Partial Characterizations of Circular-Arc Graphs

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Abstract

A circular-arc graph is the intersection graph of a family of arcs on a circle. A characterization by forbidden induced subgraphs for this class of graphs is not known, and in this work we present a partial result in this direction. We characterize circular-arc graphs by a list of minimal forbidden induced subgraphs when the graph belongs to the following classes: diamond-free graphs, P_4-free graphs, paw-free graphs, and claw-free chordal graphs.

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1 Introduction

A graph G is a *circular-arc* (CA) graph if it is the intersection graph of a set $\mathcal{S}$ of arcs on a circle, i.e., if there exists a one-to-one correspondence between the vertices of G and the arcs of $\mathcal{S}$ such that two vertices of G are adjacent if and only if the corresponding arcs in $\mathcal{S}$ intersect. Such a family of arcs is called a *circular-arc* (CA) *model* of G. CA graphs can be recognized in linear time [6]. A graph is *proper circular-arc* (PCA) if it admits a CA model in which no arc is contained in another arc. Tucker gave a characterization of PCA graphs by minimal forbidden induced subgraphs [8]. Furthermore, this subclass can be recognized in linear time [2]. A graph is *unit circular-arc* (UCA) if it admits a CA model in which all the arcs have the same length. Tucker gave a characterization by minimal forbidden induced subgraphs for this class [8]. Recently, linear and quadratic time recognition algorithms for this class were presented [5,3]. Finally, CA graphs that are complements of bipartite graphs were characterized by forbidden induced subgraphs [7].

Nevertheless, the problem of characterizing the whole class of CA graphs by forbidden induced subgraphs remains open. In this work we present some steps in this direction by providing characterizations of CA graphs by minimal forbidden subgraphs when the graph belongs to one of four different classes.

Denote by $N(v)$ the set of neighbours of $v \in V(G)$; by $G|W$ the subgraph of G induced by W, for any $W \subseteq V(G)$; by $\overline{G}$ the complement of G; and by G^* the graph obtained from G by adding an isolated vertex. If t is a nonnegative integer, then tG will denote the disjoint union of t copies of G. A graph G is a *multiple* of another graph H if G can be obtained from H by replacing each vertex x of H by a complete graph K_x and adding all possible edges between K_x and K_y if and only if x and y are adjacent in H.

The graph P_4 is an induced path on 4 vertices. A *paw* is the graph obtained from a complete K_3 by adding a vertex adjacent to exactly one of its vertices. A *diamond* is the graph obtained from a complete K_4 by removing exactly one edge. A *claw* is the complete bipartite graph $K_{1,3}$. A *hole* is an induced cycle of length at least 4. A graph is *chordal* if it does not contain any hole.

Let $A, B \subseteq V(G)$; *A is complete to B* if every vertex of A is adjacent to every vertex of B; and *A is anticomplete to B* if A is complete to B in $\overline{G}$. Let G and H be two graphs; we say that G is an *augmented* H if G is isomorphic to H or if G can be obtained from H by repeatedly adding a universal vertex; and G is a *bloomed* H if there exists a subset $W \subseteq V(G)$ such that $G|W$ is isomorphic to H and $V(G) - W$ induces in G a disjoint union of complete graphs $B_1, B_2, \ldots, B_j$ for some $j \geq 0$, and each B_i is complete to one vertex

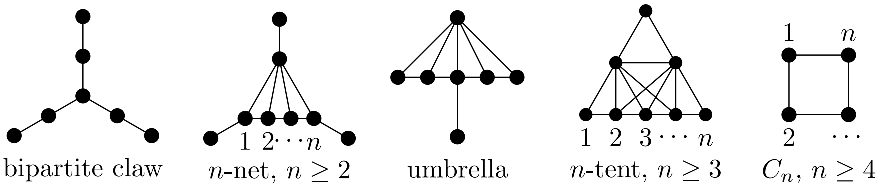

Fig. 1. Minimal forbidden induced subgraphs for the class of interval graphs

of H but anticomplete to the other vertices of H. If each vertex in W is complete to at least one of $B_1, B_2, \ldots, B_j$, we say that G is a *fully bloomed* H. The graphs $B_1, \ldots, B_j$ are the *blooms*. A bloom is *trivial* if it is composed by only one vertex.

