum length. As S is not edgeless, $t \geqslant 2$. Moreover, since S is a multitree, the endpoints of P are pendant vertices and $t \leqslant 7$ because S contains no $P_5 \cup P_3$. By maximality of P and since S contains no bipartite claw, the neighbors of $v_2, \ldots, v_{t-1}$ outside P are pendant vertices of S.

1a $t \leqslant 4$. Then, S is contained in some member of $\mathcal{B}_{15}$.

1b $t = 5$. If $\hat{d}_S(v_3) \leqslant 3$ and any edge joining v_3 to a pendant neighbor is simple, then S is contained in some member of $\mathcal{B}_{15}$. If $\hat{d}_S(v_3) > 3$ or there is a multiple edge joining v_3 to a pendant neighbor, then either $\hat{d}_S(v_2) = 2$ and v_1v_2 is simple, or $\hat{d}_S(v_4) = 2$ and v_4v_5 is simple (otherwise S would contain $3P_3$, $2P_3 \cup C_2$, $P_3 \cup 2C_2$, or $3C_2$). In either case, S is contained in some member of $\mathcal{B}_{16}$.

1c $t = 6$. If $\hat{d}_S(v_2) = \hat{d}_S(v_5) = 2$ and v_1v_2 and v_5v_6 are simple, then S is contained in some member of $\mathcal{B}_{16}$. By symmetry, assume, without loss of generality, that $\hat{d}_S(v_2) > 2$ or v_1v_2 is multiple. Then, $\hat{d}_S(v_3) = 2$ follows since S contains no $P_5 \cup P_3$ and no $P_5 \cup C_2$. In addition, $\hat{d}_S(v_5) = 2$ and v_5v_6 is simple, because S contains no braid, 1-braid, or 2-braid. Thus, also in this case, S is contained in some member of $\mathcal{B}_{16}$.

1d $t = 7$. Since S contains no $P_5 \cup P_3$ and no $P_5 \cup C_2$, $\hat{d}_S(v_2) = \hat{d}_S(v_4) = \hat{d}_S(v_6) = 2$ and the edges v_1v_2 and v_6v_7 are simple. Therefore, S is contained in some member of $\mathcal{B}_{16}$.

Case 2. *S has a longest cycle of length $\ell = 3$.*

In each subcase, we assume that the previous subcases do not hold.

2a *There is some triangle T such that all its vertices have some neighbor outside T.* Let $T = v_1v_2v_3v_1$ be such a triangle in S. By hypothesis, S has no 4-cycle and S contains no bipartite claw. Therefore, for each $i = 1, 2, 3$, each vertex $v \in N_S(v_i) \setminus V(T)$ is a pendant vertex of S. Since S contains no $3P_3$, $2P_3 \cup C_2$, $P_3 \cup 2C_2$, or $3C_2$, there are at most two vertices of T having more than one pendant neighbor or joined to a pendant neighbor by a multiple edge. Therefore, S is contained in some member of $\mathcal{B}_{15}$.

2b *There is a triangle* T *touching a* 5*-path* P *at an endpoint of* P. Let $T = v_1v_2v_3v_1$ touch $P = v_1w_1w_2w_3w_4$ at v_1. As S contains no $P_5 \cup P_3$ or $P_5 \cup C_2$ and $\ell = 3$, $\hat{d}_S(v_2) = \hat{d}_S(v_3) = 2$, $N_S(w_1) \subseteq \{v_1, w_2, w_3\}$, $N_S(w_3) \subseteq \{w_1, w_2, w_4\}$, $N_S(w_4) \subseteq \{w_2, w_3\}$, each $v \in N_S(v_1)\backslash\{v_2, v_3, w_1, w_2\}$ is pendant, each $v \in N_S(w_2)\backslash\{v_1, w_1, w_3, w_4\}$ is pendant, and the edges v_2v_3 and w_3w_4 are simple. If w_1 and w_3 are nonadjacent, then S is contained in some member of $\mathcal{B}_{16}$. So, assume, without loss of generality, that w_1 and w_3 are adjacent. Then, w_2 is nonadjacent to v_1 and to w_4 because S has no 4-cycles, and $\hat{d}_S(w_2) = 2$ because S contains no $P_5 \cup P_3$. Therefore, S is contained in some member of $\mathcal{B}_{10}$.

2c *There are two touching triangles,* say $T = v_1v_2v_3v_1$ and $T' = v_1w_1w_2v_1$. By symmetry and since 2a does not hold, we assume, without loss of generality, that $\hat{d}_S(v_2) = 2$ and $\hat{d}_S(w_1) = 2$. As S has no 4-cycles and no bipartite claw, each $v \in N_S(v_1)\backslash\{v_2, v_3, w_1, w_2\}$ is a pendant vertex. Since S has no 4-cycles and 2b does not hold, each $v \in N_S(v_3)\backslash V(T)$ is a pendant vertex. Symmetrically, each $v \in N_S(w_2)\backslash V(T')$ is also a pendant vertex.

If each of v_1 and w_2 is adjacent to some pendant neighbor, then v_2v_3 is simple and $\hat{d}_S(v_3) = 2$ (because S contains no $P_5 \cup C_2$ or $P_5 \cup P_3$), which means that S is contained in some member of $\mathcal{B}_{16}$.

So, if v_1 is adjacent to some pendant neighbor, we can assume that $\hat{d}_S(v_3) = \hat{d}_S(w_2) = 2$ and, since S contains no $P_3 \cup 2C_2$ or $3C_2$, one of the following conditions hold:

- v_1 is adjacent to exactly one pendant neighbor and the edge joining v_1 to its pendant neighbor is simple, which means that S is contained in $\mathcal{B}_1$.
- At least one of v_2v_3 and w_1w_2 is simple, which implies that S is contained in a member of $\mathcal{B}_{16}$.

So, without loss of generality, assume that v_1 is not adjacent to any pendant neighbor. If w_2 is adjacent to at least two pendant neighbors or there is a multiple edge joining w_2 to a pendant neighbor, then v_2v_3 is simple (because S contains no 1-braid or 2-braid) and, as a result, S is contained in some member of $\mathcal{B}_{16}$. If w_2 is adjacent to at most one pendant neighbor and any edge joining w_2 to a pendant vertex is simple, then, symmetrically, v_3 is adjacent to at most one pendant neighbor and any edge joining v_3 to a pendant vertex is simple and we conclude that S is contained in some member of $\mathcal{B}_2$.

2d *There is an edge touching two different triangles.* Since S has no 4-cycles and 2c does not hold, any pair of different triangles of T in S are vertex-disjoint. Let v_1w_1 be an edge touching the two triangles $T = v_1v_2v_3v_1$ and $T' = w_1w_2w_3w_1$ in S. Since S has no 4-cycle and 2b does not hold, $\hat{d}_S(w_2) = \hat{d}_S(w_3) = \hat{d}_S(v_2) = \hat{d}_S(v_3) = 2$. As S contains no bipartite claw and 2c does not hold, each $v \in N_S(v_1)\backslash\{v_2, v_3, w_1\}$ is a pendant vertex and also each $v \in N_S(w_1)\backslash\{w_2, w_3, v_1\}$ is a pendant vertex. If none of the edges v_2v_3 and w_2w_3 is multiple, S is contained in some member of $\mathcal{B}_{16}$. If v_2v_3 is multiple, then w_2w_3 is simple (because S contains no 2-braid) and $\hat{d}_S(v_1) = 3$ (because S contains no $P_5 \cup C_2$), and we conclude that S is contained in a member of $\mathcal{B}_{10}$.

2e *There is a triangle* T *touching a* 4*-path* P *at an endpoint of* P. Let $T = v_1v_2v_3v_1$ touch $P = v_1w_1w_2w_3$ at v_1. Since 2a does not hold, we assume, without loss of generality, that $\hat{d}_S(v_2) = 2$. As 2c does not hold, v_1 and w_2 are nonadjacent. Since S has no 4-cycles, v_1 and w_3 are nonadjacent. As 2d does not hold, w_1 and w_3 are nonadjacent. Since S has no 4-cycles and no 5-cycles, v_3 is nonadjacent to w_1, w_2, and w_3. So, two vertices of $V(T) \cup V(P)$ are adjacent only if they are adjacent in T or in P. Since 2b does not hold, w_3 is a pendant vertex. As S contains no $P_3 \cup P_5$ and $\ell = 3$, there is at most one vertex $v \in N_S(v_3)\backslash\{v_1, v_2\}$ and, if so, v is a pendant vertex and vv_3 is simple. Since S has no 4-cycles, 2c does not hold, and S contains no bipartite claw, each $v \in N_S(v_1)\backslash\{v_2, v_3, w_1\}$ is a pendant vertex. As 2d does not hold and S contains no bipartite claw, each $v \in N_S(w_1)\backslash\{v_1, w_2\}$ is a pendant vertex. Since 2b does not hold, each vertex $v \in N_S(w_2)\backslash\{w_1, w_3\}$ is a pendant vertex.

If w_2 has a pendant neighbor or w_2w_3 is multiple, then $\hat{d}_S(w_1) = \hat{d}_S(v_3) = 2$ and v_2v_3 is simple (otherwise S contains $P_5 \cup P_3$, $P_5 \cup C_2$, braid, 1-braid, or 2-braid) and, therefore, S is contained in some member of $\mathcal{B}_{16}$. Hence, we can assume that $\hat{d}_S(w_2) = 2$ and w_2w_3 is simple.

If $\hat{d}_S(v_3) = 3$ or v_2v_3 is multiple, then $\hat{d}_S(v_1) = 3$ (because S contains no $P_5 \cup P_3$ or $P_5 \cup C_2$) and S is contained in some member of $\mathcal{B}_{10}$. Otherwise, S is contained in some member of $\mathcal{B}_{16}$.

2f *None of the previous subcases holds.* Let $T = v_1v_2v_3v_1$ be triangle of S. Suppose, by the way of contradiction, that v_1 has two non-pendant neighbors different from v_2 and v_3. Let $w_1, w_2 \in N_G(v_1)\backslash\{v_2, v_3\}$ such that w_1 and w_2 are non-pendant. Since w_1 is non-pendant, there exists some vertex $w_3 \in N_G(w_1)\backslash\{v_1\}$. As S has no 4-cycles and 2c does not hold, $w_3 \notin V(T) \cup \{w_1, w_2\}$. Similarly, there is a vertex $w_4 \in N_G(w_2)\backslash\{v_1\}$ and $w_4 \notin V(T) \cup \{w_1, w_2, w_3\}$. But then, S contains a bipartite claw, a contradiction.

Hence, each vertex of T is adjacent to at most one non-pendant vertex not in $V(T)$. Since S has no 4-cycles, 2c and 2e do not hold, and $\ell = 3$, if w is a non-pendant neighbor of v_i for some $i \in \{1,2,3\}$, then each $v \in N_S(w) \backslash \{v_i\}$ is a pendant vertex.

Suppose that v_1 is adjacent to some non-pendant vertex w_1 such that w_1 is adjacent to two pendant neighbors or there is a multiple edge joining w_1 to a pendant neighbor. Since S contains no $P_5 \cup P_3$, $P_5 \cup C_2$, $3P_3$, $2P_3 \cup C_2$, $P_3 \cup 2C_2$, or $3C_2$, if $\hat{d}_S(v_1) \geqslant 4$, then $\hat{d}_S(v_2) = \hat{d}_S(v_3) = 2$ and one of the following holds:

- $\hat{d}_S(v_1) = 4$ and the edge joining v_1 to a pendant vertex is simple and, consequently, S is contained in some member of $\mathcal{B}_{13}$.
- v_2v_3 is simple and S is contained in some member of $\mathcal{B}_{16}$.

So, we assume that $\hat{d}_S(v_1) = 3$. Since 2a does not hold, we assume, without loss of generality, that $\hat{d}_S(v_3) = 2$. Since S contains no $P_5 \cup P_3$, $P_5 \cup C_2$, braid, 1-braid, or 2-braid, $\hat{d}_S(v_2) \leqslant 3$ and if there is $v \in N_S(v_2) \backslash \{v_1, v_3\}$, then v is pendant and v_2v is simple. We conclude that S is contained in some member of $\mathcal{B}_{10}$.

So it only remains to consider the case in which each non-pendant vertex w of v_i for some $i \in \{1,2,3\}$ satisfies that $\hat{d}_S(w) = 2$ and that, for each $w' \in N_S(w) \backslash \{v_i\}$, ww' is simple. Since 2a does not hold, S is contained in some member of $\mathcal{B}_{16}$.

Case 3. *S has a longest cycle of length* $\ell = 4$.
In each subcase, we assume that the previous subcases do not hold.

3a *There are two touching* 4*-cycles in* S, say $C = v_1v_2v_3v_4v_1$ and $C' = v_1w_2w_3w_4v_1$. Since S has no 5-cycle and contains no $P_3 \cup P_5$, $V(S) = V(C) \cup V(C')$. Since S contains no $P_5 \cup C_2$, the edges v_2v_3, v_3v_4, w_1w_2, and w_2w_3 are simple. If v_2v_4 is multiple, then there is no edge v_1v_3. Symmetrically, if w_1w_3 is multiple, then there is no edge v_1w_2. We conclude that S is contained in some member of $\mathcal{B}_2$, $\mathcal{B}_3$, or $\mathcal{B}_4$.

3b *There is a triangle* T *touching a* 4*-cycle in* S. Let $C = v_1v_2v_3v_4v_1$ touch $T = v_1w_1w_2v_1$. As S has no 5-cycle and contains no bipartite claw, we have $N_S(v_2) \subseteq \{v_1, v_3, v_4\}$, $N_S(v_4) \subseteq \{v_1, v_2, v_3\}$, and $N_S(v_3) \cap \{w_1, w_2\} = \emptyset$. This also means that $N_S(w_1) \cap V(C) = N_S(w_2) \cap V(C) = \{v_1\}$. Since S contains no $P_5 \cup P_3$ and 3a does not hold, $\hat{d}_S(w_1) \leqslant 3$ and we assume, without loss of generality, that $\hat{d}_S(w_2) = 2$.

Let us consider the case when $\hat{d}_S(w_1) = 3$ or w_1w_2 is multiple. Since S contains no $P_5 \cup P_3$ or $P_5 \cup C_2$, $N_S(v_1) \subseteq V(C) \cup V(T)$, $N_S(v_3) \subseteq V(C)$, and if there is some $w_3 \in N_S(w_1) \backslash \{v_1, w_2\}$ then w_3 is a pendant vertex of S and w_1w_3 is simple. In addition, v_2v_3 and v_3v_4 are simple because S contains no 1-braid or 2-braid. If v_2v_4 is not a multiple edge of S, then S is contained in some member of $\mathcal{B}_3$. Otherwise, v_1v_3 is not an edge of S (because S contains no 1-braid or 2-braid) and S is contained in some member of $\mathcal{B}_2$. So, from now on, we assume that $\hat{d}_S(w_1) = \hat{d}_S(w_2) = 2$ and w_1w_2 is simple.

Suppose that v_2 and v_4 are adjacent. Since S contains no bipartite claw, each $v \in N_S(v_1)\backslash(V(T) \cup V(C))$ is a pendant vertex of S. So, if $N_S(v_3) \subseteq V(C)$, then S is contained in some member of $\mathcal{B}_{12}$. Therefore, we can assume that there is some $w_3 \in N_S(v_3)\backslash V(C)$. Since S contains no bipartite claw, $P_5 \cup P_3$, or $P_5 \cup C_2$, v_1v_3 is not an edge of S, $|N_S(v_3)\backslash V(C)| = 1$, w_3 is a pendant vertex of S, and v_3w_3 is simple. We conclude that S is contained in some member of $\mathcal{B}_{11}$.

It only remains to consider the case when v_2 and v_4 are nonadjacent. Due to the first remarks of this subcase, $N_S(v_2) = N_S(v_4) = \{v_1, v_3\}$. Notice that each $v \in N_S(v_1)\backslash(V(T) \cup V(C))$ satisfies $N_S(v) \subseteq \{v_1, v_3\}$ because S contains no bipartite claw. If each $v \in N_S(v_3)\backslash\{v_1\}$ satisfies that $N_S(v) \subseteq \{v_1, v_3\}$, then S is contained in some member of $\mathcal{B}_{16}$. So, we can assume that there is some $w_3 \in N_S(v_3)\backslash\{v_1\}$ and some $w_4 \in N_S(w_3)\backslash\{v_1, v_3\}$. By construction, $w_3, w_4 \notin V(C) \cup V(T)$. Then, $N_S(w_3) = \{v_3, w_4\}$ and w_3w_4 is simple since S contains no braid or 1-braid. In addition, $N_S(w_4) \subseteq \{v_3, w_3\}$ because S contains no $P_3 \cup P_5$. Since S contains no bipartite claw, each $v \in N_S(v_3)\backslash\{v_1, v_2, v_4, w_3, w_4\}$ satisfies $N_S(v) \subseteq \{v_1, v_3\}$. Thus, S is contained in some member of $\mathcal{B}_{16}$.

3c S *contains* $K_{2,3}$. Equivalently, suppose that there are two vertices $v_1, v_3 \in V(S)$ such that $N_S(v_1) \cap N_S(v_3)$ consists of at least three vertices. Let v_2 be a vertex of $N_S(v_1) \cap N_S(v_3)$ of maximum degree in $\widehat{S}$ and let v_4 and v_5 be any two other vertices of $N_S(v_1) \cap N_S(v_3)$. Since S has no 5-cycle and contains no bipartite claw, $\{v_2, v_4, v_5\}$ is a stable set, $\hat{d}_S(v_4) = \hat{d}_S(v_5) = 2$, and each $v \in N_S(v_2)\backslash\{v_1, v_3\}$ is a pendant vertex.

Suppose that each vertex $v \in (N_S(v_1) \cup N_S(v_3))\backslash\{v_1, v_2, v_3\}$ is such that $N_S(v) \subseteq \{v_1, v_3\}$. If v_2 is adjacent to at most one pendant vertex and any edge joining v_2 to a pendant vertex is simple, then S is contained in some member of $\mathcal{B}_{15}$. So, assume, on the contrary, that v_2 is adjacent to at least two pendant vertices or v_2 is joined to a pendant vertex by a multiple edge. Then, $N_S(v_1) \subseteq \{v_2, v_3, v_4, v_5\}$ and $N_S(v_3) \subseteq \{v_1, v_2, v_4, v_5\}$ (because S contains no $P_5 \cup P_3$), each of the edges $v_1v_4, v_1v_5, v_3v_4, v_3v_5$ is simple (because S contains no 1-braid and no 2-braid) and, consequently, S is contained in some member of $\mathcal{B}_{14}$. So, we can assume that there is some vertex $w_1 \in N_S(v_1)\backslash\{v_2, v_3\}$ such that $N_S(w_1) \nsubseteq \{v_1, v_3\}$ and let $w_2 \in N_S(w_1)\backslash\{v_1, v_3\}$. Since S contains no $P_3 \cup P_5$, $\hat{d}_S(v_2) = 2$. Notice that, by construction, w_1 is nonadjacent to v_3; otherwise, $w_1 \in N_S(v_1) \cap N_S(v_3)$ and $\hat{d}_S(w_1) > 2 = \hat{d}_S(v_2)$, contradicting the choice of v_2. Since S contains no braid or 1-braid, $\hat{d}_S(w_1) = 2$ and w_1w_2 is simple. Notice that w_2 is a pendant vertex because S has no 5-cycle, contains no $P_5 \cup P_3$, and 3b does not hold. Since S contains no bipartite claw, w_1 is the only vertex $v \in N_S(v_1)\backslash\{v_2, v_3\}$ such that $N_S(v)$ is not contained in $\{v_1, v_3\}$. By symmetry, there is at most one vertex $w_3 \in N_S(v_3)\backslash\{v_1, v_2\}$ such that $N_S(w_3) \nsubseteq \{v_1, v_3\}$ and, if so, $\hat{d}_S(w_3) = 2$, the vertex $w_4 = N_S(w_3)\backslash\{v_3\}$ is a pendant vertex, and w_3w_4 is simple. Since, by construction, all vertices $v \in (N_S(v_1) \cup N_S(v_3))\backslash\{v_1, v_3, w_1, w_3\}$ are such that $N_S(v) \subseteq \{v_1, v_3\}$, S is contained in some member of $\mathcal{B}_{16}$.

3d *There is a 4-cycle $C = v_1v_2v_3v_4v_1$ such that each vertex v_i of C has a neighbor $w_i \notin V(C)$.* Since S has no 5-cycle and 3c does not hold, $N_S(v_i) \cap N_S(v_j) \subseteq V(C)$ for all i and all j. In particular, w_1, w_2, w_3, and w_4 are pairwise different. Since S contains no $P_5 \cup P_3$ or $P_5 \cup C_2$, w_i is the only vertex in $N_S(v_i)\backslash V(C)$ and v_iw_i is simple for each $i = 1, 2, 3, 4$. Moreover, w_1, w_2, w_3, and w_4 are pendant vertices as S has no 6-cycle and contains no bipartite claw. Finally, since S contains no bipartite claw, C is chordless and we conclude that S is a member of $\mathcal{B}_5$.

3e *There is a 4-cycle C touching a 4-path P at an endpoint of P.* Let $C = v_1v_2v_3v_4v_1$ touch $P = v_1w_1w_2w_3$ in v_1. Since $\ell = 4$, S contains no $P_5 \cup P_3$ or $P_5 \cup C_2$, and 3a does not hold, $N_S(w_3) \subseteq \{w_1, w_2\}$ and w_2w_3 is simple. Similarly, and since 3b does not hold, $N_S(w_2) = \{w_1, w_3\}$. Since S has no 5-cycles and 3c does not hold, $N_S(w_1) \cap V(C) = \{v_1\}$. Since $\ell = 4$ and S contains no $P_5 \cup P_3$, each $v \in N_S(w_1)\backslash\{v_1, w_2, w_3\}$ is a pendant vertex of S, $N_S(v_1) \subseteq V(C) \cup \{w_1\}$, and $N_S(v_2), N_S(v_3), N_S(v_4) \subseteq V(C)$. Notice also that v_2v_3 and v_3v_4 are simple because S contains no $P_5 \cup C_2$. Therefore, if v_2v_4 is not a multiple edge of S, then S is contained in some member of $\mathcal{B}_9$. If, on the contrary, v_2v_4 is multiple, then v_1 and v_3 are nonadjacent (because S contains no $P_5 \cup C_2$) and S is contained in some member of $\mathcal{B}_{10}$.

3f *There is a 4-cycle $C = v_1v_2v_3v_4v_1$ such that three of its vertices have a neighbor outside C*, say, v_i has a neighbor $w_i \notin V(C)$ for each $i = 1, 2, 3$. Then, $N_S(v_1)\backslash V(C)$, $N_S(v_2)\backslash V(C)$, and $N_S(v_3)\backslash V(C)$ are pairwise disjoint and each $w \in N_S(v_i)\backslash V(C)$, for some $i \in \{1, 2, 3\}$, is a pendant vertex because 3c does not hold and S has no 5-cycles or 6-cycle and contains no $P_5 \cup P_3$. Since 3d does not hold and S contains no bipartite claw, $N_S(v_4) = \{v_1, v_3\}$. Finally, w_2 is the only pendant neighbor of v_2 and v_2w_2 is simple because S contains no $P_5 \cup P_3$ or $P_5 \cup C_2$. We conclude that S is a contained in some member of $\mathcal{B}_{15}$.

3g *There is a 4-cycle $C = v_1v_2v_3v_4v_1$ where v_1 is adjacent to a non-pendant vertex $w_1 \notin V(C)$.* Let w_2 be any vertex of $N_S(w_1)\backslash\{v_1\}$. Then, $w_2 \notin V(C)$ because S contains no 5-cycle and 3c does not hold. As S has no 5-cycle or 6-cycle and 3b does not hold, $N_S(w_2) \cap V(C) = \varnothing$. Therefore, w_2 is a pendant vertex as 3e does not hold. Notice that $N_S(v_2), N_S(v_4) \subseteq V(C)$ because S contains no bipartite claw. Since w_2 is an arbitrary vertex of $N_S(w_1) \backslash \{v_1\}$, each $w \in N_S(w_1) \backslash \{v_1\}$ is a pendant vertex. Since w_1 is an arbitrary non-pendant vertex in $N_S(v_1) \backslash V(C)$, for every non-pendant vertex w_1' in $N_S(v_1) \backslash V(C)$, each $w \in N_S(w_1') \backslash \{v_1\}$ is a pendant vertex. Thus, since S contains no $P_3 \cup P_5$, w_1 is the only non-pendant vertex in $N_S(v_1) \backslash V(C)$; i.e., each $v \in N_S(v_1) \backslash \{v_2, v_3, v_4, w_1\}$ is a pendant vertex.

Suppose first that $\hat{d}_S(w_1) > 2$ or $w_1 w_2$ is multiple. Since S contains no $P_5 \cup P_3$ or $P_5 \cup C_2$, v_1 has no pendant neighbors and $N_S(v_3) \subseteq V(C)$. If $v_2 v_4$ is not a multiple edge, then S is contained in some member of $\mathcal{B}_9$, but if $v_2 v_4$ is a multiple edge, then $v_1 v_3$ is not an edge of S (because S contains no 1-braid or 2-braid) and S is contained in some member of $\mathcal{B}_{10}$. So, from now on, we assume that $\hat{d}_S(w_1) = 2$ and $w_1 w_2$ be simple.

Suppose that v_2 and v_4 are adjacent. If v_3 is adjacent to some $v \in V(S) \backslash V(C)$, then v is a pendant vertex and $v_3 v$ is simple (because S contains no $P_5 \cup P_3$ or $P_5 \cup C_2$) and v_1 is not adjacent to v_3 (because S contains no bipartite claw), so S is contained in some member of $\mathcal{B}_{11}$. Otherwise, S is contained in some member of $\mathcal{B}_{12}$. So, from now, we assume that v_2 and v_4 are nonadjacent.

If v_3 also has some non-pendant neighbor $w_3 \in V(S) \backslash V(C)$, then, reasoning with w_3 as we did with w_1, we prove that each $v \in N_S(v_3) \backslash V(C)$ different from w_3 is pendant and we can assume that $\hat{d}_S(w_3) = 2$ and, if w_4 is the only vertex of $N_S(w_3) \backslash \{v_3\}$, then $w_3 w_4$ is simple. Thus, S is contained in some member of $\mathcal{B}_{16}$, even if v_3 has no non-pendant neighbor.

3h *None of the previous subcases holds.* Since $\ell = 4$, there exists some 4-cycle $C = v_1 v_2 v_3 v_4 v_1$ in S. Since 3g does not hold, each $v \in N_S(v_i) \backslash V(C)$ is pendant, for each $i = 1, 2, 3, 4$. Since 3f does not hold, there are at most two vertices of C that are adjacent to pendant vertices. If there are less than two vertices of $V(C)$ adjacent to pendant vertices, S is contained in some member of $\mathcal{B}_{13}$. Therefore, we assume that there are two vertices of $V(C)$ adjacent to pendant vertices, say v_1 and v_j, where $j = 2$ or $j = 3$.

If each of the vertices v_1 and v_j is adjacent to two pendant vertices or joined to some pendant vertex through a multiple edge, then $j = 3$ and v_1 is nonadjacent to v_3 (because S contains no braid, 1-braid, or 2-braid). We conclude that S is contained in some member of $\mathcal{B}_{16}$.

Finally, if v_j is adjacent to only one pendant vertex through a simple edge, then S is contained in some member of $\mathcal{B}_{13}$.

Case 4. *S has a longest cycle* C *of length* $\ell = 6$*,*

Let $C = v_1v_2v_3v_4v_5v_6v_1$. Since S is connected and contains no 6-pan, the vertices of C are the only vertices of S. As S contains no 5-cycle, C has no short chords.

Suppose first that C has two multiple chords, say v_1v_4 and v_2v_5 are multiple edges. Since S contains no 2-braid, there is no edge v_3v_6 in S and each of v_2v_3, v_3v_4, v_5v_6, and v_6v_1 is simple. This means that S is a member of $\mathcal{B}_7$. So, from now on, we can assume that C has at most one multiple chord.

Since C has at most one multiple chord, S would belong to $\mathcal{B}_8$ if no edge of C were multiple. Therefore, from now on, we assume that v_1v_2 is multiple. As S contains no 2-braid, none of v_3v_4 and v_5v_6 is multiple and at most one of v_1v_6, v_2v_3, v_5v_6 is multiple. In its turn, this means that, if C has no multiple chords, then S is a member of $\mathcal{B}_7$ or $\mathcal{B}_8$. So, from now on, let C have exactly one multiple chord.

Since S contains no 2-braid, if v_3v_6 were the only multiple chord of S, then v_4v_5 would not be multiple, v_1 would be nonadjacent to v_4, v_2 would be nonadjacent to v_5, and, as a result, S is a member of $\mathcal{B}_6$, By symmetry, we assume that the only chord of S is v_1v_4. Recall that the only possible multiple edges of C are v_1v_6, v_2v_3, and v_4v_5 and that at most one of them is multiple. If v_1v_6 is multiple, then S is a member of $\mathcal{B}_8$. If v_4v_5 is multiple, then there is no edge v_3v_6 in S (because S contains no 2-braid) and, consequently, S is a member of $\mathcal{B}_7$. If v_2v_3 is multiple, then v_2v_5 and v_3v_6 are not edges of S (because S contains no 2-braid) and, consequently, S is a member of $\mathcal{B}_6$. Finally, if none of v_4v_5, v_1v_6, and v_2v_3 is multiple, then S is a member of $\mathcal{B}_8$.

In each of the Cases 1 to 4 above, we proved that the component S of H is contained in some member of $\mathcal{B}_1, \mathcal{B}_2, \ldots$, or $\mathcal{B}_{16}$. Consequently, case (b) of assertion (v) holds, which completes the proof. □

3.6.3 Recognizing balanced complements of line graphs of multigraphs

We will derive, from the above theorem, the existence of a linear-time recognition algorithm for balanced graphs within complements of line graphs of multigraphs.

Given a graph G, we define a *pruned graph* of G as any maximal induced subgraph of G having no three pairwise false twins and no universal vertices. Let $V_1, V_2, \ldots, V_q$ be the equivalent classes of the relation "is a false twin of" on the set of vertices of G. We say that the equivalent class V_i is *universal* if some vertex of V_i is a universal vertex of G. Clearly, if V_i is universal, then $|V_i| = 1$. The pruned graphs of G are those subgraphs of G induced by some set $V_1' \cup V_2' \cup \ldots \cup V_q'$ such that $V_i' \subseteq V_i$ and $|V_i'| = \beta_i$, for each $i = 1, 2, \ldots, q$, where

$$\beta_i = \begin{cases} \min(|V_i|, 2) & \text{if } V_i \text{ is not universal} \\ 0 & \text{otherwise.} \end{cases}$$

Since any two vertices that belong the same V_i are nonadjacent and have the same neighbors, the pruned graphs of G are unique up to isomorphisms and we denote any of them by $\mathcal{P}(G)$.

Lemma 3.30. *A pruned subgraph of a graph* G *can be computed in linear time.*

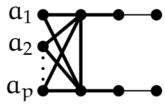
Proof. In order to compute $\mathcal{P}(G)$, we first construct the modular decomposition tree $T(G)$ of G. Then, two vertices u and v of G are false twins if and only if the leaves of the modular decomposition tree representing them are children of the same parallel node. This means that we can find a subset of vertices inducing a pruned graph of G by marking for exclusion all universal vertices of G and by performing a breadth-first search on the modular decomposition tree of G in order to mark for exclusion also the third, fourth, fifth, and so on, leaf children of each parallel node. Since the modular decomposition tree can be computed in linear time, $\mathcal{P}(G)$ can also be computed in linear time. □

The following fact about $\mathcal{P}(G)$ is crucial for our purposes.

Corollary 3.31. *Let* G *be the complement of the line graph of a multigraph. Then,* G *is balanced if and only if* $\mathcal{P}(G)$ *is balanced.*

Proof. If G is balanced, then clearly $\mathcal{P}(G)$ is also balanced (because $\mathcal{P}(G)$ is an induced subgraph of G). In order to prove the converse, we assume that G is not balanced and we will prove that $\mathcal{P}(G)$ is not balanced. Let W be a subset of vertices inducing a minimal induced subgraph of G that is not balanced. By Theorem 3.29, the subgraph of G induced by W is isomorphic to 3-sun, 2-pyramid, 3-pyramid, C_5, C_7, U_7, or V_7. In particular, there are no three pairwise false twins of G in W and there is no universal vertex of G in W. Therefore, if the equivalent classes $V_1, V_2, \ldots, V_q$ and β_i are as defined earlier and $W_i = W \cap V_i$, then $|W_i| \leqslant \beta_i$ for each $i = 1, 2, \ldots, q$. So, it is possible to find $V'_1, V'_2, \ldots, V'_q$ such that $W_i \subseteq V'_i \subseteq V_i$ and $|V'_i| = \beta_i$ for each $i = 1, 2, \ldots, q$. Then, $G' = G[V'_1 \cup V'_2 \cup \cdots \cup V'_q]$ is a pruned graph of G and G' is not balanced because $W \subseteq V(G')$ and $G'[W] = G[W]$ is not balanced. □

Let G be the complement of the line graph of a multigraph and let k be a fixed integer. According to Corollary 3.31, if $\mathcal{P}(G)$ has at most k vertices, we can decide whether G is balanced in linear time by computing $\mathcal{P}(G)$ in linear time and then deciding whether $\mathcal{P}(G)$ is balanced in constant time. (Indeed, the obvious $O(n^7)$-time recognition algorithm for balancedness among complements of line graphs of multigraphs that follows from assertion (iii) of Theorem 3.29 becomes constant-time when $n = O(1)$.) In what follows, we will fix $k = 40$ and the remainder of this subsection is devoted to proving that we can decide in linear time whether $\mathcal{P}(G)$ is balanced even if $\mathcal{P}(G)$ has more than 40 vertices.



$\mathcal{B}'_{16}$

Figure 3.3: *Multigraph family* $\mathcal{B}'_{16}$. *Light lines represent single edges, whereas bold lines represent one or more parallel edges. Parameter* p *varies over the positive integers, and* $a_1, a_2, \ldots, a_p$ *are pairwise false twins*

We denote by $L^{-1}(G)$ any multigraph H without isolated vertices such that $L(H) = G$ and whose underlying graph $\widehat{H}$ satisfies $L(\widehat{H}) = \mathcal{R}(G)$, where $\mathcal{R}(G)$ is the representative graph of G as defined in Section 3.5. Given a graph G, a multigraph $L^{-1}(G)$ can be computed in linear time of G (see [78, p. 67–68]). We say that two incident edges e_1 and e_2 of a multigraph H are *twins* if they are incident to the same edges of $E(H)$. We say that a multigraph H is *reduced* if each pair of twin edges are parallel. By construction, $H = L^{-1}(G)$ is reduced. In Figure 3.3 we introduce the multigraph family $\mathcal{B}'_{16}$.

Corollary 3.32. *Let* G *be the complement of the line graph of a multigraph and suppose that* $\mathcal{P}(G)$ *has more than* 40 *vertices. If* $H = L^{-1}(\overline{\mathcal{P}(G)})$, *then the following conditions are equivalent:*

(i) G *is balanced.*

(ii) H *is a connected submultigraph of some member of* $\mathcal{B}_{15}$ *or* $\mathcal{B}'_{16}$.

(iii) H *is connected, has exactly two vertices* v_1 *and* v_2 *that are incident to at least six edges each, and, for each* $i = 1, 2$, *there is at most one vertex* w_i *that is adjacent to* v_i *and such that there is some* $x_i \in N_H(w_i) \setminus \{v_1, v_2\}$ *and, if so, each of the following holds:* $N_H(w_i) \subseteq \{x_i, v_1, v_2\}$, *there is exactly one edge* e_i *joining* w_i *to* x_i, *and* e_i *is the only edge incident to* x_i. *(It is possible that* $w_1 = w_2$.*)*

Proof. Suppose that G is balanced and let $H = L^{-1}(\overline{\mathcal{P}(G)})$. As H has no isolated vertices and $\mathcal{P}(G)$ has no universal vertices, each component of H has at least two edges. Since G is balanced, $\mathcal{P}(G)$ is balanced; i.e., H is $\overline{L}$-balanced. So, by Theorem 3.29, either H is a connected submultigraph of some member of $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{16}$ or H has two components, each of which is contained in a member of $\mathcal{A}_1, \mathcal{A}_2$, or $\mathcal{A}_3$. But, as $\mathcal{P}(G)$ has more than 40 vertices, H has more than 40 edges. Since, by construction, $\overline{\mathcal{P}(G)}$ has no three pairwise true twins, H has no three pairwise parallel edges. Since, in addition, H is reduced, H is necessarily a connected submultigraph of $\mathcal{B}_{15}$ or $\mathcal{B}'_{16}$. Conversely, if H is a submultigraph of some member of $\mathcal{B}_{15}$ or $\mathcal{B}'_{16}$, then $\mathcal{P}(G)$ is balanced by Theorem 3.29 and, then, G is also balanced by Corollary 3.31. This concludes the proof of the equivalence between (i) and (ii).

Since clearly (iii) implies (ii), it only remains to show that (ii) implies (iii). So, assume that H is is a connected submultigraph of some member of $\mathcal{B}_{15}$ or $\mathcal{B}'_{16}$. Since $H = L^{-1}(\overline{\mathcal{P}(G)})$, H has no three pairwise parallel edges. Therefore, H has at most two vertices incident to at least six edges. Moreover, since H has at least 40 edges, H has exactly two vertices incident to at least six edges each, and (iii) clearly holds. □

The next result implies that if $\mathcal{P}(G)$ has more than 40 vertices, then we can either detect that G is not balanced or compute $L^{-1}(\overline{\mathcal{P}(G)})$ efficiently.

Corollary 3.33. *Let* G *be the complement of the line graph of a multigraph. Let* $n_{\mathcal{P}}$ *and* $m_{\mathcal{P}}$ *be the number of vertices and edges of* $\mathcal{P}(G)$ *and suppose that* $n_{\mathcal{P}} > 40$. *If*

$$m_{\mathcal{P}} \geqslant \frac{2}{9}(n_{\mathcal{P}} - 3)(n_{\mathcal{P}} - 36) \tag{3.1}$$

does not hold, then G *is not balanced. On the other hand, if* (3.1) *holds, then* $H = L^{-1}(\overline{\mathcal{P}(G)})$ *can be computed from* G *in linear time.*

Proof. Suppose first that G is balanced and let $H = L^{-1}(\overline{\mathcal{P}(G)})$. Then, H has $n_{\mathcal{P}}$ edges and satisfies condition (iii) of Corollary 3.32. Let A be the set of vertices a of H such that $N_H(a) \subseteq \{v_1, v_2\}$. Since $\overline{\mathcal{P}(G)}$ has no three pairwise true twins, H has no three pairwise parallel edges. Moreover, as H is reduced, there are at most two edges joining v_i to pendant vertices in A, for each $i = 1, 2$. Let E_i be the set of edges joining v_i to non-pendant vertices in A, for each $i = 1, 2$. Since H is a submultigraph of a member of $\mathcal{B}_{15}$ or $\mathcal{B}'_{16}$ and H is reduced, $|E_1| + |E_2| \geqslant n_{\mathcal{P}} - 12$. Without loss of generality, assume that $|E_1| \geqslant |E_2|$. Then, $\frac{1}{2}(n_{\mathcal{P}} - 12) \leqslant |E_1| \leqslant \frac{2}{3}n_{\mathcal{P}}$ because each non-pendant vertex of A is joined to v_1 by at most two edges and joined to v_2 by at least one edge. So, since each edge of E_2 is incident to at most two edges of E_1 and $\mathcal{P}(G) = \overline{L(H)}$,

$$m_{\mathcal{P}} \geqslant |E_2|(|E_1| - 2) \geqslant (n_{\mathcal{P}} - 12 - |E_1|)(|E_1| - 2) \geqslant \frac{2}{9}(n_{\mathcal{P}} - 3)(n_{\mathcal{P}} - 36).$$

This proves that if (3.1) does not hold, then G is not balanced.

Suppose now that (3.1) holds. We have seen that $\mathcal{P}(G)$ can be computed in $O(m+n)$ time, where n and m are the number of vertices and edges of G. The complement of $\mathcal{P}(G)$ can obviously be computed in $O(n_{\mathcal{P}}^2)$ time. In addition, $H = L^{-1}(\overline{\mathcal{P}(G)})$ can be computed from $\overline{\mathcal{P}(G)}$ in linear time of $\overline{\mathcal{P}(G)}$, which is again $O(n_{\mathcal{P}}^2)$. Notice that since $m_{\mathcal{P}} \leqslant m$ and we are assuming that (3.1) holds, $O(n_{\mathcal{P}}^2)$ is $O(m)$. We conclude that H can be computed from G in $O(m + n)$ time, as desired. □

Let G be the complement of the line graph of a multigraph. We know that if $\mathcal{P}(G)$ has at most 40 vertices, we can decide whether G is balanced in linear time. So, suppose that $\mathcal{P}(G)$ has more than 40 vertices and let $n_{\mathcal{P}}$ and $m_{\mathcal{P}}$ be the number of vertices and edges of $\mathcal{P}(G)$. If (3.1) does not hold, we know that G is not balanced. Otherwise, we can decide whether G is balanced in linear time by first computing $H = L^{-1}(\overline{\mathcal{P}(G)})$ and then checking the validity of condition (iii) of Corollary 3.32. As a conclusion, we have the following.

Corollary 3.34. *Given a graph G that is the complement of the line graph of a multigraph, it can be decided whether or not G is balanced in linear time.*

3.6.4 Lemmas for the proof of Theorem 3.28

This subsection is devoted to prove that each of the multigraph families $\mathcal{B}_1$, $\mathcal{B}_5$, $\mathcal{B}_6$, $\mathcal{B}_7$, $\mathcal{B}_8$, $\mathcal{B}_{13}$, $\mathcal{B}_{14}$, and $\mathcal{B}_{15}$ is $\overline{L}$-balanced.

