Exploring the complexity boundary between coloring and list-coloring

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Outline

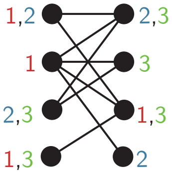
The k-coloring problem

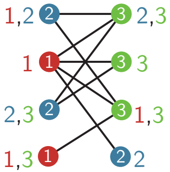
- A coloring of a graph $G = (V, E)$ is a function $f : V \rightarrow \mathbb{N}$ such that $f(v) \neq f(w)$ whenever $vw \in E$.
- A k-coloring is a coloring f such that $f(v) \leq k$ for every $v \in V$.
- The vertex coloring problem, or k-coloring problem, takes as input a graph G and a natural number k, and consists in deciding whether G is k-colorable or not.

The list-coloring problem

In order to take into account particular constraints arising in practical settings, more elaborate models of vertex coloring have been defined in the literature. One of such generalized models is the list-coloring problem, which considers a prespecified set of available colors for each vertex.

- Given a graph G and a finite list $L(v) \subseteq \mathbb{N}$ for each vertex $v \in V$, the list-coloring problem asks for a list-coloring of G, i.e., a coloring f such that $f(v) \in L(v)$ for every $v \in V$.





The list-coloring problem

- Many classes of graphs where the vertex coloring problem is polynomially solvable are known, the most prominent being the class of perfect graphs [Grötschel-Lovász-Schrijver, 1981].
- Meanwhile, the list-coloring problem is NP-complete for perfect graphs, and is also NP-complete for many subclasses of perfect graphs, including split graphs, interval graphs, and bipartite graphs.
- Trees and complete graphs are two classes of graphs where the list-coloring problem can be solved in polynomial time. In the first case it can be solved using dynamic programming techniques [Jansen-Scheffler, 1997]. In the second case, the problem can be reduced to the maximum matching problem in bipartite graphs.

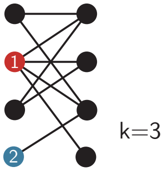
We are going to explore the complexity boundary between vertex coloring and list-coloring. Our goal is to analyze the computational complexity of coloring problems lying "between" (from a computational complexity viewpoint) these two problems.

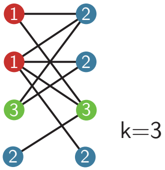
The precoloring extension problem

Some particular cases of list-coloring have been studied.

- The precoloring extension (PrExt) problem takes as input a graph $G = (V, E)$, a subset $W \subseteq V$, a coloring f' of W, and a natural number k, and consists in deciding whether G admits a k-coloring f such that $f(v) = f'(v)$ for every $v \in W$ or not [Biro-Hujter-Tuza, 1992].

In other words, a prespecified vertex subset is colored beforehand, and our task is to extend this partial coloring to a valid k-coloring of the whole graph.



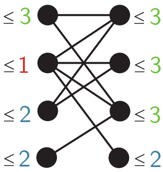


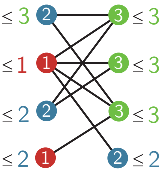
The μ-coloring problem

Another particular case of the list-coloring problem is the following.

- Given a graph G and a function $\mu : V \to \mathbb{N}$, G is μ-colorable if there exists a coloring f of G such that $f(v) \leq \mu(v)$ for every $v \in V$ [B.-Cecowski, 2005].

This model arises in the context of classroom allocation to courses, where each course must be assigned a classroom which is large enough so it fits the students taking the course.

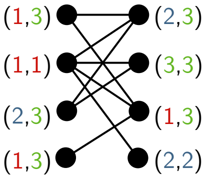


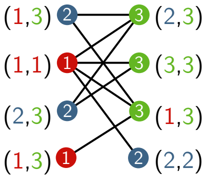


The (γ, μ)-coloring problem

We define here a new variation of this problem.