Special graphs needed throughout this work are depicted in Figures 1 and 2. We use *net* and *tent* as abbreviations for 2-net and 3-tent, respectively.

Lekkerkerker and Boland determined all the minimal forbidden induced subgraphs for the class of interval graphs, a known subclass of CA graphs.

Theorem 1.1 [4] *The minimal forbidden induced subgraphs for the class of interval graphs are: bipartite claw, n-net for every $n \geq 2$, umbrella, n-tent for every $n \geq 3$, and C_n for every $n \geq 4$ (cf. Figure 1).*

This characterization yields some minimal forbidden induced subgraphs for the class of CA graphs.

Corollary 1.2 [7] *The following graphs are minimally non-CA graphs: bipartite claw, net*, n-net for all $n \geq 3$, umbrella*, (n-tent)* for all $n \geq 3$, and C_n^* for every $n \geq 4$. Moreover, any other minimally non-CA graph is connected.*

We call these graphs *basic* minimally non-CA graphs. Any other minimally non-CA graph will be called *nonbasic*. The following result is a corollary of Theorem 1.1 and Corollary 1.2, and gives a structural property for all nonbasic minimally non-CA graphs.

Corollary 1.3 *If G is a nonbasic minimally non-CA graph, then G has an induced subgraph H which is isomorphic to an umbrella, a net, a j-tent for some $j \geq 3$, or C_j for some $j \geq 4$. In addition, each vertex v of $G - H$ is adjacent to at least one vertex of H.*

2 Partial characterizations

A *cograph* is a graph with no induced P_4. We will call *semicircular graphs* to the intersection graphs of open semicircles on a circle. By definition, semicircular graphs are UCA graphs.

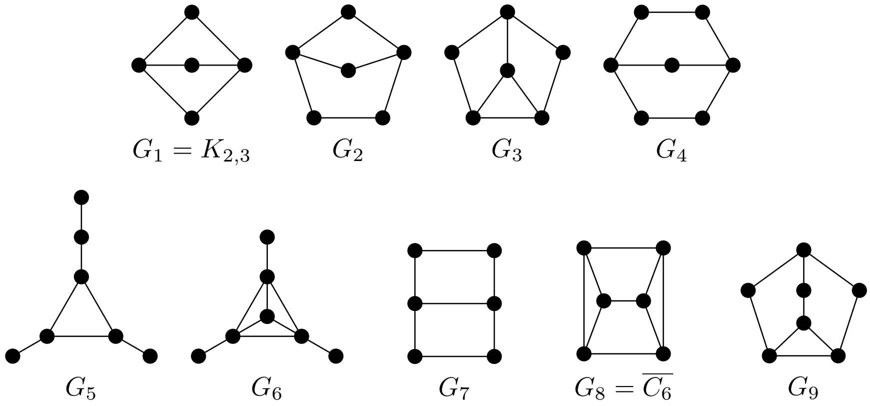

Fig. 2. Some minimally non-CA graphs.

Theorem 2.1 *Let G be a graph. The following conditions are equivalent:*

(i) *G is an augmented multiple of $\overline{tK_2}$ for some nonnegative integer t.*

(ii) *G is a semicircular graph.*

Theorem 2.2 *Let G be a cograph that contains an induced C_4, and such that all its proper induced subgraphs are CA graphs. Then, exactly one of the following conditions holds:*

(i) *G is isomorphic to G_1 or C_4^*.*

(ii) *G is an augmented multiple of $\overline{tK_2}$ for some integer $t \geq 2$.*

Corollary 2.3 *Let G be a cograph. Then, G is a CA graph if and only if G contains neither G_1 nor C_4^* as induced subgraphs.*

Proof. Suppose that H is a cograph minimally non-CA graph and H is not isomorphic to G_1 or C_4^*. Since H is not an interval graph and is P_4-free then, by Theorem 1.1, H contains an induced C_4. By Theorem 2.2, H is an augmented multiple of $\overline{tK_2}$, for some $t \geq 2$. Thus, by Theorem 2.1, H is a circular-arc graph, a contradiction. □

A *paw-free* graph is a graph with no induced paw.