A *bicoloring* of a $\{0,1\}$-matrix is a partition of its columns into red and blue columns such that every row with two or more 1's contains at least a 1 in a red column and at least a 1 in a blue column. Clearly, the edge-vertex incidence matrix of an odd cycle cannot be bicolored. Interestingly, a $\{0,1\}$-matrix is balanced if and only if each of its submatrices is bicolorable [8]. Let A be a submatrix of the matching-matrix of a multigraph H and let $\mathcal{M}$ and $\mathcal{E}$ be the sets of maximal matchings and edges of H corresponding to the rows and columns of the submatrix A, respectively. In this context, we say that a partition $\{\mathcal{E}_1, \mathcal{E}_2\}$ of $\mathcal{E}$ is a *bicoloring* of A if for each $M \in \mathcal{M}$ either $|M \cap \mathcal{E}| \leqslant 1$ or M intersects both $\mathcal{E}_1$ and $\mathcal{E}_2$.

We will make repeated use the following lemma.

Lemma 3.35. *Let H be a multigraph that is not $\overline{L}$-balanced. Then, a matching-matrix of H has some submatrix A which is an edge-vertex incidence matrix of an odd chordless cycle and let $\mathcal{E}$ be the set of edges of H corresponding to the columns of A. If X is a set of pairwise incident edges of H, there must be some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \varnothing$.*

Proof. Let $\mathcal{M}$ be the set of maximal matchings of H corresponding to the rows of the submatrix A. Since A is an edge-vertex incidence matrix of an odd chordless cycle, $|M \cap \mathcal{E}| = 2$ for each $M \in \mathcal{M}$. Since X consists of pairwise incident edges of X, $|M \cap \mathcal{E} \cap X| \leqslant 1$ for every matching M of X. As $|M \cap \mathcal{E}| = 2$ for each $M \in \mathcal{M}$, it follows that $|(M \cap \mathcal{E})\backslash X)| \geqslant 1$ for each $M \in \mathcal{M}$. Since A is not bicolorable, $\{X \cap \mathcal{E}, \mathcal{E}\backslash X\}$ is not a bicoloring of A and, necessarily, there is some $M \in \mathcal{M}$ such that $|M \cap X \cap \mathcal{E}| = 0$. $\square$

If u, v, w are three pairwise adjacent vertices of H of a multigraph, we denote by $T_H(u, v, w)$ the set of all edges of H joining any two of the vertices u, v, and w. Recall that $E_H(v)$ denote the set of edges of H incident to v.

Lemma 3.36. *The family $\mathcal{B}_1$ is $\overline{L}$-balanced.*

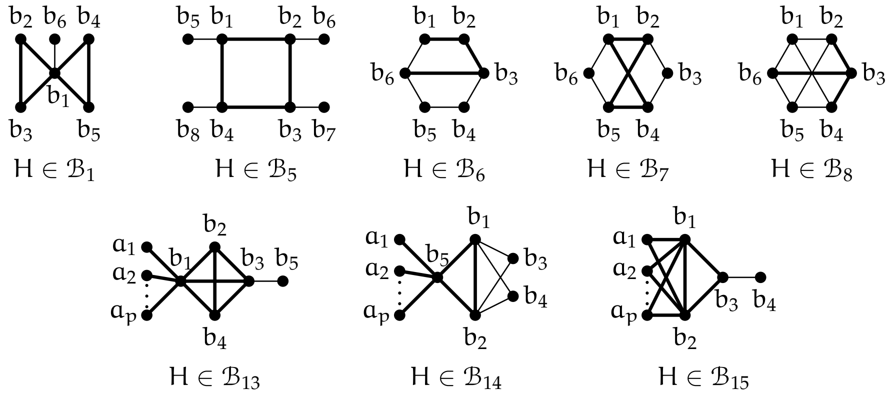


Figure 3.4: *Vertex labeling of the multigraph H for the proofs of Lemmas 3.36 to 3.43. Light lines represent single edges, whereas bold lines represent one or more parallel edges. Parameter p varies over the positive integers, and $a_1, a_2, \dots, a_p$ are pairwise false twins*

Proof. By the way of contradiction, consider a not $\overline{L}$-balanced multigraph $H \in \mathcal{B}_1$. Label its vertices as in Figure 3.4 and let A and $\mathcal{E}$ be as in Lemma 3.35.

We claim that $b_1b_6 \in \mathcal{E}$. By Lemma 3.35 applied to $X = T_H(b_1, b_2, b_3)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \emptyset$. Necessarily, $b_1b_6 \in M \cap \mathcal{E}$ and, in particular, $b_1b_6 \in \mathcal{E}$, as claimed.

By Lemma 3.35 applied to $X = E_H(b_1)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \emptyset$. Necessarily, $M \cap \mathcal{E}$ consists of one edge joining b_2 to b_3 and one edge joining b_4 to b_5 and, by the maximality of M, $b_1b_6 \in M$. So, as we proved that $b_1b_6 \in \mathcal{E}$, we conclude that $b_1b_6 \in M \cap \mathcal{E} \cap X$, which contradicts $M \cap \mathcal{E} \cap X = \emptyset$. Hence, any member of $\mathcal{B}_1$ is $\overline{L}$-balanced. □

Lemma 3.37. *The family $\mathcal{B}_5$ is $\overline{L}$-balanced.*

Proof. By the way of contradiction, consider a not $\overline{L}$-balanced multigraph $H \in \mathcal{B}_5$. Label its vertices as in Figure 3.4 and let A and $\mathcal{E}$ be as in Lemma 3.35.

By Lemma 3.35 applied to $X = E_H(b_2)$, there is a maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \emptyset$. So, necessarily, $M \cap \mathcal{E}$ contains at least one of b_1b_5 and b_3b_7. Symmetrically, $M \cap \mathcal{E}$ contains at least one of b_2b_6 and b_4b_8. So, we assume, without loss of generality, that $b_1b_5, b_2b_6 \in \mathcal{E}$.

We claim that from $b_1b_5, b_2b_6 \in \mathcal{E}$ it follows that $b_3b_7, b_4b_8 \in \mathcal{E}$. By Lemma 3.35 applied to $X = E_H(b_1)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \emptyset$. As $b_1b_5 \in \mathcal{E} \cap X$, it follows that $b_1b_5 \notin M$. Thus, by the maximality of M, M contains an edge joining b_1 to either b_2 or b_4. Then, as $M \cap \mathcal{E}$ consists of two non-incident edges and is disjoint from $E_H(b_1)$, necessarily $b_3b_7 \in M \cap \mathcal{E}$ and, in particular, $b_3b_7 \in \mathcal{E}$. Symmetrically, $b_4b_8 \in \mathcal{E}$.

Let $R = (E_H(b_2) \cup \{b_4b_8\}) \cap \mathcal{E}$ and $B = \mathcal{E} \setminus R$. Then, $\{R, B\}$ is a partition of $\mathcal{E}$ and we claim that $\{R, B\}$ is bicoloring of A. Let $\mathcal{M}$ the set of maximal matchings of H corresponding to the rows of A and let $M \in \mathcal{M}$. As A is an edge-vertex incidence matrix of an odd chordless cycle, $|M \cap \mathcal{E}| = 2$. Suppose, by the way of contradiction, that $M \cap R = \varnothing$. This meas that $M \cap \mathcal{E}$ is disjoint from $E_H(b_2) \cup \{b_4b_8\}$. So, since $|M \cap \mathcal{E}| = 2$, $M \cap \mathcal{E}$ consists of one edge incident to b_1 and one edge incident to b_3 but none of them incident to b_2 and, by the maximality of M, $b_2b_6 \in M$. Consequently, $b_2b_6 \in M \cap R$, a contradiction. This contradiction arose from assuming that $M \cap R = \varnothing$.

Suppose now that $M \cap B = \varnothing$. This means that $M \cap \mathcal{E}$ consists of two edges contained in $E_H(b_2) \cup \{b_4b_8\}$. Since M is a matching, $M \cap \mathcal{E}$ consists of b_4b_8 and one edge incident to b_2. Then, the maximality of M implies that $M \cap \{b_1b_5, b_3b_7\} \neq \varnothing$ and, consequently, $(M \cap B) \cap \{b_1b_5, b_3b_7\} \neq \varnothing$, a contradiction. This contradiction arose from assuming that $M \cap B = \varnothing$.

So, we have proved that for each $M \in \mathcal{M}$, $M \cap R \neq \varnothing$ and $M \cap B \neq \varnothing$, which proves that $\{R, B\}$ is a bicoloring of A, contradicting the choice of A. Hence, any member of $\mathcal{B}_5$ is $\overline{L}$-balanced. $\square$

Lemma 3.38. *The family $\mathcal{B}_6$ is $\overline{L}$-balanced.*

Proof. By the way of contradiction, consider a not $\overline{L}$-balanced multigraph $H \in \mathcal{B}_6$. Label its vertices as in Figure 3.4 and let A and $\mathcal{E}$ be as in Lemma 3.35.

We claim that $b_4b_5 \in \mathcal{E}$. Suppose, by the way of contradiction, that $b_4b_5 \notin \mathcal{E}$. By Lemma 3.35 applied to $X = E_H(b_3)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \varnothing$. Necessarily, $M \cap \mathcal{E}$ consists of the edge b_5b_6 and an edge joining b_1 to b_2. In particular, $b_5b_6 \in \mathcal{E}$. Similarly, by Lemma 3.35 applied to $X = E_H(b_6)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \varnothing$. Necessarily, $M \cap \mathcal{E}$ consists of b_3b_4 and an edge joining b_1 to b_2. Hence, the maximality of M implies that $b_5b_6 \in M$ and, since $b_5b_6 \in \mathcal{E}$, it follows that $b_5b_6 \in M \cap \mathcal{E} \cap X$, contradicting $M \cap \mathcal{E} \cap X = \varnothing$. This contradiction arose from assuming that $b_4b_5 \notin \mathcal{E}$ and completes the proof of the claim.

Moreover, we claim that no edge joining b_3 to b_6 belongs to $\mathcal{E}$. Suppose, by the way of contradiction there is some edge $e \in \mathcal{E}$ joining b_3 to b_6. Let $\mathcal{M}$ be the set of maximal matchings of H corresponding to the rows of A. As A is an edge-vertex incidence matrix of an odd chordless cycle, there are two different maximal matchings $M, M' \in \mathcal{M}$ such that $|M \cap \mathcal{E}| = |M \cap \mathcal{E}| = 2$ and $e \in M, M'$. Since every maximal matching of H containing e also contains b_4b_5, we conclude that $M \cap \mathcal{E} = M' \cap \mathcal{E} = \{e, b_4b_5\}$. This means that rows and columns of A corresponding to M, M' and e, b_4b_5 determine a 2×2 submatrix of A full of 1's, which contradicts the choice of A. This contradiction arose from assuming that $e \in \mathcal{E}$ and completes the proof of the claim.

We also claim that $b_5b_6 \in \mathcal{E}$. Suppose, by the way of contradiction that $b_5b_6 \notin \mathcal{E}$, by Lemma 3.35 applied to $X = E_H(b_4)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \emptyset$. As $b_5b_6 \notin \mathcal{E}$, $M \cap \mathcal{E}$ consists of b_1b_6 and an edge joining b_2 to b_3 and the maximality of M implies that $b_4b_5 \in M$. Thus, since $b_4b_5 \in \mathcal{E}$, it follows that $b_4b_5 \in M \cap \mathcal{E} \cap X$, which contradicts $M \cap \mathcal{E} \cap X = \emptyset$. This contradiction arose from assuming that $b_5b_6 \notin \mathcal{E}$. This concludes the proof of the claim.

We further claim that $b_3b_4, b_1b_6 \in \mathcal{E}$. By Lemma 3.35 applied to $X = E_H(b_5)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \emptyset$. Reasoning as in the preceding paragraph, $b_3b_4 \in M \cap \mathcal{E}$; otherwise, the maximality of M would imply that $b_4b_5 \in M$ and, since $b_4b_5 \in \mathcal{E}$, it would follow that $b_4b_5 \in M \cap \mathcal{E} \cap X$, a contradiction. Suppose, by the way of contradiction, that $b_1b_6 \notin M \cap \mathcal{E}$. Then, $M \cap \mathcal{E}$ consists of b_3b_4 and an edge joining b_1b_2 and, by maximality of M, $b_5b_6 \in M$. But then, since $b_5b_6 \in \mathcal{E}$, it follows that $b_5b_6 \in M \cap \mathcal{E} \cap X$, a contradiction. This contradiction arose from assuming that $b_1b_6 \notin M \cap \mathcal{E}$. As we proved that $b_3b_4, b_1b_6 \in M \cap \mathcal{E}$, in particular $b_3b_4, b_1b_6 \in \mathcal{E}$, as claimed.

Let $R = (E_H(b_3) \cup \{b_4b_5\}) \cap \mathcal{E}$ and $B = \mathcal{E} \backslash R$. Then, $\{R, B\}$ is a partition of $\mathcal{E}$ and we claim that $\{R, B\}$ is a bicoloring of A. Recall that $\mathcal{M}$ is the set of maximal matchings of H corresponding to the rows of A and let $M \in \mathcal{M}$. Since A is an edge-vertex incidence matrix of an odd chordless cycle, $|M \cap \mathcal{E}| = 2$. Suppose, by the way of contradiction, that $M \cap R = \emptyset$. This means that $M \cap \mathcal{E}$ is disjoint from $E_H(b_3) \cup \{b_4b_5\}$. So, since $|M \cap \mathcal{E}| = 2$, necessarily $M \cap \mathcal{E}$ consists of b_5b_6 plus an edge joining b_1 to b_2 and, by the maximality of M, $b_3b_4 \in M$, which implies $b_3b_4 \in M \cap R$, a contradiction. This contradiction arose from assuming that $M \cap R = \emptyset$.

Suppose now that $M \cap B = \emptyset$. This means that $M \cap \mathcal{E}$ consists of two edges that belong to $E_H(b_3) \cup \{b_4b_5\}$. Since there is no edge in $\mathcal{E}$ joining b_3 to b_6, $M \cap \mathcal{E}$ consists of b_4b_5 and an edge joining b_2 to b_3. Then, the maximality of M implies that $b_1b_6 \in M$ and, as $b_1b_6 \in \mathcal{E}$, $b_1b_6 \in M \cap B$, a contradiction. This contradiction arose from assuming that $M \cap B = \emptyset$.

So, we have proved that $\{R, B\}$ is a partition of $\mathcal{E}$ such that, for each $M \in \mathcal{M}$, $M \cap R \neq \emptyset$ and $M \cap B \neq \emptyset$; i.e., $\{R, B\}$ is a bicoloring of A, which contradicts the choice of A. Hence, any member of $\mathcal{B}_6$ is $\overline{L}$-balanced. □

Lemma 3.39. *The family $\mathcal{B}_7$ is $\overline{L}$-balanced.*

Proof. By the way of contradiction, consider a not $\overline{L}$-balanced multigraph $H \in \mathcal{B}_7$. Label its vertices as in Figure 3.4 and let A and $\mathcal{E}$ be as in Lemma 3.35.

By Lemma 3.35 applied to $X = E_H(b_2)$, there is some maximal matching M such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \emptyset$. Since $M \cap \mathcal{E}$ is a matching of size 2 disjoint from X, necessarily at least one of b_1b_6, b_3b_4, or b_5b_6 belongs to $M \cap \mathcal{E}$ and, in particular, to

$\mathcal{E}$. By symmetry, we assume, without loss of generality, that $b_1b_6 \in \mathcal{E}$.

We now show that our assumption that $b_1b_6 \in \mathcal{E}$ implies that $b_5b_6 \in \mathcal{E}$ and, moreover, that $b_2b_3 \in \mathcal{E}$ or $b_3b_4 \in \mathcal{E}$. By Lemma 3.35 applied to $X = E_H(b_1)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \varnothing$. Suppose, by the way of contradiction, that $b_5b_6 \notin M \cap \mathcal{E}$. Then, $M \cap \mathcal{E}$ consists either of b_2b_3 and one edge joining b_4 to b_5, or of b_3b_4 and one edge joining b_2 to b_5. In either case, the maximality of M implies that $b_1b_6 \in M$ and, since we are assuming that $b_1b_6 \in \mathcal{E}$, it follows that $b_1b_6 \in M \cap \mathcal{E} \cap X$, which contradicts $M \cap \mathcal{E} \cap X = \varnothing$. This contradiction arose from assuming that $b_5b_6 \notin \mathcal{E}$. As $M \cap \mathcal{E}$ is a matching of size 2, disjoint from X, and containing b_5b_6, necessarily $b_2b_3 \in M \cap \mathcal{E}$ or $b_3b_4 \in \mathcal{M} \cap \mathcal{E}$. This completes the proof of the claim. As we are assuming that b_1b_6, we assume further, without loss of generality, that $b_2b_3, b_5b_6 \in \mathcal{E}$.

Reasoning as in the previous paragraph, from the assumption that $b_2b_3 \in \mathcal{E}$ we can derive that $b_3b_4 \in \mathcal{E}$. We conclude that $\mathcal{E}_1 = \{b_1b_6, b_2b_3, b_3b_4, b_5b_6\}$ is contained in $\mathcal{E}$. Let $R = (E_H(b_2) \cup \{b_5b_6\}) \cap \mathcal{E}$, and $B = \mathcal{E} \backslash R$. We claim that $\{R, B\}$ is bicoloring of A. Let M be a maximal matching of H corresponding to a row of A. By construction, $|M \cap \mathcal{E}| = 2$. If $|M \cap \mathcal{E}_1| = 2$, necessarily M has an edge in R and an edge in B. Notice that if $|M \cap \mathcal{E}_1| \neq 2$, necessarily $M \cap \mathcal{E}_1 = \varnothing$ and, since $M \cap \mathcal{E}$ is a matching of size 2, M also has one edge in R and one edge in B. This shows that $\{R, B\}$ is a bicoloring of A, which contradicts the choice of A. Hence, any member of $\mathcal{B}_7$ is $\overline{L}$-balanced. □

Lemma 3.40. *The family $\mathcal{B}_8$ is $\overline{L}$-balanced.*

Proof. By the way of contradiction, consider a not $\overline{L}$-balanced multigraph $H \in \mathcal{B}_8$. Label its vertices as in Figure 3.4 and let A and $\mathcal{E}$ be as in Lemma 3.35.

By Lemma 3.35 applied to $X = E_H(b_3)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \varnothing$. By symmetry, we assume, without loss of generality, that $M \cap \mathcal{E} = \{b_1b_2, b_4b_5\}$ and, in particular, $b_1b_2, b_4b_5 \in \mathcal{E}$.

We claim that no edge joining b_3 to b_6 belongs to $\mathcal{E}$. Suppose, by the way of contradiction, that there is some edge $e \in \mathcal{E}$ joining b_3 to b_6. Since A is an edge-vertex incidence matrix of an odd chordless cycle, there are two different maximal matchings M and M' of H such that $e \in M, M'$ and $|M \cap \mathcal{E}| = |M' \cap \mathcal{E}| = 2$. But $\{e, b_1b_2, b_4b_5\}$ and $\{e, b_1b_4, b_2b_5\}$ are the only maximal matchings of H containing e and $|\{e, b_1b_2, b_4b_5\} \cap \mathcal{E}| = 3$, a contradiction. This contradiction proves the claim.

We now show that our assumption that $b_1b_2, b_4b_5 \in \mathcal{E}$ implies that $b_1b_6 \in \mathcal{E}$ or $b_5b_6 \in \mathcal{E}$. By Lemma 3.35 applied to $X = E_H(b_2)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \varnothing$. Necessarily, $M \cap \mathcal{E}$ consists of one edge incident to b_4 and one edge incident to b_6 and, since no edge joining b_3 to b_6 belongs to $\mathcal{E}$, it follows that $M \cap \mathcal{E}$ contains b_1b_6 or b_5b_6. In particular, $b_1b_6 \in \mathcal{E}$ or $b_5b_6 \in \mathcal{E}$.

By symmetry, we assume, without loss of generality that $b_1b_6 \in \mathcal{E}$.

We further claim that $b_5b_6 \in \mathcal{E}$. By Lemma 3.35 applied to $X = E_H(b_1)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ and $M \cap \mathcal{E} \cap X = \emptyset$. Suppose, by the way of contradiction, that $b_5b_6 \notin \mathcal{E}$. In particular, $b_5b_6 \notin M \cap \mathcal{E}$. As we proved that no edge joining b_3 to b_6 belongs to $\mathcal{E}$, we conclude that $M \cap \mathcal{E}$ consists either of the edge b_2b_5 and an edge joining b_3 to b_4, or of the edge b_4b_5 and an edge joining b_2 to b_3. In either case, the maximality of M implies that $b_1b_6 \in M$ and, since $b_1b_6 \in \mathcal{E}$, it follows that $b_1b_6 \in M \cap \mathcal{E} \cap X$, contradicting $M \cap \mathcal{E} \cap X = \emptyset$. This contradiction arose from assuming that $b_5b_6 \notin \mathcal{E}$. This concludes the proof of the claim.

As $b_1b_2, b_4b_5 \in \mathcal{E}$ implies that $b_1b_6, b_5b_6 \in \mathcal{E}$, by symmetry, $b_1b_2, b_5b_6 \in \mathcal{E}$ implies that $b_1b_4 \in \mathcal{E}$. Similarly, from $b_1b_6, b_4b_5 \in \mathcal{E}$ follows that $b_2b_5 \in \mathcal{E}$. We infer that $E_H(b_1) \cup E_H(b_5) \subseteq \mathcal{E}$. Let $R = E_H(b_1)$ and $B = \mathcal{E} \backslash R$. Then, $\{R, B\}$ is a partition of $\mathcal{E}$ and we claim that $\{R, B\}$ is a bicoloring of A. Indeed, given any maximal matching M of H, it contains one edge incident to b_1, one edge incident to b_3, and one edge incident to b_5. As $E_H(b_1) = R$ and $E_H(b_5) \subseteq B$, M contains one edge from R and at least one edge from B. This proves that $\{R, B\}$ is a bicoloring of A, contradicting the choice of A. Hence, any member of $\mathcal{B}_8$ is $\overline{L}$-balanced. □

Lemma 3.41. *The family $\mathcal{B}_{13}$ is $\overline{L}$-balanced.*

Proof. By the way of contradiction, consider a not $\overline{L}$-balanced multigraph $H \in \mathcal{B}_{13}$. Label its vertices as in Figure 3.4 and let A and $\mathcal{E}$ be as in Lemma 3.35.

We claim that $b_3b_4 \in \mathcal{E}$. By Lemma 3.35 applied to $X = E_H(b_1)$, there is some maximal matching of M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \emptyset$. Necessarily $b_3b_4 \in M \cap \mathcal{E}$ and, in particular, $b_3b_4 \in \mathcal{E}$.

By Lemma 3.35 applied to $X = E_H(b_3)$, there is some maximal matching of M such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \emptyset$. Necessarily, $b_2b_4 \in M \cap \mathcal{E}$ and, by the maximality of M, $b_3b_4 \in M$. Hence, as $b_3b_4 \in \mathcal{E}$, we conclude that $b_3b_4 \in M \cap \mathcal{E} \cap X$, which contradicts $M \cap \mathcal{E} \cap X = \emptyset$. Hence, any member of $\mathcal{B}_{13}$ is $\overline{L}$-balanced. □

Lemma 3.42. *The family $\mathcal{B}_{14}$ is $\overline{L}$-balanced.*

Proof. By the way of contradiction, consider a not $\overline{L}$-balanced multigraph $H \in \mathcal{B}_{14}$. Label its vertices as in Figure 3.4 and let A and $\mathcal{E}$ be as in Lemma 3.35.

By Lemma 3.35 applied to $X = E_H(b_1)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \emptyset$. Necessarily, $b_2b_3 \in M \cap \mathcal{E}$ or $b_2b_4 \in M \cap \mathcal{E}$. By symmetry, we assume, without loss of generality, that $b_2b_3 \in M \cap \mathcal{E}$. Then, the maximality of M implies that $b_1b_4 \in M$ and, necessarily $b_1b_4 \notin \mathcal{E}$ (otherwise, b_1b_4 would belong to $M \cap \mathcal{E} \cap X$).

By Lemma 3.35 applied to $X = E_H(b_3)$, there is some maximal matching M such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \emptyset$. Since $b_1b_4 \notin \mathcal{E}$, necessarily $b_1b_3 \in M \cap \mathcal{E}$. Then, the maximality of M implies that $b_2b_4 \in M$ and, necessarily $b_2b_4 \notin \mathcal{E}$ (otherwise, b_2b_4 would belong to $M \cap \mathcal{E} \cap X$.)

By Lemma 3.35 applied to $X = T_H(b_1, b_2, b_3)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \varnothing$. Necessarily, $M \cap \mathcal{E}$ contains an edge incident to b_4 and, in particular, $b_1b_4 \in \mathcal{E}$ or $b_2b_4 \in \mathcal{E}$, which contradicts the conclusion of the preceding two paragraphs.

Hence, any member of $\mathcal{B}_{14}$ is $\overline{L}$-balanced. □

Lemma 3.43. *The family $\mathcal{B}_{15}$ is $\overline{L}$-balanced.*

Proof. By the way of contradiction, consider a not $\overline{L}$-balanced multigraph $H \in \mathcal{B}_{15}$. Label its vertices as in Figure 3.4 and let A and $\mathcal{E}$ be as in Lemma 3.35.

We claim that $b_3b_4 \in \mathcal{E}$. By Lemma 3.35 applied to $R = E_H(b_1)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \varnothing$. Necessarily $b_3b_4 \in M \cap \mathcal{E}$ and, in particular, $b_3b_4 \in \mathcal{E}$.

By Lemma 3.35 applied to $X = E_H(b_3)$, there is some maximal matching M of H such that $|M \cap \mathcal{E}| = 2$ but $M \cap \mathcal{E} \cap X = \varnothing$. Necessarily, $M \cap \mathcal{E}$ consists one edge incident to b_1 and one edge incident to b_2, but none of them incident to b_3. Hence, by the maximality of M, $b_3b_4 \in M$ and, as we proved that $b_3b_4 \in \mathcal{E}$, we conclude that $b_3b_4 \in M \cap \mathcal{E} \cap X$, which contradicts $M \cap \mathcal{E} \cap X = \varnothing$.

Hence, any member of $\mathcal{B}_{15}$ is $\overline{L}$-balanced. □

3.7 Balancedness of a superclass of Helly circular-arc graphs

In this section, we give a minimal forbidden induced subgraph characterization of balancedness for a superclass of Helly circular-arc graphs. In order to do so, we introduce the graph families below, which are schematically represented in Figure 3.5.

- For each $t \geqslant 2$ and each p even such that $2 \leqslant p \leqslant 2t$, the graph V_p^{2t+1} has vertex set $\{v_1, v_2, \dots, v_{2t+1}, u_1, u_2\}$, $v_1v_2 \dots v_{2t+1}v_1$ is a cycle whose only chord is v_1v_3, $N(u_1) = \{v_1, v_2\}$, and $N(u_2) = \{v_2, v_3, \dots, v_{p+1}\}$.

- For each $t \geqslant 2$, let D^{2t+1} be the graph with $\{v_1, v_2, \dots, v_{2t+1}, u_1, u_2, u_3\}$ as vertex set such that $v_1v_2 \dots v_{2t+1}v_1$ is a cycle whose only chords are $v_{2t+1}v_2$ and v_1v_3, $N(u_1) = \{v_{2t+1}, v_1\}$, $N(u_2) = \{v_2, v_3\}$, and $N(u_3) = \{v_1, v_2\}$.

- For each $t \geqslant 2$ and each even p with $4 \leqslant p \leqslant 2t$, let X_p^{2t+1} be the graph with vertex set $\{v_1, v_2, \dots, v_{2t+1}, u_1, u_2, u_3, u_4\}$ such that $v_1v_2 \dots v_{2t+1}v_1$ is a cycle whose only chords are $v_{2t+1}v_2$ and v_1v_3, $N(u_1) = \{v_{2t+1}, v_1\}$, $N(u_2) = \{v_2, v_3, u_4\}$, $N(u_3) = \{v_{2t+1}, v_1, v_2, u_4\}$, and $N(u_4) = \{v_1, v_2, v_3, \dots, v_p, u_2, u_3\}$.

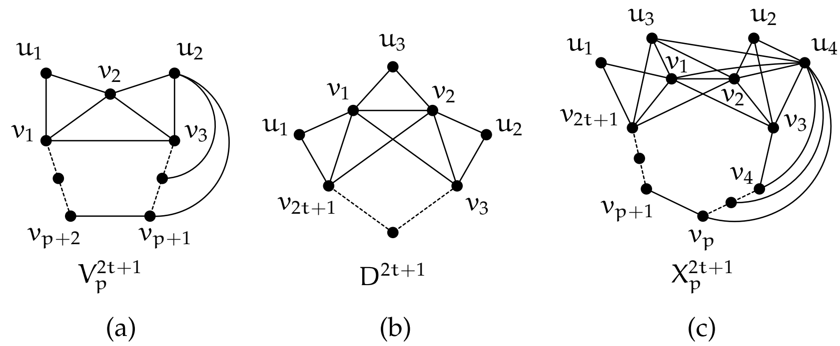


Figure 3.5: *Families of minimally not balanced Helly circular-arc graphs: (a) Family V_p^{2t+1}: The dotted paths joining v_3 and v_{p+1} resp. v_{p+2} and v_1 represent chordless even paths, not simultaneously empty. All vertices of the dotted path joining v_3 to v_{p+1} are adjacent to u_2. (b) Family D^{2t+1}: The dotted path joining v_3 and v_{2t+1} represents a nonempty even path of length $2t-2$. (c) Family X_p^{2t+1}: The dotted paths joining v_4 and v_p resp. v_{p+1} and v_{2t+1} represent any chordless even paths, both of them possibly empty, even simultaneously. The vertices of the dotted path joining v_4 to v_p are all adjacent to u_4.*

In the three families of graphs above, $C = v_1v_2\dots v_{2t+1}v_1$ is an unbalanced cycle and consequently all their members are not balanced. In fact, we will see later that all these graphs are minimally not balanced (see Corollary 3.45).

Our first result below is a minimal forbidden induced subgraph characterization of balanced graphs restricted to Helly circular-arc graphs.

Theorem 3.44. *Let G be a Helly circular-arc graph. Then, G is balanced if and only if G has no odd holes and contains no induced 3-sun, 1-pyramid, 2-pyramid, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, or X_p^{2t+1} for any $t \geqslant 2$ and any valid p.*

Proof. The 'only if' part is clear because the class of balanced graphs is hereditary. Conversely, suppose that G is not balanced. Then, G contains some induced subgraph H that is minimally not balanced. Since G is a Helly circular-arc graph, H is also so. The proof will be complete as soon as we prove that H is a 3-sun, 1-pyramid, 2-pyramid, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, or X_p^{2t+1} for some $t \geqslant 2$ and some valid p.

Since H is not balanced, a clique-matrix of H contains some square submatrix that is an edge-vertex incidence matrix of an odd chordless cycle. Therefore, there are some cliques $Q_1, Q_2, \dots, Q_{2t+1}$ and some pairwise different vertices $v_1, v_2, \dots, v_{2t+1}$ of H such that $\{v_1, v_2, \dots, v_{2t+1}\} \cap Q_i = \{v_i, v_{i+1}\}$ for each $i = 1, 2, \dots, 2t+1$ (all along the proof, subindices are to be understood modulo $2t+1$) for some $t \geqslant 1$. It is easy to verify that $C = v_1v_2\dots v_{2t+1}v_1$ is an unbalanced cycle by setting $W_e := Q_i \setminus \{v_i, v_{i+1}\}$ for each edge $e = v_iv_{i+1}$ of C.

If $t = 1$, Theorem 2.5 implies that H contains an induced pyramid. This implies that H itself is a pyramid because H is minimally not balanced. So, if $t = 1$, then H equals the 3-sun, 1-pyramid or 2-pyramid (because the 3-pyramid is not a Helly circular-arc graph). So, from now on, we assume, without loss of generality, that $t \geqslant 2$.

Let $\mathcal{A}$ be a Helly circular-arc model of H on a circle $\mathcal{C}$. Denote by A_i the arc of $\mathcal{A}$ corresponding to the vertex v_i for each $i = 1, 2, \ldots, 2t+1$. Fix an anchor p_j of the clique Q_j for each $j = 1, \ldots, 2t+1$. By construction, $p_j \in A_i$ if and only if $v_i \in Q_j$. Therefore, by hypothesis, $\{p_1, p_2 \ldots, p_{2t+1}\} \cap A_i = \{p_{i-1}, p_i\}$ for each $i = 1, \ldots, 2t+1$. Since $A_1, A_2, \ldots, A_{2t+1}$ are arcs of $\mathcal{C}$, there are only two possible orders for the anchors when traversing $\mathcal{C}$ in clockwise direction, either $p_1, p_2, \ldots, p_{2t+1}$ or $p_{2t+1}, \ldots, p_2, p_1$. So, we can assume, without loss of generality, that the anchors $p_1, p_2, \ldots, p_{2t+1}$ appear exactly in that order when traversing $\mathcal{C}$ in clockwise direction. Hence, $A_i \cap \{p_1, p_2, \ldots, p_{2t+1}\} = \{p_{i-1}, p_i\}$ implies that A_i is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_{i-2} and ends in p_{i+1} for each $i = 1, \ldots, 2t+1$. We now prove the following three claims about C.

Claim 1. *All chords of* C *are short.*

Proof of the claim. If $t = 2$, all possible chords of C are short. So, suppose that $t \geqslant 3$. Since A_i is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_{i-2} and ends in p_{i+1} for each $i \in \{1, \ldots, 2t+1\}$, it follows that if the arc A_i intersects A_j for some $j \in \{1, \ldots, 2t+1\}$ then $i = j-2, j-1, j, j+1$, or $j+2$ (modulo $2t+1$). We conclude that each chord of C is short, as claimed. □

Claim 2. *Any set of three vertices of* C *that induces a triangle in* H *consists of three consecutive vertices of* C.

Proof of the claim. Suppose, by the way of contradiction, that there is some set S of three vertices of C that induces a triangle T in H but, nevertheless, S does not consist of three consecutive vertices of C. Notice that if each vertex of S were consecutive in C to some other vertex of S, then S would consist of three consecutive vertices of C. So, necessarily, there must be some vertex s_1 of S such that s_1 is not consecutive in C to any vertex of $S \setminus \{s_1\}$. By symmetry, we can assume that $s_1 = v_1$ and, since all chords of C are short, $S = \{v_1, v_3, v_{2t}\}$. Being C odd and each of its chords short, necessarily $t = 2$. Consequently, $S = \{v_1, v_3, v_4\}$ is contained in some clique of H, that should have some anchor q. Nevertheless, since A_1 is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_4 and ends in p_2, A_3 is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_1 and ends in p_4, and A_4 is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_3 and ends in p_1, there is no suitable position in $\mathcal{C}$ for q. This contradiction proves that indeed any set of three vertices of C that induces a triangle in H consists of three consecutive vertices of C, as claimed. □

Claim 3. *Every two chords of* C *are crossing.*

Proof of the claim. Suppose, by the way of contradiction, that C has two different chords $e_i = v_{i-1}v_{i+1}$ and $e_j = v_{j-1}v_{j+1}$ that are not crossing. Notice that it is possible that e_i and e_j share one endpoint. We will show that $H - \{v_i, v_j\}$ is not balanced. Indeed, consider the cycle $C' = v_1v_2 \dots v_{i-1}v_{i+1} \dots v_{j-1}v_{j+1} \dots v_{2t+1}v_1$. For each edge e of C', define $W'_e = \emptyset$, if $e = e_i$ or e_j; and $W'_e = W_e$, otherwise. Since all the triangles of C are induced by three consecutive vertices of C, by Claim 2, C' and the W'_e's satisfy the definition of unbalanced cycle. Indeed, for each edge e of C, either $W'_e = W_e$ and $W'_e \cap N(e) \cap V(C') \subseteq W_e \cap N(e) \cap V(C) = \emptyset$, or $e = e_k$ for $k \in \{i, j\}$ and $N(W'_e) \cap N(e) \cap V(C') \subseteq N(e) \cap (V(C) \backslash \{v_k\}) = \emptyset$ because, by Claim 2, the only vertex of C with which vertices v_{k-1} and v_{k+1} can form a triangle in H is v_k. Therefore, $H - \{v_i, v_j\}$ is not balanced, a contradiction with the minimality of H. This contradiction shows that indeed every two chords of C are crossing, as claimed. □

With the help of the three previous claims, we complete the proof of Theorem 3.44. Notice that if C has no chords, then, by the minimality of H, $H = C_{2t+1}$, as required. Therefore, we will assume that C contains at least one chord. Since all chords of C are short and crossing by Claims 1 and 3, either C has exactly one chord that is short or C has two chords that are short and are crossing. We divide the remaining proof into two parts corresponding to the former and the latter case.

Case 1. C *has exactly one chord that is short.*

Without loss of generality, let v_1v_3 be the only chord of C. Since C is an unbalanced cycle, there exists $u_1 \in N_H(v_1v_2) \backslash V(C)$ such that u_1 is not adjacent to v_3. Analogously, there exists $u_2 \in N_H(v_2v_3) \backslash V(C)$ such that u_2 is not adjacent to v_1. By minimality, $V(H) = V(C) \cup \{u_1, u_2\}$. Let $p = |N_H(u_2) \cap V(C)|$ and $q = |N_H(u_1) \cap V(C)|$. By construction, $2 \leqslant p, q \leqslant 2t$. By Lemma 2.8 applied to the hole induced by $V(C) \backslash \{v_2\}$, $N_H(u_2) \cap V(C) = \{v_2, v_3, v_4, \dots, v_{p+1}\}$ and, by symmetry, $N_H(u_1) \cap V(C) = \{v_2, v_1, v_{2t+1}, v_{2t}, \dots, v_{2t-q+4}\}$ (where for $q = 2$, we mean that $N_H(u_1) \cap V(C) = \{v_2, v_1\}$).

Suppose, by the way of contradiction, that u_1 is adjacent to u_2. If u_2 were adjacent to v_{2t+1}, then either $\{v_{2t+1}, v_1, v_2, v_3, u_1, u_2\}$ would induce a proper 2-pyramid in H or $\{v_{2t+1}, v_1, v_3, u_1, u_2\}$ would induce a $K_{2,3}$ in H, depending on whether u_1 is adjacent v_{2t+1} or not, respectively. Since H is a minimally not balanced circular-arc graph and $K_{2,3}$ is not a circular-arc graph, we conclude that u_2 is not adjacent to v_{2t+1}. If u_1 were adjacent to v_{2t+1}, then $\{v_{2t+1}, v_1, v_2, v_3, u_1, u_2\}$ would induce a proper 1-pyramid in H. This contradiction shows that u_1 is not adjacent to v_{2t+1}, and this means that $q = 2$. Symmetrically, $p = 2$. But then, $\{v_1, v_3, u_2, u_1, v_5\}$ induces a $C_4 \cup K_1$ in H, which is not a circular-arc graph, a contradiction. This contradiction arose from assuming that u_1 and u_2 were adjacent, so we conclude that u_1 is not adjacent to u_2.

If p were odd, then $u_2v_{p+1}v_{p+2} \dots v_{2t+1}v_1v_2u_2$ would be an odd hole in H, contradicting the minimality of H. Thus, p is even and, by symmetry, q is also even. If $t = 2$, then, up to symmetry, either $p = q = 4$ and $H = \overline{C_7}$, or $q = 2$ and $H = V_p^5$ for some $p \in \{2, 4\}$, as desired. So, without loss of generality, assume that $t \geqslant 3$. If $N_H(u_1) \cap N_H(u_2) \neq \{v_2\}$, then, since p and q are even, there would exist some k such that $5 \leqslant k \leqslant 2t$ and $v_k \in N_H(u_1) \cap N_H(u_2)$; but then, $\{v_1, u_1, v_k, u_2, v_3\}$ would induce a C_5 in H, in contradiction with the minimality of H. This contradiction shows that $N_H(u_1) \cap N_H(u_2) = \{v_2\}$. If $p \neq 2$ and $q \neq 2$, then $u_2 v_{p+1} v_{p+2} \ldots v_{2t-q+4} u_1 v_2 u_2$ would be an odd hole in H, contradicting the minimality of H. Therefore, we can assume that $q = 2$, and finally $H = V_p^{2t+1}$ for some p even such that $2 \leqslant p \leqslant 2t$.

Case 2. C *has exactly two chords that are short and are crossing.*

Since the two chords are crossing, we assume, without loss of generality, that the chords of C are $v_1 v_3$ and $v_{2t+1} v_2$. Since C is an unbalanced cycle, there is some $u_1 \in N_H(v_{2t+1} v_1) \backslash V(C)$ such that u_1 is not adjacent to v_2 and there is some $u_2 \in N_H(v_2 v_3) \backslash V(C)$ such that u_2 is not adjacent to v_1.

Let $r = |N_H(u_2) \cap V(C)|$. By construction, $2 \leqslant r \leqslant 2t$ and, by Lemma 2.8 applied to the hole induced by $V(C) \backslash \{v_1\}$, $N_H(u_2) \cap V(C) = \{v_2, v_3, v_4, \ldots, v_{r+1}\}$. If $r = 2t$, then $\{v_{2t+1}, v_1, v_2, v_3, u_1, u_2\}$ would induce a proper 1-, 2- or 3-pyramid in H (depending on the existence or not of the edges $u_1 u_2$ and $u_1 v_3$), a contradiction with the minimality of H. If r is even and $2 < r < 2t$, then $u_2 v_{r+1} v_{r+2} \ldots v_{2t} v_{2t+1} v_2 u_2$ would be a proper odd hole in H, a contradiction. If r were odd and $r \neq 3$, then the cycle $u_2 v_{r+1} v_{r+2} \ldots v_{2t} v_{2t+1} v_1 v_3 u_2$ would be a proper odd hole in H, a contradiction. So, $r = 2$ or 3. Symmetrically, if $s = |N_H(u_1) \cap V(C)|$, then $s = 2$ or 3 and, by Lemma 2.8 applied to the hole induced by $V(C) \backslash \{v_2\}$, $N_H(u_1) \cap V(C) = \{v_{2t+1}, v_1\}$ or $\{v_{2t}, v_{2t+1}, v_1\}$, respectively.