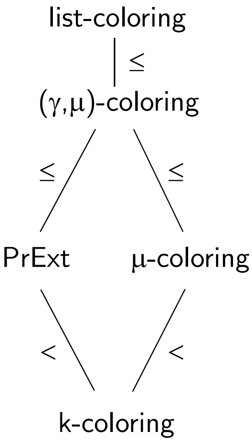
- Given a graph G and functions $\gamma, \mu : V \to \mathbb{N}$ such that $\gamma(v) \leq \mu(v)$ for every $v \in V$, we say that G is (γ, μ)-colorable if there exists a coloring f of G such that $\gamma(v) \leq f(v) \leq \mu(v)$ for every $v \in V$.





Hierarchy of coloring problems

- The classical vertex coloring problem is clearly a special case of μ-coloring and precoloring extension, which in turn are special cases of (γ, μ)-coloring.
- Furthermore, (γ, μ)-coloring is a particular case of list-coloring.
- These observations imply that all the problems in this hierarchy are polynomially solvable in those graph classes where list-coloring is polynomial and, on the other hand, all the problems are NP-complete in those graph classes where vertex coloring is NP-complete.



In this work, we are interested in the computational complexity of these problems over subclasses of perfect graphs where vertex coloring is polynomially solvable and list-coloring is NP-complete.

Interval graphs

- A graph is an interval graph if it is the intersection graph of a set of intervals over the real line. A unit interval graph is the intersection graph of a set of intervals of length one.

- Since interval graphs are perfect, vertex coloring over interval and unit interval graphs is polynomially solvable. On the other hand, precoloring extension over unit interval graphs is NP-complete [Marx, 2004], implying that (γ, μ)-coloring and list-coloring are NP-complete over this class and over interval graphs.

Split graphs

- A split graph is a graph whose vertex set can be partitioned into a complete graph K and a stable set S. A split graph is said to be complete if its edge set includes all possible edges between K and S.
- It is trivial to color a split graph in polynomial time, and it is a known result that precoloring extension is also solvable in polynomial time on split graphs [Hujter-Tuza, 1996], whereas list-coloring is known to be NP-complete even over complete split graphs [Jansen-Scheffler, 1997].

Bipartite graphs

- A bipartite graph is a graph whose vertex set can be partitioned into two independent sets V_1 and V_2. A bipartite graph is said to be complete if its edge set includes all possible edges between V_1 and V_2.
- Again, the vertex coloring problem over bipartite graphs is trivial, whereas precoloring extension [Hujter-Tuza, 1993] and μ-coloring [B.-Cecowski, 2005] are known to be NP-complete over bipartite graphs, implying that (γ, μ)-coloring and list-coloring over this class are also NP-complete.
- Moreover, list-coloring is NP-complete even over complete bipartite graphs [Jansen-Scheffler, 1997].

Complements of bipartite graphs and cographs

- For complements of bipartite graphs, precoloring extension can be solved in polynomial time [Hujter-Tuza, 1996], but list-coloring is NP-complete [Jansen, 1997].
- The same happens for cographs, graphs with no induced P_4 [Hujter-Tuza, Jansen-Scheffler, 1996]. For this class of graphs, μ-coloring is polynomial [B.-Cecowski, 2005].

Line graphs

- The line graph of a graph is the intersection graph of its edges. The edge coloring problem (equivalent to coloring the line graph) is NP-complete in general [Holyer, 1981], but it can be solved in polynomial-time for complete graphs and bipartite graphs [König, 1916].
- It is known that precoloring extension is NP-complete on line graphs of complete bipartite graphs $K_{n,n}$ [Colbourn, 1984], and list-coloring is NP-complete on line graphs of complete graphs [Kubale, 1992].

