The Baum-Connes conjecture for the dual of $SU_q(2)$

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What happens if G is a locally compact *quantum* group?

Recapitulation and motivation

Meyer and Nest have reformulated the Baum-Connes conjecture using the language of triangulated categories and derived functors. This yields

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In particular, Meyer and Nest formulate and prove a generalization of the Baum-Connes conjecture for duals of compact groups.

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Why is one interested in such a result?

- serves as an interesting "test case" for the machinery of Meyer-Nest
- yields a conceptual approach to compute certain K-theory groups
- might lead to new insights in the theory of quantum groups

Locally compact quantum groups

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Historically, the operator algebra approach to quantum groups grew out of attempts to generalize the Pontrjagin duality theorem to non-abelian locally compact groups.

Locally compact quantum groups

- *Duality for compact groups* - Tannaka (1938)
- *Kac algebras* - Kac-Vainerman, Enock-Schwartz (1973)
- $SU_q(2)$ - Woronowicz (1987)
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Definition (Kustermans-Vaes (1999))

A *locally compact quantum group* is a C^*-algebra H together with a comultiplication $\Delta : H \to M(H \otimes H)$ and left and right Haar integrals.

Locally compact quantum groups

Examples

- A basic example is the algebra $C_0(G)$ of functions on a locally compact group G. The comultiplication $\Delta : C_0(G) \to C_b(G \times G)$ is given by

  $$\Delta(f)(s,t) = f(st)$$

  and the integrals are given by left/right Haar measure.

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and the integrals are given by left/right Haar measure.

- Another basic example is the reduced group C^*-algebra $C^*_{\mathrm{red}}(G)$ of G. The comultiplication is given by

$$\Delta(\lambda_t) = \lambda_t \otimes \lambda_t$$

and the left and right Haar integral ϕ satisfies $\phi(f) = f(e)$ where $f \in C_c(G) \subset C^*_{\mathrm{red}}(G)$ and $e \in G$ is the identity.

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Remark

If G is a locally compact group then the quantum groups $C_0(G)$ and $C^*_{\mathrm{red}}(G)$ are dual to each other.
If in addition G is abelian then $C^*_{\mathrm{red}}(G) \cong C_0(\hat{G})$ where $\hat{G}$ is the dual group.

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- its algebra $\mathcal{O}(SU_q(2))$ of polynomial functions
- its C^*-algebra $C(SU_q(2))$ of continuous functions

The quantum group $SU_q(2)$

Definition

Fix $q \in (0,1]$. The unital $*$-algebra $\mathcal{O}(SU_q(2))$ (over $\mathbb{C}$) is generated by elements α and γ satisfying the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

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These relations are equivalent to saying that the fundamental matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary.

The quantum group $SU_q(2)$

The *comultiplication* $\Delta : \mathcal{O}(SU_q(2)) \to \mathcal{O}(SU_q(2)) \otimes \mathcal{O}(SU_q(2))$ is defined by

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The *counit* ϵ and the *antipode* S of $\mathcal{O}(SU_q(2))$ are defined by the formulas

$$\epsilon \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha^* & \gamma^* \\ -q\gamma & \alpha \end{pmatrix}$$

on generators.

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on generators.
In this way $\mathcal{O}(SU_q(2))$ becomes a *Hopf-$*$-algebra*.

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- For $q = 1$ one obtains in this way the algebras $\mathcal{O}(SU(2))$ and $C(SU(2))$ of polynomial and continuous functions on $SU(2)$, respectively.
- The antipode does not extend to $C(SU_q(2))$ for $q \neq 1$

In addition to $SU_q(2)$ we need...

The Podleś sphere

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The *maximal torus* $T = S^1 \subset SU_q(2)$ is given by the projection $\pi : C(SU_q(2)) \to C(T)$ given by

$$\pi \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

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The (standard) *Podleś sphere* is the homogenous space $SU_q(2)/T$ given by the algebra of coinvariants

$$C(SU_q(2)/T) = \{x \in C(SU_q(2)) | (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$$

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under right translations.
We remark that for $q \in (0,1)$ one has $C(SU_q(2)/T) \cong \mathbb{K}^+$. There is an algebraic version $\mathcal{O}(SU_q(2)/T)$ as well.