Suppose, by the way of contradiction, that u_1 and u_2 are adjacent. Then, the set $\{u_1, v_1, v_2, u_2\}$ induces a C_4 in H, which must be dominating because H is a circular-arc graph. If $t = 2$, then at least one of u_1 and u_2 should be adjacent to v_4 and $V(C) \cup \{u_1, u_2\}$ would induce a proper V_4^5 or $\overline{C_7}$ in H. (Notice that indeed $V(C) \cup \{u_1, u_2\} \neq V(H)$ because, by definition of unbalanced cycle, $W_{v_1 v_2} \subseteq V(H) \backslash V(C)$ and $N_H(W_{v_1 v_2}) \cap \{v_3, v_4\} = \varnothing$, which implies $W_{v_1 v_2} \neq \varnothing$ and, by construction, $W_{v_1 v_2} \cap (V(C) \cup \{u_1, u_2\}) = \varnothing$.) If $t \geqslant 3$, then u_1 must be adjacent to v_{2t} and $\{v_{2t}, v_{2t+1}, v_1, v_2, v_3, u_1, u_2\}$ would induce a proper V_4^5 in H. So, in all cases we reach a contradiction with the minimality of H. These contradictions prove that u_1 and u_2 are nonadjacent.

We claim that $r = s = 2$. Indeed, if $r = s = 3$, then $v_1 v_2 u_2 v_4 v_5 \ldots v_{2t} u_1 v_1$ would be an odd hole in H, a contradiction. Alternatively, if $r = 3$ and $s = 2$, then $C' = v_1 v_2 u_2 v_4 v_5 \ldots v_{2t+1} v_1$ would be a cycle whose only chord is $v_{2t+1} v_2$, $N_H(u_1) \cap V(C') = \{v_{2t+1}, v_1\}$, $N_H(v_3) \cap V(C') = \{v_1, v_2, u_2, v_4\}$ and, therefore, $V(C) \cup \{u_1, u_2\}$ would induce a proper V_4^{2t+1} in H, a contradiction. (Recall that $V(C) \cup \{u_1, u_2\} \neq V(H)$ from the discussion in the paragraph above.) The case $r = 2$ and $s = 3$ is symmetric. We conclude that our claim, $r = s = 2$, is true; in other words, $N_H(u_1) \cap V(C) = \{v_{2t+1}, v_1\}$ and $N_H(u_2) \cap V(C) = \{v_2, v_3\}$.

Suppose that

$$\text{there is some } u_3 \in N_H(v_1v_2) \backslash V(C) \text{ such that } u_3v_{2t+1}, u_3v_3 \notin E(H). \tag{3.2}$$

Then, by minimality, $V(H) = V(C) \cup \{u_1, u_2, u_3\}$. By Lemma 2.8 applied to the holes induced by $V(C) \backslash \{v_1\}$ and $V(C) \backslash \{v_2\}$, $N_H(u_3) \cap V(C) = \{v_1, v_2\}$. If u_1 were adjacent to u_3, then either $t = 2$ and $\{v_2, v_3, v_4, v_5, u_1, u_3\}$ would induce a domino, or $t \geqslant 3$ and $\{u_1, v_{2t+1}, v_1, u_3, v_5\}$ would induce $C_4 \cup K_1$ in H, which are not circular-arcgraphs, a contradiction. So, u_1 is nonadjacent to u_3 and, symmetrically, u_2 is nonadjacent to u_3. We conclude that, if (3.2) holds, $H = D^{2t+1}$, as desired.

It only remains to consider the case when (3.2) does not hold. Since C is an unbalanced cycle, this means that there are two adjacent vertices u_3 and u_4 such that $u_3, u_4 \in N_H(v_1v_2) \backslash V(C)$, u_3 is adjacent to v_{2t+1} but not to v_3, and u_4 is adjacent to v_3 but not to v_{2t+1}.

Suppose, by the way of contradiction, that $N_H(u_3) \cap N_H(u_4) \cap V(C) \neq \{v_1, v_2\}$. Then, there exists some k such that $4 \leqslant k \leqslant 2t$ and $v_k \in N_H(u_3) \cap N_H(u_4)$. If $k = 4$, then $\{v_{2t+1}, v_1, v_3, v_4, u_3, u_4\}$ would induce a proper 1- or 2-pyramid in H depending on whether $t \geqslant 3$ or $t = 2$, respectively, contradicting the minimality of H. So, $k \neq 4$ and, symmetrically, $k \neq 2t$. But then, $\{v_{2t+1}, v_1, v_3, u_3, u_4, v_k\}$ induces a proper 3-sun in H, a contradiction. We conclude that $N_H(u_3) \cap N_H(u_4) \cap V(C) = \{v_1, v_2\}$.

Let $p = |N_H(u_4) \cap V(C)|$ and $q = |N_H(u_3) \cap V(C)|$. By construction, $3 \leqslant p, q \leqslant 2t$. By Lemma 2.8 applied to the hole induced by $V(C) \backslash \{v_2\}$, it follows that $N_H(u_4) \cap V(C) = \{v_1, v_2, v_3, \ldots, v_p\}$ and $N_H(u_3) \cap V(C) = \{v_2, v_1, v_{2t+1}, \ldots, v_{2t-q+4}\}$. If p were odd and $p \neq 3$, then $v_1u_4v_pv_{p+1} \ldots v_{2t+1}v_1$ would be a proper odd hole in H, a contradiction. So, $p = 3$ or p is even. Symmetrically, $q = 3$ or q is even. If p and q had the same parity, then $u_3u_4v_pv_{p+1} \ldots v_{2t-q+4}u_3$ would be a proper odd hole of H (recall that $N_H(u_3) \cap N_H(u_4) \cap V(C) = \{v_1, v_2\}$), a contradiction. By symmetry, we will assume, without loss of generality, that p is even, $p \geqslant 4$, and $q = 3$. In particular, u_4 is adjacent to v_4.

Notice that u_2 is not adjacent to u_3, since otherwise $u_2v_3v_4 \ldots v_{2t+1}u_3u_2$ would be a proper odd hole of H. This, in its turn, implies that u_2 is adjacent to u_4, since otherwise $\{v_2, v_3, v_4, u_3, u_4, u_2\}$ would induce a proper 3-sun in H. So, $N_H(u_2) = \{v_2, v_3, u_4\}$. (Recall that we already proved that u_1 and u_2 are nonadjacent.)

If u_1 were adjacent to u_4, then $\{v_{2t+1}, v_1, v_2, u_1, u_2, u_4\}$ would induce a proper 1-pyramid in H, contradicting the minimality of H. So, u_1 is nonadjacent to u_4. Finally, if

u_1 were adjacent to u_3, then $C' = u_3v_2v_3\dots v_{2t+1}u_3$ would be a cycle whose only chord is $v_{2t+1}v_2$, $N_H(u_1) \cap V(C') = \{v_{2t+1}, u_3\}$, $N_H(u_4) \cap V(C') = \{u_3, v_2, v_3, \dots, v_p\}$ and, therefore, since u_1 and u_4 are nonadjacent, $V(C') \cup \{u_1, u_4\}$ would induce a proper V_p^{2t+1} in H, a contradiction. This contradiction shows that u_1 is nonadjacent to u_3 and we conclude that $N_H(u_1) = \{v_{2t+1}, v_1\}$. We proved that $H = X_p^{2t+1}$ where p is even and $4 \leqslant p \leqslant 2t$, as required. □

It is easy to see that among the forbidden induced subgraphs that characterize balancedness in Theorem 3.44 there are no two of them such that one is a proper induced subgraph of the other. Therefore, Theorem 3.44 is indeed a characterization by *minimal* forbidden induced subgraphs. In particular, we obtain the following result.

Corollary 3.45. *The graphs* V_p^{2t+1}, D^{2t+1}, *and* X_p^{2t+1} *are minimally not balanced for any* $t \geqslant 2$ *and any valid* p.

We will extend Theorem 3.44 to a superclass of Helly circular-arc graphs; namely, the class of {net,U_4,S_4}-free circular-arc graphs (see Figure 2.1). This extension will also serve as a basis for the characterizations in the following two sections.

For that, let us firstly present the forbidden induced subgraph characterization of those circular-arc graphs that are Helly circular-arc graphs given in [75]. Let an *obstacle* be a graph H containing a clique $Q = \{v_1, v_2, \dots, v_t\}$ where $t \geqslant 3$ and such that for each $i = 1, \dots, t$, at least one of the following assertions holds (where in both assertions, w_{t+1} means w_1):

($\mathcal{O}_1$) $N(w_i) \cap Q = Q \setminus \{v_i, v_{i+1}\}$, for some $w_i \in V(H) \setminus Q$.

($\mathcal{O}_2$) $N(u_i) \cap Q = Q \setminus \{v_i\}$ and $N(z_i) \cap Q = Q \setminus \{v_{i+1}\}$, for some adjacent vertices $u_i, z_i \in V(H) \setminus Q$.

With this definition, the characterization of those circular-arc graphs that are Helly circular-arc graphs runs as follows.

Theorem 3.46 ([75]). *Let* G *be a circular-arc graph. Then,* G *is a Helly circular-arc graph if and only if* G *contains no induced obstacle.*

Notice that obstacles are not necessarily *minimal*; i.e., there are obstacles that contain proper induced obstacles. For instance, $\overline{2C_5}$ is an obstacle and contains a proper induced $\overline{2P_4}$, which is also an obstacle. In addition, there are minimal obstacles that are not circular-arc graphs; e.g., antenna and $\overline{C_6}$ are minimal obstacles that are not circular-arc graphs. Our next result determines all the {1-pyramid,2-pyramid}-free minimal obstacles that are circular-arc graphs. Recall that for each $t \geqslant 3$, S_t denotes the complete t-sun.

Theorem 3.47. *Let* H *be a {1-pyramid,2-pyramid}-free minimal obstacle which is a circular-arc graph. Then,* H *is 3-pyramid,* U_4*, or* $\overline{S_t}$ *for some* $t \geqslant 3$.

Proof. Let $Q = \{v_1, \ldots, v_t\}$, the w_i's, the u_i's, and the z_i's as in the definition of an obstacle. All along the proof, subindices should be understood modulo t.

Let us consider first the case where $t = 3$. Suppose that ($\mathcal{O}_2$) holds for at least two values of i, say $i = 1$ and $i = 2$. Then, $\{u_1, z_1, z_2\}$ is a complete and $\{v_1, v_2, v_3, u_1, z_1, z_2\}$ induces a 3-pyramid, since otherwise $\{v_1, v_2, v_3, u_1, z_1, z_2\}$ would induce a 1-pyramid or a 2-pyramid. Hence, by minimality, $H =$ 3-pyramid. Consider now the case where ($\mathcal{O}_2$) holds for exactly one value of i, say $i = 1$, and, consequently, ($\mathcal{O}_1$) holds for $i = 2$ and $i = 3$. We claim that $\{u_1, z_1\}$ is anticomplete to w_2. Indeed, if w_2 were adjacent to z_1, then $\{v_1, v_2, v_3, w_2, z_1, u_1\}$ would induce a 1-pyramid or a 2-pyramid in H, a contradiction. In addition, if w_2 were adjacent to u_1, then $\{v_1, v_2, u_1, z_1, w_2\}$ would induce a $K_{2,3}$ in G, which is not a circular-arc graph. We proved that $\{u_1, z_1\}$ is anticomplete to w_2 and, symmetrically, to w_3. Also notice that w_2 and w_3 are nonadjacent, since otherwise $\{v_1, v_2, w_2, w_3, u_1, z_1\}$ would induce a domino, which is not a circular-arc graph. Then, by minimality, $H = U_4$, as desired. Finally, assume that ($\mathcal{O}_1$) holds for each $i = 1, 2, 3$. Necessarily $\{w_1, w_2, w_3\}$ is a stable set, since otherwise G would contain an induced $C_4 \cup K_1$, G_3 (see Figure 2.3), or $\overline{C_6}$ which are not circular-arc graphs. By minimality, $H = \text{net} = \overline{S_3}$, as desired.

From now on, we assume that $t \geqslant 4$. Suppose, by the way of contradiction, that ($\mathcal{O}_2$) holds for some i, say $i = 1$. On the one hand, if ($\mathcal{O}_1$) held for $i = 3$, then $\{v_1, v_2, v_3, u_1, z_1, w_3\}$ would induce a 1-pyramid, 2-pyramid, or a proper 3-pyramid in H, a contradiction. On the other hand, if ($\mathcal{O}_2$) held for $i = 3$, then $\{v_1, v_2, v_3, u_1, z_1, u_3\}$ would induce a 1-pyramid, 2-pyramid or a proper 3-pyramid in H, a contradiction. These contradictions arose from assuming that ($\mathcal{O}_2$) held for some i. We conclude that, if $t \geqslant 4$, then ($\mathcal{O}_2$) does not hold for any $i = 1, \ldots, t$ and, by definition of an obstacle, ($\mathcal{O}_1$) holds for each $i = 1, \ldots, t$. By minimality, the vertices of H are $Q \cup W$ where $W = \{w_1, w_2, \ldots, w_t\}$. We claim that W is a stable set and, consequently, $H = \overline{S_t}$. We divide the proof of the claim into two cases: $t = 4$ and $t \geqslant 5$.

Assume that $t = 4$. Suppose, by the way of contradiction, that W is not a stable set. Suppose first that w_i is adjacent to w_{i+1} for some i, say w_3 is adjacent to w_4. Necessarily w_1 is nonadjacent to w_4, since otherwise $\{v_1, v_2, v_3, w_1, w_3, w_4\}$ would induce a 1-pyramid or a 2-pyramid in H (depending on the adjacency between w_1 and w_3), a contradiction. In addition, w_1 is nonadjacent to w_3, since otherwise $\{w_1, v_1, w_4, v_3, w_3\}$ would induce a $K_{2,3}$, which is not a circular-arc graph. Symmetrically, w_2 is nonadjacent to w_3 and w_4. On the one hand, if w_1 and w_2 are adjacent, $\{w_2, v_1, w_3, w_4, v_3, w_1\}$ induces a domino in G, which is not a circular-arc graph. On the other hand, if w_1 and w_2 are nonadjacent, then $\{v_1, v_2, v_3, w_1, w_2, w_3, w_4\}$ induces a proper U_4 in H, a contradiction with the minimality of H. These contradictions prove that w_i is not adjacent to w_{i+1} for any i. Notice that also w_i and w_{i+2} are nonadjacent, since otherwise $\{v_i, w_i, w_{i+2}, v_{i+3}, w_{i+3}\}$ would induce $K_4 \cup K_1$ in G, which is not a circular-arc graph. We conclude that W is a stable set and $H = \overline{S_4}$, as claimed.

It only remains to consider the case where $t \geqslant 5$. Let S be any unordered pair of vertices from W. Since $t \geqslant 5$, S can be extended to a set $S' = \{w_i, w_j, w_{j+1}\}$ of three vertices where i and j are not consecutive modulo t and neither are i and $j+1$. Notice that S' is a stable set in H, since otherwise $\{v_i, v_j, v_{j+2}, w_i, w_j, w_{j+1}\}$ would induce a 1-pyramid, a 2-pyramid, or a proper 3-pyramid in H, a contradiction. Since S' is a stable set, so is S. Since S is any pair of vertices from W, W is a stable set and $H = \overline{S_t}$, as claimed.

Finally, notice that 3-pyramid, U_4 and $\overline{S_t}$ for $t \geqslant 3$ are obstacles, are circular-arc graphs, and none of them is a proper induced subgraph of any of the others. □

As a corollary of Theorems 3.46 and 3.47, we obtain a minimal forbidden induced subgraph characterization of Helly circular-arc graphs within {1-pyramid,2-pyramid}-free circular-arc graphs.

Corollary 3.48. *Let G be a {1-pyramid,2-pyramid}-free circular-arc graph. Then, G is a Helly circular-arc graph if and only if it contains no induced 3-pyramid, U_4, or $\overline{S_t}$ for any $t \geqslant 3$.*

Since net, U_4, and S_4 are obstacles, the class of {net,U_4,S_4}-free circular-arc graphs is indeed a superclass of Helly circular-arc graphs. We now prove the main result of this section, which is an extension of the characterization of Theorem 3.44 to the class of {net,U_4,S_4}-free circular-arc graphs.

Corollary 3.49. *Let G be a {net,U_4,S_4}-free circular-arc graph. Then, G is balanced if and only if G has no odd holes and contains no induced pyramid, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, or X_p^{2t+1} for any $t \geqslant 2$ and any valid p.*

Proof. If G is a Helly circular-arc graph, the result reduces to Theorem 3.44. So, assume that G is not a Helly circular-arc graph. Then, by Corollary 3.48 and since G is {net,U_4,S_4}-free, G contains an induced 1-pyramid, 2-pyramid, or 3-pyramid or an induced $\overline{S_t}$ for some $t \geqslant 5$ (notice that $\overline{S_3} =$ net and $\overline{S_4} = S_4$). Since $\overline{S_t}$ contains an induced 3-sun for every $t \geqslant 5$, we conclude that G is not balanced and contains an induced pyramid. □

3.8 Balancedness of claw-free circular-arc graphs

In this section we will characterize, by minimal forbidden induced subgraphs, those claw-free circular-arc graphs that are balanced. A *proper circular-arc graph* is a circular-arc graph admitting a circular-arc model in which no arc properly contains another.

The class of claw-free circular-arc graphs is a superclass of the class of proper circular-arc graphs, as follows from the forbidden induced subgraph characterization of proper circular-arc graphs in [118].

By Corollary 3.49, in order to characterize those claw-free circular-arc graphs that are balanced, it will be enough to study the balancedness of those claw-free circular-arc graphs containing an induced net (because claw-free graphs contain neither induced U_4's nor induced S_4's). The following lemma will be of help in analyzing the structure of claw-free circular-arc graphs containing an induced net.

Lemma 3.50 ([18]). *Let G be a claw-free circular-arc graph containing a net induced by the set $W = \{t_1, t_2, t_3, s_1, s_2, s_3\}$, where $\{t_1, t_2, t_3\}$ induces a triangle and s_i is adjacent to t_i for $i = 1, 2, 3$. If v is a vertex of $G - W$, then $N_G(v) \cap W$ is either $\{s_i, t_i\}$, or $\{t_1, t_2, t_3, s_i\}$, or $\{s_{i+1}, t_{i+1}, t_{i+2}, s_{i+2}\}$, for some $i \in \{1, 2, 3\}$ (subindices should be understood modulo 3).*

A graph G is a *multiple* of another graph H if G arises from H by successively adding true twins to H; i.e., if G arises from H by replacing each vertex x of H by a nonempty complete graph M_x and adding all possible edges between M_x and M_y if and only if x and y are adjacent in H. In [18], a slightly stronger variant of the above lemma is used to study the structure of chordal claw-free circular-arc graphs containing an induced net. The proof in [18] can be easily adapted to prove the following related result in which chordality is not required. For the sake of completeness, we give the adapted proof.

Theorem 3.51 ([18]). *If G is a claw-free circular-arc graph containing an induced net and containing no induced 3-sun, then G is a multiple of a net.*

Proof. The proof will be by induction on the number of vertices of G. If $|V(G)| = 6$, G equals a net, which is a trivial multiple of a net. So, assume that $|V(G)| > 6$. Then, there is some vertex v of G such that $G - \{v\}$ contains an induced net. Since $G - \{v\}$ is also a claw-free graph containing an induced net and containing no induced 3-sun, by induction hypothesis, $G - \{v\}$ is the multiple of a net; i.e., the vertices of $V(G - \{v\})$ can be partitioned into nonempty completes S_1, S_2, S_3, T_1, T_2, T_3 such that T_1, T_2, T_3 are mutually complete and T_i is complete to S_i and anticomplete to S_{i+1} and S_{i+2}, for each $i = 1, 2, 3$ (where subindices along the proof should be understood modulo 3). By Lemma 3.50, $N_G(v) \cap H = \{s_i, t_i\}$ or $N_G(v) \cap H = \{t_1, t_2, t_3, s_i\}$ for some $i \in \{1, 2, 3\}$. (Notice that the fact that G contains no induced 3-sun prevents $N_G(v) \cap H = \{t_{i+1}, s_{i+1}, t_{i+2}, s_{i+2}\}$ from holding.)

Suppose first that $N_G(v) \cap H = \{t_i, s_i\}$ for some $i \in \{1, 2, 3\}$. Let $j \in \{1, 2, 3\}$, $s'_j \in S_j$ and H' be the net induced by $\{t_1, t_2, t_3, s_1, s_{j+1}, s_{j+2}\}$. Applying Lemma 3.50 to H', it follows that v is adjacent to s_j if and only if $i = j$. Thus, v is complete to S_i

and anticomplete to S_{i+1} and S_{i+2}. Using the same strategy, we can prove that v is complete to T_i and anticomplete to T_{i+1} and T_{i+2}. Thus, we can obtain a partition of the vertices of G showing that G as a multiple of a net by replacing S_i by S_{i+1}.

Finally, consider that $N_G(v) \cap H = \{t_1, t_2, t_3, s_i\}$. Reasoning as in the above paragraph, it follows that v is complete to T_1, T_2, T_3, and S_i, and v is anticomplete to S_{i+1} and S_{i+2}. Thus, we obtain a partition of the vertices of G showing that G is a multiple of a net by replacing T_i by $T_i \cup \{v\}$. □

Now, we state and prove the main result of this section.

Theorem 3.52. *Let* G *be a claw-free circular-arc graph. Then,* G *is balanced if and only if* G *has no odd holes and contains no induced pyramids and no induced* $\overline{C_7}$*.*

Proof. The 'only if' part is clear. In order to prove the 'if' part, suppose that G is not balanced. Then, G contains some induced subgraph H that is minimally not balanced. Since G is a claw-free circular-arc graph, H also is so. The proof will be complete if we prove that H is an odd hole, a pyramid, or $\overline{C_7}$. Suppose, by the way of contradiction, that H is not net-free. By Theorem 3.51, H is a net, has true twins, or contains an induced 3-sun. Since the net is balanced and since minimally not balanced graphs have no true twins (Lemma 3.7), G contains an induced 3-sun. By minimality, H is a 3-sun, a contradiction with the fact that H is not net-free. This contradiction proves that H is net-free. Since U_4 and S_4 are not claw-free, H is {net,U_4,S_4}-free and Corollary 3.49 implies that H has an odd hole or contains an induced pyramid or $\overline{C_7}$ (because each of X_p^{2t+1}, D^{2t+1}, and X_p^{2t+1} contains an induced claw for each $t \geqslant 2$ and each valid p). By the minimality of H, we conclude that H is an odd hole, a pyramid, or $\overline{C_7}$, as required. □

As proper circular-arc graphs are claw-free, and the odd holes, the pyramids, and $\overline{C_7}$ are all proper circular-arc graphs, the minimal forbidden induced subgraphs for balancedness within proper circular-arc graphs are the same as those within claw-free circular-arc graphs.

3.9 Balancedness of gem-free circular-arc graphs

In this section, we will give a minimal forbidden induced subgraph characterization of those gem-free circular-arc graphs that are balanced.

Lemma 3.53. *Let* G *be a gem-free circular-arc graph that contains an induced net or an induced* U_4*. Then,* G *either has true twins or has a cutpoint.*

Proof. Assume that G has no true twins. We will prove that G has a cutpoint.

Consider first the case where G contains an induced U_4. That is, there is some chordless cycle $C = u_1u_2u_3u_4u_1$ in G, some vertex z that is complete to $V(C)$, and a pair of nonadjacent vertices p_1, p_2 of G such that $N_G(p_i) \cap (V(C) \cup \{z\}) = \{u_i\}$ for each $i = 1, 2$. Since G is a circular-arc graph, $V(C)$ is a dominating set of G. Let v be a vertex of G not in $V(C) \cup \{p_1, p_2\}$. We will analyze the possibilities for the nonempty set $N_G(v) \cap V(C)$.

Suppose, by the way of contradiction, that the neighbors of v in C are two. Then, they are consecutive vertices of C by Lemma 2.8. So, $N_G(v) \cap V(C) = \{u_i, u_{i+1}\}$ for some $i \in \{1, 2, 3, 4\}$ (from now on, subindices should be understood modulo 4). If v were not adjacent to z, then $\{v, u_i, z, u_{i+2}, u_{i+1}\}$ would induce a gem in G. If v were adjacent to z, then $\{v, u_{i+1}, u_{i+2}, u_{i+3}, z\}$ would induce a gem in G. Since G is gem-free, we conclude that $|N_G(v) \cap V(C)| \neq 2$.

Now, for each $i = 1, \ldots, 4$, let V_i be the set of vertices not in $V(C)$ whose only neighbor in C is u_i. In particular, $p_i \in V_i$ for each $i = 1, 2$. Let Z be the set of vertices not in $V(C)$ that are complete to $V(C)$, so $z \in Z$. Finally, for each $i = 1, \ldots, 4$, let $\bar{V}_i$ be the set of vertices not in $V(C)$ whose only non-neighbor in C is u_i.

Claim 1. *V_i is anticomplete to V_j for every $i \neq j$.*

Proof of the claim. Indeed, if $v_i \in V_i$ and $v_j \in V_j$ were adjacent, then $V(C) \cup \{v_i, v_j\}$ would induce either a domino or the graph G_2 in Figure 2.3, which are not circular-arc graphs, a contradiction. □

Claim 2. *V_i is anticomplete to Z for every $1 \leqslant i \leqslant 4$.*

Proof of the claim. Indeed, if $v_i \in V_i$ were adjacent to $w \in Z$, then $\{v_i, u_i, u_{i+1}, u_{i+2}, w\}$ would induce a gem in G, a contradiction. □

Claim 3. *Z is a complete.*

Proof of the claim. Indeed, if w, w' in Z were nonadjacent, then, by the previous claim, both of them would be nonadjacent to p_2 and $\{u_1, w, u_3, w', p_2\}$ would induce $C_4 \cup K_1$ in G, which is not a circular-arc graph, a contradiction. □

Claim 4. *$\bar{V}_i$ is a complete and is complete to Z for every $1 \leqslant i \leqslant 4$.*

Proof of the claim. Indeed, if $\bar{v}_i, \bar{v}'_i$ in $\bar{V}_i$ were nonadjacent, then $\{\bar{v}_i, \bar{v}'_i, u_i, u_{i-1}, u_{i+1}\}$ would induce $K_{2,3}$ in G, which is not a circular-arc graph, a contradiction. And, if $\bar{v}_i \in \bar{V}_i$ and $w \in Z$ were nonadjacent, then $\{\bar{v}_i, u_{i+2}, w, u_i, u_{i+1}\}$ would induce a gem in G, also a contradiction. □

By the previous claims, all the vertices in Z are true twins. So, since G has no true twins, we conclude that $Z = \{z\}$.

Claim 5. *$\bar{V}_i$ is complete to $\bar{V}_{i+1}$ and anticomplete to $\bar{V}_{i+2}$ for every $1 \leqslant i \leqslant 4$.*

Proof of the claim. Let $\bar{v}_i \in \bar{V}_i$ and $\bar{v}_{i+1} \in \bar{V}_{i+1}$. By Claim 4, z is adjacent to both of them. So, if $\bar{v}_i$ and $\bar{v}_{i+1}$ were nonadjacent, then $\{\bar{v}_i, u_{i+1}, u_i, \bar{v}_{i+1}, z\}$ would induce a gem in G, a contradiction. Now, let $\bar{v}_{i+2} \in \bar{V}_{i+2}$. If $\bar{v}_{i+2}$ were adjacent to $\bar{v}_i$, then $\{u_i, \bar{v}_{i+2}, \bar{v}_i, u_{i+2}, u_{i+1}\}$ would induce a gem in G, a contradiction. □

Claim 6. *$\bar{V}_i$ is anticomplete to V_j for every $j \neq i+2$.*

Proof of the claim. Let $\bar{v}_i \in \bar{V}_i$ and $v_j \in V_j$ and suppose, by the way of contradiction, they are adjacent. If $j = i$, then $\{u_i, \bar{v}_i, u_{i+1}, u_{i+3}, v_j\}$ induces a $K_{2,3}$ in G, that is not a circular-arc graph, a contradiction. If $j = i \pm 1$, then $\{v_j, u_{i+1}, u_{i+2}, u_{i+3}, \bar{v}_i\}$ induces a gem in G, also a contradiction. These contradictions prove that $\bar{v}_i$ and v_j are nonadjacent unless $j = i + 2$. □

Claim 7. *V_i is empty for every $1 \leqslant i \leqslant 4$.*

Proof of the claim. Suppose, by the way of contradiction, that $\bar{V}_i$ is nonempty for some $i \in \{1, 2, 3, 4\}$ and let $\bar{v}_i \in \bar{V}_i$. Since $\bar{v}_i$ is not a true twin of v_{i+2}, by the previous claims, there must be a vertex v_{i+2} in V_{i+2} nonadjacent to $\bar{v}_i$. But then, $\{\bar{v}_i, u_{i+3}, u_i, u_{i+1}, v_{i+2}\}$ induces a $C_4 \cup K_1$ in G, that is not a circular-arc graph, a contradiction. □

By the above claims, u_1 and u_2 are cutpoints of G, as required. This completes the proof when G contains an induced U_4.

It only remains to consider the case where G contains no induced U_4 but a net induced by $H = T \cup S$ where $T = \{t_1, t_2, t_3\}$ is a complete, $S = \{s_1, s_2, s_3\}$ is a stable set and $N_G(s_i) \cap T = \{t_i\}$ for each $i = 1, 2, 3$. Let v be a vertex of G not in H. Then, $N_G(v) \cap H$ is nonempty because net$\cup K_1$ is not a circular-arc graph. If $|N_G(v) \cap H| \geqslant 5$, then G would contain an induced gem, so $|N_G(v) \cap H| \leqslant 4$.

Suppose that $|N_G(v) \cap H| = 4$. If $|N_G(v) \cap S| = 3$ then G would contain the graph G_3 in Figure 2.3 as induced subgraph, which is not a circular-arc graph. If $|N_G(v) \cap S| = 2$, then G would contain an induced gem. So, if $|N_G(v) \cap H| = 4$, then $|N_G(v) \cap S| = 1$.

Suppose, by the way of contradiction, that $|N_G(v) \cap H| = 3$. If $|N_G(v) \cap S| = 3$, then G would contain the graph G_9 in Figure 2.3 as induced subgraph, which is not a circular-arc graph. If $|N_G(v) \cap S| = 2$, then G would contain either $C_5 \cup K_1$ or $C_4 \cup K_1$ as induced subgraph, and none of them is a circular-arc graph. If $|N_G(v) \cap S| = 1$, then G would contain either a gem or $C_4 \cup K_1$ as induced subgraph. If $|N_G(v) \cap S| = 0$, then G would contain the graph G_6 in Figure 2.3 as induced subgraph, which is not a circular-arc graph. We conclude that $|N_G(v) \cap S| \neq 3$.

Suppose now that $|N_G(v) \cap H| = 2$. If $|N_G(v) \cap S| = 2$, then G would contain $C_5 \cup K_1$ as induced subgraph, which is not a circular-arc graph. If $|N_G(v) \cap S| = 1$ and the neighbors of v in H were nonadjacent, then G would contain $C_4 \cup K_1$ as induced subgraph. So, if $|N_G(v) \cap H| = 2$, then either $N_G(v) \cap H \subseteq T$ or $N_G(v) \cap H = \{t_i, s_i\}$ for some $i \in \{1, 2, 3\}$.

Finally, if $|N_G(v) \cap H| = 1$, then the neighbor of v in H belongs to T; since otherwise G would contain the graph G_5 in Figure 2.3 as induced subgraph, and it is not a circular-arc graph.

Let S_i be the set of vertices in $G - H$ whose only neighbor in T is t_i (i.e., the set of neighbors in H is either $\{t_i\}$ or $\{t_i, s_i\}$), T_i be the set of vertices in $G - H$ whose neighbors in H are $\{t_1, t_2, t_3, s_i\}$, and Z_i be the set of vertices in $G - H$ whose neighbors in H are $T - \{t_i\}$. Since G is gem-free, at most one of the Z_i's is nonempty. So, without loss of generality, assume that Z_2 and Z_3 are empty.

Claim 8. *S_i is anticomplete to S_j for $i \neq j$.*

Proof of the claim. Indeed, if $v \in S_i$ were adjacent to $w \in S_j$ and $i \neq j$, $\{v, t_i, t_j, w, s_{6-i-j}\}$ would induce a $C_4 \cup K_1$ in G, which is not a circular-arc graph, a contradiction. □

Claim 9. *For each $i = 1, 2, 3$, S_i is complete to T_i and anticomplete to T_j for every $j \neq i$.*

Proof of the claim. If $v \in S_i$ and $w \in T_i$ were nonadjacent, then $(H \backslash \{s_1\}) \cup \{v, w\}$ would induce the graph G_6 in Figure 2.3, which is not a circular-arc graph, a contradiction. If $v \in S_i$ were adjacent to $w \in T_j$ and $j \neq i$, then $\{s_j, t_j, t_i, v, w\}$ would induce a gem in G, a contradiction. □

Claim 10. *For each $i = 1, 2, 3$, T_i is a complete and T_i is complete to T_j for every $j \neq i$.*

Proof of the claim. Indeed, if $w, w' \in T_i$ were nonadjacent, then $\{w, s_i, w', t_{i+1}, s_{i+2}\}$ would induce $C_4 \cup K_1$ in G, which is not a circular-arc graph, a contradiction. Also, if $w_i \in T_i$ were nonadjacent to $w_j \in T_j$ and $j \neq i$, then $\{s_j, w_j, t_i, w_i, t_j\}$ would induce a gem in G, a contradiction. □

Claim 11. *For each $i = 1, 2, 3$, S_i is anticomplete to Z_1.*

Proof of the claim. Indeed, if $v \in S_i$ were adjacent to $z_1 \in Z_1$, then either $i = 1$ and $\{v, t_1, t_2, z_1, s_3\}$ would induce $C_4 \cup K_1$ in G, or $i \neq 1$ and $\{t_1, t_{5-i}, z_1, v, t_i\}$ would induce gem in G, and in both cases we would reach a contradiction. □

Claim 12. *T_1 is anticomplete to Z_1.*

Proof of the claim. Indeed, if $w_1 \in T_1$ were adjacent to $z_1 \in Z_1$, then $\{s_1, t_1, t_2, z_1, w_1\}$ would induce a gem in G, a contradiction. □

By the previous claims, every vertex in T_1 is a true twin of t_1 and, since there are no true twins in G, T_1 is empty. Since the claims also prove that $S_1 \cup \{s_1\}$ is anticomplete to $V(G - \{t_1\}) \backslash (S_1 \cup \{s_1\})$, t_1 is a cutpoint of G, as required. □

Now we are ready to characterize balanced graphs among gem-free circular-arc graphs.

Theorem 3.54. *Let* G *be a gem-free circular-arc graph. Then,* G *is balanced if and only if* G *has no odd holes and contains no induced 3-pyramid.*

Proof. The 'only if' part is clear. In order to prove the 'if' part, suppose that G is not balanced. Then, G contains some induced subgraph H that is minimally not balanced. Clearly, H is a gem-free circular-arc graph because G is so. The proof will be complete as soon as we prove that H is an odd hole or a 3-pyramid. Suppose, by the way of contradiction, that H is not {net,U_4,S_4}-free. Since H is gem-free, H contains an induced net or an induced U_4. By Lemma 3.53, H has true twins or has a cutpoint, a contradiction with the minimality of H (Lemma 3.7). This contradiction proves that H is {net,U_4,S_4}-free and Corollary 3.49 implies that H has an odd hole or contains an induced 3-pyramid (because each of 3-sun, 1-pyramid, 2-pyramid, $\overline{C_7}$, X_p^{2t+1}, D^{2t+1}, and X_p^{2t+1}, for each $t \geqslant 2$ and each valid p, contains an induced gem). The minimality of H ensures that H is an odd hole or 3-pyramid, which concludes the proof. □

Chapter 4

Clique-perfect graphs

This chapter is organized as follows.

- In Section 4.1, we give some background about clique-perfect graphs and introduce two further superclasses of balanced graphs: coordinated graphs and hereditary K-perfect graphs. In Subsection 4.1.3, we give a brief account on the connections between these four graph classes and with some notions studied in hypergraph theory.

- In Section 4.2, we characterize clique-perfect graphs by minimal forbidden induced subgraphs within complements of line graphs. This characterization leads to an $O(n^2)$-time algorithm for deciding whether or not a given complement of line graph having n vertices is clique-perfect and, if affirmative, finding a minimum clique-transversal. Our results follows from a characterization by minimal forbidden subgraphs of *matching-perfect* graphs, which we define to be those graphs such that, in each of its subgraphs, the maximal matchings have the Kőnig property (i.e., the minimum number of edges needed to meet every maximal matching equals the maximum number of edge-disjoint maximal matchings). On the way to the proof, we also describe a simple linear and circular structure for graphs containing no bipartite claw that help us give a structural characterization of all Class 2 graphs with respect to edge-coloring within graphs containing no bipartite claw.

  The results of this section appeared in [24].

- In Section 4.3, we show that a gem-free circular-arc graph is clique-perfect if and only if it has no odd holes. This means that clique-perfect graphs coincide with perfect graphs within gem-free circular-arc graphs. Moreover, we show that, within gem-free circular-arc graphs, clique-perfect graphs coincide also with coordinated graphs and hereditary K-perfect graphs.

4.1 Background

4.1.1 Clique-perfect graphs

A graph G is *clique-perfect* if and only if $\alpha_c(G') = \tau_c(G')$ for each induced subgraph G' of G, where α_c and τ_c are the *clique-independence number* and τ_c is the *clique-transversal number* defined in the Introduction. While the name 'clique-perfect' was introduced in 2000 by Guruswami and Pandu Rangan [64], the equality between α_c and τ_c was studied long before. Recall from the Introduction that Kőnig's matching theorem [52, 77] is easily seen to be equivalent to the fact that $\alpha_c(G) = \tau_c(G)$ holds for every bipartite graph G and that Berge and Las Vergnas [12] generalized this result by proving that $\alpha_c(G) = \tau_c(G)$ remains true for all balanced graphs G. In [2], the equality $\alpha_c(G) = \tau_c(G)$ was shown to hold for all comparability graphs G, which form another superclass of bipartite graphs [59]. As the classes of balanced graphs and comparability graphs are hereditary, all the graphs in these classes are clique-perfect. Recall from the Introduction that dually chordal graphs G defined in [29] satisfy $\alpha_c(G) = \tau_c(G)$ but are not clique-perfect in general because they are not closed under taking induced subgraphs. More recently, it was shown that complements of forests and distance-hereditary graphs are clique-perfect [15, 87]. Balanced graphs, comparability graphs, complements of forests, and distance-hereditary graphs, are perfect. However, clique-perfect graphs are not necessarily perfect and perfect graphs are not necessarily clique-perfect, as the following result holds.

Theorem 4.1 ([64] and Reed (2001), see [50])**.** *A hole is clique-perfect if and only if it is even. An antihole is clique-imperfect if and only if its number of vertices is a multiple of* 3*.*

So, the odd holes and the antiholes whose number of vertices are not multiples of 3 are forbidden induced subgraphs for the class of clique-perfect graphs. In fact, all these graphs are minimal forbidden induced subgraphs for clique-perfectness [15].

Odd generalized suns [20] are a family of forbidden subgraph for the class of clique-perfect graphs that properly contain the odd suns and the odd holes, and are defined as follows. Let G be a graph and C be a cycle of G. An edge $e \in E(C)$ is *non-proper* (or *improper*) if it forms a triangle with some vertex of C; i.e., if $N(e) \cap V(C) \neq \varnothing$. For each $t \geqslant 3$, a t-*generalized sun*, is a graph G whose vertex set can be partitioned into two sets: a cycle C of t vertices whose set of non-proper edges is $\{e_j\}_{j \in J}$ (J is permitted to be an empty set) and a stable set $U = \{u_j\}_{j \in J}$ such that, for each $j \in J$, u_j is adjacent exactly to the endpoints of e_j. A t-generalized sun is *odd* if t is odd. A cycle is said *proper* if none of its edges is improper. By definition, proper odd cycles are odd generalized suns. What interest us about odd generalized suns is that they are not clique-perfect.

Theorem 4.2 ([20])**.** *Odd generalized suns are not clique-perfect.*

Unfortunately, as the extended odd suns in Figure 1.2 are also odd generalized suns, odd generalized suns are not necessarily minimal forbidden induced subgraphs for the class of clique-perfect graphs. Some odd generalized suns that are minimally not clique-perfect are the odd holes and the odd complete suns.

The following characterization of clique-perfect graphs within chordal graphs follows from Theorem 3.5 because balanced graphs are clique-perfect and because the odd suns are not clique-perfect.

Theorem 4.3 ([12, 88])**.** *Let* G *be a chordal graph. Then,* G *is clique-perfect if and only if it contains no induced odd sun.*

In other words, a chordal graph is clique-perfect if and only if it is balanced. So, the situation regarding Theorem 4.3 is the same as that regarding Theorem 3.5: characterizing clique-perfect graphs (or equivalently, balanced graphs) by minimal forbidden induced subgraphs is open even within chordal graphs. Moreover, Corollary 3.6 remains true if 'balanced' is replaced by 'clique-perfect' and the resulting characterization of clique-perfect graphs within pseudo-split graphs is by minimal forbidden induced subgraph. This also meas that Lemma 3.14 remains true if 'balancedness of any given split graph' is replaced by 'clique-perfectness of any given split graph'. However, the problem of determining the complexity of the recognition problem of clique-perfect graphs in general is still open, as balanced graphs and clique-perfect graphs do not coincide in general (see Figure 4.3 on page 81).

A graph G is called *clique-complete* [96] if each pair of its cliques has nonempty intersection; i.e., if $\alpha_c(G) = 1$. In [96], the clique-complete graphs G without universal vertices (i.e., such that $\tau_c(G) > 1$) that are minimal with respect to taking induced subgraphs were identified to be those graphs $\mathcal{Q}_{2n+1}$ ($n \geqslant 1$) having $4n+2$ vertices $u_1, u_2, \dots, u_{2n+1}, v_1, v_2, \dots, v_{2n+1}$ such that $\mathcal{Q}_{2n+1}[\{v_1, v_2, \dots, v_n\}] = \overline{C_{2n+1}}$ and $N_{\mathcal{Q}_{2n+1}}(u_i) = V(\mathcal{Q}_{2n+1}) \setminus \{v_i\}$, for each $i = 1, 2, \dots, 2n+1$.