Review of known results

Class	coloring	PrExt	μ-col.	(γ, μ)-col.	list-col.
Complete bipartite	P	P	?	?	NP-c
Bipartite	P	NP-c	NP-c	NP-c	NP-c
Cographs	P	P	P	?	NP-c
Interval	P	NP-c	?	NP-c	NP-c
Unit interval	P	NP-c	?	NP-c	NP-c
Complete split	P	P	?	?	NP-c
Split	P	P	?	?	NP-c
Line of $K_{n,n}$	P	NP-c	?	NP-c	NP-c
Line of K_n	P	?	?	?	NP-c
Complement of bipartite	P	P	?	?	NP-c

"NP-c": NP-complete problem, "P": polynomial problem, "?": open problem.

Interval graphs

Theorem

The μ-coloring problem over interval graphs is NP-complete.

This result implies that (γ, μ)-coloring over interval graphs is also NP-complete.

Its proof is based on the NP-completeness of the coloring problem on circular-arc graphs [Garey-Johnson-Miller-Papadimitriou, 1980].

Complete bipartite and split graphs

Theorem

The (γ, μ)-coloring problem in complete bipartite graphs and complete split graphs can be solved in polynomial time.

Combinatorial arguments are used to prove that the (γ, μ)-coloring problem is polynomial on complete bipartite graphs, whereas integer programming techniques are employed to prove the polynomiality for complete split graphs.

This theorem implies that μ-coloring and precoloring extension over complete bipartite graphs and complete split graphs can be solved in polynomial time.

Split graphs

Theorem

The μ-coloring problem over split graphs is NP-complete.

The proof of this result is based on the NP-completeness of finding a dominating set on split graphs [Bertossi, Corneil-Perl, 1984].

At this moment, this is the only class that we know where the computational complexity of μ-coloring and precoloring extension is different, unless $\mathrm{P} = \mathrm{NP}$.

Line graphs

Considering these coloring variations applied to edge coloring, we have the following results.

Theorem

The μ-coloring problem over line graphs of complete graphs and complete bipartite graphs is NP-complete.

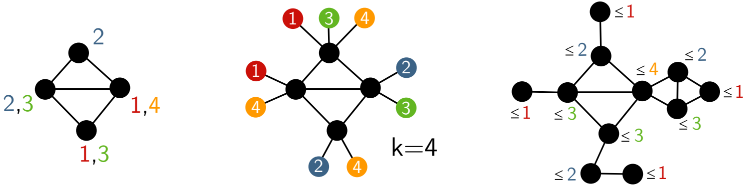
Theorem

The precoloring extension problem over line graphs of complete graphs is NP-complete.

All these proofs are based on the NP-completeness of precoloring extension on line graphs of complete bipartite graphs.

General results

Since all the problems are NP-complete in the general case, there are also polynomial-time reductions from list-coloring to precoloring extension and μ-coloring. An example is shown in the figure, where we can see a list-coloring instance and its corresponding precoloring extension and μ-coloring instances.



These reductions involve changes in the graph, but are closed within some graph classes. This fact allows us to prove the following general results.

General results

Theorem

Let $\mathcal{F}$ be a family of graphs with minimum degree at least two. Then list-coloring, (γ, μ)-coloring and precoloring extension are polynomially equivalent in the class of $\mathcal{F}$-free graphs.

Theorem

Let $\mathcal{F}$ be a family of graphs satisfying the following property: for every graph G in $\mathcal{F}$, no connected component of G is complete, and for every vertex v of G, no connected component of $G \setminus v$ is complete. Then list-coloring, (γ, μ)-coloring, μ-coloring and precoloring extension are polynomially equivalent in the class of $\mathcal{F}$-free graphs.

Since odd holes and antiholes satisfy the conditions of the theorems above, these theorems are applicable for many subclasses of perfect graphs.