Theorem 2.4 *Let G be a paw-free graph that contains an induced C_4 and such that all its proper induced subgraphs are CA graphs. Then, at least one of the following conditions holds:*

(i) *G is isomorphic to G_1, G_2, G_7, or C_4^*.*

(ii) *G is a bloomed C_4 with trivial blooms.*

(iii) *G is an augmented multiple of $\overline{tK_2}$ for some $t \geq 2$.*

Theorem 2.5 *Let G be a paw-free graph that contains an induced C_j for some $j \geq 5$, and such that all its proper induced subgraphs are CA graphs. Then, exactly one of the following conditions holds:*

(i) *G is isomorphic to G_2, G_4, or C_j^*.*

(ii) *G is a bloomed C_j with trivial blooms.*

Corollary 2.6 *Let G be a paw-free graph. Then G is a CA graph if and only if G contains no induced bipartite claw, G_1, G_2, G_4, G_7, or C_n^* $(n \geq 4)$.*

Proof. Suppose that H is not any of those graphs but it is still a paw-free minimally non-CA graph. In particular, H is not basic. Since H is paw-free then, by Corollary 1.3, H contains an induced C_j for some $j \geq 4$. By Theorem 2.4 and Theorem 2.5, H is an augmented multiple of $\overline{tK_2}$ for some $t \geq 2$ or H is a bloomed C_j with trivial blooms. In both cases H would be a CA graph, a contradiction. □

A graph is *claw-free chordal* if it is chordal and contains no induced claw.

Theorem 2.7 *Let G be a claw-free chordal graph that contains an induced net, and such that all its proper induced subgraphs are CA graphs. Then, exactly one of the following conditions holds:*

(i) *G is isomorphic to a net*, G_5 or G_6.*

(ii) *G is a CA graph.*

Theorem 2.8 [1] *Let G be a connected graph which contains no induced claw, net, C_4, or C_5. If G contains an induced tent, then G is a multiple of a tent.*

Corollary 2.9 *Let G be a claw-free chordal graph. Then, G is CA if and only if G contains no induced tent*, net*, G_5 or G_6.*

Proof. Suppose that H is not any of those graphs but it is still a claw-free chordal minimally non-CA graph. In particular, H is not basic. By Corollary 1.3, H contains an induced net or an induced tent. If H contains an induced net then, by Theorem 2.7, H would be isomorphic to a net*, G_5 or G_6, a contradiction. Thus H contains no induced net but an induced tent. If H is connected, by Theorem 2.8, H is a multiple of a tent and, in particular, a CA graph. Otherwise, H contains a tent*, a contradiction. □

A *diamond-free* graph is a graph with no induced diamond.

Theorem 2.10 *Let G be a diamond-free graph that contains a hole, and such that all its proper induced subgraphs are CA graphs. Then, exactly one of the following conditions holds:*

(i) G *is isomorphic to* G_1, G_2, G_3, G_4, G_7, G_8, G_9, *or* C_j^* *for some* $j \geq 4$.

(ii) G *is a* CA *graph.*

Theorem 2.11 *Let* G *be a diamond-free chordal graph that contains an induced net, and such that all its proper induced subgraphs are* CA *graphs. Then, exactly one of the following conditions holds:*

(i) G *is isomorphic to a net**, G_5, *or* G_6.

(ii) G *is a fully bloomed triangle, and in consequence, it is a* CA *graph.*

Corollary 2.12 *Let* G *be a diamond-free graph.* G *is* CA *if and only if* G *contains no induced bipartite claw, net**, G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, *or* C_n^* *for every* $n \geq 4$.

Proof. Suppose that H is not isomorphic to any of those graphs but it is still a diamond-free minimally non-CA graph. Since H is not an interval graph and it is diamond-free, by Theorem 1.1, H contains either a hole or an induced net. If H contains a hole, it contradicts Theorem 2.10. If H is chordal, it contains an induced net, and so H contradicts Theorem 2.11. □

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