Theorem 4.4 ([96])**.** *For each* $n \geqslant 1$, $\alpha_c(\mathcal{Q}_{2n+1}) = 1$ *and* $\tau_c(\mathcal{Q}_{2n+1}) = 2$. *Moreover, if* G *is a graph such that* $\alpha_c(G) = 1$ *but* $\tau_c(G) > 1$, *then* G *contains an induced* $\mathcal{Q}_{2n+1}$ *for some* $n \geqslant 1$.

In [15], it was shown that $\mathcal{Q}_{2n+1}$ is minimally clique-imperfect if and only if $n \equiv 1 \mod 3$. Yet, forbidding induced odd generalized suns, clique-imperfect antiholes, and clique-imperfect $\mathcal{Q}_{2n+1}$ graphs is not sufficient to ensure clique-perfectness in general. For instance, the following holds.

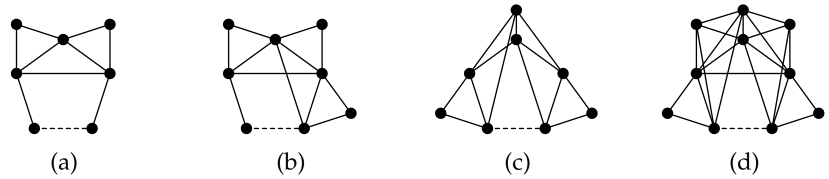


Figure 4.1: *Four families of minimal forbidden induced subgraphs for the class of clique-perfect graphs within the class of Helly circular-arc graphs. Dashed lines represent induced paths of length* $2t-3$ *for each* $t \geqslant 2$.

Theorem 4.5 ([17]). *Let* G *be a Helly circular-arc graph. Then,* G *is clique-perfect if and only if it has no odd holes and it contains no induced* 3*-sun,* $\overline{C_7}$*, or any graph belonging to any of the four families depicted in Figure 4.1.*

Here, the graphs of the families (c) and (d) of Figure 4.1 are neither odd generalized suns, nor antiholes, nor $\mathcal{Q}_{2n+1}$ graphs for any $n \geqslant 1$. Although there is no known forbidden induced subgraph characterization of clique-perfect graphs in general, there are some more graph classes within which clique-perfect graphs were characterized by forbidden induced subgraphs [16, 17, 25]: diamond-free graphs, line graphs, hereditary clique-Helly claw-free graphs, paw-free graphs, and {gem,W_4,bull}-free graphs (see, for instance, Theorems 4.6 and 4.15). For each of the graph classes within which clique-perfect graphs were characterized by forbidden induced subgraphs, also a polynomial-time or even linear-time algorithm for the recognition of clique-perfectness within the class was devised, with the only exception of diamond-free graphs. In [17], the following characterization of those diamond-free graphs that are clique-perfect was given.

Theorem 4.6 ([17]). *Let* G *be a diamond-free graph. Then,* G *is clique-perfect if and only if* G *contains no induced odd generalized sun.*

In [17], also the question of whether there is a polynomial-time algorithm for deciding whether a given diamond-free graph is clique-perfect was posed. Interestingly, the answer can be shown to be affirmative by reducing the problem to that of deciding balancedness, as follows.

Corollary 4.7. *Let* G *be a diamond-free graph. Then,* G *is clique-perfect if and only if* G *is balanced.*

Proof. Since balanced graphs are clique-perfect, we only need to prove that diamond-free clique-perfect graphs are balanced, or equivalently, that a diamond-free graph that is not balanced is not clique-perfect. Let G be a diamond-free graph that is not balanced. By Theorem 3.4, G contains an unbalanced cycle C, that is, an odd cycle C.

Notice that if u and v are two consecutive vertices of C, then $N_G(uv) \cap V(C) = \emptyset$. Indeed, if $N_G(uv) \cap V(C) \neq \emptyset$, then, as $N_G(W_{uv}) \cap N_G(uv) \cap V(C) = \emptyset$, for each $v \in N_G(uv)$ there is some $w \in W_e \subseteq N_G(e)$ such that w is nonadjacent to x and, in particular, $\{u, v, x, w\}$ induces a diamond in G. Since $N_G(uv) \cap V(C) = \emptyset$ for each two consecutive vertices u and v of C, $V(C)$ induces an odd generalized sun in G and, by Theorem 4.2, G is not clique-perfect, as desired. □

Notice also that if G is a diamond-free graph, the problem of deciding whether G is a minimal odd generalized sun can be solved in polynomial time (it suffices to verify that G is not clique-perfect but $G - v$ is clique-perfect for every vertex v of G). Rather surprisingly, the problem of deciding whether a graph is an odd generalized sun (not necessarily minimal) is NP-complete even if G is a triangle-free graph [83]. Indeed, an odd cycle in a triangle-free graph cannot have improper edges. Hence, if G is a triangle-free graph with an odd number of vertices, then G is an odd generalized sun if and only if G has a Hamiltonian cycle, and the Hamiltonian cycle problem on triangle-free graphs with an odd number of vertices is NP-complete [60, pp. 56–60].

4.1.2 Coordinated graphs and hereditary K-perfect graphs

Coordinated graphs and K-perfect graphs were introduced while looking for characterizations of clique-perfect graphs and the three classes are strongly related [19, 20].

Let $\mathcal{F}$ be a family of nonempty sets. The *chromatic index* $\gamma(\mathcal{F})$ of $\mathcal{F}$ is the minimum number of colors necessary to color the members of $\mathcal{F}$ such that any two intersecting members are colored with different colors. For each $x \in \bigcup \mathcal{F}$, let $d_{\mathcal{F}}(x)$ be the number of members of $\mathcal{F}$ to which x belongs and let the *maximum degree* $\Delta(\mathcal{F}) = \max_{x \in \bigcup \mathcal{F}} d_{\mathcal{F}}(x)$. Clearly, $\Delta(\mathcal{F}) \leqslant \gamma(\mathcal{F})$ and $\mathcal{F}$ is said to have the *edge-coloring property* [9] if equality $\Delta(\mathcal{F}) = \gamma(\mathcal{F})$ holds. The edge coloring property has its origins in a celebrated theorem of Kőnig [76] that states that the number of colors needed to color the edges of a bipartite graph in such a way that incident edges receive different colors equals the maximum degree of the graph. This result is known as *Kőnig's edge-coloring theorem*.

Let the *clique-chromatic index* $\gamma_c(G)$ of a graph G be the minimum number of colors needed to assign different colors to intersecting cliques of G and let the *maximum clique-degree* $\Delta_c(G)$ be the maximum cardinality of a family of cliques having at least one vertex of G in common. Then, $\Delta_c(G) \leqslant \gamma_c(G)$ holds for every graph G and a graph G is called *coordinated* [19] if $\Delta_c(G') = \gamma_c(G')$ for each induced subgraph G' of G. Equivalently, a graph is coordinated if, in every induced subgraph, the *rows* of a clique-matrix have the edge coloring property. Interestingly, the edge coloring property is connected to the equality $\omega = \chi$ in such a way that a graph is perfect if and only

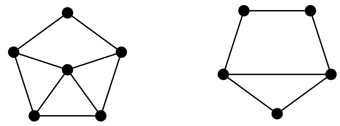


Figure 4.2: *The graph* N_1 *and its clique graph*

if, in every induced subgraph, the *columns* of its clique-matrix have the edge-coloring property. Moreover, in [19], coordinated graphs were proved to form a subclass of the class of perfect graphs. In [25] and [26], coordination was characterized by forbidden induced subgraphs within graphs belonging to different graph classes: line graphs, paw-free graphs, {gem,W_4,bull}-free graphs, and complements of forests. No complete characterization of coordinated graphs by forbidden induced subgraphs is known, but it is known that the recognition problem is NP-hard [110] and the number of minimal forbidden induced subgraphs for the class grows exponentially with the number of vertices and edges [109].

The *clique graph* $K(G)$ of a graph G is the intersection graph of the family of cliques of G. A graph G is called K-*perfect* [20] if $K(G)$ is perfect. Notice that the class of K-perfect graphs is not hereditary. For instance, the graph N_1 of Figure 4.2 is K-perfect but it contains an induced C_5 and $K(C_5) = C_5$ is imperfect. We introduce here the following terminology: a graph will be said *hereditary* K-*perfect graph* if all its induced subgraphs are K-perfect. It turns out that hereditary K-perfect graphs are perfect, as implied by the Strong Perfect Graph Theorem (Theorem 2.3) together with the following lemma.

Lemma 4.8. *A hereditary* K-*perfect graph has no odd holes and has no antiholes with more than* 6 *vertices.*

Proof. Hereditary K-perfect graphs have no odd holes since odd holes are clearly K-imperfect. Along the proof, C_n will denote the graph such that $V(C_n) = \{0, 1, \ldots, n-1\}$ and $E(C_n) = \{01, 12, 23, \ldots, (n-1)0\}$. Assume that $n \geqslant 5$ and $n \neq 6, 7, 9, 12$. By elementary number theory, $n = 5a + 3b$ for some $a \geqslant 1$ and some $b \geqslant 0$. This implies that there exists a sequence $a_1, \ldots, a_k$ of integers taken from the set $\{2, 3\}$ that satisfies the following conditions: (1) $a_1 + \cdots + a_k = n$; (2) $a_i = 2$ for some $i \in \{1, \ldots, k\}$; and (3) for each $j = 1, \ldots, k$, $a_j = 2$ implies $a_{j+1} = 3$ (where a_{k+1} stands for a_1). Assume that such a sequence $\{a_i\}$ is given and define b_i equal to $a_1 + \cdots + a_i$ modulo n for each $i = 1, \ldots, k$. In particular, $b_k = 0$. Let $Q_1 = \{b_1, b_2, \ldots, b_k\}$, $Q_2 = Q_1 + 2$, $Q_3 = Q_1 + 4$, $Q_4 = Q_1 + 1$, and $Q_5 = Q_1 + 3$, where $A + p = \{a + p : a \in A\}$ and the sum is taken modulo n. Then, Q_i is a clique of $\overline{C_n}$ for $i = 1, 2, \ldots, 5$ and, by construction, $Q_1 Q_2 \ldots Q_5 Q_1$ is an odd hole in $K(\overline{C_n})$. Finally, observe that $K(\overline{C_7}) = \overline{C_7}$;

that if $Q_1 = \{0, 2, 4, 6\}$ then $\{Q_1, Q_1 + 1, Q_1 + 2, \ldots, Q_1 + 8\}$ induces a $\overline{C_9}$ in $K(\overline{C_9})$; and that if $Q_1 = \{0, 2, 5, 7, 9\}$ and $Q_2 = \{1, 3, 5, 7, 10\}$ then $\{Q_1, Q_1 + 1, Q_1 + 2, Q_1 + 3, Q_1 + 9, Q_2, Q_2 + 1, Q_2 + 2, Q_2 + 3\}$ induces a $\overline{C_9}$ in $K(\overline{C_{12}})$. □

Interestingly, a careful reading of the proofs in [16, 17, 25] reveals that hereditary K-perfectness was implicitly characterized when restricted to different graph classes: line graphs, Helly circular-arc graphs, hereditary clique-Helly claw-free graphs, paw-free graphs, and {gem,W_4,bull}-free graphs.

In the next subsection, we will show how coordinated and hereditary K-perfect graphs relate to balanced and clique-perfect graph, with the help of some results in hypergraph theory.

4.1.3 Connection with hypergraph theory

A *hypergraph* H is an ordered pair $(X, \mathcal{E})$ where X is a finite set and $\mathcal{E}$ is a family of nonempty subsets of X such that $X = \bigcup \mathcal{E}$. The elements of X are the *vertices* of H and the elements of $\mathcal{E}$ are the *hyperedges* of H. If $x_1, \ldots, x_n$ are the vertices of H and $E_1, \ldots, E_m$ are the hyperedges of H, then a *hyperedge-vertex incidence matrix* of H is a $m \times n$ matrix $A = (a_{ij})$ where a_{ij} is 1 if $x_j \in E_i$ and 0 otherwise. The *dual hypergraph* H^* of a hypergraph $H = (X, \mathcal{E})$ has $\mathcal{E}$ as vertex set and its hyperedges are the sets $\mathcal{E}_x = \{E \in \mathcal{E} : x \in E\}|$ for each $x \in X$. This means that a hyperedge-vertex incidence matrix of H^* is the transpose of one of H.

We will be mostly interested in clique hypergraphs of graphs. Namely, the *clique hypergraph* of a graph G is the hypergraph $\mathcal{K}(G) = (X, \mathcal{E})$ where X is the set of vertices of G and $\mathcal{E}$ is the family of cliques of G. A hyperedge-vertex incidence matrix of $\mathcal{K}(G)$ is a clique-matrix of G.

A hypergraph has the *Kőnig property* if the family of its hyperedges have the Kőnig property. A hypergraph has the *dual Kőnig property* if its dual has the Kőnig property. As discussed in the Introduction, for every graph G, $\alpha_c(G) = \tau_c(G)$ is equivalent to the Kőnig property for $\mathcal{K}(G)$ and $\theta(G) = \alpha(G)$ is equivalent to the dual Kőnig property for $\mathcal{K}(G)$. Therefore, the following holds.

Remark 4.9. *Let G be a graph. Then:*

- G *is perfect if and only if* $\mathcal{K}(G')$ *has the dual Kőnig property for every induced subgraph* G' *of* G
- G *is clique-perfect if and only if* $\mathcal{K}(G')$ *has the Kőnig property for every induced subgraph* G' *of* G.

A hypergraph has the *edge-coloring property* if its hyperedges have the edge coloring property. A hypergraph has the *dual edge-coloring property* if its dual has the edge coloring property. It is easy to see that the equality $\omega(G) = \chi(G)$ for a graph G is equivalent to the dual edge coloring property for $\mathcal{K}(G)$. Therefore, the following holds.

Remark 4.10. *Let G be a graph. Then:*

- G *is perfect if and only if* $\mathcal{K}(G')$ *has the dual edge coloring property for every induced subgraph* G' *of* G
- G *is coordinated if and only if* $\mathcal{K}(G')$ *has the edge coloring property for every induced subgraph* G' *of* G.

A *partial hypergraph* of a hypergraph $H = (X, \mathcal{E})$ is any hypergraph having as hyperedge set a subset of $\mathcal{E}$. A hypergraph has the *Helly property* if the family $\mathcal{E}$ of its hyperedges has the Helly property. So, a graph G is clique-Helly if and only if $\mathcal{K}(G)$ has the Helly property. The *line graph* (or *representative graph*) of a hypergraph $H = (X, \mathcal{E})$, denoted by $L(H)$, is the intersection graph of the family $\mathcal{E}$. The line graph relates clique graphs and clique hypergraphs in the following way: $K(G) = L(\mathcal{K}(G))$. The Kőnig property, the edge coloring property, the Helly property, and perfectness are related in the following way.

Theorem 4.11 ([36, 92])**.** *Let H be a hypergraph, A_H be the hyperedge-vertex incidence matrix of H, and A_H^T be its transpose. Then, the following assertions are equivalent:*

(i) *Every partial hypergraph of* H *has the Kőnig property.*

(ii) *Every partial hypergraph of* H *has the colored edge property.*

(iii) H *has the Helly property and* $L(H)$ *is perfect.*

(iv) *The matrix* A_H^T *is perfect.*

Lovász defined the hypergraphs satisfying the above assertions to be *normal* [92]. Since $K(G) = L(\mathcal{K}(G))$ and recalling Remarks 4.9 and 4.10, Theorem 4.11 implies the following.

Corollary 4.12. *If G is hereditary-clique Helly and hereditary K-perfect, then G is clique-perfect and coordinated.*

Berge defined a hypergraph to be *balanced* [7] if its hyperedge-vertex incidence matrix is balanced. So, a graph is balanced if its clique hypergraph is balanced. In [12], Berge and Las Vergnas proved that balanced hypergraphs had the Kőnig property and, since the partial hypergraphs of a balanced hypergraph are balanced by definition, the following holds.

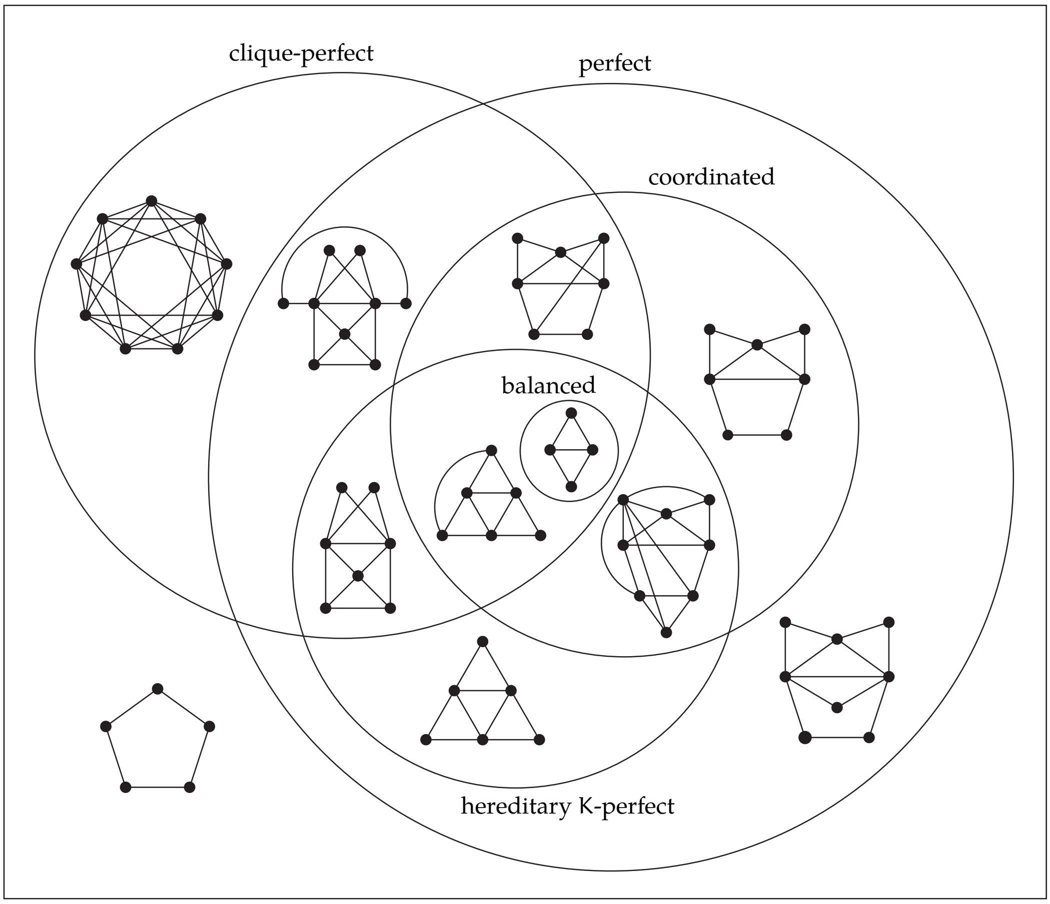


Figure 4.3: *Containment and intersections among the classes of balanced, perfect, clique-perfect, coordinated, and hereditary* K*-perfect graphs.*

Theorem 4.13 ([12])**.** *Every balanced hypergraph is normal.*

In light of Theorem 4.11, the above theorem implies that every balanced graph is clique-Helly and K-perfect. As the class of balanced graphs is hereditary, we have the following.

Corollary 4.14 ([12, 92])**.** *Balanced graphs are hereditary clique-Helly and hereditary* K*-perfect. In particular, balanced graphs are clique-perfect and coordinated.*

Figure 4.3 illustrates the containment relations and intersections among balanced, perfect, clique-perfect, coordinated, and hereditary K-perfect graphs by exhibiting one graph in each possible intersection.

4.2 Clique-perfectness of complements of line graphs

In [16], clique-perfect graphs were characterized by minimal forbidden induced subgraphs within the class of line graphs, as follows.

Theorem 4.15 ([16])**.** *Let* G *be a line graph. Then,* G *is clique-perfect if and only if* G *contains no induced 3-sun and has no odd hole.*

Nevertheless, as clique-perfect graphs are not closed by complementation, this result does not tell us which complements of line graphs are clique-perfect. Precisely, the main result of this section is the following characterization of clique-perfect graphs within complements of line graphs by means of minimal forbidden induced subgraphs.

Theorem 4.16. *Let* G *be the complement of a line graph. Then,* G *is clique-perfect if and only if* G *contains no induced 3-sun and has no antihole* $\overline{C_k}$ *for any* $k \geqslant 5$ *such that* k *is not a multiple of* 3*.*

Let G be the complement of the line graph of a graph H. In order to prove Theorem 4.16, we profit from the fact that the cliques of G are precisely the maximal matchings of H. We call a *matching-transversal* of H any set of edges meeting all the maximal matchings of H and *matching-independent set* of H any collection of edge-disjoint maximal matchings of H. We define the *matching-transversal number* $\tau_m(H)$ of H as the minimum size of a matching-transversal of H and the *matching-independence number* $\alpha_m(H)$ of H as the maximum size of a matching-independent set of H. Clearly, $\alpha_c(G) = \alpha_m(H)$ and $\tau_c(G) = \tau_m(H)$. Finally, we say that H is *matching-perfect* if $\alpha_m(H') = \tau_m(H')$ for every subgraph H' (induced or not) of H. Hence, G is clique-perfect if and only if H is matching-perfect, and Theorem 4.16 can be reformulated as follows.

Theorem 4.17. *Let* H *be a graph. Then,* H *is matching-perfect if and only if* H *contains no bipartite claw and the length of each cycle of* H *is at most* 4 *or is a multiple of* 3*.*

Recall that 'H contains no bipartite claw' means H contains neither induced nor non-induced subgraphs isomorphic to the bipartite claw. To prove Theorem 4.17, it suffices to show that, if H is a graph containing no bipartite claw and the length of each cycle of H is at most 4 or is a multiple of 3, then $\alpha_m(H) = \tau_m(H)$. In addition, we can assume that H is connected because clearly $\alpha_m(H)$ (resp. $\tau_m(H)$) is the minimum of $\alpha_m(H')$ (resp. $\tau_m(H')$) among the components H' of H. The proof splits into two parts according to whether or not H has some cycle of length at least 5. In both cases, we obtain an upper bound on $\tau_m(H)$ and exhibit a collection of edge-disjoint maximal matchings of the same size, which means that $\alpha_m(H) = \tau_m(H)$. To produce these collections of edge-disjoint maximal matchings, we ultimately rely on edge-coloring (via Theorem 4.30) tailored subgraphs of H.

The structure of this section is as follows. In Subsection 4.2.1, we present a structure theorem for graphs containing no bipartite claw that is used all along this section. In Subsection 4.2.2, we completely describe those graphs not containing bipartite claw that are Class 2 with respect to edge-coloring. In Subsection 4.2.3, we prove the main results of this section (Theorems 4.16 and 4.17). Finally, in Subsection 4.2.4, we show a linear-time recognition algorithm for matching-perfect graphs and a quadratic-time one for clique-perfect graphs that follow from our main results.

4.2.1 Linear and circular structure of graphs containing no bipartite claw

In this subsection, we present a structure theorem for graphs containing no bipartite claw that will prove very useful to us all along this section. In [30], the linear and circular structure of net-free$\cap$claw-free graphs is studied. As the line graphs of graphs containing no bipartite claw are the net-free $\cap$ line graphs, the main result of this subsection (Theorem 4.25 on page 93) can be regarded as describing a more explicit linear and circular structure for the more restricted class of net-free $\cap$ line graphs.

Our structure theorem will be stated in terms of linear and circular concatenations of two-terminal graphs that we now introduce. A *two-terminal graph* is a triple $\Gamma = (H, s, t)$, where H is a graph and s and t are two different vertices of H, called the *terminals* of Γ. We now introduce in some detail the two-terminal graphs depicted in Figure 4.4. For each $m \geqslant 0$, the m-*crown* is the two-terminal graph (H, s, t) where $V(H) = \{s, t, a_1, a_2, \ldots, a_m\}$ and $E(H) = \{st\} \cup \{sa_i : 1 \leqslant i \leqslant m\} \cup \{ta_i : 1 \leqslant i \leqslant m\}$. The 0-crown and the 1-crown are called *edge* and *triangle*, respectively. For each $m \geqslant 2$, the m-*fold* is the two-terminal graph (H, s, t) where $V(H) = \{s, t, a_1, a_2, \ldots, a_m\}$ and $E(H) = \{sa_i : 1 \leqslant i \leqslant m\} \cup \{ta_i : 1 \leqslant i \leqslant m\}$. The 2-fold is also called *square*. By a *crown* we mean an m-crown for some $m \geqslant 0$, and by a *fold* we mean an m-fold for some $m \geqslant 2$. Finally, K_4 will also denote the two-terminal graph (K_4, s, t) for any two vertices s and t of the K_4. We will refer to the crowns, folds, rhombus, and K_4 as the *basic two-terminal graphs*.

If $\Gamma = (H, s, t)$ is a two-terminal graph, H is called the *underlying* graph of Γ, s is its *source*, and t its *sink*. If $\Gamma_1 = (H_1, s_1, t_1)$ and $\Gamma_2 = (H_2, s_2, t_2)$ are two-terminal graphs, the p-*concatenation* $\Gamma_1 \&_p \Gamma_2$ is the two-terminal graph (H, s_1, t_2) where H arises from $H_1 \cup H_2$ by identifying t_1 and s_2 into one vertex u and attaching p pendant vertices adjacent to u. The 0-concatenation $\Gamma_1 \&_0 \Gamma_2$ is denoted simply by $\Gamma_1 \& \Gamma_2$. If a two-terminal graph $\Gamma = (H, s, t)$ is such that $N_H[s] \cap N_H[t] = \varnothing$, we define its p-*closure*, denoted $\Gamma \&_p \circlearrowright$, as the graph that arises by identifying s and t into one vertex u and then attaching p pendant vertices adjacent to u. The 0-closure of Γ is simply denoted

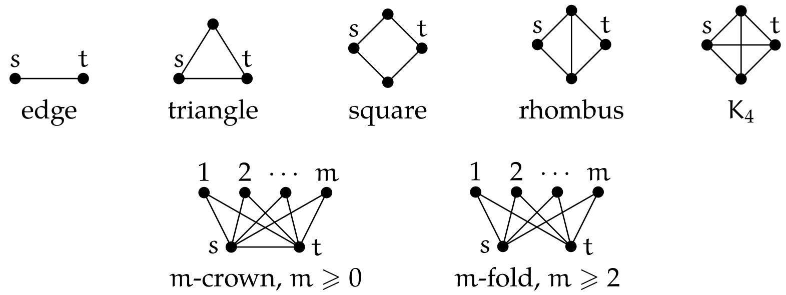


Figure 4.4: *Basic two-terminal graphs with terminals* s *and* t

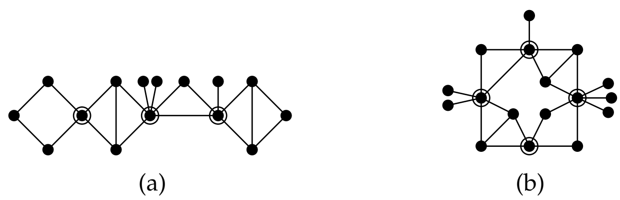


Figure 4.5: *A linear and a circular concatenation the sequence* $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ *of two-terminal graphs, where* Γ_1 *is a square,* Γ_2 *and* Γ_4 *are rhombi, and* Γ_3 *is a triangle: (a) Underlying graph of* $\Gamma_1 \,\&\, \Gamma_2 \,\&_2\, \Gamma_3 \,\&_1\, \Gamma_4$ *and (b)* $\Gamma_1 \,\&\, \Gamma_2 \,\&_2\, \Gamma_3 \,\&_1\, \Gamma_4 \,\&_3 \circlearrowleft$*. Concatenation vertices are circled.*

by $\Gamma \,\&\circlearrowleft$.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be a sequence of two-terminal graphs. A *linear concatenation* of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is the underlying graph of the two-terminal graph $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_2} \cdots \&_{p_{n-1}} \Gamma_n$ for some nonnegative integers $p_1, p_2, \ldots, p_{n-1}$. The two-terminal graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are called the *links* of the linear concatenation. The *concatenation vertices* of such a linear concatenation are the $n-1$ vertices that arise by identifying the sink of Γ_i with the source of Γ_{i+1} for each $i = 1, 2, \ldots, n-1$. The two links Γ_i and Γ_{i+1} are called *adjacent* in the linear concatenation, for each $i = 1, 2, \ldots, n-1$. The graph K_1 will be regarded as the linear concatenation of an empty sequence of two-terminal graphs. See Figure 4.5(a) for an example of a linear concatenation. A *circular concatenation* of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is any graph $\Gamma_1 \,\&_{p_1}\, \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}}\, \Gamma_n \,\&_{p_n} \circlearrowleft$ for some nonnegative integers $p_1, p_2, \ldots, p_{n-1}$. The two-terminal graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are called the *links* of the circular concatenation. The *concatenation vertices* of such a circular concatenation are the $n-1$ vertices that arise by identifying the sink of Γ_i with the source of Γ_{i+1} for each $i = 1, 2, \ldots, n-1$, as well as the vertex that arises by identifying the sink of Γ_n with the source of Γ_1. The two links Γ_i and Γ_{i+1} are called *adjacent* in the circular concatenation, for each $i = 1, 2, \ldots, n-1$, as well as the links Γ_n and Γ_1. See Figure 4.5(b) for an example of a circular concatenation. Each of the Γ_i's is called a *link* of either the linear or the circular concatenation.

4.2.1.1 Structure of fat caterpillars

A *caterpillar* [69] is a connected graph containing no bipartite claw and having no cycle. We call *fat caterpillars* to those connected graphs containing no bipartite claw and having no cycle of length greater than 4. The fact that caterpillars have edge-dominating paths gives them a very simple linear structure; namely, they are linear concatenations (in our sense) of edge links [68]. We will show that fat caterpillars containing no A and no net are linear concatenations of basic two-terminal graphs, like the graph depicted in Figure 4.5(a). This result will be the last in the following sequence of three lemmas.

Lemma 4.18. *Let H be a fat caterpillar containing no A and no net. Then, H has an edge-dominating path $P = u_0u_1 \dots u_\ell$ having no long chords and no three consecutive short chords, and such that each vertex $v \in V(H)\backslash V(P)$ satisfies one the following assertions:*

(i) *v is a pendant vertex and the only neighbor of v is neither an endpoint of P nor the midpoint of any short chord of P.*

(ii) *v has degree 2 and is a false twin of u_j for some $j \in \{1, 2, \dots, \ell - 1\}$.*

(iii) *v has degree 3 and is a true twin of u_j for some $j \in \{1, \ell - 1\}$ such that u_{j-1} is adjacent to u_{j+1}.*

Proof. If H is the underlying graph of an m-crown for some $m \geqslant 3$, then the lemma holds trivially by letting P be any path of H of length 2 whose endpoints are the two vertices of H of degree $m+1$. Therefore, without loss of generality, we will assume that H is not the underlying graph of an m-crown for any $m \geqslant 3$. Among the longest paths of H without long chords, let us choose some path $P = u_0u_1u_2 \dots u_\ell$ that maximizes $d_H(u_0) + d_H(u_\ell)$ and, among those with maximal $d_H(u_0) + d_H(u_\ell)$, we choose one that minimizes $\min\{d_H(u_0), d_H(u_\ell)\}$. We will show that P satisfies the thesis of the lemma. Notice that P has no long chords by construction and that P has no three consecutive short chords simply because H has no 5-cycle. The lemma follows from the following four claims.

Claim 1. P *is edge-dominating.*

Proof of the claim. Suppose, by the way of contradiction, that P is not edge-dominating. Since H is connected, there is some edge vw of H such that none of v and w is a vertex of P and v is adjacent to u_j for some $j \in \{0, 1, 2, \dots, \ell\}$. Since H contains no bipartite claw, $j \in \{0, 1, \ell - 1, \ell\}$. Let us consider first the case $j = 0$. Then, the path vP must have some long chord because it is longer than P. Since P has no long chords and H has no cycle of length greater than 4, necessarily v is adjacent to u_2. So, as H contains no A, $\ell = 2$. Then, as $P' = u_1u_0vw$ is a path longer than P, P' must have some long chord; i.e., w is adjacent to u_1. In addition, $\{u_0, u_2, w\}$ is a stable set because H has no 5-cycles.

Moreover, $N_H(u_0) = N_H(u_2) = N_H(w) = \{u_1, v\}$ because H contains no A. Now, $P'' = u_1u_0v$ is a path of the same length than P but the sum of the degrees of the endpoints of P'' is $d_H(u_1) + d_H(v) > 4 = d_H(u_0) + d_H(u_2)$, which contradicts the choice of P. The contradiction arose from assuming that $j = 0$. So, $j \neq 0$ and, symmetrically, $j \neq \ell$. Therefore, also by symmetry, we assume, without loss of generality, that $j = 1$. As $P''' = wvu_1u_2 \ldots u_\ell$ is longer than P, P''' must have some long chord. So, as H is a fat caterpillar containing no A and no net, this means that w is adjacent to u_2 and $\ell = 2$. But then, we find ourselves in the case $j = \ell$ by letting w play the role of v and vice versa, which leads again to a contradiction. As this contradiction arose from assuming that P was not edge-dominating, Claim 1 follows. □

Claim 2. *If $v \in V(H) \backslash V(P)$ is pendant, then (i) holds.*

Proof of the claim. Suppose that $v \in V(H) \backslash V(P)$ is pendant. As P is edge-dominating, $N_H(v) = \{u_j\}$ for some $j \in \{0, 1, 2, \ldots, \ell\}$. If $j = 0$, then vP would be a path longer than P and without long chords, contradicting the choice of P. This contradiction proves that $j \neq 0$ and, by symmetry, $j \neq \ell$. Suppose, by the way of contradiction, that u_j is the midpoint of some short chord of P; i.e., u_{j-1} is adjacent to u_{j+1}. Since H contains no net and by symmetry, we assume, without loss of generality, that $j = 1$. As $vu_1u_0u_2u_3 \ldots u_\ell$ is longer than P, it must have some long chord; i.e., u_1 is adjacent to u_3. Then, as H contains no A and P has no long chords, $\ell = 3$ and $d_H(u_0) = d_H(u_3) = 2$. So, $P' = vu_1u_0u_2$ is a path of the same length that P without long chords and such that $d_H(v) + d_H(u_2) \geqslant 4 = d_H(u_0) + d_H(u_3)$ and $\min\{d_H(v), d_H(u_2)\} = 1 < \min\{d_H(u_0), d_H(u_3)\}$, which contradicts the choice of P. This contradiction arose from assuming that v was adjacent to the midpoint of some short chord of P. Now, Claim 2 follows. □

Claim 3. *If $v \in V(H) \backslash V(P)$ has degree 2, then (ii) holds.*

Proof of the claim. Let $v \in V(H) \backslash V(P)$ of degree 2 and suppose, by the way of contradiction, that v is adjacent to two consecutive vertices of P; i.e., $N_H(v) = \{u_j, u_{j+1}\}$ for some $j \in \{0, 1, 2, \ldots, \ell - 1\}$. If $j = 0$, then vP would be a path without long chords and longer than P, contradicting the choice of P. Therefore, $j \geqslant 1$ and, by symmetry, $j \leqslant \ell - 1$. The path $u_0u_1 \ldots u_jvu_{j+1}u_{j+2} \ldots u_\ell$ must have some long chord because it is longer than P and, as P has no long chords, this means that u_ju_{j+2} or $u_{j+1}u_{j-1}$ is a chord of P. By symmetry, suppose, without loss of generality, that u_ju_{j+2} is a chord of P. Then, $j = \ell - 2$ since otherwise H would contain A. In addition, $N_H(u_\ell) = \{u_{\ell-2}, u_{\ell-1}\}$ because P has no long chords and H contains no A. Hence, $d_H(u_\ell) = 2 < d_H(u_{\ell-1})$. Now, $P' = u_0u_1 \ldots u_{\ell-2}vu_{\ell-1}$ is a path of the same length than P but $d_H(u_0) + d_H(u_{\ell-1}) > d_H(u_0) + d_H(u_\ell)$. Because of the choice of P, P' must have some long chord and, necessarily, u_{j+1} is adjacent to u_{j-1}. As u_j adjacent to u_{j+2} implies $j = \ell - 2$ and $d_H(u_\ell) = 2$, u_{j+1} adjacent to u_{j-1} implies $j = 1$ and $d_H(u_0) = 2$. Therefore, $\ell = 3$, $d_H(u_0) = d_H(u_\ell) = 2$, and $N_H(v) = \{u_1, u_2\}$. Hence, as H is connected and P is edge-dominating, every vertex $v \in V(H)\setminus V(P)$ is adjacent to u_1 and/or to u_2 only. If some vertex $w \in V(H)\setminus V(P)$ were adjacent to u_1 but not to u_2, then $P'' = wu_1u_0u_2$ would be a path without long chords of the same length than P and such that $d_H(w) + d_H(u_2) > 4 = d_H(u_0) + d_H(u_3)$, contradicting the choice of P. This proves that each vertex $w \in V(H)\setminus V(P)$ satisfies $N_H(w) = \{u_1, u_2\}$. We conclude that H is the underlying graph of an m-crown for some $m \geqslant 3$, which contradicts our initial hypothesis. This contradiction arose from assuming that v was adjacent to two consecutive vertices of P. So, as P is edge-dominating and H has no cycle of length greater than 4, necessarily $N_H(v) = \{v_{j-1}, v_{j+1}\}$ for some $j \in \{1, 2, \ldots, \ell-1\}$. Suppose, by the way of contradiction, that $d_H(u_j) > 2$ and let w be a neighbor of u_j different from u_{j-1} and u_{j+1}. Then, as H contains no A and has no 5-cycle, $\ell = 2$ and $j = 1$. But then, wu_1u_2v is a path longer than P and without long chords, contradicting the choice of P. This contradiction arose from assuming that $d_H(u_j) > 2$. Consequently, u_j is a false twin of v and (ii) holds. Hence, Claim 3 follows. □

Claim 4. *If $v \in V(H)\setminus V(P)$ has degree at least 3, then (iii) holds.*

Proof of the claim. Let $v \in V(H)\setminus V(P)$ of degree at least 3. As P is edge-dominating and H has no cycles of length greater than 4, $N_H(v) = \{u_{j-1}, u_j, u_{j+1}\}$ for some $j \in \{1, 2, \ldots, \ell-1\}$. As $u_0u_1 \ldots u_{j-1}vu_ju_{j+1} \ldots u_\ell$ and $u_0u_1 \ldots u_{j-1}u_jvu_{j+1} \ldots u_\ell$ are longer than P, they have at least one long chord each. So, if u_{j-1} were nonadjacent to u_{j+1}, then u_j would be adjacent to u_{j-2} and to u_{j+2} and, therefore, $vu_{j+1}u_{j+2}u_ju_{j-2}u_{j-1}v$ would be a 6-cycle of H, a contradiction. Therefore, u_{j-1} is adjacent to u_{j+1}. As H contains no A, $j = 1$ or $j = \ell-1$. By symmetry, assume that $N_H(v) = \{u_0, u_1, u_2\}$. Suppose, by the way of contradiction, that u_1 is not a true twin of v. Then, there is some $w \in N_H(u_1)\setminus\{v, u_0, u_2\}$ and, as P is edge-dominating and H has no cycle of length greater than 4, w is pendant. Then, $wu_1u_0u_2u_3 \ldots u_\ell$ is a path longer than P and without long chords, a contradiction with the choice of P. This contradiction proves that v is a true twin of u_1 and (iii) holds. This completes the proof of Claim 4 and of the lemma. □

Lemma 4.19. *Let H be a fat caterpillar containing no A and no net, let $P = u_0u_1 \ldots u_\ell$ as in the statement of Lemma 4.18, and suppose that $\ell \geqslant 1$. Then, H is the underlying graph of $\Gamma_1 \,\&_{p_1}\, \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}}\, \Gamma_n$ for some basic two-terminal graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ and some nonnegative integers $p_1, p_2, \ldots, p_{n-1}$ such that the source of Γ_1 is u_0 and the sink of Γ_n is u_ℓ.*

Proof. The proof will be by induction on ℓ. If $\ell = 1$, H is the underlying graph of an edge link with source u_0 and sink u_1. Let $\ell \geqslant 2$ and assume that the claim holds whenever P has length less than ℓ. We will define a two-terminal graph Γ_1 by considering several cases. In each case, we assume that the preceding cases do not hold.

Case 1. u_0 *is adjacent to some vertex* $v \in V(H)\backslash V(P)$ *of degree* 3.

Then, by assertions (i)–(iii) of Lemma 4.18, we have that v is a true twin of u_1 and $N_H(u_0) = \{v, u_1, u_2\}$. We define Γ_1 to be the two-terminal graph with source u_0 and sink u_2 and whose underlying graph is the subgraph of H induced by $N_H[v]$. In particular, Γ_1 is a K_4.

Case 2. u_0 *is adjacent to some vertex in* $v \in V(H)\backslash V(P)$ *of degree* 2.

Then, by assertions (i)–(iii) of Lemma 4.18, we have that v is a false twin of u_1 and each neighbor of u_0 in $V(H)\backslash V(P)$ is also a false twin of u_1. We define Γ_1 as the two-terminal graph with source u_0 and sink u_2, and whose underlying graph is the subgraph of H induced by $N_H[u_0] \cup \{u_2\}$. Notice that Γ_1 is a crown or a fold, depending on whether or not u_0 is adjacent to u_2.

As (i)–(iii) of Lemma 4.18 imply that each neighbor of u_0 in $V(H)\backslash V(P)$ has degree 2 or 3, in the cases below we are assuming that u_0 has no neighbors in $V(H)\backslash V(P)$.

Case 3. u_0 *is adjacent to* u_2 *and* u_1 *is adjacent to* u_3.

Then, by assertions (i)–(iii) of Lemma 4.18, $d_H(u_0) = 2$ and $d_H(u_1) = d_H(u_2) = 3$. Let Γ_1 be the two-terminal graph with source u_0 and sink u_3, and whose underlying graph is the subgraph of H induced by $\{u_0, u_1, u_2, u_3\}$. Then, Γ_1 is a rhombus.

Case 4. u_0 *is adjacent to* u_2 *and* u_1 *is nonadjacent to* u_3.