Review: complexity table for coloring problems

Class	coloring	PrExt	μ-col.	(γ, μ)-col.	list-col.
Complete bipartite	P	P	**P**	**P**	NP-c
Bipartite	P	NP-c	NP-c	NP-c	NP-c
Cographs	P	P	P	?	NP-c
Interval	P	NP-c	**NP-c**	NP-c	NP-c
Unit interval	P	NP-c	?	NP-c	NP-c
Complete split	P	P	**P**	**P**	NP-c
Split	P	P	**NP-c**	**NP-c**	NP-c
Line of $K_{n,n}$	P	NP-c	**NP-c**	NP-c	NP-c
Line of K_n	P	**NP-c**	**NP-c**	**NP-c**	NP-c
Complement of bipartite	P	P	?	?	NP-c

"NP-c": NP-complete problem, "P": polynomial problem, "?": open problem.

Review: complexity table for coloring problems

Class	coloring	PrExt	μ-col.	(γ, μ)-col.	list-col.
Complete bipartite	P	P	**P**	**P**	NP-c
Bipartite	P	NP-c	NP-c	NP-c	NP-c
Cographs	P	P	P	?	NP-c
Interval	P	NP-c	**NP-c**	NP-c	NP-c
Unit interval	P	NP-c	?	NP-c	NP-c
Complete split	P	P	**P**	**P**	NP-c
Split	P	P	**NP-c**	**NP-c**	NP-c
Line of $K_{n,n}$	P	NP-c	**NP-c**	NP-c	NP-c
Line of K_n	P	**NP-c**	**NP-c**	**NP-c**	NP-c
Complement of bipartite	P	P	?	?	NP-c

"NP-c": NP-complete problem, "P": polynomial problem, "?": open problem.

As this table shows, unless P $=$ NP, μ-coloring and precoloring extension are strictly more difficult than vertex coloring, list-coloring is strictly more difficult than (γ, μ)-coloring and (γ, μ)-coloring is strictly more difficult than precoloring extension.

Review: complexity table for coloring problems

Class	coloring	PrExt	μ-col.	(γ, μ)-col.	list-col.
Complete bipartite	P	P	**P**	**P**	NP-c
Bipartite	P	NP-c	NP-c	NP-c	NP-c
Cographs	P	P	P	?	NP-c
Interval	P	NP-c	**NP-c**	NP-c	NP-c
Unit interval	P	NP-c	?	NP-c	NP-c
Complete split	P	P	**P**	**P**	NP-c
Split	P	P	**NP-c**	**NP-c**	NP-c
Line of $K_{n,n}$	P	NP-c	**NP-c**	NP-c	NP-c
Line of K_n	P	**NP-c**	**NP-c**	**NP-c**	NP-c
Complement of bipartite	P	P	?	?	NP-c

"NP-c": NP-complete problem, "P": polynomial problem, "?": open problem.

It remains as an open problem to know if there exists any class of graphs such that (γ, μ)-coloring is NP-complete and μ-coloring can be solved in polynomial time. Among the classes considered in this work, the candidate classes are cographs, unit interval and complement of bipartite.

Review: complexity table for coloring problems

Class	coloring	PrExt	μ-col.	(γ, μ)-col.	list-col.
Complete bipartite	P	P	**P**	**P**	NP-c
Bipartite	P	NP-c	NP-c	NP-c	NP-c
Cographs	P	P	P	?	NP-c
Interval	P	NP-c	**NP-c**	NP-c	NP-c
Unit interval	P	NP-c	?	NP-c	NP-c
Complete split	P	P	**P**	**P**	NP-c
Split	P	P	**NP-c**	**NP-c**	NP-c
Line of $K_{n,n}$	P	NP-c	**NP-c**	NP-c	NP-c
Line of K_n	P	**NP-c**	**NP-c**	**NP-c**	NP-c
Complement of bipartite	P	P	?	?	NP-c

"NP-c": NP-complete problem, "P": polynomial problem, "?": open problem.

For split graphs, precoloring extension can be solved in polynomial time, whereas μ-coloring is NP-complete. It remains as an open problem to find a class of graphs where the converse holds. Among the classes considered in this work, the candidate class is unit interval.

Review: hierarchy of coloring problems