As u_1 is the midpoint of the short chord u_0u_2 and we are assuming that u_0 has no neighbors in $V(H)\backslash V(P)$, assertions (i)–(iii) of Lemma 4.18 imply that u_1 has no neighbors in $V(H)\backslash V(P)$. Therefore, as u_1 is nonadjacent to u_3, $d_H(u_1) = 2$. Let Γ_1 be the two-terminal graph whose source is u_0 and sink u_2, and whose underlying graph is the subgraph of H induced by $\{u_0, u_1, u_2\}$. Then, Γ_1 is a triangle.

Case 5. u_0 *is nonadjacent to* u_2.

In this case, we define Γ_1 as the two-terminal graph with source u_0, sink u_1, and whose underlying graph is the induced subgraph of H induced by $\{u_0, u_1\}$. Then, Γ_1 is an edge.

Once defined Γ_1 as prescribed in Cases 1 to 5 above, we let j be such that u_j is the sink of Γ_1, $v_1, v_2, \ldots, v_{p_1}$ be the pendant vertices adjacent to u_j, $P' = u_j u_{j+1} \ldots u_\ell$, and $H' = H - ((V(\Gamma_1)\backslash\{u_j\}) \cup \{v_1, \ldots, v_{p_1}\})$. By construction, H' and P' satisfy the statement of Lemma 4.18 by letting H' and P' play the roles of H and P, respectively. If $j = \ell$, then H is the underlying graph of Γ_1 with source u_0 and sink u_ℓ and the lemma holds for H. If $j < \ell$, by induction hypothesis, H' is the underlying graph of some $\Gamma_2 \&_{p_2} \Gamma_3 \&_{p_3} \cdots \&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graphs and each

$p_i \geqslant 0$, the source of Γ_2 is u_j, and the sink of Γ_n is u_ℓ. So, H is the underlying graph of $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_2} \Gamma_3 \&_{p_3} \cdots \&_{p_{n-1}} \Gamma_n$ where u_0 is the source of Γ_1 and u_ℓ is the sink of Γ_n. Now, Lemma 4.19 follows by induction. □

As a consequence of the two above results, we now prove a structural characterization for fat caterpillars containing no A and no net.

Lemma 4.20. *Let* H *be a graph. Then,* H *is a fat caterpillar containing no* A *and no net if and only if* H *is a linear concatenation of crowns, folds, rhombi, and* K_4*'s where the* K_4 *links may occur only as the first and/or last links of the concatenation.*

Proof. Suppose that H is a linear concatenation of a sequence $\Gamma_1, \ldots, \Gamma_n$ of basic two-terminal graphs such that if Γ_j is a K_4 then $j \in \{1, n\}$. Then, H contains no A and no net because each 4-cycle of H has two nonconsecutive vertices adjacent to vertices of the 4-cycle only and each triangle of H has at least one vertex of degree 2. Moreover, H has no cycle of length greater than 4 because each cycle of H is contained in one of the links and, by construction, the links are basic. Suppose, by the way of contradiction, that H contains a bipartite claw B. Let b_0 be the center of B and let b_1, b_2, and b_3 be the neighbors of b_0 in B. As b_0 has degree at least 3 in H, b_0 is a concatenation vertex of H or a non-terminal vertex of a rhombus link. If b_0 were the non-terminal vertex of a rhombus links and, without loss of generality, b_1 were the remaining non-terminal vertex of the same link, then $N_H(b_1) = \{b_0, b_2, b_3\}$, which contradicts the choice of b_0, b_1, b_2, and b_3. Therefore, b_0 is necessarily a concatenation vertex of H. As each of b_1, b_2, and b_3 is a non-pendant vertex, at least two of them belong to the same link of H. Hence, we assume, without loss of generality, that b_0 is a terminal vertex of Γ_j for some $j \in \{1, 2, \ldots, n\}$ and b_1 and b_2 are two other vertices of Γ_j. By construction, $b_1, b_2 \in N_H(b_0)$, $N_H(b_1) \setminus \{b_0, b_2\} \neq \varnothing$, $N_H(b_2) \setminus \{b_0, b_1\} \neq \varnothing$, and $|(N_H(b_1) \cup N_H(b_2)) \setminus \{b_0, b_1, b_2\}| \geqslant 2$. So, since Γ_j is basic, necessarily Γ_j is a K_4 and either b_1 or b_2 is also a concatenation vertex of H. By symmetry, we assume, without loss of generality, that $j = 1$, b_0 is the source of Γ_1, b_1 is the sink of Γ_1, and b_2 and b_3 are the non-terminal vertices of H. Then, $N_H[b_2] = N_H[b_3] = \{b_0, b_1, b_2, b_3\}$, contradicting the choice of b_0, b_1, b_2, and b_3. This contradiction shows that H contains no bipartite claw and we conclude that H is a fat caterpillar.

Conversely, let H be a fat caterpillar containing no A and no net. If $H = K_1$, H is the linear concatenation of an empty sequence of two-terminal graphs. Otherwise, there is some path $P = u_0 u_1 \ldots u_\ell$ as in Lemma 4.18 and $\ell \geqslant 1$. Then, by Lemma 4.19, H is the linear concatenation of basic two-terminal graphs. Moreover, as H contains no A, the K_4 links, if any, occur as first and/or last links of the concatenation, which completes the proof of Lemma 4.20. □

The following two lemmas describe the structure of the remaining fat caterpillars; i.e., those containing A or net.

Lemma 4.21. *Let H be graph. Then, H is a fat caterpillar containing A if and only if H has an edge-dominating 4-cycle $C = v_1v_2v_3v_4v_1$ and two different vertices $x_1, x_2 \in V(H)\backslash V(C)$ such that x_i is adjacent to v_i for $i = 1, 2$, each non-pendant vertex in $V(H)\backslash V(C)$ is a false twin of v_4 of degree 2, and one of the following holds:*

(i) C *is chordless.*

(ii) v_1v_3 *is the only chord of* C *and* $d_H(v_4) = 2$.

(iii) C *has two chords and* $d_H(v_3) = d_H(v_4) = 3$.

Proof. The 'if' part is clear. In order to prove the 'only if', suppose that H is a fat caterpillar containing A. Then, there is some 4-cycle $C = v_1v_2v_3v_4v_1$ and two different vertices $x_1, x_2 \in V(H)\backslash V(C)$ such that x_i is adjacent to v_i for $i = 1, 2$. As H contains no bipartite claw and H is connected, C is edge-dominating in H. Therefore, as H has no 5-cycle, each vertex in $V(H)\backslash V(C)$ is pendant or has exactly two neighbors which are nonconsecutive vertices of C. If there are two non-pendant vertices $w_1, w_2 \in V(H)\backslash V(C)$, then w_1 and w_2 are false twins because H contains no bipartite claw. Therefore, we assume, without loss of generality, that each non-pendant vertex in $V(H)\backslash V(C)$ is adjacent precisely to v_1 and v_3. Thus, if there is some non-pendant vertex $w \in V(H)\backslash V(C)$, then w is a false twin of v_4 because H contains no bipartite claw and has no 5-cycle. If C is chordless, then (i) holds. If C has two chords, then, as H contains no bipartite claw, $d_H(v_3) = d_H(v_4) = 3$ and (iii) holds. Suppose that C has exactly one chord and assume, without loss of generality, that v_1v_3 is the only chord of C. As H has no 5-cycle and contains no bipartite claw, $d_H(v_4) = 2$ and (ii) holds. □

Lemma 4.22. *Let H be a graph. Then, H is a fat caterpillar containing net but containing no A if and only if H has some edge-dominating triangle C such that each vertex in $V(H)\backslash V(C)$ is pendant.*

Proof. The 'if' part is clear. For the converse, suppose that H contains no bipartite claw. Since H contains net, there are six different vertices $v_1, v_2, v_3, x_1, x_2, x_3$ such that v_1, v_2, v_3 are pairwise adjacent and v_i is adjacent to x_i for $i = 1, 2, 3$. As H contains no bipartite claw and H is connected, $C = v_1v_2v_3v_1$ is edge-dominating in H. In addition, as H contains no A, each vertex in $V(H)\backslash V(C)$ is pendant. □

We close this sub-subsection with the following result that summarizes the structure of fat caterpillars.

Theorem 4.23. *A graph* H *is a fat caterpillar if and only if exactly one of the following conditions holds:*

(i) H *is a linear concatenation of crowns, folds, rhombi, and* K_4*'s where the* K_4 *links may occur only as the first and/or last links of the concatenation.*

(ii) H *is the circular concatenation* $\text{edge} \,\&_{p_1}\, \text{edge} \,\&_{p_2}\, \text{edge} \,\&_{p_3}\, \text{edge} \,\&_{p_4} \circlearrowright$ *for some nonnegative integers* p_1, p_2, p_3, p_4 *such that* $p_1, p_2 \geqslant 1$.

(iii) H *is the circular concatenation* $\text{edge} \,\&_{p_1}\, \text{edge} \,\&_{p_2}\, m\text{-fold} \,\&_{p_3} \circlearrowright$ *for some* $m \geqslant 2$ *and some nonnegative integers* p_1, p_2, p_3, p_4 *such that* $p_1, p_2 \geqslant 1$.

(iv) H *is the circular concatenation* $\text{edge} \,\&_{p_1}\, \text{edge} \,\&_{p_2}\, m\text{-crown} \,\&_{p_3} \circlearrowright$ *for some* $m \geqslant 1$ *and some nonnegative integers* p_1, p_2, p_3, p_4 *such that* $p_1, p_2 \geqslant 1$.

(v) H *is the underlying graph of* $\text{edge}\&_{p_1} K_4 \&_{p_2} \text{edge}$ *for some nonnegative integers* p_1, p_2.

(vi) H *is the circular concatenation* $\text{edge}\&_{p_1}\text{edge}\&_{p_2}\text{edge}\&_{p_3}\circlearrowright$ *for some positive integers* p_1, p_2, p_3.

4.2.1.2 Structure theorem for graphs containing no bipartite claw

To prove our structure theorem, we need to prove first the following lemma.

Lemma 4.24. *Let* H *be a connected graph containing no bipartite claw and having some cycle of length at least* 5*. Assume further that the* 5*-cycles of* H *are chordless and the* 6*-cycles of* H *have no long chords and no three consecutive short chords. If* $C = u_1u_2 \dots u_\ell u_1$ *is a longest cycle of* H*, then* C *has no long chords and no three consecutive short chords and, for each vertex* $v \in V(H)\backslash V(C)$*, one of the following assertions holds:*

(i) v *is pendant and its only neighbor is not the midpoint of any short chord of* C.

(ii) v *has degree* 2 *and is a false twin of* u_j *for some* $j \in \{1, 2, \dots, \ell\}$.

As a result, H *is a circular concatenation of crowns, folds, and rhombi.*

Proof. C has length at least 5 by hypothesis and C is edge-dominating in H because H contains no bipartite claw. If C had a long chord, then C would have length at least 7 (because we are assuming that the 6-cycles have no long chords) and, as a consequence, H would contain a bipartite claw. Hence, C has no long chords. If C had three consecutive short chords, then C would have length at least 7 (because we are assuming that the 5-cycles are chordless and the 6-cycles have no three consecutive short chords) and would imply that H contains a bipartite claw. This means that C has no three consecutive short chords.

Let $v \in V(H) \setminus V(C)$. As C is edge-dominating and H is connected, $d_H(v) \geqslant 1$. Assume first that v is pendant. If the only neighbor of v were the midpoint of some short chord of C, then C should have length at least 6 (because we are assuming that 5-cycles are chordless) and, therefore, H would contain a bipartite claw, a contradiction. Therefore, if v is pendant, then (i) holds. Assume now that v is non-pendant. As C is a longest cycle of H, no two consecutive vertex of C are adjacent to v. Moreover, as H contains no bipartite claw, v has no two neighbors at distance larger than 2 within C. This means that if v had at least three neighbors, then C would be a 6-cycle and v would be adjacent to every second vertex of C, but then H would contain a bipartite claw. We conclude that v has exactly two neighbors and that this two neighbors are at distance 2 within C; i.e., $N_H(v) = \{u_{j-1}, u_{j+1}\}$ for some $j \in \{1, \ldots, \ell\}$ (from this point on, subindices should be understood modulo ℓ) and, due to the fact that H contains no bipartite claw and its 5-cycles are chordless, u_j is a false twin of v. This proves that if v is not pendant, then (ii) holds.

It only remains to prove that H is a circular concatenation of crowns, folds, and rhombi. We claim that there is some $k \in \{1, 2, \ldots, \ell\}$ such that u_k is neither the midpoint of any short chord of C nor a false twin of any vertex outside $V(C)$. Indeed, if no vertex of C is a false twin of a vertex outside $V(C)$, the existence of k is guaranteed by the fact that C has no three consecutive short chords. Suppose that, on the contrary, there is some $j \in \{1, \ldots, \ell\}$ such that u_j is a false twin of a vertex outside $V(C)$. Then, as C is a longest cycle of H, u_{j-1} is not the midpoint of a short chord of C and u_{j-1} is not the false twin of any vertex outside $V(C)$ because $d_H(u_{j-1}) > 2$. Then, the claim holds by letting $k = j - 1$. This concludes the proof of the claim.

Assume, without loss of generality, that u_ℓ is neither the midpoint of any short chord nor a false twin of any vertex outside $V(C)$. Let $v_1, v_2, \ldots, v_q$ be the pendant vertices of H incident to u_ℓ. We create a new vertex u_0 and we add the edge u_0u_1 and the edges joining u_0 to every false twin of u_1 outside $V(C)$ (if any). If u_ℓ is adjacent to u_2, then we also add an edge joining u_0 to u_2. Finally, we remove every edge joining u_ℓ to a neighbor of u_0. Let H' be the graph that arises this way and let $P' = u_0u_1u_2 \ldots u_\ell$. Clearly, H' and P' satisfy Lemma 4.18 by letting H' and P' play the roles of H and P, respectively. So, by Lemma 4.19 and its proof, H' is the underlying graph of some $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \& \cdots \&_{p_{n-1}} \Gamma_n$ where each Γ_i is a crown, a fold, or a rhombus, and each $p_i \geqslant 0$. (Indeed, no Γ_i is a K_4 because no vertex $v \in V(H') \setminus V(P')$ has degree 3.) Finally, H is the circular concatenation $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \& \cdots \&_{p_{n-1}} \Gamma_n \,\&_q \circlearrowleft$, where each link is a crown, a fold, or a rhombus. $\square$

The next theorem is the main result of this subsection and proves that, except for a few sporadic cases (assertions (i), (ii), and (iii) below), connected graphs containing no bipartite claw are linear and circular concatenations of basic two-terminal graphs (assertion (iv)).

Theorem 4.25. *Let* H *be a connected graph. Then,* H *contains no bipartite claw if and only if at least one of the following assertions holds:*

(i) H *is spanned by a* 6*-cycle having a long chord or three consecutive short chords.*

(ii) H *has a* 5*-cycle* C *and a vertex* $u \in V(C)$ *such that:* (1) *each* $v \in V(H)\setminus V(C)$ *is a pendant vertex adjacent to* u *and* (2) C *has three consecutive short chords or* u *is the midpoint of a chord of* C*.*

(iii) H *has a complete set* Q *of size* 4 *and there are two vertices* $q_1, q_2 \in Q$ *such that:* (1) *each* $v \in V(H)\setminus V(Q)$ *is a pendant vertex adjacent to* q_1 *or* q_2 *and* (2) *there is at least one pendant vertex adjacent to* q_i *for* $i = 1, 2$*.*

(iv) H *is a linear or circular concatenation of crowns, folds, rhombi, and* K_4*'s, where the* K_4 *links may occur only in the case of linear concatenation and only as the first and/or last links of the concatenation.*

Proof. Suppose that H contains no bipartite claw and we will prove that at least one of the assertions (i)–(iv) holds. Since H contains no bipartite claw and H is connected, every cycle of H of length at least 5 is edge-dominating in H.

If H contains a 6-cycle C having a long chord or three consecutive short chords, then, as H contains no bipartite claw, H is spanned by C and assertion (i) holds. So, from now on, we assume, without loss of generality, that H contains no 6-cycle having a long chord or three consecutive short chords.

Suppose now that H contains antenna. Then, H has some 5-cycle $C = v_1v_2v_3v_4v_5v_1$ and some vertex $v \in V(H)\setminus V(C)$ such that v is adjacent to v_2 and v_1 is adjacent to v_3. If v were adjacent to any vertex of C different from v_2, then H would have a 6-cycle having a long chord, contradicting our assumption. If any vertex of C different from v_2 were adjacent to some vertex outside $V(C)$ different from v, then H would contain a bipartite claw. Therefore, as H is connected and C is edge-dominating, each vertex $v \in V(H)\setminus V(C)$ is a pendant vertex adjacent to v_2. Thus, (ii) holds. So, from now on, we assume, without loss of generality, that H contains no antenna.

Suppose now H has a 5-cycle C with three consecutive short chords. If there were any vertex $v \in V(H)\setminus V(C)$ adjacent to the two vertices v_1 and v_2 of C that are no midpoints of any of these three short chords, then H would have a 6-cycle with three consecutive short chords, contradicting our assumption. Since H contains no antenna, the midpoints of the chords of C have neighbors in $V(C)$ only. Therefore, as C is edge-dominating, each $v \in V(H)\setminus V(C)$ is a pendant vertex adjacent to v_1 or v_2. If there were two different vertices $u_1, u_2 \in V(H)\setminus V(C)$ such that u_i is adjacent to v_i for

$i = 1, 2$, then H would contain a bipartite claw. Hence, without loss of generality, each $v \in V(H)\setminus V(C)$ is a pendant vertex adjacent to v_1 and (ii) holds. From now on, we assume, without loss of generality that H has no 5-cycle with three consecutive short chords.

Suppose now that H has a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ with at least three chords. Then, by hypothesis, C has exactly three chords and, without loss of generality, the chords of C are v_1v_3, v_1v_4, and v_3v_5. As C is edge-dominating and H contains no antenna, each vertex $v \in V(H)\setminus V(C)$ is adjacent to v_1 and/or to v_3 only. Then, $H =$ rhombus $\&_{p_1}$ m-crown $\&_{p_2} \circlearrowleft$ for some $p_1, p_2 \geqslant 0$ and some $m \geqslant 1$ and, in particular, (iv) holds. So, from now on, we assume, without loss of generality, that each 5-cycle of H has at most two chords.

Suppose that H has a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ with two crossing chords. Without loss of generality, assume that v_2v_4 and v_3v_5 are the chords of C. As H contains no antenna, v_3 and v_4 have neighbors in $V(C)$ only. Suppose that there is some vertex $v \in V(H)\setminus V(C)$ such that v is adjacent simultaneously to v_1, v_2, and v_5. Since H contains no bipartite claw, it follows that the only neighbors of v_1 are v, v_2, and v_5, and the only vertex outside $V(C)$ adjacent simultaneously to v_2 and v_5 is v. So, since C is edge-dominating, we conclude that $H =$ rhombus $\&_{p_1}$ rhombus $\&_{p_2} \circlearrowleft$ for some $p_1, p_2 \geqslant 0$ and, in particular, (iv) holds. So, without loss of generality, assume that there is no vertex outside $V(C)$ adjacent to v_1, v_2, and v_5 simultaneously. Suppose now that there is some vertex $v \in V(H)\setminus V(C)$ which is adjacent to v_2 and v_5 and nonadjacent to v_1. Since H contains no bipartite claw, v_1 has no neighbors apart from v_2 and v_5. So, since C is edge-dominating, we conclude that $H =$ rhombus $\&_{p_1}$ m-fold $\&_{p_2} \circlearrowleft$ for some $p_1, p_2 \geqslant 0$ and $m \geqslant 2$ and, in particular, (iv) holds. Finally, assume, without loss of generality, that there is no vertex $v \in V(H)\setminus V(C)$ adjacent to v_2 and v_5 simultaneously. Then, since C is edge-dominating, $H =$ rhombus $\&_{p_1}$ m_1-crown $\&_{p_2}$ m_2-crown $\&_{p_3} \circlearrowleft$ for some $p_1, p_2, p_3, m_1, m_2 \geqslant 0$ and (iv) holds.

Suppose that H has a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ with two noncrossing chords. Without loss of generality, assume that v_1v_3 and v_1v_4 are the chords of C. Since H contains no antenna, vertices v_2 and v_5 have neighbors in $V(C)$ only. If there were a vertex outside $V(C)$ which were adjacent to v_1, v_3, and v_4, then H would have a 6-cycle with a long chord, contradicting our assumption. Therefore, as C is edge-dominating, $H = m_1$-crown $\&_{p_1}$ m_2-crown $\&_{p_2}$ m_3-crown $\&_{p_3} \circlearrowleft$ for some $p_1, p_2, p_3, m_1 \geqslant 0$ and some $m_2, m_3 \geqslant 1$ and (iv) holds. Therefore, from now on, we assume, without loss of generality, that each 5-cycle of H has at most one chord.

Suppose now that H has a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ with exactly one chord. Without loss of generality, assume that the only chord is v_1v_3. Since H has no antenna, no vertex outside $V(C)$ is adjacent to v_2. If there were some vertex outside $V(C)$ adjacent to at least three vertices of C, then H would have a 5-cycle with at least two chords, contradicting our hypothesis. Suppose that there is some vertex $v \in V(H) \setminus V(C)$ which is adjacent to two nonconsecutive vertices of C different from v_1 and v_3. Without loss of generality, assume that the two neighbors of v are v_1 and v_4. Since H contains no bipartite claw, v_5 has no neighbors outside $V(C)$. As C is edge-dominating, we conclude that $H = m_1$-fold $\&_{p_1}$ m_2-crown $\&_{p_2}$ m_3-crown $\&_{p_3} \circlearrowright$ for some $m_1 \geqslant 2$, $m_2 \geqslant 1$, and some $m_3, p_1, p_2, p_3 \geqslant 0$. If, on the contrary, there is no vertex in $V(H) \setminus V(C)$ adjacent to two nonconsecutive vertices of C different from v_1 and v_3, then $H = m_1$-crown $\&_{p_1}$ m_2-crown $\&_{p_2}$ m_3-crown $\&_{p_3}$ m_4-crown $\&_{p_4} \circlearrowright$ for some $m_1 \geqslant 1$ and some $m_2, m_3, m_4, p_1, p_2, p_3, p_4 \geqslant 0$. In both cases, (iv) holds. Hence, from now on, we assume that every 5-cycle of H is chordless.

As we are assuming that H has no 6-cycle having a long chord or three consecutive short chords and that each 5-cycle of H is chordless, if H has a cycle of length at least 5, then, by Lemma 4.24, H is a circular concatenation of crowns, folds, and rhombi, which means that (iv) holds. So, we assume, without loss of generality, that each cycle of H has length at most 4. But then, H is a fat caterpillar and assertion (iii) or (iv) holds by virtue of Theorem 4.23.

Conversely, if H satisfies one of the assertions (i)–(iii), then clearly H contains no bipartite claw. Finally, if H satisfies assertion (iv), then also H contains no biparite claw by reasoning as in the first part of the proof of Lemma 4.20. □

Notice that, although those graphs satisfying (iii) are the underlying graphs of edge $\&_{p_1}$ K_4 $\&_{p_2}$ edge for positive integers p_1, p_2, we prefer to consider (iii) a sporadic case.

4.2.2 Edge-coloring graphs containing no bipartite claw

The *chromatic index* $\chi'(H)$ of a graph H is the minimum number of colors needed to color all the edges of H so that no two incident edges receive the same color. Clearly, $\chi'(H) \geqslant \Delta(H)$. In fact, Vizing [122] proved that for every graph H either $\chi'(H) = \Delta(H)$ or $\chi'(H) = \Delta(H) + 1$. The problem of deciding whether a graph H satisfies $\chi'(H) = \Delta(H)$ is NP-complete even for graphs having only vertices of degree 3 [74]. Interestingly, the problem of deciding whether or not $\chi'(H) = \Delta(H)$ can be solved in linear time if H is contains no bipartite claw. Indeed, as H contains no bipartite claw as a minor, it has bounded tree-width [106], which means that $\chi'(H)$ can determined via the algorithm devised in [129] (for the undefined notions see, e.g., Chapter 12 of [46]). In this subsection, we give a structural characterization of those graphs having no bipartite claw that satisfy $\chi' \neq \Delta$.

We need to introduce some terminology related to edge-coloring. A *major vertex* of

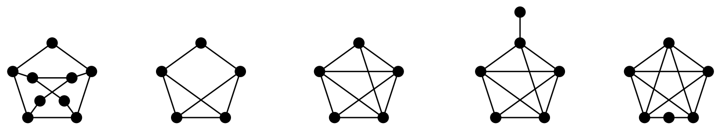


Figure 4.6: *Graphs* P^*, SK_4, $K_5 - e$, L_5, *and* SK_5

a graph is a vertex of maximum degree. If H is a graph, the *core* H_Δ of H is the subgraph of H induced by the major vertices of H. Graphs H for which $\chi'(H) = \Delta(H)$ are *Class 1*, and otherwise they are *Class 2*. A graph H is *critical* if H is Class 2, connected, and $\chi'(H - e) < \chi'(H)$ for each $e \in E(H)$. Some graphs needed in what follows are introduced in Figure 4.6.

We rely on the following results.

Theorem 4.26 ([73])**.** *If* H *is a connected Class 2 graph with* $\Delta(H_\Delta) \leqslant 2$*, then the following conditions hold:*

(i) H *is critical.*

(ii) $\delta(H_\Delta) = 2$.

(iii) $\delta(H) = \Delta(H) - 1$*, unless* H *is an odd chordless cycle.*

(iv) *Every vertex of* H *is adjacent to some major vertex of* H.

Theorem 4.27 ([31])**.** *Let* H *be a connected graph such that* $\Delta(H_\Delta) \leqslant 2$ *and* $\Delta(H) = 3$*. Then,* H *is Class 1, unless* $H = P^*$.

Theorem 4.28 ([123])**.** *If* H *is a graph of Class 2, then* H *contains a critical subgraph of maximum degree* k *for each* k *such that* $2 \leqslant k \leqslant \Delta(H)$.

Theorem 4.29 ([3])**.** *There are no critical graphs having* 4 *or* 6 *vertices. The only critical graphs having* 5 *vertices are* C_5, SK_4*, and* $K_5 - e$.

By exploiting our structure theorem for graphs containing no bipartite claw (Theorem 4.25) and the results above, we give a structural characterization of all connected Class 2 graphs within graphs containing no bipartite claw, as follows.

Theorem 4.30. *Let* H *be a connected graph containing no bipartite claw. Then,* $\chi'(H) = \Delta(H)$ *if and only if none of the following statements holds:*

(i) $\Delta(H) = 2$ *and* H *is an odd chordless cycle.*

(ii) $\Delta(H) = 3$ *and* H *is the circular concatenation of a sequence of edges, triangles, and rhombi, where the number of edge links equals one plus the number of rhombus links.*

(iii) $\Delta(H) = 4$ *and* $H = K_5 - e$, K_5, L_5, *or* SK_5.

Proof. Let H be a connected graph containing no bipartite claw and such that $\chi'(H) \neq \Delta(H)$. We need to prove that H satisfies (i), (ii), or (iii). Since the result holds trivially if $\Delta(H) \leqslant 2$, we assume, without loss of generality, that $\Delta(H) \geqslant 3$. The proof splits into three cases.

Case 1. $\Delta(H_\Delta) \leqslant 2$.

We claim that $H = K_5 - e$. Since P^* contains a bipartite claw, if $\Delta(H) = 3$ then H would be Class 1 by Theorem 4.27, contradicting the hypothesis. Thus, $\Delta(H) \geqslant 4$. By Theorem 4.26, $\delta(H_\Delta) = 2$ and $\delta(H) = \Delta(H) - 1 \geqslant 3$. Suppose, by the way of contradiction, that assertion (iv) of Theorem 4.25 holds for H. Since the vertices of H that are not concatenation vertices have degree at most 3, all major vertices of H are concatenation vertices. Since $\delta(H_\Delta) = 2$, H is necessarily a circular concatenation of crowns. Finally, since $\delta(H) \geqslant 3$, each of the crowns of the concatenation is an edge and H has no pendant vertices; i.e., H is a chordless cycle, contradicting $\Delta(H) \geqslant 4$. This contradiction proves that assertion (iv) of Theorem 4.25 does not hold for H. Thus, assertion (i), (ii), or (iii) of Theorem 4.25 holds for H. As $\delta(H) \geqslant 3$, H has no pendant vertices and necessarily $|V(H)| = 5$ or 6. So, since H is critical and $\Delta(H) \geqslant 4$, it follows from Theorem 4.29 that $H = K_5 - e$, as claimed.

Case 2. $\Delta(H_\Delta) \geqslant 3$ *and* $\Delta(H) \geqslant 4$.

Suppose that H has a 6-cycle C having a long chord. This implies that C is spanning in H because H is connected and contains no bipartite claw. In particular, $|V(H)| \leqslant 6$. Then, as we are assuming that $\Delta(H) \geqslant 4$, Theorems 4.28 and 4.29 imply that H contains $K_5 - e$ and $\Delta(H) = 4$. Therefore, as H has a spanning 6-cycle, H arises from $K_5 - e$ by adding one vertex adjacent precisely to the two vertices of degree 3 of the $K_5 - e$; i.e., $H = SK_5$. So, for the remaining of this case, we assume that H has no 6-cycle having a long chord.

As $\Delta(H_\Delta) \geqslant 3$, there is some major vertex w_0 of H that is adjacent in H to three other major vertices w_1, w_2, w_3 of H. Let $W = \{w_0, w_1, w_2, w_3\}$.

Suppose, by the way of contradiction, that $|N_H(w_i) \backslash W| \geqslant 2$ for each $i = 1, 2, 3$. If $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) \backslash W| \geqslant 3$, then, by Hall's Theorem, H would contain a bipartite claw, a contradiction. We conclude that there are two vertices $x_1, x_2 \in V(H) \backslash W$ such that $x_1 \neq x_2$ and $N_H(w_i) \backslash W = \{x_1, x_2\}$ for each $i = 1, 2, 3$. Then, $w_0 w_1 x_1 w_2 x_2 w_3 w_0$ is a 6-cycle having three long chords, a contradiction. As this contradiction arose from assuming that $|N_H(w_i) \backslash W| \geqslant 2$ for each $i = 1, 2, 3$, there is some $j \in \{1, 2, 3\}$ such that $|N_H(w_j) \backslash W| \leqslant 1$ and, in particular, $\Delta(H) = 4$.

Suppose now that $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) \backslash W| \geqslant 2$. Then, by Hall's Theorem and by symmetry, we assume, without loss of generality, that there are two different vertices $x_1, x_2 \in V(H)\setminus W$ such that x_i is adjacent to w_i, for $i = 1, 2$, and $|N_H(w_3)\setminus W| \leqslant 1$. As w_3 is a major vertex, w_3 is necessarily adjacent to w_1 and w_2. As $\Delta(H) = 4$ and H contains no bipartite claw, for each of w_0 and w_3, its only neighbor outside W is either x_1 or x_2. By symmetry, we assume, without loss of generality, that $N_H[w_3] = W \cup \{x_1\}$. Then, as H contains no bipartite claw and has no 6-cycle having a long chord, $N_H[w_0] = W \cup \{x_1\}$, $N_H[w_1] = W \cup \{x_1\}$, $N_H[w_2] = W \cup \{x_2\}$, $N_H(x_1) = \{w_0, w_1, w_3\}$, and $N_H(x_2) = \{w_2\}$. We conclude that $H = L_5$.

Finally, suppose that $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3))\setminus W| = 1$ and, consequently, $N_H[w_1] = N_H[w_2] = N_H[w_3] = W \cup \{x\}$ for some $x \in V(H)\setminus W$. If w_0 is adjacent to x, then $H = K_5$. If, on the contrary, the neighbor of w_0 outside W is $x' \neq x$, then, as H contains no bipartite claw and has no 6-cycle having a long chord, $H = L_5$.

Case 3. $\Delta(H_\Delta) \geqslant 3$ *and* $\Delta(H) = 3$.

As $\Delta(H) = 3$, (iii) of Theorem 4.25 does not hold. Suppose, by the way of contradiction, that (i) or (ii) of Theorem 4.25 holds for H. Then, $|V(H)| = 5$ or 6 and, by Theorems 4.28 and 4.29, H contains a SK_4. Therefore, as H contains no bipartite claw, H is connected, and $\Delta(H) = 3$, it follows that either $H = SK_4$ or H arises from SK_4 by adding a pendant vertex adjacent to the vertex of degree 2 of the SK_4, contradicting the assumption that (i) or (ii) of Theorem 4.25 holds. We conclude that, necessarily, H is a linear or circular concatenation as described in (iv) of Theorem 4.25. As $\Delta(H) = 3$, no link of the linear or circular concatenation is an m-crown for any $m \geqslant 3$ or an m-fold for any $m \geqslant 4$. Moreover, if any of the links in the linear or circular concatenation were a 2-crown, 3-fold, or K_4, then H would be precisely the underlying graph of a 2-crown, 3-fold, or K_4, and H would be Class 1, a contradiction. Therefore, H is a linear or circular concatenation of edges, triangles, squares, and rhombi. As $\Delta(H) = 3$, if any link of the concatenation is a triangle, square, or rhombus, then its adjacent links in the concatenation are edges. Then, it is clear that there is a 3-edge-coloring of H if and only if there is a coloring of only the edge links of H such that:

(1) Each two edge links that are adjacent to the same triangle link are colored with different colors.

(2) Each two edge links that are adjacent to the same rhombus link are colored with the same color.

(3) Each two adjacent edge links are colored with different colors.

So, if H is a linear concatenation, a greedy coloring of the edge links following the order of their occurrence in the linear concatenation and following rules (1)–(3) above, ends up successfully, implying that H has a 3-edge-coloring, a contradiction with the fact that H is Class 2. So, H is a circular concatenation. Suppose, by the way of contradiction, that some link of the circular concatenation is a square. Then, $H = \text{edge} \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}} \text{edge} \,\&\, \text{square} \circlearrowright$ and, as H is not 3-edge-colorable, $\text{edge} \,\&_1 \text{edge} \,\&_{p_1} \&\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}} \text{edge} \,\&_1 \text{edge}$ is a linear concatenation of edges, triangle, squares, and rhombi that is no 3-edge-colorable, a contradiction to what we have just shown. This contradiction proves that H is a circular concatenation of edges, triangles, and rhombi only.

We will now prove that if H is a circular concatenation of edges, triangles, and rhombi such that $\Delta(H_\Delta) \geqslant 3$ and $\Delta(H) = 3$, then H is Class 2 if and only if H has exactly one more edge links than rhombus links. As $\Delta(H_\Delta) \geqslant 3$, H has at last one rhombus link. So, without loss of generality, $H = \text{edge} \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}} \text{edge} \,\&\, \text{rhombus} \,\&\circlearrowright$. Notice that H is Class 2 if and only if there is no 3-edge-coloring of the edge links of $H' = \text{edge} \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}} \text{edge}$ satisfying rules (1)–(3) above and such that the first and the last link of H' are colored with the same color. Moreover, H' is not 3-edge-colorable satisfying rules (1)–(3) above if and only if the graph H'', that arises from H' by contracting each triangle link to a vertex and contracting each pair formed by a rhombus link followed by an edge also to a vertex, consists of precisely two edges; i.e., H' has two more edge links than rhombus links. We conclude that H has exactly one more edge links than rhombus links; i.e., (ii) holds. This completes Case 3 and the proof of the 'only if' part of the theorem.

Notice also that we have just proved that if (ii) holds for H, then by the analysis in Case 3, H is Class 2. As a result, the 'if' part of the theorem is also proved because, if (i) or (iii) holds for H, then H is clearly Class 2. □

Corollary 4.31. *The critical graphs containing no bipartite claw are the odd cycles, $K_5 - e$, and those graphs H satisfying $\Delta(H) = 3$ that are circular concatenations of edges, triangles, and rhombi having exactly one more edge links than rhombus links and without pendant edges.*

4.2.3 Matching-perfect graphs

As mentioned in the beginning of this section, in order to prove Theorems 4.16 and 4.17, it suffices to prove the theorem below, which is the main result of this subsection.

Theorem 4.32. *Let H be a connected graph containing no bipartite claw and such that the length of each cycle of H is at most 4 or is a multiple of 3. Then, $\alpha_m(H) = \tau_m(H)$.*

To prove that $\alpha_m(H) = \tau_m(H)$ in Theorem 4.32, we combine upper bounds on $\tau_m(H)$ with lower bounds on $\alpha_m(H)$. For instance, the next lemma states a simple yet useful upper bound on $\tau_m(H)$.

Lemma 4.33. *If H is a graph and v_1 and v_2 are two adjacent vertices of H, then the set of edges of H that are incident to v_1 and/or to v_2 is a matching-transversal of H. In particular, $\tau_m(H) \leqslant d_H(v_1) + d_H(v_2) - 1$.*

Proof. No matching M of H disjoint from $E_H(v_1) \cup E_H(v_2)$ is maximum because $M \cup \{v_1v_2\}$ is a larger matching of H. $\square$

A *partial* k-*edge-coloring* of a graph H is a map $\phi : E(H) \to \{0, 1, 2, \ldots, k\}$ such that, for each pair of incident edges e_1, e_2 of H, $\phi(e_1) = \phi(e_2)$ implies $\phi(e_1) = \phi(e_2) = 0$. If $\phi(e) \neq 0$, e is said to be *colored with color* $\phi(e)$; otherwise, e is said to be *uncolored*. A k-*edge-coloring* of H is a partial k-edge-coloring that colors all edges of H. The *color classes* of a partial k-edge-coloring are the sets $\xi_1, \xi_2, \ldots, \xi_k$ where ξ_j is the set of edges of H colored by ϕ with color j, for each $j = 1, 2, \ldots, k$.

We complement the upper bounds on τ_m with lower bounds on α_m obtained with the help of a special kind of partial edge-colorings that we call profuse-colorings. A k-*profuse-coloring* of a graph H is a partial k-edge-coloring $\phi : E(H) \to \{0, 1, 2, \ldots, k\}$ such that, for each edge e of H (either colored or not), there are edges of H incident to e that are colored with at least $k - 1$ different colors. We say that a k-profuse-coloring ϕ is *maximal* if, for each uncolored edge, there are edges incident to it that are colored with the k different colors (i.e., no uncolored edge can be colored while keeping the coloring a k-profuse-coloring). We now show that the maximum value of k for which a graph H has a k-profuse coloring is precisely $\alpha_m(H)$. Hence, in order to prove that $\alpha_m(H) \geqslant k$ it will suffice to exhibit a k-profuse-coloring of H.

Lemma 4.34. *Let H be a graph. Then, the following assertions are equivalent:*

(i) $\alpha_m(H) \geqslant k$.

(ii) H *has a* k-*profuse-coloring.*

(iii) H *has a maximal* k-*profuse-coloring.*

Indeed, the collection of color classes of a maximal k-*profuse-coloring of* H *is a matching-independent set of size* k.

Proof. Let us prove first that (i) $\Rightarrow$ (iii). Suppose that $\alpha_m(H) \geqslant k$. Then, there is a collection $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ of k pairwise disjoint maximal matchings of H. Let $\phi_{\mathcal{M}} : E(H) \to \{0, 1, 2, \ldots, k\}$ be defined by

$$\phi_{\mathcal{M}}(e) = i \text{ if and only if } e \in M_i, \quad \text{for each } e \in E(H) \text{ and each } i = 1, \ldots, k.$$

Notice that $\phi_{\mathcal{M}}(e) = 0$ if and only if $e \notin M_1 \cup M_2 \cup \cdots \cup M_k$. We claim that $\phi_{\mathcal{M}}$ is a maximal k-profuse-coloring of H. Since each M_i is a matching, $\phi_{\mathcal{M}}$ is a partial edge-coloring of H. Let e be any edge of H. Assume first that $e \in M_j$ for some $j \in \{1, 2, \ldots, k\}$. For each $i = 1, 2, \ldots, k$ and each $i \neq j$, the maximality of M_i implies that there is some edge e_i of H incident to e such that $\phi_{\mathcal{M}}(e_i) = i$. So, the set $\{e_i : i \neq j\}$ consists of

$k-1$ edges incident to e that are colored with $k-1$ different colors. Assume now that $e \notin M_1 \cup M_2 \cup \cdots \cup M_k$. For each each $i = 1, 2, \ldots, k$, the maximality of M_i implies that there is some edge e_i of H incident to e such that $\phi_{\mathcal{M}}(e_i) = i$. We conclude that $\phi_{\mathcal{M}}$ is a maximal k-profuse-coloring of H and (iii) holds.

We now prove that (ii) $\Rightarrow$ (i). Suppose (ii) holds and let $\phi : E(H) \to \{0, 1, 2, \ldots, k\}$ be a k-profuse coloring of H. Then, for each $i = 1, 2, \ldots, k$, the color class $\xi_i = \phi^{-1}(i)$ is a matching of H. For each $i = 1, 2, \ldots, k$, let M_i be any maximal matching of H containing ξ_i and let e be any edge of H. As ϕ is a k-profuse-coloring, there are $k-1$ edges $e_1, e_2, \ldots, e_k$ of H incident to e such that $\phi(e_1), \phi(e_2), \ldots, \phi(e_{k-1})$ are positive and pairwise different. So, as $e_i \in \xi_{\phi(e_i)}$ and $M_{\phi(e_i)}$ is a matching containing $\xi_{\phi(e_i)}$, $e \notin M_{\phi(e_i)}$ for each $i = 1, 2, \ldots, k-1$. This proves that each edge e of H belongs to at most one of M_1, M_2, …, M_k. Thus, by construction, $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ is a collection of k disjoint maximal matchings of H and $\alpha_m(H) \geqslant k$; i.e., (i) holds, as desired. Since (iii) trivially implies (ii), this completes the proof the equivalence among (i)–(iii). Finally, notice that if ϕ is maximal, then $M_i = \xi_i$ because each $e \in E(H) \backslash \xi_i$ is incident to some edge in ξ_i. Therefore, if ϕ is maximal, then $\{\xi_1, \ldots, \xi_k\}$ is a collection of k disjoint maximal matchings, proving the last assertion of Lemma 4.34. □

We state the following immediate consequence of Lemma 4.34 for future reference.

Corollary 4.35. *Let H be a graph and let ϕ be a maximal k-profuse-coloring of H. Then, every matching-transversal of H has at least one edge colored with color i for each $i = 1, 2, \ldots, k$.*

More upper bounds on τ_m and lower bounds on α_m will be proved later in this subsection. Some of them depend on the degrees of what we call hubs. The *hubs* of a graph are the vertices of degree at least 3. The *minimum hub degree* $\delta_h(H)$ of a graph H is the infimum of the degrees of the hubs of H. Notice that $\delta_h(H) \geqslant 3$ for any graph H and that $\delta_h(H) = +\infty$ if and only if H has no hubs. A hub is *minimum* if its degree is the minimum hub degree. An edge of a graph is *hub-covered* if at least one of its endpoints is a hub. A graph H is *hub-covered* if each of its edges is hub-covered. Equivalently, H is hub-covered if and only if its hub set is edge-dominating. A graph is *hub-regular* if all its hubs have the same degree. Equivalently, a graph H is hub-regular if and only if $\delta_h(H) = \Delta(H)$ or $\delta_h(H) = +\infty$.

The proof of Theorem 4.32 splits into two parts. In Sub-subsection 4.2.3.1, we consider the case when H has some cycle of length greater than 4 (which is necessarily a cycle of length $3k$ for some $k \geqslant 2$). Later, in Sub-subsection 4.2.3.2, we show how to deal with the case when H has no cycle of length greater than 4.

4.2.3.1 Graphs having some cycle of length $3k$ for some $k \geqslant 2$

The main result of this sub-subsection is the theorem below, which is the restriction of Theorem 4.32 to graphs containing some cycle of length $3k$ for some $k \geqslant 2$.

Theorem 4.36. *Let H be a connected graph containing no bipartite claw and such that the length of each cycle of H is at most 4 or is a multiple of 3. If H has some cycle of length $3k$ for some $k \geqslant 2$, then $\alpha_m(H) = \tau_m(H)$.*

Theorem 4.36 will follow by considering separately the cases when the graph is hub-covered (Lemma 4.42) and when it is not hub-covered (Lemma 4.43).

From the structure lemma below, whose proof is immediate, it follows that if a graph H containing no bipartite claw is such that the length of each of its cycles is at most 4 or is a multiple of 3 and H contains a cycle of length $3k$ for some $k \geqslant 2$, then H is triangle-free.

Lemma 4.37. *Let H be a connected graph containing no bipartite claw such that the length of each cycle is at most 4 or is a multiple of 3. If H contains some cycle C of length $3k$ for some $k \geqslant 2$, then one of the following conditions holds:*

(i) *H arises from C_6 by adding 1, 2, or 3 long chords.*

(ii) *C is chordless and each vertex $v \in V(H) \setminus V(C)$ is either: (1) a false twin of a vertex of C of degree 2 in H or (2) a pendant vertex adjacent to a vertex of C.*

In particular, H is triangle-free.

We begin the case of hub-covered graphs with the following upper bound on τ_m.

Lemma 4.38. *Let H be a triangle-free graph containing no bipartite claw. If v is any hub of H, then the set of edges of H incident to v is a matching-transversal of H. In particular, $\tau_m(H) \leqslant \delta_h(H)$.*

Proof. Let v be any minimum hub of H and let w_1, w_2 and w_3 be three of its neighbors in H. If $E_H(v)$ were not a matching-transversal of H, there would be a maximal matching M of H disjoint from $E_H(v)$. Then, for each $i = 1, 2, 3$, there would be some $e_i \in M$ incident to w_i and non-incident to v. As H is triangle-free, w_i would be the only endpoint of e_i in $\{w_1, w_2, w_3\}$, for each $i = 1, 2, 3$. But then, $\{vw_1, vw_2, vw_3, e_1, e_2, e_3\}$ would be the edge set of a bipartite claw contained in H, a contradiction. This contradiction proves that $E_H(v)$ is a matching-transversal of H and that $\tau_m(H) \leqslant \delta_h(H)$. □

The counterpart of the above upper bound on $\tau_m(H)$ is the following lemma from which we deduce sufficient conditions for $\delta_h(H)$ being also a lower bound on $\alpha_m(H)$.

Lemma 4.39. *Let H be a triangle-free graph containing no bipartite claw. Then, there exists a set F of hub-covered edges of H such that the graph $H' = H \backslash F$ is hub-regular and has the same hub set and the same minimum hub degree as H.*

Proof. Let H be a counterexample to the lemma with minimum number of edges. If H were hub-regular, the lemma would hold by letting $F = \emptyset$. So, H is not hub-regular; i.e., $\Delta(H) > \delta_h(H)$. Let v be any hub of H that is not minimum.

We claim that v has some neighbor w in H which is not a minimum hub. Suppose, by the way of contradiction, that all the neighbors of v are minimum hubs. By construction, v has at least four neighbors w_1, w_2, w_3, w_4 and let $W = \{v, w_1, w_2, w_3, w_4\}$. As H is triangle-free and w_i is a hub, $|N_H(w_i) \backslash W| \geqslant \delta_h(H) - 1$ for each $i = 1, 2, 3$. Then, $\delta_h(H) = 3$, since otherwise, Hall's Theorem would imply that v is the center of a bipartite claw contained in H. Similarly, Hall's Theorem forces $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) \backslash W| \leqslant 2$. So, $\delta_h(H) = 3$ and there are two different vertices x_1, x_2 outside W such that $N_H(w_1) = N_H(w_2) = N_H(w_3) = \{v, x_1, x_2\}$ and, by symmetry, also $N_H(w_4) = \{v, x_1, x_2\}$. Then, H contains a bipartite claw, a contradiction. This contradiction proves that v has some neighbor w which is not a minimum hub, as claimed.

Let w be a neighbor of v which is not a minimum hub of H. Then, vw is a hub-covered edge of H and $H_1 = H \backslash \{vw\}$ has the same hub set and the same minimum hub degree as H. By minimality of the counterexample H, the lemma holds for H_1. Hence, there exists a set F_1 of hub-covered edges of H_1 such that $H' = H_1 \backslash F_1$ is hub-regular and has the same hub set and the same minimum hub-degree as H_1. By construction, $F = F_1 \cup \{vw\}$ is a set of hub-covered edges of H such that $H' = H \backslash F$ is hub-regular and H' has the same hub set and the same minimum hub degree as H. So, the lemma holds for H, contradicting the choice of H. This contradiction proves the lemma. □

Lemma 4.40. *Let H be a triangle-free graph containing no bipartite claw. If H is hub-covered and has at least one edge, then $\alpha_m(H) \geqslant \delta_h(H)$.*

Proof. By Lemma 4.39, there exists a set F of hub-covered edges of H such that $H' = H \backslash F$ is hub-regular and has the same hub set and the same minimum hub degree as H. Since H has at least one edge and H is hub-covered, H has at least one hub; i.e., $3 \leqslant \delta_h(H) < +\infty$. By construction, H' is also hub-covered and $\Delta(H') = \delta_h(H') = \delta_h(H) \geqslant 3$. Since H' is a subgraph of H, H' is also triangle-free and contains no bipartite claw. By Theorem 4.30, $\chi'(H') = \Delta(H')$; i.e., there is an edge-coloring ϕ' of H' using $\Delta(H') = \delta_h(H)$ colors. Let $\phi : E(H) \to \{0, 1, 2, \ldots, \delta_h(H)\}$ be defined by $\phi(e) = \phi'(e)$ for each $e \in E(H')$ and $\phi(e) = 0$ for each $e \in E(H) \backslash E(H')$. Since H is hub-covered, ϕ is a $\delta_h(H)$-profuse-coloring of H by construction. Thus, by Lemma 4.34, $\alpha_m(H) \geqslant \delta_h(H)$. □

From Lemmas 4.38 and 4.40, we can determine α_m and τ_m for all connected hub-covered triangle-free graphs containing no bipartite claw.

Lemma 4.41. *If* H *is a connected hub-covered triangle-free graph containing no bipartite claw and having at least one edge, then* $\alpha_m(H) = \tau_m(H) = \delta_h(H)$*.*

By Lemma 4.37 and the above lemma, we settle Theorem 4.36 for hub-covered graphs, as follows.

Lemma 4.42. *Let* H *be a connected graph containing no bipartite claw and such that the length of each cycle of* H *is at most* 4 *or is a multiple of* 3*. If* H *has a cycle of length* $3k$ *for some* $k \geqslant 2$ *and* H *is hub-covered, then* $\alpha_m(H) = \tau_m(H) = \delta_h(H)$*.*

Finally, we also settle Theorem 4.36 for graphs that are not hub-covered.

Lemma 4.43. *Let* H *be a connected graph containing no bipartite claw and such that the length of each cycle of* H *is at most* 4 *or is a multiple of* 3*. If* H *has a cycle of length* $3k$ *for some* $k \geqslant 2$ *and* H *is not hub-covered, then* $\alpha_m(H) = \tau_m(H) = 3$*.*

Proof. As H is not hub-covered and has at least one edge, Lemma 4.33 implies $\tau_m(H) \leqslant 3$. So, we just need to prove that $\alpha_m(H) \geqslant 3$. Since the length of C is a multiple of 3, there is a 3-edge-coloring of C, $\phi' : E(C) \to \{1, 2, 3\}$ such that each three consecutive edges of C are colored with three different colors by ϕ'. Let $\phi : E(H) \to \{0, 1, 2, 3\}$ be defined by $\phi(e) = \phi'(e)$ for each $e \in E(C)$ and $\phi(e) = 0$ for each $e \in E(H) \setminus E(C)$. Since H is connected and contains no bipartite claw, C is edge-dominating in H and, consequently, ϕ is a 3-profuse-coloring of H. By virtue of Lemma 4.34, $\alpha_m(H) \geqslant 3$, as needed. □

Clearly, Lemmas 4.42 and 4.43 together imply Theorem 4.36.

4.2.3.2 Graphs having no cycle of length greater than 4

As Theorem 4.36 is now proved, to complete the proof of Theorem 4.32, it only remains to prove the theorem below, which is the main result of this sub-subsection.

Theorem 4.44. *If* H *is a fat caterpillar, then* $\alpha_m(H) = \tau_m(H)$*.*

To begin with, the next lemma provides several upper bounds on τ_m.

Lemma 4.45. *Let* H *be a graph containing no bipartite claw and having no* 5*-cycle and let* v *be a hub of* H*. Then:*

(i) *If* v *has degree at least* 5 *in* H*, then* $E_H(v)$ *is a matching-transversal of* H *and, in particular,* $\tau_m(H) \leqslant d_H(v)$*.*

(ii) *Suppose that* v *has degree* 4 *in* H*. Then,* $\tau_m(H) \leqslant 5$*. If* $N_H(v)$ *does not induce* $2K_2$ *in* H*, then* $E_H(v)$ *is a matching-transversal of* H *and, in particular,* $\tau_m(H) \leqslant 4$*.*

(iii) *Suppose that* v *has degree* 3 *in* H*. Then,* $\tau_m(H) \leqslant 5$*. If* $N_H(v)$ *induces* $3K_1$ *in* H*, then* $E_H(v)$ *is a matching-transversal of* H *and, in particular,* $\tau_m(H) \leqslant 3$*. If* $N_H(v)$ *induces* $K_2 \cup K_1$ *in* H*, then* $\tau_m(H) \leqslant 4$*.*

Proof. If $E_H(v)$ is a matching-transversal of H, then $\tau_m(H) \leqslant d_H(v)$ and there is nothing left to prove. Therefore, we assume, without loss of generality, that $E_H(v)$ is not a matching-transversal of H. Therefore, there exists a maximal matching M of H such that $M \cap E_H(v) = \varnothing$. Because of the maximality of M, for each neighbor w of v there is exactly one edge $e_w \in M$ that is incident to w. Notice that there could be two different neighbors w_1 and w_2 of v such that $e_{w_1} = e_{w_2}$.

We claim that $|\{e_w \mid w \in N_H(v)\}| \leqslant 2$. Suppose, by the way of contradiction, that there are three different edges $e_{w_1}, e_{w_2}, e_{w_3}$ for some $w_1, w_2, w_3 \in N_H(v)$. Then, v is the center of a bipartite claw contained in H with edge set $\{vw_1, e_{w_1}, vw_2, e_{w_2}, vw_3, e_{w_3}\}$, a contradiction. This contradiction proves the claim. Therefore, as each edge e_w is incident to at most two vertices of $N_H(v)$, in particular, $d_H(v) \leqslant 4$. So far, we have proved (i).

Suppose that $d_H(v) = 3$ and let $N_H(v) = \{w_1, w_2, w_3\}$. Suppose, by the way of contradiction, that $E_H(v) \cup F_H(v)$ is not a matching-transversal of H. Then, there is some maximal matching M' such that $M' \cap (E_H(v) \cup F_H(v)) = \varnothing$. Because of the maximality of M', for each $i = 1, 2, 3$, there is an edge $e'_{w_i} \in M'$. Then, v is the center of a bipartite claw whose edge set is $\{vw_1, e'_{w_1}, vw_2, e'_{w_2}, vw_3, e'_{w_3}\}$, a contradiction. This contradiction proves that $E_H(v) \cup F_H(v)$ is a matching-transversal of H. In particular, $\tau_m(H) \leqslant 3 + |F_H(v)|$. This proves (iii) when $N_H(v)$ is not a complete. So, assume that $N_H(v)$ is a complete. Since H has no 5-cycle, every vertex $x \in V(H) \setminus N_H[v]$ having at least one neighbor in $N_H(v)$, has exactly one neighbor in $N_H(v)$. So, since H contains no bipartite claw, there is at least one vertex in $N_H(v)$ that has degree 3 in H. Assume, without loss of generality, that w_1 has degree 3 in H. Then, by Lemma 4.33, $\tau_m(H) \leqslant d_H(v) + d_H(w_1) - 1 = 5$. This completes the proof of (iii).

Finally, we consider the case $d_H(v) = 4$. Since $|\{e_w \mid w \in N_H(v)\}| \leqslant 2$ and each edge e_w is incident to at most two neighbors of v, we assume, without loss of generality, that $e_{w_1} = e_{w_2} = w_1w_2$ and $e_{w_3} = e_{w_4} = w_3w_4$. In particular, the graph induced by $N_H(v)$ contains $2K_2$. Moreover, since H has no 5-cycle, $N_H(v)$ induces $2K_2$. To complete the proof of (ii) it only remains to prove that $\tau_m(H) \leqslant 5$. Suppose, by the way of contradiction, that $E_H(v) \cup \{w_1w_2\}$ is not a matching-transversal. Then, there is maximal matching M' of H such that $M' \cap (E_H(v) \cup \{w_1w_2\}) = \varnothing$. Because of the maximality of M', for each $w \in N_H(v)$, there is some edge $e'_w \in M'$ incident to w. Since

$w_1w_2 \notin M'$, $e_{w_1} \neq e_{w_2}$. Since w_3 is nonadjacent to w_1 and w_2, e_{w_3} is different from e_{w_1} and e_{w_2}. We conclude that v is the center of a bipartite claw contained in H whose edge set $\{vw_1, e'_{w_1}, vw_2, e'_{w_2}, vw_3, e'_{w_3}\}$. This contradiction proves that $E_H(v) \cup \{w_1w_2\}$ is a matching-transversal, which means that $\tau_m(H) \leqslant 5$. This completes the proof of (ii) and of the lemma. □

We now prove a lower bound on α_m (Lemma 4.48), which is the last of the next three lemmas.

Lemma 4.46. *Let H be a graph. If v is a vertex of H that is neither the center of a bipartite claw nor a vertex of a 5-cycle, at most two of the neighbors of v have degree at least 4 each.*

Proof. Suppose, by the way of contradiction, that there exists some vertex v of H that is neither the center of a bipartite claw nor a vertex of 5-cycle and such that v has three different neighbors w_1, w_2, w_3 in H such that $d_H(w_i) \geqslant 4$ for each $i = 1, 2, 3$. Since $d_H(w_i) \geqslant 4$ for each $i = 1, 2, 3$, each w_i is adjacent to at least one vertex x_i different from v, w_1, w_2, w_3.

We claim that $\{w_1, w_2, w_3\}$ is a stable set of H. Suppose, by the way of contradiction, that $\{w_1, w_2, w_3\}$ is not a stable set of H. By symmetry, we assume, without loss of generality, that w_1 is adjacent to w_2. Since there is no 5-cycle passing through v, x_3 is different from x_1 and x_2. Thus, $x_1 = x_2$ and $N_H(w_1) \subseteq \{v, w_2, w_3, x_1\}$ because v is not the center of a bipartite claw. So, as $d_H(v_1) \geqslant 4$, necessarily w_1 is adjacent to w_3 and $w_1x_1w_2vw_3w_1$ is a 5-cycle of H passing through v, which is a contradiction. This contradiction proves that $\{w_1, w_2, w_3\}$ is a stable set of H.

Since $\{w_1, w_2, w_3\}$ is a stable set and $d_H(w_i) \geqslant 4$, there are three pairwise different vertices $x_{i1}, x_{i2}, x_{i3} \in N_H(w_i) \setminus \{v, w_1, w_2, w_3\}$, for each $i = 1, 2, 3$. By Hall's Theorem, there are some $j_1, j_2, j_3 \in \{1, 2, 3\}$ such that $M = \{w_1x_{1j_1}, w_2x_{2j_2}, w_3x_{3j_3}\}$ is a matching of H of size 3. Then, $\{vw_1, vw_2, vw_3\} \cup M$ is the edge set of a bipartite claw with center v, a contradiction. This contradiction completes the proof of the lemma. □

Lemma 4.47. *Let H be a graph containing no bipartite claw and having no 5-cycle. If $\delta_h(H) \geqslant 4$, then there exists a set F of hub-covered edges of H such that the graph $H' = H \setminus F$ is hub-regular and has the same hub set and the same minimum hub degree as H.*

Proof. Suppose, by the way of contradiction, that the lemma is false and let H be a counterexample to the lemma with minimum number of edges. If H were hub-regular, then the lemma would hold for H by letting $F = \emptyset$, a contradiction. Hence, H is not hub-regular; i.e., $\Delta(G) > \delta_h(G)$. Let v be a hub of H that is not minimum. As $\delta_h(G) \geqslant 4$, the vertex v has at least 5 neighbors. So, since H contains no bipartite claw and has no 5-cycle, Lemma 4.46 implies that v has some neighbor w that is not a hub (because $\delta_h(H) \geqslant 4$). Then, since vw is not incident to any minimum hub of

H, $H_1 = H \backslash \{vw\}$ has the same hub set and the same minimum hub degree as H. The proof ends exactly as the one of Lemma 4.39. □

Lemma 4.48. *Let H be a graph containing no bipartite claw and having no 5-cycle. If H is hub-covered, has at least one edge, and $\delta_h(H) \geqslant 4$, then $\alpha_m(H) \geqslant \delta_h(H)$.*

Proof. By Lemma 4.47, there exists a set F of hub-covered edges of H such that $H' = H \backslash F$ is hub-regular and has the same hub set and the same minimum hub degree as H. As H is hub-covered and has at least one edge, $\delta_h(H) < +\infty$. Then, H' is also hub-covered and $\Delta(H') = \delta_h(H') = \delta_h(H) \geqslant 4$. Since H' is a subgraph of H, H' contains no bipartite claw and has no 5-cycle. Therefore, by Theorem 4.30, $\chi'(H') = \Delta(H')$; i.e., there is an edge-coloring ϕ' of H' using $\Delta(H') = \delta_h(H)$ colors. Let $\phi : E(H) \to \{0, 1, 2, \ldots, \delta_h(H)\}$ be such that $\phi(e) = \phi'(e)$ for each $e \in E(H')$ and $\phi(e) = 0$ for each $e \in E(H) \backslash E(H')$. Since H is hub-covered, ϕ is a $\delta_h(H)$-profuse-coloring of H by construction. Thus, by Lemma 4.34, $\alpha_m(H) \geqslant \delta_h(H)$. □

The next two lemmas settle Theorem 4.44 for fat caterpillars containing A or net.

Lemma 4.49. *Let H be a fat caterpillar containing A. Then, $\alpha_m(H) = \tau_m(H)$. More precisely, there are some $C = v_1v_2v_3v_4v_1$ and $x_1, x_2 \in V(H) \backslash V(C)$ as in the statement of Lemma 4.21 and one of the following assertions holds:*

(i) C *is chordless and*

$$\alpha_m(G) = \tau_m(G) = \begin{cases} 3 & \textit{if } d_H(v_3) = d_H(v_4) = 2 \\ \delta_h(H) & \textit{otherwise.} \end{cases}$$

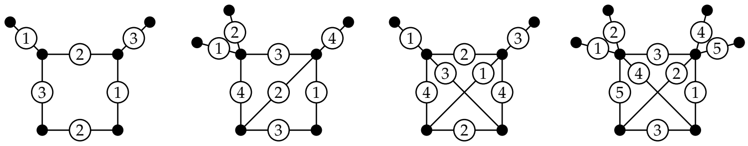
(ii) v_1v_3 *is the only chord of* C, $d_H(v_4) = 2$, *and*

$$\alpha_m(G) = \tau_m(G) = \begin{cases} 4 & \textit{if } d_H(v_2) \geqslant 4 \textit{ and } \delta_h(H) = 3 \\ \delta_h(H) & \textit{otherwise.} \end{cases}$$

(iii) C *has two chords,* $d_H(v_3) = d_H(v_4) = 3$, *and*

$$\alpha_m(G) = \tau_m(G) = \begin{cases} 5 & \textit{if each of } v_1 \textit{ and } v_2 \textit{ has degree at least } 5 \\ 4 & \textit{otherwise.} \end{cases}$$

Proof. Let $C = v_1v_2v_3v_4v_1$ and $x_1, x_2 \in V(H) \backslash V(C)$ as in the statement of Lemma 4.21. In particular, each non-pendant vertex in $V(H) \backslash V(C)$ is a false twin of v_4 of degree 2. Notice that $\alpha_m(H) \geqslant 3$ because a 3-profuse-coloring of H arises by coloring the



(a) (b) (c) (d)

Figure 4.7: *Some profuse-colorings for the proof of Lemma 4.49*

edges in $E(C) \cup \{v_1x_1, v_2x_2\}$ as in Figure 4.7(a) and leaving the remaining edges of H uncolored.

We claim that if $\delta_h(H) \geqslant 4$ then $\tau_m(H) \leqslant \delta_h(H)$. On the one hand, if some minimum hub of H is adjacent to some pendant vertex, then $\tau_m(H) \leqslant \delta_h(H)$ because of Lemma 4.33. On the other hand, if $\delta_h(H) \geqslant 4$ and the minimum hubs of H are adjacent to non-pendant vertices only, then v_3 is the only minimum hub of H and Lemma 4.45 implies that $\tau_m(H) \leqslant \delta_h(H)$ because $d_H(v_3) = \delta_h(H) \geqslant 4$ and $N_H(v_3)$ does not induce $2K_2$. Hence, the claim follows.

The proof splits into three cases corresponding to assertions (i)–(iii) of Lemma 4.21.

Case 1. C *is chordless.*

Suppose first that $d_H(v_3) = d_H(v_4) = 2$ or $\delta_h(H) = 3$. If $d_H(v_3) = d_H(v_4) = 2$ or some vertex of degree 3 is adjacent to a pendant vertex, then $\alpha_m(H) = \tau_m(H) = 3$ because $\tau_m(H) \leqslant 3$ by Lemma 4.33 and we have seen that $\alpha_m(H) \geqslant 3$. Otherwise, the only minimum hub is v_3 and $N_H(v_3)$ induces $3K_1$ which also leads to $\alpha_m(H) = \tau_m(H) = 3$ because $\tau_m(H) \leqslant 3$ by Lemma 4.45 and we have seen that $\alpha_m(H) \geqslant 3$. So, if $d_H(v_3) = d_H(v_4) = 2$ or $\delta_h(H) = 3$, then (i) holds.

Suppose now that neither $d_H(v_3) = d_H(v_4) = 2$ nor $\delta_h(H) = 3$ holds. Then, H is hub-covered and $\delta_h(H) \geqslant 4$ which implies that $\alpha_m(H) = \tau_m(H) = \delta_h(H)$ because $\alpha_m(H) \geqslant \delta_h(H)$ by Lemma 4.48 and $\tau_m(H) \leqslant \delta_h(H)$. So, also in this case, (i) holds.

Case 2. v_1v_3 *is the only chord of* C *and* $d_H(v_4) = 2$.

Assume first that $d_H(v_2) \geqslant 4$ and $\delta_h(H) = 3$. Necessarily, $d_H(v_3) = 3$. Hence, as $d_H(v_4) = 2$, Lemma 4.33 implies that $\tau_m(H) \leqslant 4$. Let y_2 be a neighbor of v_2 outside $V(C)$ different from x_2. Then, $\alpha_m(H) \geqslant 4$ because a 4-profuse-coloring of H arises by coloring the subgraph of H induced by $V(C) \cup \{x_1, x_2, y_2\}$ as in Figure 4.7(b) and leaving the remaining edges of H uncolored. We have proved that, if $d_H(v_2) \geqslant 4$ and $\delta_h(H) = 3$, then $\alpha_m(H) = \tau_m(H) = 4$ and, in particular, (ii) holds.

Assume now that, on the contrary, $d_H(v_2) = 3$ or $\delta_h(H) \geqslant 4$. If the former holds, then $\alpha_m(H) = \tau_m(H) = 3 = \delta_h(H)$ because we know that $\alpha_m(H) \geqslant 3$ and Lemma 4.33 would imply that $\tau_m(H) \leqslant 3$. If the latter holds, then $\alpha_m(H) = \tau_m(H) = \delta_h(H)$ because H is hub-covered and Lemma 4.48 would imply that $\alpha_m(H) \geqslant \delta_h(H)$ and because we have proved that $\tau_m(H) \leqslant \delta_h(H)$ whenever $\delta_h(H) \geqslant 4$. We conclude that if $d_H(v_1) = 3$ or $\delta_h(H) \geqslant 4$, then $\alpha_m(H) = \tau_m(H) = \delta_h(H)$ and (ii) holds.

Case 3. C *has two chords and* $d_H(v_3) = d_H(v_4) = 3$.

Assume v_1 or v_2 has degree 4. Then, Lemma 4.45 implies that $\tau_m(H) \leqslant 4$. In addition, a 4-profuse-coloring of H arises by coloring the edges of the subgraph of H induced by $V(C) \cup \{x_1, x_2\}$ as in Figure 4.7(c) and leaving all the remaining edges of H uncolored. In particular, $\alpha_m(H) \geqslant 4$. So, in this case, $\alpha_m(H) = \tau_m(H) = 4$ and (iii) holds.

Assume now that each of v_1 and v_2 has degree at least 5 and, for each $i = 1, 2$, let y_i be a neighbor of v_i outside $V(C)$ different from x_i. As $d_H(v_3) = d_H(v_4) = 3$, Lemma 4.33 implies that $\tau_m(H) \leqslant 5$. In addition, $\alpha_m(H) \geqslant 5$ because a 5-profuse-coloring of H arises by coloring the subgraph of H induced by $V(C) \cup \{x_1, x_2, y_1, y_2\}$ as in Figure 4.7(d) and leaving the remaining edges of H uncolored. Hence, in this case, $\alpha_m(H) = \tau_m(H) = 5$ and (iii) holds. □

Lemma 4.50. *Let* H *be a fat caterpillar containing net but containing no* A*. Then,* H *has some edge-dominating triangle* C *such that each* $v \in V(H)\backslash V(C)$ *is pendant and* $\alpha_m(H) = \tau_m(H) = \delta_h(H)$.

Proof. That H has an edge-dominating cycle C such that each $v \in V(H)\backslash V(C)$ is pendant follows from Lemma 4.22. As the hubs of H are the vertices of C and each of them is adjacent to some pendant vertex, Lemma 4.33 implies that $\tau_m(H) \leqslant \delta_h(H)$. For the proof of the lemma to be complete, it suffices to show that $\alpha_m(H) \geqslant \delta_h(H)$. If $\delta_h(H) \geqslant 4$, then, as H is hub-covered, $\alpha_m(H) \geqslant \delta_h(H)$ by Lemma 4.48. Finally, if $\delta_h(H) = 3$, then $\alpha_m(H) \geqslant 3$ because a 3-profuse-coloring of H arises by 3-edge-coloring the net induced in H by $\{v_1, v_2, v_3, u_1, u_2, u_3\}$ and leaving the remaining edges of H uncolored. □

In order to settle Theorem 4.44, it only remains to prove the next result.

Theorem 4.51. *Let* H *be a fat caterpillar containing no* A *and no net. Then, for each* $k \geqslant 1$, $\alpha_m(H) \geqslant k$ *if and only if* $\tau_m(H) \geqslant k$.

By Lemma 4.20, fat caterpillars containing no A and not net are certain linear concatenations of basic two-terminal graphs. To begin with, the following lemma, whose proof is straightforward, enumerates the values of α_m and τ_m for the underlying graphs of each of the basic two-terminal graphs.

Lemma 4.52. *The underlying graphs of each of the basic two-terminal graphs satisfy* $\alpha_m = \tau_m$*. Moreover:*

- *For the underlying graph of the edge,* $\alpha_m = \tau_m = 1$.
- *For the underlying graph of the triangle, the rhombus, and the* K_4, $\alpha_m = \tau_m = 3$.
- *For the underlying graph of the* m*-crown,* $\alpha_m = \tau_m = m + 1$, *for each* $m \geqslant 2$.
- *For the underlying graph of the* m*-fold,* $\alpha_m = \tau_m = m$, *for each* $m \geqslant 2$.

Our proof of Theorem 4.51 is indirect. The theorem clearly holds for $k = 1$. In the remaining of this sub-subsection, we deal separately with cases $k = 2$, $k = 3$, $k = 4$, $k = 5$, and finally $k \geqslant 6$. Case $k = 2$ of Theorem 4.51 can be derived from Theorem 4.4, as follows.

Lemma 4.53. *Let* H *be a fat caterpillar. Then,* $\alpha_m(H) \geqslant 2$ *if and only if* $\tau_m(H) \geqslant 2$.

Proof. The 'only if' part is trivial. For the converse, suppose, by the way of contradiction, that $\tau_m(H) \geqslant 2$ but $\alpha_m(H) \leqslant 1$. So, if $G = \overline{L(H)}$, then $\tau_c(G) \geqslant 2$ and $\tau_c(G) \leqslant 1$. Hence, by Theorem 4.4, G contains an induced Q_{2n+1} for some $n \geqslant 1$. As G is the complement of a line graph, necessarily G contains an induced Q_3 (= 3-sun) and, as a result, H contains a bipartite claw, a contradiction. This contradiction proves the 'if' part and the lemma follows. □

Case $k = 3$ can be dealt as follows.

Lemma 4.54. *Let* H *be a fat caterpillar containing no* A *and no net and having at least one edge. Then,* $\alpha_m(H) \geqslant 3$ *if and only if* $\tau_m(H) \geqslant 3$. *In fact, both inequalities hold if and only if* H *satisfies all of the following assertions:*

(i) *For each pair of adjacent vertices* v_1 *and* v_2, $d_H(v_1) + d_H(v_2) - 1 \geqslant 3$.

(ii) *Each* 4*-cycle of* H *has at most two vertices of degree* 2 *in* H.

(iii) H *is not the underlying graph of* triangle $\&_p$ triangle *for any* $p \geqslant 0$.

Proof. Since $\alpha_m(H) \leqslant \tau_m(H)$, clearly $\alpha_m(H) \geqslant 3$ implies $\tau_m(H) \geqslant 3$. Suppose that $\tau_m(H) \geqslant 3$. Then, (i) holds because of Lemma 4.33. If there were some 4-cycle $C = v_1v_2v_3v_4v_1$ such that $d_H(v_1) = d_H(v_2) = d_H(v_3) = 2$, then $\{v_1v_2, v_2v_3\}$ would be a matching-transversal of H, contradicting $\tau_m(H) \geqslant 3$. Similarly, if H were the underlying graph of triangle $\&_p$ triangle for some $p \geqslant 0$, then the two edges of H that are not incident to the concatenation vertex are a matching-transversal of H, another contradiction. These contradictions prove that (ii) and (iii) also hold.

To complete the proof of the lemma, let us assume that (i)–(iii) hold and we will prove that $\alpha_m(H) \geqslant 3$, or, equivalently, that H has a 3-profuse-coloring. As H is a fat caterpillar containing no A and no net, Lemma 4.20 implies that H is the underlying graph of $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_2} \cdots \&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph and each $p_i \geqslant 0$. If $n = 1$, then H is the underlying graph of some two-terminal graph different from an edge and a square and H admits a 3-profuse-coloring by Lemma 4.52. So, assume that $n \geqslant 2$.

Case 1. H *is the underlying graph of* $\Gamma_1 \&_p \Gamma_2$ *where each of* Γ_1 *and* Γ_2 *is an edge or a triangle and* $p \geqslant 0$.

By (iii), assume, without loss of generality, that Γ_1 is an edge. If Γ_2 is also an edge, then (i) implies that $p \geqslant 1$ and clearly $\alpha_m(H) \geqslant 3$ because a 3-profuse-coloring of H arises by coloring with three different colors any three edges of H and leaving the remaining edges of H uncolored. If, on the contrary, Γ_2 is a triangle, then also $\alpha_m(H) \geqslant 3$ because a 3-profuse-coloring of H arises by coloring the edge of Γ_1 and the two edges of Γ_2 incident to the concatenation vertex with three different colors and leaving the remaining edges of H uncolored.

Case 2. H *does not fulfils Case 1.*

For each $i = 1, \ldots, n$, let P_i be some shortest path in Γ_i joining its two terminal vertices. Then, $P = P_1 P_2 \ldots P_n$ is a chordless path in H and let $P = u_0 u_1 \ldots u_\ell$ where u_0 is the source of Γ_1 and u_ℓ is the sink of Γ_n. Consider a coloring of the edges of P with the colors 1, 2, and 3, such that any three consecutive edges of P receive three different colors. As P is edge-dominating, every edge of H is incident to at least two differently colored edges, except for the edges incident to u_0 and u_ℓ. Assume without loss of generality that $u_0 u_1$ is colored with color 1 and $u_1 u_2$ with color 2. We make the edges incident to u_0 adjacent to at least two differently colored edges as follows:

(1) If there are at least two edges joining u_0 to vertices outside P, we color two of these edges using colors 2 and 3.

(2) If there is exactly one vertex u' outside P adjacent to u_0, then Γ_1 is a triangle or a rhombus (because (ii) ensures that Γ_1 is not a square). In particular, u_1 is also adjacent to u' and we color $u_1 u'$ with color 3.

(3) If there is no vertex outside P adjacent to u_0, then Γ_1 is an edge and, by (i), u_1 is adjacent to some vertex u' outside P. We color $u_1 u'$ with color 3.

Symmetrically, let x be the color of $u_{\ell-1} u_\ell$, y be the color of $u_{\ell-2} u_{\ell-1}$, and $z \in \{1,2,3\} \setminus \{x,y\}$. We make the edges incident to u_ℓ adjacent to at least two differently colored edges as follows:

(1′) If there are at least two edges joining u_ℓ to vertices outside P, we color two of these edges using colors y and z.

(2′) If there is exactly one vertex u'' outside P adjacent to u_ℓ, then u'' is adjacent to $u_{\ell-1}$ (as in (2)). If there were an edge incident to $u_{\ell-1}$ colored with color z, then, $n = 2$, Γ_2 is a triangle, and either Γ_1 is an triangle or an edge, contradicting the hypothesis. So, we color the edge $u_{\ell-1}u''$ with color z.

(3′) If there is no vertex outside P adjacent to u_ℓ, then Γ_n is an edge and $u_{\ell-1}$ is adjacent to some vertex u'' outside P (as in (3)). If there were some edge incident to $u_{\ell-1}$ colored with color z, then $n = 2$ and Γ_1 is an edge or a triangle, which would contradict our hypothesis because Γ_2 is a triangle. So, we color $u_{\ell-1}u''$ with color z.

The resulting partial 3-edge-coloring is a 3-profuse-coloring of H because each edge of H is incident to at least two differently colored edges. Hence, $\alpha_m(H) \geqslant 3$, as needed. □

For case $k = 4$, we prove the following.

Lemma 4.55. *Let H be a fat caterpillar containing no net and no A and having at least one edge. Then, $\alpha_m(H) \geqslant 4$ if and only if $\tau_m(H) \geqslant 4$. In fact, both inequalities hold if and only H satisfies all of the following conditions:*

(i) *For each pair of adjacent vertices v_1 and v_2, $d_H(v_1) + d_H(v_2) - 1 \geqslant 4$.*

(ii) *No block of H is a complete of four vertices.*

(iii) *Each vertex of degree 3 that is not a cutpoint has only neighbors of degree at least 3.*

(iv) *The neighborhood of each cutpoint of degree 3 induces $K_2 \cup K_1$ in H.*

Proof. By Lemma 4.20, H is the underlying graph of some $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_2} \cdots \&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph and each $p_i \geqslant 0$. For each $i = 1, 2, \ldots, n-1$, let v_i be the concatenation vertex of H that arises by identifying the sink of Γ_i with the source of Γ_{i+1} and let v_0 be the source of Γ_1 and v_n be the sink of Γ_n. Clearly, the cutpoints of H are the concatenation vertices $v_1, v_2, \ldots, v_{n-1}$ and the underlying graph of each Γ_i is a block of H.

Since $\alpha_m(H) \leqslant \tau_m(H)$, $\alpha_m(H) \geqslant 4$ implies that $\tau_m(H) \geqslant 4$. Suppose now that H satisfies $\tau_m(H) \geqslant 4$. Then, H satisfies (i) because of Lemma 4.33. If some block of H were a complete of size four, this block would have at least three vertices of degree 3 in H (because H contains no A and has no 5-cycle) and the edges of the K_3 induced by these three vertices would be a matching-transversal of H. So, since $\tau_m(H) \geqslant 4$, H satisfies (ii). If there were a vertex v of H of degree 3 that were not a cutpoint and had a neighbor of degree less than 3, then, up to symmetry, either: (1) v is a non-terminal vertex of Γ_1 and Γ_1 is a rhombus, or (2) v is the source of Γ_1 and Γ_1 is a 2-crown or a 3-fold.

If (1) holds, the edges of the triangle induced by $N_H[v_0]$ form a matching-transversal of H of size 3. If (2) holds, $E_H(v_0)$ is a matching-transversal of H of size 3. In either case, we reach a contradiction to $\tau_m(H) \geqslant 4$. This contradiction proves that H satisfies (iii). Finally, if v is a cutpoint of H of degree 3, then $N_H(v)$ induces a disconnected graph with three vertices; i.e., $N_H(v)$ induces $3K_1$ or $K_2 \cup K_1$. But, if $N_H(v)$ induces $3K_1$, then, by Lemma 4.45, $\tau_m(H) \leqslant 3$. This proves that H satisfies (iv). Altogether, we have proved that, if $\tau_m(H) \geqslant 4$, then H satisfies conditions (i)–(iv).

To complete the proof of the lemma, we assume that H satisfies conditions (i)–(iv) and we will prove that $\alpha_m(H) \geqslant 4$, or, equivalently, by Lemma 4.34, that H has a 4-profuse-coloring. To begin with, we prove the following claims about H.

Claim 1. *Each of Γ_1 and Γ_n is either an edge, m-crown for some $m \geqslant 3$, or m-fold for some $m \geqslant 4$.*

Proof of the claim. Indeed, each of Γ_1 and Γ_n is different from triangle and square because of (i), different from 2-crown, 3-fold, and rhombus because of (iii), and different from K_4 because of (ii). As each of Γ_1 and Γ_n is basic, the claim follows. □

Claim 2. *If there is a maximal 4-profuse-coloring ϕ of H and there are at least three edges of Γ_j incident to the same terminal vertex of Γ_j, then each terminal vertex of Γ_j is incident to four edges of H colored by ϕ.*

Proof of the claim. Without loss of generality, assume that there are at least three edges of Γ_j incident to v_j. As Γ_j is basic, there are also at least three edges of Γ_j incident to v_{j-1} and Γ_j is either an m-crown for some $m \geqslant 2$ or an m-fold for some $m \geqslant 3$. So, if $d_H(v_j) = 3$, then $j = n$ and either Γ_n would be a 2-crown or a 3-fold, contradicting Claim 1. Therefore, $d_H(v_j) \geqslant 4$ and, symmetrically, $d_H(v_{j-1}) \geqslant 4$. In addition, neither $N_H(v_j)$ nor $N_H(v_{j-1})$ induces $2K_2$ in H and, by Lemma 4.45, $E_H(v_j)$ and $E_H(v_{j-1})$ are matching-transversals of H. Hence, by Corollary 4.35, the maximality of ϕ implies that each of v_j and v_{j-1} is incident to four edges of H colored by ϕ. □

Claim 3. *If $n \geqslant 2$, Γ_{n-1} and Γ_n are both edges, $p_{n-1} = 2$, and there is some 4-profuse-coloring of H, then either $n = 2$ or there is some 4-profuse-coloring of H that colors at least two of the edges incident to v_{n-2}.*

Proof of the claim. Suppose that $n \geqslant 3$ and we have to prove that there is a 4-profuse-coloring of H that colors at least two edges incident to v_{n-2}. Let ϕ be a 4-profuse-coloring of H that maximizes the number of colored edges incident to v_{n-2} and, without loss of generality, assume that ϕ is maximal. Suppose, by the way of contradiction, that ϕ colors at most one edge incident to v_{n-2}. As ϕ is maximal, the four edges incident to v_{n-1} are colored by ϕ and, in particular, $v_{n-2}v_{n-1}$ is colored. So, by hypothesis, all edges incident to v_{n-2} different from $v_{n-2}v_{n-1}$ are uncolored. If there were an edge joining v_{n-2} to some non-cutpoint vertex of H, then this edge would be uncolored and, at the same time, incident to at most three colored edges, contradicting the maximality of ϕ. Therefore, $p_{n-1} = 0$ and Γ_{n-2} is an edge. As $v_{n-3}v_{n-2}$ is uncolored and $v_{n-2}v_{n-1}$ is the only colored edge incident to v_{n-2}, there are at least three colored edges incident to v_{n-3} such that each of them is colored differently from $v_{n-2}v_{n-1}$. If there were some pendant edge p incident to v_{n-3} and colored differently from $v_{n-2}v_{n-1}$, then, by coloring $v_{n-3}v_{n-2}$ with the color of p and uncoloring p, a new 4-profuse-coloring of H arises that colors at least two edges incident to v_{n-2}, a contradiction with the choice of ϕ. This contradiction shows that there are at least three colored edges of Γ_{n-2} incident to v_{n-3}. So, by Claim 2, v_{n-4} is incident to four colored edges. Let e be any of the colored edges incident to v_{n-3} but not to v_{n-4} such that e is colored differently from $v_{n-2}v_{n-1}$. Then, coloring $v_{n-3}v_{n-2}$ with the color of e and uncoloring e, a new 4-profuse-coloring of H arises that colors two of the edges edges incident to v_{n-2}, contradicting the choice of ϕ. This contradiction arose from assuming that ϕ does not color at least two edges incident to v_{n-2}. So, the claim follows. $\square$

Claim 4. *If H has a 4-profuse-coloring, Γ_1 is an edge, $n \geqslant 2$, $p_1 = 1$, and $N_H(v_1)$ induces $K_2 \cup 2K_1$ in H, then there is a 4-profuse-coloring ϕ of H that colors the only edge of H joining two neighbors of v_1.*

Proof of the claim. Let ϕ' be a maximal 4-profuse-coloring of H and let e be the only edge of H joining two vertices in $N_H(v_1)$. As $d_H(v_1) = 4$ and $N_H(v_1)$ does not induce $2K_2$, Lemma 4.45 implies that $E_H(v_1)$ is a matching-transversal of H and the four edges incident to v_1 are colored by ϕ' because of the maximality of ϕ' and because of Corollary 4.35. If ϕ' colors e, the claim holds by letting $\phi = \phi'$. So, suppose that e is not colored by ϕ'. Then, the maximality of ϕ' implies that e is incident to at least four other edges of H.

Suppose first that e is incident to exactly four edges of H; i.e., either Γ_2 is triangle and $d_H(v_2) = 4$, or Γ_2 is rhombus. Let w be an endpoint of e different from v_2 and let $e' = v_1w$. Let e'' be a pendant edge incident to v_1 and colored differently from each of the colored edges incident to w. Notice that the maximality of ϕ, Lemma 4.33, and Corollary 4.35 imply that the four edges of H incident to e are colored by ϕ' using four different colors. So, if we define $\phi : E(H) \to \{0, 1, 2, 3, 4\}$ to be as ϕ' except that ϕ colors e and e'' with color $\phi'(e')$ and e' with color $\phi'(e'')$, then ϕ is a 4-profuse-coloring of H that colors e, as claimed.

It only remains to consider the case when e is incident to more than four edges of H. Necessarily, Γ_2 is a triangle and $d_H(v_2) \geqslant 5$. Let w be the non-terminal vertex of Γ_2. Suppose that there is some pendant edge p incident to v_2 that is colored by ϕ'. By permuting, if necessary, the colors of the edges of H incident to v_1 that are different from v_1v_2, we assume, without loss of generality, that v_1w is colored differently from p. Then, by coloring e with the color of p and uncoloring p, a new 4-profuse-coloring of H arises that colors e, as claimed. So, from now on, we assume, without loss of generality, that there is no pendant edge incident to v_2 colored by ϕ'. So, as $d_H(v_2) \geqslant 5$, Lemma 4.45 and Corollary 4.35 imply that there are four edges incident to v_2 colored by ϕ' and, necessarily, three of them are edges of Γ_3. By Claim 2, there are four colored edges incident to v_3. Therefore, if we let e' be any edge of Γ_3 incident to v_2 but not to v_3 and colored by ϕ' differently from v_1w, then by coloring e with the color of e' and uncoloring e', a new 4-profuse-coloring of H arises that colors e, as claimed. □

Claim 5. *If H has a 4-profuse-coloring, Γ_1 = edge, $n \geqslant 2$, and $p_1 \geqslant 1$, then there is a 4-profuse-coloring of H that colors at least two pendant edges incident to v_1.*

Proof of the claim. Suppose, by the way of contradiction, that ϕ is a 4-profuse-coloring of H that maximizes the number of colored pendant edges incident to v_1 and that, nevertheless, ϕ colors at most one pendant edge incident to v_1. Since $p_1 \geqslant 1$, there is at least one uncolored pendant edge incident to v_1. Then, the maximality of ϕ means that there are four colored edges incident to v_1. As Γ_1 is an edge and there is at most one colored pendant edge incident to v_1, there are at least three colored edges of Γ_2 incident to v_1. Then, by Claim 2, there are four colored edges incident to v_2. Let e be any colored edge of Γ_2 incident to v_1 but not to v_2 and let p be any of the uncolored pendant edges incident to v_1. If we color p with the color of e and uncolor e, a new 4-profuse-coloring of H arises that colors one more pendant edge incident to v_1 than ϕ, contradicting the choice of ϕ. This contradiction proves that the claim holds. □

We turn back to the proof of the lemma. The proof proceeds by induction on the number of cutpoints of H. Clearly, the cutpoints of H are the $n-1$ vertices $v_1, \ldots, v_{n-1}$. Consider first the case when H has no cutpoints; i.e., $n = 1$. Then, H is the underlying graph of Γ_1 which, by Claim 1, is an edge, m-crown for some $n \geqslant 3$, or m-fold for some $m \geqslant 2$. If Γ_1 were an edge, then $d_H(v_0) + d_H(v_1) - 1 = 1$, which contradicts (i). Therefore, if $n = 1$, then H is m-crown for some $m \geqslant 3$ or m-fold for some $m \geqslant 4$ and, by Lemma 4.52, $\alpha_m(H) \geqslant 4$.

Assume that $n \geqslant 2$ and that the lemma holds for graphs with less than $n-1$ cutpoints. Suppose that H has some cutpoint of degree 3; i.e., there is some $j \in \{1, 2, \ldots, n-1\}$ such that v_j has degree 3 in H. By (iv), $N_H(v_j)$ induces $K_2 \cup K_1$ in H. Therefore, $p_j = 0$ and, by symmetry, assume, without loss of generality, that Γ_j is an edge and Γ_{j+1} is either a triangle or a rhombus. Let H_1 be the graph that arises from H by first removing all vertices and edges from $\Gamma_{j+1}, \Gamma_{j+2}, \ldots, \Gamma_n$, except for the vertices of $N_H[v_j]$ and the edges incident to v_j, and, then, adding one pendant edge p incident to v_j. Notice that H_1 can be regarded as the underlying graph of $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_1} \ldots \&_{p_{j-1}} \Gamma_j \&_2$ edge. Clearly, H_1 satisfies (i)–(iv) and, by induction hypothesis, there is a 4-profuse-coloring of H_1. By Claim 3, there is a 4-profuse-coloring ϕ_1 of H_1 that colors at least two of the edges of H_1 incident to v_{j-1}. So, by permuting, if necessary, the pendant edges incident to v_j in H_1, we assume, without loss of generality, that ϕ_1 colors some edge incident to v_{j-1} with color $\phi_1(p)$. Let H_2 be the graph that arises from H by first removing all vertices an edges of $\Gamma_1, \Gamma_2, \ldots, \Gamma_j$, except for the vertices of $N_H[v_j]$ and the edges incident to v_j, and, then, adding one pendant edge incident to v_j. The graph H_2 can also be regarded as the underlying graph of edge $\&_1 \Gamma_{j+1} \&_{p_{j+1}} \Gamma_{j+2} \&_{p_{j+2}} \cdots \&_{p_{n-1}} \Gamma_n$. By Claim 4, there is a maximal 4-profuse-coloring ϕ_2 of H_2 that colors the only edge e joining two neighbors of v_j. By permuting, if necessary, the pendant edges incident to v_j, we assume, without loss of generality, that ϕ_2 colors e differently from the edge of Γ_j. Moreover, by permuting, if necessary, the colors of ϕ_2, we assume without loss of generality, that ϕ_1 and ϕ_2 color the edge of Γ_j and each of the edges of Γ_{j+1} incident to v_j in exactly the same way. Thus, there is no edge of H where ϕ_1 and ϕ_2 differ and the partial edge-coloring ϕ that results by merging ϕ_1 and ϕ_2 is easily seen to be 4-profuse-coloring of H, as desired. Therefore, from now on, we assume, without loss of generality, that H has no cutpoint of degree 3.

Suppose now that there is some $j \in \{1, 2, \ldots, n\}$ such that Γ_j is a rhombus. Let H_1 be the graph that arises from H by removing all the vertices and edges from $\Gamma_j, \Gamma_{j+1}, \ldots, \Gamma_n$ except for the vertices of $N_H[v_{j-1}]$ and the edges incident to v_{j-1}, and let H_2 the graph that arises from H by removing all vertices and edges from $\Gamma_1, \Gamma_2, \ldots, \Gamma_j$ except for the vertices of $N_H[v_j]$ and the edges incident to v_j. Moreover, as H has no cutpoint of degree 3, $d_{H_1}(v_{j-1}) \geqslant 4$, from which it follows that H_1 satisfies (i)–(iv) and, by induction hypothesis, H_1 admits a 4-profuse-coloring ϕ_1. Similarly, $d_{H_2}(v_{j+1}) \geqslant 4$ and H_2 admits a 4-profuse-coloring ϕ_2. By Claim 5, we assume, without loss of generality, that ϕ_i colors both edges of Γ_j that belong to H_i, for $i = 1, 2$. By permuting, if necessary, the colors of ϕ_2, we assume, without loss of generality, that ϕ_1 and ϕ_2 color the four edges of Γ_j that belong to H_1 or H_2 using 4 different colors. Then, let $\phi : E(H) \to \{0, 1, 2, 3, 4\}$ defined as ϕ_1 in $E(H_1)$, as ϕ_2 in $E(H_2)$, and that leaves the only edge of Γ_j that belongs neither to H_1 nor to H_2 uncolored. Clearly, ϕ is a 4-profuse-coloring of H, as desired.

It only remains to consider the case when H has no cutpoins of degree 3 and no Γ_j is a rhombus; i.e., the case when $\delta_h(H) \geqslant 4$. Then, as (i) ensures that H is hub-covered and since H has at least one edge, Lemma 4.48 implies that $\alpha_m(H) \geqslant \delta_h(H) \geqslant 4$, which completes the proof of the lemma. □

The following lemma settles case $k = 5$.

Lemma 4.56. *Let* H *be a fat caterpillar containing no* A *and no net and having at least one edge. Then,* $\alpha_m(H) \geqslant 5$ *if and only if* $\tau_m(H) \geqslant 5$. *In fact, both inequalities hold if and only if*

H *satisfies all of the following assertions:*

(i) *For each pair of adjacent vertices* v_1 *and* v_2, $d_H(v_1) + d_H(v_2) - 1 \geqslant 5$.

(ii) *No block of* H *is a complete of four vertices.*

(iii) *No cutpoint of* H *has degree* 3 *in* H.

(iv) *The neighborhood of each vertex of degree* 4 *induces* $2K_2$ *in* H.

Proof. Since $\alpha_m(H) \leqslant \tau_m(H)$, $\alpha_m(H) \geqslant 5$ implies $\tau_m(H) \geqslant 5$. Suppose now that H satisfies $\tau_m(H) \geqslant 5$. Then, H satisfies (i) because of Lemma 4.33. If there were some block of H of size four, it would have at least three vertices of degree 3 in H (because H contains no A and has no 5-cycle) and the edges of the K_3 induced by these three vertices would be a matching-transversal of H, contradicting $\tau_m(H) \geqslant 5$. So, H satisfies (ii). Since the neighborhood of a cutpoint induces a disconnected graph, if H had some cutpoint of degree 3, then, by Lemma 4.45, $\tau_m(H) \leqslant 4$. Hence, H satisfies (iii). Finally, Lemma 4.45 implies that H satisfies (iv). Hence, we have proved that if $\tau_m(H) \geqslant 5$, then H satisfies (i)–(iv). To complete the proof of the lemma, we assume that H satisfies conditions (i)–(iv) and we will show that $\alpha_m(H) \geqslant 5$, or, equivalently, by Lemma 4.34, that H has a 5-profuse-coloring.

By virtue of Lemma 4.20, H is the underlying graph of some $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_2} \cdots \&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph and each $p_i \geqslant 0$. Clearly, the underlying graph of each Γ_i is a block of H. Therefore, because of (ii), none of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is a K_4. For each $i = 1, 2, \ldots, n-1$, let v_i be the concatenation vertex of H that arises by identifying the sink of Γ_i with the source of Γ_{i+1}. Let v_0 be the source of Γ_1 and let v_n be the sink of Γ_n. We make the following claims.

Claim 1. *Each of* Γ_1 *and* Γ_n *is either an edge,* m*-crown for some* $m \geqslant 4$*, or* m*-fold for some* $m \geqslant 5$.

Proof of the claim. Indeed, each of Γ_1 and Γ_n is different from triangle, square, 2-crown, 3-fold, and rhombus because of (i), different from 3-crown and 4-fold because of (iv), and different from K_4 because of (ii). The claim follows. □

Claim 2. *If there is a maximal* 5*-profuse-coloring* ϕ *of* H *and there are at least three edges of* Γ_j *incident to the same terminal vertex of* Γ_j*, then each terminal vertex of* Γ_j *is incident to five edges of* H *colored by* ϕ.

Proof of the claim. Without loss of generality, suppose that there are at least three edges of Γ_j incident to v_j. As Γ_j is basic, there are also at least three edges of Γ_j incident to v_{j-1} and Γ_j is either and m-crown for some $m \geqslant 2$ or an m-fold for some $m \geqslant 3$. If $d_H(v_j) = 3$, then $j = n$ and Γ_n is either a 3-crown or a 4-fold, contradicting Claim 1.

So, $d_H(v_j) \geqslant 4$ and, symmetrically, $d_H(v_{j-1}) \geqslant 4$. In addition, neither $N_H(v_j)$ nor $N_H(v_{j-1})$ induces $2K_2$ and, by (iv), $d_H(v_j) \geqslant 5$ and $d_H(v_{j-1}) \geqslant 5$. Hence, Lemma 4.45, Corollary 4.35, and the maximality of ϕ imply that each of v_j and v_{j-1} is incident to five edges colored of H by ϕ, as claimed. □

Claim 3. *If* H *has a* 5*-profuse-coloring and* Γ_j *is a triangle of* H*, then there is a* 5*-profuse-coloring of* H *that colors the three edges of* Γ_j.

Proof of the claim. By the way of contradiction, assume that the claim is false. Hence, there is some link Γ_j that is a triangle and some 5-profuse-coloring ϕ of H that maximizes the number of colored edges of Γ_j such that, nevertheless, ϕ does not color the three edges of Γ_j. Without loss of generality, assume that ϕ is maximal. Let w be the non-terminal vertex of Γ_j. By Claim 1 and (iii), $d_H(v_{j-1}) \geqslant 4$ and $d_H(v_j) \geqslant 4$. Suppose, by the way of contradiction, that $d_H(v_j) = 4$. Then, Lemma 4.33 implies that the set of five edges $E_H(v_j) \cup E_H(w)$ is a matching-transversal of H and, by the maximality of ϕ and Corollary 4.35, these five edges are colored by ϕ, contradicting the fact that not all the edges of Γ_j are colored. So, necessarily $d_H(v_j) \geqslant 5$ and, symmetrically, $d_H(v_{j-1}) \geqslant 5$. Let e be any uncolored edge of Γ_j and assume, without loss of generality, that e is incident to v_j. As $d_H(v_j) \geqslant 5$, there are five colored edges incident to v_j because of Lemma 4.45, Corollary 4.35, and the maximality of ϕ. If there were some pendant edge p incident to v_j and colored differently from $v_{j-1}w$ (if colored), then, by coloring e with the color of p and uncoloring p, a new 5-profuse-coloring of H that colors one more edge of Γ_j would arise, contradicting the choice of ϕ. This contradiction proves that among the colored edges incident to v_j, there are at least three of them that are edges of Γ_j. Therefore, by Claim 2, there are five colored edges incident to v_{j+1}. Symmetrically, if e were incident to v_{j-1}, then there would be five colored edges incident to v_{j-2}. Finally, let $c \in \{1,2,3,4,5\}$ different from the colors of the colored edges of Γ_j and different from the colors of v_jv_{j+1} (if present and colored) and $v_{j-2}v_{j-1}$ (if present and colored). Let ϕ' be the partial edge-coloring of H defined as ϕ except that ϕ' colors e with color c and uncolors the edge of H incident to e colored by ϕ with color c. By construction, ϕ' is a 5-profuse-coloring of H and ϕ' colors one more edge of Γ_j than ϕ, a contradiction with the choice of ϕ. This contradiction proves that ϕ colors all the edges of Γ_j and the claim holds. □

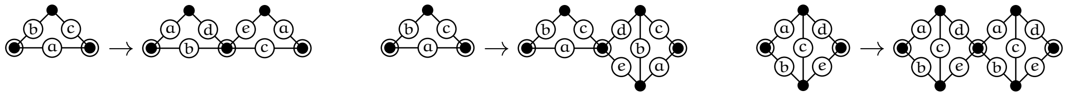
Claim 4. *If* H *has a* 5*-profuse-coloring,* Γ_1 *is an edge,* $n \geqslant 2$*, and* $p_1 \geqslant 1$*, then there is a* 5*-profuse-coloring* ϕ *of* H *that colors at least two pendant edges incident to* v_1.

Proof of the claim. By the way of contradiction, suppose that there is 5-profuse-coloring ϕ of H that maximizes the number of colored pendant edges incident to v_1 and that, nevertheless, ϕ colors at most one pendant edge incident to v_1. Without loss of generality, assume that ϕ is maximal. Since $p_1 \geqslant 1$, there is still at least one uncolored pendant edge incident to v_1. Then, the maximality of ϕ implies that there are five colored edges incident to v_1 and, as there is at most one pendant colored edge incident to v_1, there are at least four colored edges of Γ_2 incident to v_1. By Claim 2, there are five colored edges incident to v_2. Let e be any of the colored edges of Γ_2 incident to v_1 but not to v_2 and let p be any of the uncolored pendant edges incident to v_1. If we color p with the color of e and uncolor e, a new 5-profuse-coloring of H arises that colors one more pendant edge incident to v_1 than ϕ, contradicting the choice of ϕ. This contradiction proves the claim. □

We turn back to the proof of the lemma. The proof proceeds by induction on the number of cutpoints of H. Consider the case H has no cutpoints; i.e., $n = 1$. Then H is the underlying graph of Γ_1 which, by Claim 1, is an edge, m-crown for some $m \geqslant 4$, or m-fold for some $m \geqslant 5$. If H were an edge, v_0 and v_1 would be two adjacent pendant vertices of H and $d_H(v_0)+d_H(v_1)-1 = 1$, which would contradict (i). So, H is m-crown for some $m \geqslant 4$ or m-fold for some $m \geqslant 5$ and, by Lemma 4.52, $\alpha_m(H) \geqslant 5$.

Assume now that $n \geqslant 2$ and that the lemma holds for graphs with less than $n-1$ cutpoints. Suppose first that H has a cutpoint of degree 4 and let $j \in \{1, 2, 3, \ldots, n-1\}$ such that $d_H(v_j) = 4$. Because of (iv), $N_H(v_j)$ induces $2K_2$ in H. Therefore, $p_j = 0$ and each of Γ_j and Γ_{j+1} is a triangle or a rhombus. If one of Γ_j and Γ_{j+1} is a triangle and the other is a rhombus, we assume, without loss of generality, that Γ_j is the one that is a triangle. Let H' be the graph that arises from H by contracting Γ_{j+1} to a vertex. Then, H' is the underlying graph of $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_2} \cdots \&_{p_{j-1}} \Gamma_j \&_{p_{j+1}} \Gamma_{j+2} \&_{p_{j+2}} \cdots \&_{p_{n-1}} \Gamma_n$ and H' satisfies (i)–(iv). By induction hypothesis, H' has a 5-profuse-coloring ϕ'. Without loss of generality, assume that ϕ' is maximal. If Γ_j is a rhombus, the maximality of ϕ' implies that ϕ' colors all the edges of Γ_j. If, instead, Γ_j is a triangle, then Claim 3 allows us to assume that ϕ' colors all the edges of Γ_j. Then, we define a new partial 5-edge-coloring $\phi : E(H) \to \{0, 1, 2, 3, 4, 5\}$ as follows. Let ϕ coincide with ϕ' in those edges of H that are neither of Γ_j nor of Γ_{j+1} and we define ϕ on the edges of Γ_j and Γ_{j+1} depending on how ϕ' colors the edges of Γ_j as described in Figure 4.8, where a, b, c, d, e is a permutation of the colors $1, 2, 3, 4, 5$. Clearly, ϕ is a 5-profuse-coloring of H and $\alpha_m(H) \geqslant 5$, as desired. Therefore, from now on, we assume that $d_H(v_i) \geqslant 5$ for each $i = 1, 2, \ldots, n-1$.

Next, we assume that Γ_j is a rhombus for some j. As Claim 1 implies that neither Γ_1 nor Γ_n is rhombus, $2 \leqslant j \leqslant n-1$. Let H_1 be the graph that arises from H by removing all the vertices and edges of $\Gamma_j, \Gamma_{j+1}, \ldots, \Gamma_n$ except for the vertices of $N_H[v_{j-1}]$ and the edges incident to v_{j-1}. Let H_2 be the graph that arises from H by removing all the vertices and edges of $\Gamma_1, \Gamma_2, \ldots, \Gamma_j$ except the vertices of $N_H[v_j]$ and the edges incident to v_j. Then, we can regard H_1 as the underlying graph of $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_2} \cdots \&_{p_{j-2}}$ $\Gamma_{j-1} \&_{p_{j-1}+1}$ edge and H_2 as the underlying graph of edge $\&_{p_j+1} \Gamma_{j+1} \&_{p_{j+1}} \Gamma_{j+2} \&_{p_{j+2}}$



ϕ' on Γ_j ϕ on Γ_j and Γ_{j+1} (a) ϕ' on Γ_j ϕ on Γ_j and Γ_{j+1} (b) ϕ' on Γ_j ϕ on Γ_j and Γ_{j+1} (c)

Figure 4.8: *Rules for transforming ϕ' into ϕ in the proof of Lemma 4.56. Here a, b, c, d, e represents any permutation of the colors $1, 2, 3, 4, 5$ and rule (a), (b), or (c) apply depending on whether each of Γ_j and Γ_{j+1} is a triangle or a rhombus.*

$\cdots \&_{p_{n-1}} \Gamma_n$. Since we are assuming that $d_H(v_{j-1}) \geqslant 5$ and $d_H(v_j) \geqslant 5$, H_1 and H_2 satisfy conditions (i)–(iv). By induction hypothesis, there are 5-profuse colorings of H_1 and H_2. By Claim 4, we can assume that the 5-profuse-colorings of H_1 and H_2 are such that the two edges of Γ_j incident to v_{j-1} are colored by the 5-profuse-coloring of H_1 and the two edges of Γ_j incident to v_j are colored by the 5-profuse-coloring of H_2. By permuting, if necessary, the colors in the 5-profuse-coloring of H_2, we can assume that the four edges of Γ_j that are incident to some terminal vertex of Γ_j are colored by these profuse colorings using four different colors. So, a 5-profuse-coloring of H arises by merging the profuse-colorings of H_1 and H_2 and letting the edge joining the two non-terminal vertices of Γ_j uncolored. Thus, by Lemma 4.34, $\alpha_m(H) \geqslant 5$. So, from this point on, we assume that no Γ_i is a rhombus.

Because of (iii) and because we are assuming that no cutpoint of H has degree 4, each of the vertices $v_1, v_2, \ldots, v_{n-1}$ has either degree 2 or degree at least 5. In addition, since each of Γ_1 and Γ_n is either an edge, m-crown for some $m \geqslant 4$, or m-fold for some $m \geqslant 5$, each of v_0 and v_n has degree 1 or at least 5. Finally, since no Γ_i is rhombus or K_4, each vertex of H different from $v_0, v_1, \ldots, v_n$ has degree at most 2. So, $\delta_h(H) \geqslant 5$. Since H has at least one edge and H is hub-covered (because of (i)), Lemma 4.48 implies that $\alpha_m(H) \geqslant \delta_h(H) \geqslant 5$, which completes the proof. □

Finally, for the $k \geqslant 6$ we prove the following.

Lemma 4.57. *Let H be a fat caterpillar containing no A and no net and having at least one edge. If $k \geqslant 6$, then the following assertions are equivalent:*

(i) $\alpha_m(H) \geqslant k$.

(ii) $\tau_m(H) \geqslant k$.

(iii) H *is hub-covered and* $\delta_h(H) \geqslant k$.

Proof. Clearly, (i) implies (ii) because $\alpha_m(H) \leqslant \tau_m(H)$. As $k \geqslant 6$ and H has at least one edge, Lemma 4.48 shows that (iii) implies (i). For the proof to be complete, it suffices to show that (ii) implies (iii). Suppose that $\tau_m(H) \geqslant k$. Since $k \geqslant 6$, H is hub-covered because of Lemma 4.33. By virtue of Lemma 4.20, H is the underlying graph of some $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph and each $p_i \geqslant 0$. If there were some $i \in \{1, 2, \ldots, n\}$ such that Γ_i is a rhombus or K_4, then the two non-terminal vertices of Γ_i would be two adjacent vertices of degree 3 and Lemma 4.33 would imply that $\tau_m(H) \leqslant 5$, a contradiction. Therefore, each Γ_i is an m-crown for some $m \geqslant 0$ or an m-fold for some $m \geqslant 2$. Let v_i the vertex of H that arises by identifying the sink of Γ_i and the source of Γ_{i+1} and let v_0 be the source of Γ_1 and v_n be the sink of Γ_n. Then, each v_i has degree 2 in H or has a neighbor in H of degree 2 in H. Therefore, for each $i = 1, 2, \ldots, n$, either $d_H(v_i) = 2$ or $d_H(v_i) \geqslant k - 1$ because given any neighbor w of degree 2 of v_i the inequality $d_H(v_i) + 1 = d_H(w) + d_H(v_i) - 1 \geqslant k$ must hold because of Lemma 4.33. Notice also that, since Γ_1 is a crown or a fold, either $d_H(v_0) = 1$ or $d_H(v_0) \geqslant k$ because if v_0 is not pendant then $E_H(v_0)$ is clearly a matching-transversal of H. Symmetrically, either $d_H(v_n) = 1$ or $d_H(v_n) \geqslant k$. Finally, all vertices of H different from $v_0, v_1, \ldots, v_n$ are vertices of degree 2 because no block of H is a rhombus or K_4. We conclude that $\delta_h(H) \geqslant k - 1$. Since $k - 1 \geqslant 5$, Lemma 4.45 implies that $\tau_m(H) \leqslant \delta_h(H)$. Since we are assuming $\tau_m(H) \geqslant k$, $\delta_h(H) \geqslant k$. Thus, (ii) implies (iii) and the proof is complete. □

As we have proved Lemmas 4.49 and 4.50 and all the cases of Theorem 4.51, now Theorem 4.44 follows. This, together with Theorem 4.36, imply Theorem 4.32, from which the main results of this section (Theorems 4.16 and 4.17) follow.

4.2.4 Recognition algorithm and computing the parameters

The reader acquainted with the theory of tree-width and second order logic may notice the following. Since forbidding the bipartite claw as a subgraph or as a minor are equivalent, graphs containing no bipartite claw have bounded tree-width [106] and have a linear-time recognition algorithm [14]. Moreover, as the characterization in Theorem 4.17 can be expressed in counting monadic-second order logic with edge set quantifications (see [39]), its validity can be verified in linear time within any graph class of bounded tree-width [27, 38]. In particular, matching-perfect graphs can be recognized in linear time. Nevertheless, the resulting algorithm is not elementary. Instead, below we propose a very elementary linear-time recognition algorithm for matching-perfect graphs which relies on depth-first search only.

Let H be a graph. We denote by H_1 the graph that arises from H by removing all vertices that are pendant in H. We denote by H_2 some maximal induced subgraph of H having no vertices that are pendant in H and no two vertices that are false twins of degree 2 in H. Finally, we denote by H_3 some maximal induced subgraph of H

having no two vertices that are false twins of degree 2 in H. We claim that there is an elementary linear-time algorithm that either computes H_3 or determines that H contains a bipartite claw. Let us consider an algorithm that keeps a list $L(v)$ for each vertex v of H and that stores at each vertex v of H a boolean variable indicating whether or not the vertex is marked for deletion. Initially, all the list are empty and no vertex is marked for deletion. The algorithm proceeds by visiting every vertex v of H and, for each neighbor $u \in N_H(v)$ that was not marked for deletion and such that $N_H(u) = \{v, w\}$ for some $w \in V(H)$, we do the following: if w is already in the list of $L(v)$, then we mark u for deletion, otherwise we add w to $L(v)$. To make the algorithm linear-time, we stop whenever we attempt to add a third vertex to any of the lists $L(v)$ as this means that v is the center of a bipartite claw. If all vertices of H are visited and no bipartite claw is detected, then we output as H_3 the subgraph of H induced by those vertices not marked for deletion. The algorithm is clearly correct and linear-time. So, it follows that there is an elementary algorithm that either computes H_1, H_2, and H_3 in linear time or detects that H contains a bipartite claw.

We now claim that there is also an elementary linear-time algorithm to decide whether a given graph is a fat caterpillar and, if affirmative, compute a matching-transversal of minimum size. To begin with, we proceed as in the preceding paragraph in order to either compute H_1, H_2, and H_3, or detect that H contains a bipartite claw. If the latter occurs, we can be certain that H is not a fat caterpillar and stop. So, without loss of generality, assume that H_1, H_2, and H_3 were successfully computed in linear time. If H_1 is a triangle and each vertex of H_1 has some neighbor in H outside H_1, then Lemma 4.50 implies that H is a fat caterpillar and the set of edges incident to any minimum hub of H is a matching-transversal of minimum size. Suppose now that H_2 is spanned by a 4-cycle C having at least two consecutive vertices adjacent in H to some vertex outside H_2. Let $C = v_1v_2v_3v_4v_1$ where v_1 and v_2 are adjacent to some vertex outside H_2 and v_4 is the only vertex of H_2 that may have false twins of degree 2 in H. In this case, it is straightforward to determine whether or not H is a fat caterpillar and, if affirmative, compute a matching-transversal of minimum size in linear time thanks to Lemma 4.49. Assume now that neither H_1 is a triangle such that each vertex of H_1 is adjacent in H to some vertex outside H_1, nor H_2 is spanned by a 4-cycle having at least two consecutive vertices adjacent in H to vertices outside H_2. Then, by Lemmas 4.49 and 4.50, H is a fat caterpillar if and only if H is a fat caterpillar containing no A and no net. Therefore, by Lemma 4.20, H is a fat caterpillar if and only if H is a linear concatenation of basic two-terminal graphs where the K_4 links may occur only as the first and/or last links of the concatenation. So, H is a fat caterpillar if and only if H_3 is a linear concatenation of edge, triangle, rhombus, and K_4 links where the K_4 links may occur only as the first/and or last link of the concatenation and no vertex of a rhombus link has a false twin of degree 2 in H. Equivalently, H is a fat caterpillar if and only if H_3 satisfies each of the following conditions:

(1) Each of the blocks of H_3 is an edge, a triangle, a diamond, or a K_4

(2) Each block of H_3 has at most two cutpoints

(3) The cutpoints of the diamond blocks are vertices of degree 2 in the diamond.

(4) Each K_4 block has at most one cutpoint.

(5) Each cutpoint of H_3 belongs to at most two blocks of H_3 that are not pendant edges.

(6) No vertex of a diamond block of H_3 of degree 2 in H has a false twin in H.

All these conditions can be easily verified in linear time once the blocks and the cutpoints of H_3 are determined, which in its turn can be done in linear time by performing a depth-first search [112]. Finally, if all these conditions are met, H is a fat caterpillar containing no A and no net and a matching-transversal of H of minimum size can be determined in linear time as follows from the characterizations given in Lemmas 4.53 to 4.57.

Suppose now that we need to determine whether a given graph H is matching-perfect and assume, without loss of generality, that H has more than 6 vertices. We begin by deciding whether H is a fat caterpillar as in the preceding discussion. If H is found to be a fat caterpillar, we are done because we know that H is matching-perfect and stop. Therefore, assume without loss of generality that H is not a fat caterpillar. Then, H is matching-perfect if and only if H is matching-perfect and contains a cycle of length $3k$ for some $k \geqslant 2$. So, by Lemma 4.37, if H is matching-perfect, then H_3 is a chordless cycle of length $3k$ for some $k \geqslant 2$. Conversely, if H_3 is a chordless cycle of length $3k$ for some $k \geqslant 3$, clearly H is matching-perfect by Theorem 4.17. This shows that we can decide in linear time whether H is matching-perfect. Finally, if there is any edge $e = uv$ of H_3 that is not hub-covered in H, then $E_H(u) \cup E_H(v)$ is a matching-transversal of H of minimum size by Lemma 4.43; otherwise, if v is any minimum hub v of H, then $E_H(v)$ is a matching-transversal of H of minimum size by Lemma 4.42.

Theorem 4.58. *There is a simple linear-time algorithm that decides whether a given graph H is matching-perfect and, if affirmative, computes a matching-transversal of H of minimum size within the same time bound.*

In particular, if H is matching-perfect, we can also determine the common value of $\alpha_m(H)$ and $\tau_m(H)$ in linear time. We do not know if it is possible to also compute a matching-independent set of maximum size within the same time bound. Notice however that the only non-constructive argument used in the proofs of Subsection 4.2.3 is the existence of optimal edge-colorings for some Class 1 graphs containing no bipartite claw. This meas that, using an algorithm such the as the one given in [129] to produce the necessary edge-colorings, our proofs in Subsection 4.2.3 can actually be turned into a procedure to compute a matching-independent set of maximum size for any given matching-perfect graph.

Let G be graph on n vertices which is the complement of a line graph. We can compute a root graph H of $\overline{G}$ in $O(n^2)$ time by relying on [89, 107] and then decide whether G is clique-perfect by determining whether H is matching-perfect as above. Thus, we conclude the following.

Theorem 4.59. *There is an $O(n^2)$-time algorithm that given a graph G, which is the complement of a line graph, decides whether or not G is clique-perfect and, if affirmative, computes a minimum clique-transversal of G within the same time bound.*

Notice that the bottleneck of the algorithm is computing a root graph H of $\overline{G}$.

4.3 Clique-perfectness of gem-free circular-arc graphs

In [17], clique-perfect graphs were characterized within Helly circular-arc graphs (Theorem 4.5 on page 76). The problem of charactering which circular-arc graphs are clique-perfect is still open. In this section, we characterize clique-perfect graphs by minimal forbidden induced subgraphs within gem-free circular-arc graphs. In fact, we show that, within gem-free circular-arc graphs, being perfect, clique-perfect, coordinated, or hereditary K-perfect, are all equivalent.

Theorem 4.60. *Let G be a gem-free circular-arc graph. Then, the following statements are equivalent:*

(i) G *is clique-perfect.*

(ii) G *is coordinated.*

(iii) G *is hereditary* K*-perfect.*

(iv) G *is perfect.*

(v) G *has no odd holes.*

Proof. Along this proof, denote by $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, the families of minimally not clique-perfect, minimally not coordinated, and minimally not hereditary K-perfect graphs, respectively, and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. Clearly, odd holes are in $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$. If we prove that the odd holes are the only graphs in $\mathcal{C}$, then the equivalence among (i), (ii), (iii), and (v) follows. The equivalence between (iv) and (v) is an immediate consequence of the Strong Perfect Graph Theorem (Theorem 2.3).

Suppose, by the way of contradiction, that there exists a graph H in $\mathcal{C}$ that is not an odd hole. As $H \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, H is not balanced. Hence, by Theorem 3.54, H has an odd hole or contains an induced 3-pyramid. If H had an odd hole, then the minimality of the graphs in $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ would imply that H is an odd hole, contradicting the hypothesis. Therefore, H contains an induced 3-pyramid. Let $P \subseteq V(H)$ such that P induces a 3-pyramid in H and let $W \subseteq P$ such that W induces a C_4 in H.

We claim that $V(H) \setminus W$ is complete to W in H. Indeed, let $w_1w_2w_3w_4w_1$ be the hole induced by the vertices of W in H and let $P \setminus W = \{u_1, u_2\}$. Let v be an arbitrary vertex of $V(H) \setminus W$. If $v \in P \setminus W$, then v is complete to W by construction. So, without loss of generality, suppose that $v \in V(H) \setminus P$. Let $k = |N_H(v) \cap W|$. By Lemma 2.8 and symmetry, we can assume, without loss of generality, that $N_H(v) \cap W = \{u_i : 1 \leqslant i \leqslant k\}$. If $k = 0$ or $k = 1$, $\{u_1, w_2, u_2, w_4, v\}$ would induce $C_4 \cup K_1$ in H, which is not a circular-arc graph, a contradiction. If $k = 2$ or $k = 3$, $\{v, w_2, u_1, w_4, w_1\}$ would induce a gem in H, another contradiction. We conclude that $k = 4$, which proves that $V(H) \setminus P$ is complete to W in H, as claimed.

Since H is gem-free, $H - W$ is P_4-free. Since H is $K_{2,3}$-free, $H - W$ is $3K_1$-free. So, $\overline{H - W}$ is a P_4-free bipartite graph and, as we saw in the proof of Corollary 3.16, this means that each component of $\overline{H - W}$ is a complete bipartite graph. Since $\overline{H}[W] = \overline{C_4} = 2K_2$ and W is anticomplete to $V(H) \setminus W$ in $\overline{H}$, $\overline{H}$ is the disjoint union of at least three complete bipartite graphs.

We claim that $H \notin \mathcal{C}_1$. In fact, as H has disconnected complement, let H_1 and H_2 be two graphs having at least one vertex each such that $H = H_1 + H_2$. Then, as noted in [85, 87], $\alpha_c(H) = \min\{\alpha_c(H_1), \alpha_c(H_2)\}$ and $\tau_c(H) = \min\{\tau_c(H_1), \tau_c(H_2)\}$. So, if $H \in \mathcal{C}_1$, the minimality of H would ensure that $\alpha_c(H_i) = \tau_c(H_i)$ for each $i = 1, 2$ and the conclusion would be that $\alpha_c(H) = \tau_c(H)$, contradicting $H \in \mathcal{C}_1$. This proves the claim.

So, necessarily, $H \in \mathcal{C}_2 \cup \mathcal{C}_3$; i.e., H is minimally not coordinated or minimally not hereditary K-perfect. In particular, $\gamma_c(H) \neq \Delta_c(H)$ or $K(H)$ is imperfect and, in either case, H has no universal vertices; i.e., each component of $\overline{H}$ has at least two vertices. Let $\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_t$ be the components of $\overline{H}$ and, for each $i = 1, 2, \ldots, t$, let $\{A_i^1, A_i^2\}$ be the bipartition of the complete bipartite graph $\overline{H}_i$. Then, the cliques of H are of the form $A_1^{j_1} \cup A_2^{j_2} \cup \cdots \cup A_t^{j_t}$ where $j_1, \ldots, j_t \in \{1, 2\}$. Notice that $\gamma_c(H) = 2^{t-1}$ and $\Delta_c(H) = 2^{t-1}$ (indeed, each vertex of H belongs to 2^{t-1} cliques of H), which contradicts the fact that $H \in \mathcal{C}_2$, and that $K(H) = \overline{2^t K_2}$ which is a cograph and, in particular, perfect, which contradicts $H \in \mathcal{C}_3$, as desired. □

Chapter 5

Graphs having the Kőnig property and edge-perfect graphs

This chapter is organized as follows.

- In Section 5.1, we give some background about graphs having the Kőnig property and about edge-perfect graphs.
- In Section 5.2, we prove a characterization of graphs having the Kőnig property in terms of forbidden strongly splitting subgraphs, which is a strengthened version of a characterization due to Korach, Nguyen, and Peis [82] by forbidden configurations: (1) First, we show that one of their forbidden configurations is redundant and can be omitted; (2) then, we reformulate the resulting characterization in terms of forbidden subgraphs; (3) finally, we strengthen the formulation by restricting the way in which the forbidden subgraphs may occur.
- In Section 5.3, we use our characterization of graphs having the Kőnig property in order to prove a characterization of edge-perfect graphs by forbidden edge-subgraphs.

The results of this chapter appeared in [49].

5.1 Background

5.1.1 Graphs having the Kőnig property

Recall from the Introduction that a graph G has the *Kőnig property* is its matching number $\nu(G)$ equals its transversal number $\tau(G)$. This means that Kőnig's matching theorem [77] can be regarded as asserting that bipartite graphs have the Kőnig

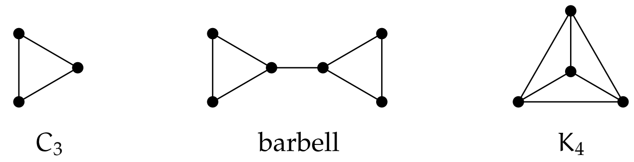


Figure 5.1: C_3, *barbell, and* K_4

property. Graphs having the Kőnig property have received considerable attention [28, 44, 81, 82, 86, 90, 93, 94, 99, 103, 111]. The study of graphs having the Kőnig property from a structural point of view has its origins in the works of Sterboul [111] and Deming [44] who, independently, gave the first structural characterization for these graphs. In [51], Edmonds devised the first polynomial-time algorithm for maximum matching in general graphs, for which he introduced the notions of blossoms, stems, and flowers. Let G be a graph and let M be a matching of G. An M-*blossom* is an odd cycle of length $2k+1$ for some $k \geqslant 1$ such that k of its edges are edges of M. An M-*stem* is either an exposed vertex or an even M-alternating path having an M-unsaturated vertex in one end and an edge of M in the other; the M-unsaturated vertex and the vertex at the other end are called, respectively, the *root* and the *tip* of the stem. An M-*flower* consists of a blossom and a stem whose only common vertex are the base of the blossom and the tip of the stem. In [111], Sterboul defined an M-*posy* to consist of two (not necessarily disjoint) blossoms joined by an odd M-alternating path that starts and ends in edges of M and whose endpoints are the bases of the two blossoms. He observes that if an M-posy exists, one M-posy can be found whose only vertex of each blossom belonging to the path is its base. The characterization is as follows.

Theorem 5.1 ([44, 111])**.** *Let G be a graph. The following assertions are equivalent:*

(i) G *has the Kőnig property (i.e.,* $\tau(G) = \nu(G)$*).*

(ii) *For every maximum matching* M*, there exists an* M*-flower or an* M*-posy.*

(iii) *For some maximum matching* M*, there is an* M*-flower or an* M*-posy.*

Deming [44] continues the analysis and also devises a polynomial-time algorithm for recognizing graphs having the Kőnig property and, if affirmative, computing a maximum independent set. Nevertheless, the fact that the two blossoms that define an M-posy may intersect does not give a simple forbidden subgraph characterization of graph having the Kőnig property.

In [93], Lovász proved a characterization of graphs having the Kőnig property, restricted to graphs having a perfect matching by means of what he called *nice subgraphs*. An *even subdivision* of an edge uv consists in replacing the edge uv by two new vertices w_1 and w_2 together with three edges uw_1, w_1w_2, and w_2v. An *even subdivision* of

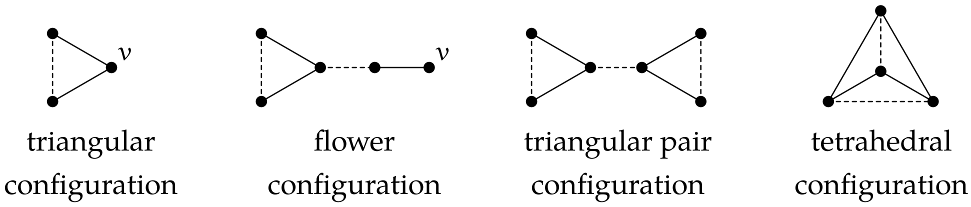


Figure 5.2: *Forbidden configurations for graphs having the Kőnig property*

a graph G is either the graph G itself or any of the graphs that arise from G by successive application of even subdivisions. A subgraph H of a graph G is *nice* if $G - V(H)$ has a perfect matching. The aforementioned characterization is stated below. For the *barbell* graph, see Figure 5.1.

Theorem 5.2 ([93]). *A graph with a perfect matching has the Kőnig property if and only if it has no even subdivision of barbell or* K_4 *as a nice subgraph.*

In [82], Korach, Nguyen, and Peis extended Lovász's result to a characterization of all graphs having the Kőnig property by, what we call, *forbidden configurations*. A *configuration* of a graph G is an ordered pair $\xi = (S, M)$ where S is a subgraph of G, M is a maximum matching of G, and S belongs to one of the four families of graphs represented in Figure 5.2, where dashed edges stand for M-alternating paths starting and ending in edges of M, solid edges stand for M-alternating paths starting and ending in edges not belonging to M, and the vertex v is M-unsaturated. The graph S is said the *underlying graph* of ξ. The characterization by Korach et al. by forbidden configurations is the following.

Theorem 5.3 ([82]). *A graph has the Kőnig property if and only if it has none of the configurations in Figure 5.2.*

Notice that if we require that each induced subgraph of a graph G have the Kőnig property, then G should be bipartite because the chordless odd cycles do not have the Kőnig property. Recall from the Introduction that, instead, edge-perfect graphs are those graphs such that the Kőnig property holds for each of their 'edge subgraphs'. If F is any set of edges, we will denote by $V(F)$ the set of endpoints of the edges belonging to F; i.e., $V(F) = \bigcup_{e \in F} e$ by regarding each edge e as the set of its endpoints. With this notation, the *edge-subgraphs* of a graph G are the induced subgraphs $G - V(F)$ for some $F \subseteq E(G)$. Clearly, edge-perfect graphs form a superclass of the class of bipartite graphs and a subclass of the class of graphs having the Kőnig property. Moreover, both inclusions are proper, as shown by the paw (which is edge-perfect but not bipartite) and the graph that arises from C_6 by adding a short chord (which has the Kőnig property but is not edge-perfect).

If C is a chordless odd cycle of a graph G, let a *savior* of C be a vertex v of $V(G) \setminus V(C)$ such that $N_G(v) \subseteq V(C)$. Let a *two-twin pair* be a pair of false twins of degree 2 and let $\mathcal{N}(G)$ be the family of the neighborhoods of the vertices in each two-twin pair; i.e., $\mathcal{N}(G) = \{N_G(v) : v \text{ has degree 2 and has a false twin in } G\}$. Finally, let G_P be the edge-subgraph of G that arises by removing the endpoints of all the pendant edges of G. In [47] and [48], edge-perfect graphs were characterized by the presence of saviors and by the absence in G_P of chordless odd cycles with forbidden pairs.

Theorem 5.4 ([47, 48])**.** *A graph* G *is edge-perfect if and only if each chordless odd cycle of* G *has a savior that is either a pendant vertex or belongs to some two-twin pair or, equivalently, if and only if* G_P *has no chordless odd cycle containing at most one vertex from each pair in* $\mathcal{N}(G)$.

These characterizations were used to identify some graph classes within which there are polynomial-time recognition algorithms for edge-perfect graphs [47] and to prove that the problem of recognizing edge-perfect graphs is NP-hard in general [48]. Originally, edge-perfect graphs were defined in [53] in connection with packing and covering games introduced in [45]. In fact, in [48], based on the NP-hardness of the recognition of edge-perfect graphs, it is deduced that the recognition of matrices defining totally balanced packing games is NP-hard, answering a question raised in [45]. This is in contrast with the case of matrices defining totally balanced covering games, which can be recognized in polynomial time [121].

5.2 The Kőnig property in terms of forbidden subgraphs

We will first show that it is not possible to extend Theorem 5.2 to a characterization of all graph having the Kőnig property by forbidden nice subgraphs. That is, we cannot drop the hypothesis that G has a perfect matching by adding some extra forbidden nice subgraphs. It is not possible to do so because, while the relation "is a nice subgraph of" is clearly transitive, the Kőnig property is not always inherited by the nice subgraphs (as the example given in Figure 5.3 shows). Suppose, by the way of contradiction, that it were possible to characterize the whole class of graphs having the Kőnig property by forbidden nice subgraphs. Consider Figure 5.3, where a graph is displayed on the left and a nice subgraph of it on the right. Since the graph on the right does not have the Kőnig property, it should have some nice subgraph Φ which is forbidden in the characterization whose existence we are assuming. Then, by transitivity, the forbidden nice subgraph Φ would also be a nice subgraph of the graph on the left, which would contradict the fact that the graph on the left does have the Kőnig property. This contradiction proves that Theorem 5.2 cannot be extended to a

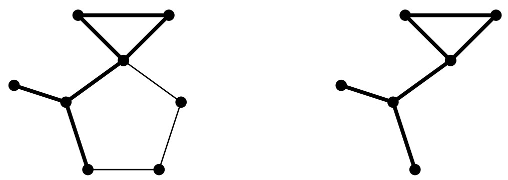


Figure 5.3: *The Kőnig property is not always inherited by the nice subgraphs. The graph on the left has the Kőnig property while its bold edges correspond to a nice subgraph of it (depicted also on the right) that does not have the Kőnig property.*

characterization by forbidden nice subgraphs of all graphs having the Kőnig property. Instead, our approach towards obtaining a similar result holding for all graphs will be to replace nice subgraphs by *splitting subgraphs* (to be defined after Lemma 5.5) and later by *strongly splitting subgraphs* (to be defined on page 133).

Let G be a graph and let X be a subset of $V(G)$. We say that X is a *splitting set* of G if and only if there is some maximum matching M of G such that no edge of M joins a vertex of X to a vertex of $G - X$. If so, we say that M *is split by* X. The next lemma gives a sufficient condition for a subgraph of a graph having the Kőnig property to also have the Kőnig property.

Lemma 5.5. *Let G be a graph having the Kőnig property and let H be a subgraph of G. If $V(H)$ is a splitting set of G and $\nu(H) = \nu(G[V(H)])$, then H also has the Kőnig property.*

Proof. Suppose that $V(H)$ is a splitting set of G and $\nu(H) = \nu(G[V(H)])$. Let M be a maximum matching of G split by $V(H)$; i.e., there is no edge of M joining a vertex of H to a vertex of $G - V(H)$. Let M_H be the set of edges of M joining two vertices of $V(H)$ and let $M_{G-V(H)}$ be the set of edges of M joining two vertices of $G - V(H)$. Since M is a maximum matching of G and M is split by $V(H)$, M_H is a maximum matching of $G[V(H)]$. Since $\nu(H) = \nu(G[V(H)])$, there is maximum matching M'_H of H such that $|M'_H| = |M_H|$. Therefore, $M' = M'_H \cup M_{G-V(H)}$ is a maximum matching of G. Then,

$$\nu(G) = \nu(H) + \nu(G - V(H)) \leqslant \tau(H) + \tau(G - V(H)) \leqslant \tau(G). \tag{5.1}$$

Since G has the Kőnig property, both inequalities in (5.1) hold with equality and, necessarily, $\nu(H) = \tau(H)$ and $\nu(G - V(H)) = \tau(G - V(H))$. This proves that H has the Kőnig property. □

The above lemma leads us to introduce the notion of *splitting subgraphs* as follows. Let G be a graph and let H be a subgraph of G. We will say that H is a *splitting subgraph* of G if and only if $V(H)$ is a splitting set of G and H has a perfect or near-perfect matching. Notice that if H has a perfect or near-perfect matching, $\nu(H) = \nu(G[V(H)])$ holds trivially. Therefore, we have the following corollary of Lemma 5.5 showing that, contrary to the case of nice subgraphs, the Kőnig property is always inherited by the splitting subgraphs.

Corollary 5.6. *If a graph has the Kőnig property, then each of its splitting subgraphs has the Kőnig property.*

Notice that if G has a perfect matching, then H is a splitting subgraph of G if and only if H has a perfect matching and H is a nice subgraph of G. Since all the graphs involved in Theorem 5.2 have perfect matchings, the result still holds if we replace 'nice subgraphs' by 'splitting subgraphs':

Theorem 5.2 in terms of splitting subgraphs ([93]). A graph with a perfect matching has the Kőnig property if and only if it has no even subdivision of barbell or K_4 as a splitting subgraph.

We will show that, contrary to the case of nice subgraphs, the whole class of graph having the Kőnig property can be characterized by means of splitting subgraphs. That is, when Theorem 5.2 is reformulated in terms of forbidden splitting subgraphs as above, the hypothesis that G has a perfect matching can be dropped by simply adding some extra forbidden splitting subgraphs. The characterization of the graphs having the Kőnig property by forbidden splitting subgraphs will follow from the characterization by Korach et al. (Theorem 5.3). To begin with, the lemma below shows that it is not essential to forbid the flower configurations in Theorem 5.3 because forbidding triangular configurations prevents both triangular and flower configurations from occurring.

Lemma 5.7. *If a graph has a flower configuration, then it also has a triangular configuration.*

Proof. Assume that a graph G has some flower configuration $\xi = (S, M)$. Let v be the M-unsaturated vertex of S and let w be the vertex of S of degree 3 in S. Let P be the path of S joining v to w and let C be the only cycle of S. Notice that $M' = M \triangle E(P)$ is also a maximum matching of G because P is an M-alternating even path of G and v is M-unsaturated. Therefore, (C, M') is a triangular configuration of G, which completes the proof. □

Next we observe that the occurrence of the three remaining configurations coincides with the occurrence of their underlying graphs as splitting subgraphs.

Lemma 5.8. *Let G be a graph and let S be a subgraph of G. Then, S is the underlying graph of a triangular, triangular pair, or tetrahedral configuration of G if and only if S is a splitting subgraph of G which is an even subdivision of C_3, barbell, or K_4, respectively.*

Proof. Assume that there is some splitting subgraph S of G which is an even subdivision of C_3, barbell, or K_4. By definition, $V(S)$ is a splitting set of G; i.e., there is a maximum matching M of G such that no edge of M joins a vertex of S with a vertex of $G - V(S)$. Let M_S be the set of edges of M that join two vertices of S and let $M_{G-V(S)}$ be the set of edges of M that join two vertices of $G - V(S)$. By construction, M_S is a maximum matching of $G[V(S)]$. Since S is an even subdivision of C_3, barbell, or K_4, there is a perfect or near-perfect matching R_S of S. Notice that R_S is unique up to isomorphisms of S. As S is a spanning subgraph of $G[V(S)]$, $|R_S| = |M_S|$. Then, $M' = R_S \cup M_{G-V(S)}$ is a maximum matching of G. By construction, (S, M') is a triangular, triangular pair, or tetrahedral configuration of G depending on whether S is an even subdivision of C_3, barbell, or K_4, respectively.

Conversely, assume that S is the underlying graph of a triangular, triangular pair, or tetrahedral configuration $\xi = (S, M)$ of G. By definition, $V(S)$ is a splitting set of G and S has a perfect or near-perfect matching. Thus, S is a splitting subgraph of G. We conclude that S is a splitting subgraph of G which is an even subdivision of C_3, barbell, or K_4 depending on whether ξ is a triangular, triangular pair, or tetrahedral configuration, respectively. □

Therefore, the characterization by Korach et al. can be reformulated in terms of splitting subgraphs:

Theorem 5.3 in terms of splitting subgraphs. A graph has the Kőnig property if and only if it has no even subdivision of any of the graphs in Figure 5.1 as a splitting subgraph.

Notice that the above statement is precisely a characterization of the whole class of graphs having the Kőnig property in terms of splitting subgraphs of the kind that we were looking for. Indeed, it arises from the reformulation of Theorem 5.2 in terms of forbidden splitting subgraphs by dropping the hypothesis that G has a perfect matching and adding the even subdivisions of C_3 as the extra forbidden splitting subgraphs.

Finally, we will prove Theorem 5.9, which is a strengthened characterization of graphs with the Kőnig property obtained by restricting the way in which the forbidden subgraphs may occur. For the purpose of formulating our characterization, we introduce the notion of *strongly splitting subgraphs* as follows. Let G be a graph. A subset X of $V(G)$ is a *strongly splitting set* if there is a maximum matching M of G such that no edge of M joins a vertex of X to a vertex of $G - X$ and *no vertex of* X *is adjacent to any* M*-unsaturated vertex of* $G - X$. A subgraph H of G is a *strongly splitting subgraph* if $V(H)$ is a strongly splitting set of G and H has a perfect or near-perfect matching.

Clearly, strongly splitting sets are splitting sets, and strongly splitting subgraphs are splitting subgraphs. Moreover, the notion of strongly splitting subgraph is indeed more restrictive than that of splitting subgraph. For instance, K_5 has K_4 as splitting subgraph but not as strongly splitting subgraph. More generally, if H has a perfect matching, then $H + K_1$ has H as splitting subgraph but not as strongly splitting subgraph.

The theorem below is the main result of this section and shows that the forbidden splitting subgraphs for the class of graphs having the Kőnig property can be forced to occur as strongly splitting subgraphs.

Theorem 5.9. *A graph has the Kőnig property if and only if it has no even subdivision of any of the graphs in Figure 5.1 as a strongly splitting subgraph.*

Proof. Since strongly splitting subgraphs are splitting subgraphs, Corollary 5.6 implies that if G has the Kőnig property then no strongly splitting subgraph of G is an even subdivision of any of the graphs in Figure 5.1. Therefore, it suffices to prove that if G does not have the Kőnig property then G has a strongly splitting subgraph which is an even subdivision of one of the graphs in Figure 5.1.

Suppose that G does not have the Kőnig property. By Theorem 5.3 and Lemma 5.7, G has a triangular, triangular pair, or tetrahedral configuration $\xi = (S, M)$. Denote by U the set of M-unsaturated vertices of $G - V(S)$.

Case 1. $\xi = (S, M)$ *is a triangular configuration.*

Let v be the M-unsaturated vertex of S and suppose, by the way of contradiction, that there is a vertex $s \in V(S)$ adjacent to some vertex $u \in U$. Since M is maximum and u is M-unsaturated, s is M-saturated. In particular, $s \neq v$. Since S is a chordless odd cycle, there is exactly one even path P in S joining s to v. By construction, uP is an M-alternating path joining the M-unsaturated vertices u and v; i.e., uP is an M-augmenting path, a contradiction with the fact that M is maximum. This contradiction proves that there is no edge joining a vertex of S and a vertex of U. We conclude that if G has a triangular configuration $\xi = (S, M)$ then S is a strongly splitting subgraph of G which is an even subdivision of C_3. From now on, we assume, without loss of generality, that G has no triangular configuration.

Case 2. $\xi = (S, M)$ *is a triangular pair configuration.*

Suppose, by the way of contradiction, that there is a vertex $s \in V(S)$ adjacent to some vertex $u \in U$. Let w_1 and w_2 be the two vertices of S of degree 3 in S. Let P be the path in S joining w_1 to w_2 and let C^i be the cycle of S through w_i for $i = 1, 2$. If $s \in V(P)$, let Q be the subpath of P that joins s to w_1 and, by symmetry, we can assume that Q is odd. If, on the contrary, $s \in V(S) \setminus V(P)$, we can assume without loss of generality that $s \in V(C^2) \setminus \{w_2\}$ and let Q be the odd path in S joining s to w_1 (which exists because C^2 is odd). In both cases, uQ is an M-alternating even path of G where

u is not saturated by M. Therefore, $M' = M \triangle E(uQ)$ is a maximum matching of G and (C^1, M') is a triangular configuration of G, a contradiction. This contradiction proves that there is no edge joining a vertex of S and a vertex of U. We conclude that if G has a triangular pair configuration $\xi = (S, M)$ and G has no triangular configuration, then S is a strongly splitting subgraph of G which is an even subdivision of barbell.

Case 3. $\xi = (S, M)$ *is a tetrahedral configuration.*

Let w_1, w_2, w_3, w_4 be the set of vertices of S of degree 3 in S. For each $i, j \in \{1, 2, 3, 4\}$ such that $i \neq j$, let $P^{i,j}$ be the path of S joining w_i to w_j but not passing through w_k for any $k \neq i, j$. Without loss of generality, we assume that the vertices w_1, w_2, w_3, w_4 are labeled in such a way that the path $P^{i,i+1}$ starts and ends in edges not belonging to M for each $i = 1, 2, 3, 4$ (superindices should be understood modulo 4). For each pairwise different $i, j, k \in \{1, 2, 3, 4\}$, let $C^{i,j,k}$ be the cycle of S passing through w_i, w_j, and w_k but not passing through w_ℓ where $\ell \neq i, j, k$. Suppose, by the way of contradiction, that there is a vertex $s \in V(S)$ that is adjacent to some vertex $u \in U$. By symmetry, we can assume that $s \in V(P^{1,2})$ or $s \in V(P^{1,3})$. Suppose first that $s \in V(P^{1,2})$. Since $P^{1,2}$ is odd, there is an even subpath Q of $P^{1,2}$ joining s to w_j for $j = 1$ or $j = 2$. (Eventually P is the empty path starting and ending in w_j.) Without loss of generality, assume that Q joins s to w_1. Since $uQP^{1,3}$ is an M-alternating even path and u is M-unsaturated, $M' = M \triangle E(uQP^{1,3})$ is a maximum matching of G and $(C^{2,3,4}, M')$ is a triangular configuration of G, a contradiction. Necessarily, $s \in V(P^{1,3})$. Since $P^{1,3}$ is odd, there is an odd subpath Q of P joining s to w_1 or w_3. Without loss of generality assume that Q joins s to w_1. Since uQ is an M-alternating even path and u is M-unsaturated, $M' = M \triangle E(uQ)$ is a maximum matching of G and $(C^{1,2,4}, M')$ is a triangular configuration of G, a contradiction. This contradiction proves that there is no edge between $V(S)$ and U. We conclude that if G has a tetrahedral configuration (S, M) and G has no triangular configuration, then S is a strongly splitting subgraph of G which is an even subdivision of K_4.

We proved that if G does not have the Kőnig property then G has a strongly splitting subgraph which is an even subdivision of C_3, barbell, or K_4, which concludes the proof. □

Notice that if G is a graph having a perfect matching and H is a strongly splitting subgraph of G, then H is a nice subgraph of G and H has a perfect matching. In addition, the even subdivisions of C_3 clearly do not have perfect matchings (because they have an odd number of vertices). Therefore, for graphs with a perfect matching, Theorem 5.9 reduces precisely to Lovász's characterization (Theorem 5.2).

The aim of our characterization is not to address the recognition problem, which was already addressed in [44]. Instead, the usefulness of our characterization is on the structural side: given that a graph does not have the Kőnig property, our result ensures that an even subdivision of C_3, barbell, or K_4 occurs as a *strongly splitting subgraph*. As an example of this, in the next section, we use Theorem 5.9 to derive a characterization of edge-perfect graphs by forbidden edge-subgraphs.

5.3 Edge-perfectness and forbidden edge-subgraphs

Notice that the class of edge-perfect graphs is not closed by taking induced subgraphs. Indeed, the paw is edge-perfect but contains an induced C_3 which is not edge-perfect. This simple example shows that the class of edge-perfect graphs cannot be characterized by forbidden induced subgraphs. Instead, we will characterize edge-perfect graphs by forbidden edge-subgraphs. Before turning into the proof of the characterization, we observe the following two facts.

Lemma 5.10. *If* F *is an edge-subgraph of* H *and* H *is an edge-subgraph of* G*, then* F *is an edge-subgraph of* G*.*

Proof. Let E_1 be a set of edges of H such that $H - V(E_1) = F$ and let E_2 be a set of edges of G such that $G - V(E_2) = H$. Then, $G - V(E_1 \cup E_2) = F$ where $E_1 \cup E_2$ is a set of edges of G because H is a subgraph of G. □

Lemma 5.11. *Let* G *be a graph. If* G *has an odd cycle whose vertex set induces an edge-subgraph of* G*, then* G *has an edge-subgraph which is a chordless odd cycle.*

Proof. Suppose that G has an odd cycle whose vertex set induces an edge-subgraph of G and let C be the shortest such odd cycle. It suffices to prove that C is chordless. Suppose, by the way of contradiction, that C has some chord $e = xy$. Since C is odd, its vertices can be labeled in such a way that $C = v_1 v_2 \ldots v_{2k+1} v_1$, where $v_1 = x$ and $v_{2p+1} = y$ for some $p \in \{1, 2, 3, \ldots, k-1\}$. Now $C' = v_1 v_2 v_3 \ldots v_{2p+1} v_1$ is an odd cycle of G and $G[V(C')]$ is an edge-subgraph of $G[V(C)]$ because $G[V(C')] = G[V(C)] - V(\{x_j x_{j+1} \mid 2p+2 \leqslant j \leqslant 2k\})$. Since $G[V(C')]$ is an edge-subgraph of $G[V(C)]$ and $G[V(C)]$ is an edge-subgraph of G, by Lemma 5.10, $G[V(C')]$ is an edge-subgraph of G. Therefore, C' is an odd cycle of G that induces an edge-subgraph of G and C' is shorter than C, a contradiction with the choice of C. This contradiction arose by assuming that C had some chord. So, $G[V(C)]$ is an edge-subgraph of G which is a chordless odd cycle, which completes the proof. □

The chordless odd cycles and K_4 are not edge-perfect because they do not even have the Kőnig property. Therefore, these graphs cannot be edge-subgraphs of any edge-perfect graph. The following result shows that, conversely, if a graph without isolated vertices is not edge-perfect, it is because it contains a chordless odd cycle or K_4 as an edge-subgraph.

Theorem 5.12. *A graph with no isolated vertices is edge-perfect if and only if it has neither a chordless odd cycle nor* K_4 *as an edge-subgraph.*

Proof. As we have just discussed, if a graph is edge-perfect then no edge-subgraph of it can be a chordless odd odd cycle or K_4. Conversely, let G be a graph with no isolated vertices that is not edge-perfect. Then, G has at least one edge-subgraph that does not have the Kőnig property. Let H be an edge-subgraph of G with minimum number of vertices that does not have the Kőnig property. As H does not have the Kőnig property, there is some component H' of H that does not have the Kőnig property. In particular, H' consists of at least two vertices.

We claim that H' is the only component of H having at least two vertices. Suppose, by the way of contradiction, that H has some other component H'' having at least two vertices. If $E_{H''}$ is the set of edges of H joining vertices of H'', then $H - V(E_{H''}) = H - V(H'')$ is an edge-subgraph of H that does not have the Kőnig property because one of its components is still H'. By Lemma 5.10, $H - V(H'')$ is also an edge-subgraph of G. Since $H - V(H'')$ does not have the Kőnig property and has less vertices than H, this contradicts the minimality of H. This contradiction shows that H' is the only component of H having at least two vertices.

We now show that the fact that G has no isolated vertices implies that H is connected; i.e., $H = H'$. Indeed, since G has no isolated vertices, for each isolated vertex v of H (i.e., $v \in V(H) \backslash V(H')$) there is some edge $e_v \in E(G)$ that is incident to v. If e_v were incident to some vertex of H' then e_v would be an edge of H, which would contradict the fact that v does not belong to the component H' of H. Therefore, e_v is not incident to any vertex of H' for any $v \in V(H) \backslash V(H')$. Since H is an edge-subgraph of G, there is some $E_H \subseteq E(G)$ such that $G - V(E_H) = H$. So, if $E_I = \{e_v : v \in V(H) \backslash V(H')\}$ then $G - V(E_H \cup E_I) = H'$, which proves that H' is an edge-subgraph of G. Since H' does not have the Kőnig property, the minimality of H implies that $H = H'$, as claimed.

Since H does not have the Kőnig property, Theorem 5.9 ensures that there is a strongly splitting subgraph S of H which is an even subdivision of C_3, barbell, or K_4. We claim that $H[V(S)]$ is an edge-subgraph of H. Indeed, since S is a strongly splitting subgraph of H, there is a maximum matching M of H such that no edge of M joins a vertex of S with a vertex of $H - V(S)$ and such that no vertex of S is adjacent to an M-unsaturated vertex of $H - V(S)$. Let E_1 be the set of edges of M joining two vertices of $H - V(S)$, and let E_2 be the set of edges of H incident to some M-unsaturated vertex of $H - V(S)$. Since H is connected, for each M-unsaturated vertex of $H - V(S)$ there is at least one edge incident to it in E_2. Also notice that since S is strongly splitting subgraph, no edge of E_2 is incident to a vertex of S. We conclude that $H[V(S)] = H - V(E_1 \cup E_2)$, which shows that $H[V(S)]$ is an edge-subgraph of H, as claimed.

Finally, we claim that H has a chordless odd cycle or K_4 as an edge-subgraph.

Case 1. *S is an even subdivision of C_3.*

Then, S is an odd cycle of H whose vertex set induces an edge-subgraph of H. By Lemma 5.11, H has an edge-subgraph which is a chordless odd cycle, as claimed.

Case 2. *S is an even subdivision of barbell.*

Let w_1 and w_2 be the vertices of S of degree 3 in S, let C^i be the cycle of S through w_i for $i = 1, 2$ and let P be the path of S joining w_1 to w_2. Let $P = x_1x_2x_3 \dots x_{2k+1}$ where $x_1 = w_1$ and $x_{2k+1} = w_2$. Let $E_3 = E(C^2)$ and let $E_4 = \{x_jx_{j+1} \mid 2 \leqslant j \leqslant 2k\}$. Then, $H[V(C^1)]$ is an edge-subgraph of $H[V(S)]$ because $H[V(C^1)] = H[V(S)] - V(E_3 \cup E_4)$. Since $H[V(S)]$ is an edge-subgraph of H, by Lemma 5.10, $H[V(C^1)]$ is an edge-subgraph of H. Thus, C^1 is an odd cycle of H whose vertex set induces an edge-subgraph of H and, by Lemma 5.11, H has an edge-subgraph which is a chordless odd cycle, as claimed.

Case 3. *S is an even subdivision of K_4.*

Let W be the set of vertices of S of degree 3 in S. For each $w, w' \in W$, let $P^{w,w'}$ be the path in S joining w to w' and not passing through any vertex of $W \backslash \{w, w'\}$. If $P^{w,w'}$ has length 1 for each $w, w' \in W$, then $S = H[V(S)]$ is an edge-subgraph of H which is a K_4, and the claim holds. Therefore, we assume without loss of generality that there are two vertices $w_1, w_2 \in W$ such that P^{w_1,w_2} has length greater than 1. Let w_3 and w_4 be the remaining two vertices of W. Let C be the cycle of S through w_2, w_3, and w_4, but not through w_1. For each $i = 2, 3, 4$, let $P^{w_1w_i} = y_1^iy_2^iy_3^i \dots y_{2k_i+1}^i$ where $y_1^i = w_1$ and $y_{2k_i+1}^i = w_i$ and let $F_i = \{y_j^iy_{j+1}^i \mid 1 \leqslant j \leqslant 2k_i - 1\}$. Notice that $H[V(C)]$ is an edge-subgraph of $H[V(S)]$ because $H[V(C)] = H[V(S)] - V(F_2 \cup F_3 \cup F_4)$. Since $H[V(S)]$ is an edge-subgraph of H, by Lemma 5.10, $H[V(C)]$ is an edge-subgraph of H and, by Lemma 5.11, H has an edge-subgraph which is a chordless odd cycle, as claimed.

Thus, we proved that H has a chordless odd cycle or K_4 as an edge-subgraph. Since H is an edge-subgraph of G, Lemma 5.10 implies that G has a chordless odd cycle or K_4 as edge-subgraphs, which completes the proof. □

We would like to draw attention to the role played by our characterization of graphs having the Kőnig property (Theorem 5.9) in the above proof. Indeed, the fact that S is a strongly splitting subgraph of H was key in the proof of the claim that $H[V(S)]$ is an edge-subgraph of H, because it guarantees that there is no M-unsaturated vertex of S such that each edge incident to it were also incident to some vertex of S and, in particular, no edge of E_2 is incident to a vertex of S.

Finally, we present the characterization of edge-perfectness by forbidden edge-subgraphs also for graphs that may have isolated vertices. Notice that when taking an edge-subgraph H of a graph G, the isolated vertices of G are never removed. Therefore, H has at least as many isolated vertices as G. That explains why in the theorem below we must forbid edge-subgraphs with an arbitrary number of isolated vertices.

Theorem 5.13. *A graph is edge-perfect if and only it has neither $K_4 \cup tK_1$ nor $C_{2k+1} \cup tK_1$ as an edge-subgraph for any $k \geqslant 1$ and any $t \geqslant 0$.*

Proof. If G is edge-perfect then all its edge-subgraphs have the Kőnig property and, in particular, G has neither $K_4 \cup tK_1$ nor $C_{2k+1} \cup tK_1$ as an edge-subgraph for any $k \geqslant 1$ and any $t \geqslant 0$.

Conversely, assume that G is not edge-perfect. Let t be the number of isolated vertices of G. Then, the graph G' that arises from G by removing its t isolated vertices is also not edge-perfect. By Theorem 5.13, G' has K_4 or C_{2k+1} for some $k \geqslant 1$ as an edge-subgraph. So, G has $K_4 \cup tK_1$ or $C_{2k+1} \cup tK_1$ for some $k \geqslant 1$ as an edge-subgraph. □

Chapter 6

Final remarks

In Chapter 3, we studied the problem of characterizing balanced graphs by minimal forbidden induced subgraphs within different graph classes. The main results of the chapter are summarized in Table 6.1.

Sections 3.4 to 3.6 were devoted to address the problem when restricted to the classes of complements of bipartite graphs, line graphs of multigraphs, and complements of line graphs of multigraphs, and to show that balanced graphs are recognizable in linear time within each of these graph classes. We observed that the characterization of balanced graphs within line graphs leads naturally to the characterization within line graphs of multigraphs because adding true twins preserves balancedness. Nevertheless, the same does not hold for complements of line graphs of multigraphs because adding false twins does not always preserve balancedness. Indeed, for each multigraph H in Figure 3.2, light lines are those that correspond to vertices in $\overline{L(H)}$ for which adding a false twin may destroy balancedness. It would be interesting to have a general criterion to decide when a false twin of a vertex can be added to a graph while preserving balancedness. Such a result would have somewhat simplified our proof of the characterization of balancedness within complements of line graphs of multigraphs. In order to be able to characterize balanced graphs within larger graph classes, we may need to develop more convenient tools to prove balancedness of graphs. For instance, although the proofs in Subsection 3.6.4 are very similar among themselves, each of them had to be addressed separately. A natural step towards generalizing our results within line graphs of multigraphs and their complements would be to attempt to characterize balanced graphs within claw-free graphs and their complements. The decomposition of claw-free graphs proposed in [35] could prove useful in such an attempt.

In Sections 3.7 to 3.9, we considered the problem of characterizing balanced graphs by minimal forbidden subgraphs within three subclasses of the class of circular-arc

Graph class	Minimal forbidden induced subgraphs for balancedness	Reference
complements of bipartite graphs	1-pyramid, 2-pyramid, and 3-pyramid	Theorem 3.15
line graphs of multigraphs	odd holes, 3-sun, 1-pyramid, and 3-pyramid	Theorem 3.23
complement of line graphs of multigraphs	3-sun, 2-pyramid, 3-pyramid, C_5, $\overline{C_7}$, U_7, and V_7.	Theorem 3.29
{net,U_4,S_4}-free circular-arc graphs (contains all Helly circular-arc graphs)	odd holes, pyramids, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, and X_p^{2t+1}	Corollary 3.49
claw-free circular-arc graphs (contains all proper circular-arc graphs)	odd holes, pyramids, and $\overline{C_7}$	Theorem 3.52
gem-free circular-arc graphs	odd holes and 3-pyramid	Theorem 3.54

Table 6.1: *Minimal forbidden induced subgraphs for balancedness within the graph classes studied in Chapter 3*

graphs, including a superclass of each of two of the most studied subclasses of circular-arc graphs: the class of Helly circular-arc graphs and the class of proper circular-arc graphs. Interestingly, a careful reading of the proof of Theorem 3.44 reveals that the hypothesis that the graph is a Helly circular-arc graph (and not merely a circular-arc graph) is only used in the proofs of Claim 1 and Claim 2, and in the latter case only for $t = 2$. So, along the proof we indeed identified all circular-arc graphs that are minimally not balanced and whose unbalanced cycles have length at least 7 and have only short chords. Therefore, a possible road towards extending the proof of Theorem 3.44 to the entire class of circular-arc graphs could be that of finding some property of the chords of the unbalanced cycles within circular-arc graphs in general that could serve as a substitute for Claim 1. A different approach would be to take Theorem 3.47 as a starting point and study the balancedness of circular-arc graphs containing net, U_4, or S_4 as induced subgraph. We managed to do so when restricting ourselves to claw-free and gem-free graphs. A better understanding of the structure of circular-arc graphs would be of help to overcome these restrictions. The complete characterization of balanced graphs by minimal forbidden induced subgraphs within circular-arc graphs in general, remains unknown. The sun S_5 is an example of circular-arc graph that is minimally not balanced but that does not belong to any of the classes of circular-arc graphs discussed in Chapter 3. We do not know if there are further

Graph class	Minimal forbidden induced subgraphs for clique-perfectness	Reference
complements of line graphs	3-sun and $\overline{C_k}$ for each $k \geqslant 5$ that is not a multiple of 3	Theorem 4.16
gem-free circular-arc graphs	odd holes	Theorem 4.60

Table 6.2: *Minimal forbidden induced subgraphs for clique-perfectness within the graph classes studied in Chapter 4*

examples of such graphs. Notice that the complete suns S_t with t odd and $t \geqslant 7$ are not circular-arc graphs. We remark that the problem of characterizing balanced graphs by minimal forbidden induced subgraphs remains unsolved even within chordal graphs.

In Chapter 4, we considered the problem of characterizing clique-perfect by minimal forbidden induced subgraphs. The main results are summarized in Table 6.2.

We devoted Section 4.2 to characterizing clique-perfect graphs by minimal forbidden induced subgraphs within complements of line graphs and showed that clique-perfect graphs can be recognized in $O(n^2)$ time within the same class, where n is the number of vertices of the input graph. Similarly to the case of balanced graphs, the characterization of clique-perfect graphs within line graphs proved in [16] (stated here as Theorem 4.15) extends naturally to complements of line graphs because adding true twins preserves clique-perfectness. Nevertheless, as adding false twins does not always preserve clique-perfectness, from our characterization within complement of line graphs it does not immediately follow a characterization of clique-perfect graphs within complements of line graphs of multigraphs. In general, the problem of characterizing clique-perfect graphs within claw-free graphs and their complements remains unsolved. A different partial answer was given in [16], where clique-perfect graphs were characterized within claw-free hereditary clique-Helly graphs. In Subsection 4.2.1, we gave a structure theorem for graphs containing no bipartite-claw, which are precisely those graphs whose line graphs are net-free. It would be interesting to study to which extent a characterization of the same type can be formulated for some superclass of net-free line graphs.

In Section 4.3, we proved that clique-perfect graphs coincide with perfect graphs within gem-free circular-arc graphs. This means that clique-perfect graphs can be recognized in polynomial time within gem-free circular-arc graphs. In [17], a minimal forbidden induced subgraph characterization of clique-perfect graphs within Helly circular-arc graphs was given (see Theorem 4.5). It is easy to see that the approach used in Section 3.7 to extend Theorem 3.44 to Corollary 3.49, works also for extending the characterization of clique-perfect graphs within Helly circular-arc graphs in

Graph class	Forbidden subgraphs	Reference
graphs having the Kőnig property	even subdivisions of C_3, barbell, and K_4 as strongly splitting subgraphs	Theorem 5.9
edge-perfect graphs	$K_4 \cup tK_1$ and $C_{2k+1} \cup tK_1$ as edge subgraphs for $k \geqslant 1$, $t \geqslant 0$	Theorem 5.13

Table 6.3: *Main results of Chapter 5*

Theorem 4.5 to a characterization of clique-perfect graphs within {net,U_4,S_4}-free $\cap$ {1-pyramid,2-pyramid,3-pyramid}-free circular-arc graphs. Nevertheless, characterizing clique-perfect graphs by forbidden induced subgraphs within all {net,U_4,S_4}-free circular-arc graphs seems more difficult. The characterization of clique-perfect graphs by forbidden induced subgraphs is open even within proper circular-arc graphs. For coordinated graphs, the characterization remains unresolved even within both Helly circular-arc graphs and proper circular-arc graphs. It is not hard to see that the results in [17] imply a characterization of hereditary K-perfect graphs within Helly circular-arc graphs. As circular-arc graphs are a natural generalization of interval graphs and interval graphs are known to have perfect clique graph [70], we feel that it would be interesting to study hereditary K-perfect graphs further within circular-arc graphs like, for instance, proper circular-arc graphs.

The problem of determining the complexity of the recognition problem of clique-perfect graphs remains open in general. Neither it is known to be polynomial-time solvable nor was it shown to belong to any class of problems considered to be hard.

In Chapter 5, we studied the problem of characterizing graphs having the Kőnig property and edge-perfect graphs by means of certain types of forbidden subgraphs. In Section 5.2, we proved a characterization of graphs having the Kőnig property by means of strongly splitting subgraphs. In Section 5.3, we used this result to prove a characterization of edge-perfect graphs by forbidden edge-subgraphs. Edge-perfect graphs are those graphs whose edge-subgraphs have the Kőnig property. It would be interesting to know if a simple characterization as the one we proved for the edge-perfect graphs is also possible for those graphs whose edge-subgraphs are Class 1 (i.e., satisfy the edge-coloring property for edges). The results of Chapter 5 are summarized in Table 6.3.

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Glossary of notation

$\|S\|$	size of a set S, 9
$X \triangle Y$	symmetric difference of the sets X and Y, 9
$\overline{G}$	complement of G, 9
$G + vw$	graph G plus the edge vw, 9
$G - v$	graph G minus the vertex v, 9
$G - W$	graph G minus the vertex set W, 9
$G - e$	graph G minus the edge e, 9
$G \backslash F$	graph G minus the edge set F, 9
$G[W]$	subgraph of G induced by W, 9
$G_1 + G_2$	join of G_1 and G_2, 11
$H_1 \cup H_2$	disjoint union of the graphs or multigraphs H_1 and H_2, 13
tH	disjoint union of t copies of a graph or hypergraph H, 13
$G_1 \triangle_{AB} G_2$	merging of t-blooms A and B of G_1 and G_2, 31
$\widehat{H}$	underlying graph of the multigraph H, 13
$\Gamma_1 \&_p \Gamma_2$	p-concatenation of two-terminal graphs Γ_1 and Γ_2, 83
$\Gamma \&_p \circlearrowright$	p-closure of a two-terminal graph Γ, 83
$\alpha(G)$	stability number of G, 4
$\alpha_c(G)$	clique-independence number of G, 5
$\alpha_m(G)$	matching-independence number of G, 82
$\delta(G)$	minimum degree of G, 10
$\delta_h(G)$	minimum hub degree of G, 101
$\Delta(G)$	maximum degree of G, 10
$\Delta(\mathcal{F})$	maximum degree of a family $\mathcal{F}$ of sets, 77
$\Delta_c(G)$	maximum clique-degree of G, 77
G_Δ	core of G, 96
$\gamma(\mathcal{F})$	chromatic index of a family $\mathcal{F}$ of sets, 77
$\gamma_c(G)$	clique-chromatic index of G, 77
$\theta(G)$	clique covering number, 4

$\nu(G)$	matching number of G, 4
$\tau(G)$	transversal number of G, 4
$\tau_c(G)$	clique-transversal number of G, 5
$\tau_m(G)$	matching-transversal number of G, 82
$\chi(G)$	chromatic number of G, 3
$\chi'(G)$	chromatic index of G, 95
$\omega(G)$	clique number of G, 3
$\overline{G}^{bip}$	bipartite complement of G, 30
C_n	chordless cycle on n vertices, 10
$d_G(v)$	degree of v in G, 10
$\hat{d}_H(v)$	underlying degree of v in a multigraph H, 13
$E_G(v)$	set of edges incident to v in G, 10
K_n	complete graph on n vertices, 10
$L(R)$	line graph of a graph or multigraph R, 13
$L(H)$	line graph (or representative graph) of a multigraph H, 80
$N_G(v)$	neighborhood of v in G, 10
$N_G[v]$	closed neighborhood of v in G, 10
$N_G(W)$	common neighborhood of the set of vertices W in G, 10
$N_G(e)$	common neighborhood of the edge e in G, 10
P_n	chordless path on n vertices, 10
$\mathcal{P}(G)$	pruned graph of G, 49
$\mathcal{R}(G)$	representative graph of G, 36
W_n	wheel on n vertices, 10

Index