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The formulation of a Baum-Connes conjecture for quantum groups is based on the work of Meyer and Nest.

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...what is the Baum-Connes conjecture in this situation?

The equivariant Kasparov category

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- The (inverse of the) *suspension* $\Sigma(A) = C_0(\mathbb{R}) \otimes A$ yields the translation functor.
- Distinguished *triangles* are all triangles isomorphic to mapping cone triangles

$$\Sigma(B) \to C_f \to A \to B$$

for equivariant $*$-homomorphisms $f : A \to B$.

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The discrete quantum group $\hat{G}$ is *torsion-free*. The *proper homogenous* $\hat{G}$-algebra corresponding to the trivial subgroup is $C^*(G)$. We write $\mathcal{P}$ for the localizing subcategory of $KK^{\hat{G}}$ generated by algebras of the form $C^*(G) \otimes A$ where A is some C^*-algebra and the coaction inherited from $C^*(G)$.

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Theorem

Let $G = SU_q(2)$. Then $\hat{G}$ satisfies the strong Baum-Connes conjecture.

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Theorem

We have $\mathbb{C} \in \mathcal{P} \subset KK^{\hat{G}}$.

Theorem (Baaj-Skandalis)

The reduced crossed product functor $KK^{\hat{G}} \to KK^{G}$ is an equivalence of categories.

As a consequence, in order to prove $\mathbb{C} \in \mathcal{P}$ it suffices to show $C(G) \in \langle \mathbb{C} \rangle \in KK^{G}$.

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We have $C(G) \in \langle C(G/T) \rangle$ in KK^G - this follows from (the validity of) the Baum-Connes conjecture for $\hat{T}$ and induction. Hence it suffices to show

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We have $C(G/T) \cong \mathbb{C} \oplus \mathbb{C}$ in KK^G.

In the case $q = 1$ this follows from equivariant *Poincaré duality* for G/T. This duality result is no longer available for $q \neq 1$.

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We need two results concerning the equivariant K-theory and K-homology of the Podleś sphere G/T.

K-theory of the Podleś sphere

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Natural elements in $K_0^T(C(G/T))$ are given by the $\mathcal{O}(G/T)$-modules

$$\Gamma(G \times_T \mathbb{C}_k) = \{x \in \mathcal{O}(SU_q(2)) | (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes z^{-k}\}$$

for $k \in \mathbb{Z}$.

It follows from *Hopf-Galois theory* that these modules are finitely generated and projective.

Geometrically, $\Gamma(G \times_T \mathbb{C}_k)$ corresponds to an *induced bundle* on G/T.

K-theory of the Podleś sphere

Theorem

For $G = SU_q(2)$ there is a commutative diagram

$$\begin{tikzcd} R(T) \otimes_{R(G)} R(T) \arrow[r, "\lambda"] \arrow[d, "E"] & K_*^T(C(G/T)) \arrow[d, "i"] \\ \bigoplus_{w \in W} R(T) \arrow[r, "\cong"] & K_*^T(C(W)) \end{tikzcd}$$

where the upper horizontal map λ is an isomorphism.

Here $W = \mathbb{Z}/2\mathbb{Z}$ is the (classical) *Weyl group*. The map λ in this diagram is the left $R(T)$-linear map defined by

$$\lambda(1 \otimes z^k) = \Gamma(G \times_T \mathbb{C}_k)$$

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The proof is a (lengthy) computation involving the equivariant Chern character $ch_0^T : K_0^T \to HP_0^T$.

Remark

- It follows that the group $K_0^T(C(G/T))$ is a free $R(T)$-module generated by $\Gamma(G \times_T \mathbb{C}_0)$ and $\Gamma(G \times_T \mathbb{C}_{-1})$.

Remark

- It follows that the group $K_0^T(C(G/T))$ is a free $R(T)$-module generated by $\Gamma(G \times_T \mathbb{C}_0)$ and $\Gamma(G \times_T \mathbb{C}_{-1})$.
- The Weyl group W acts on $K_*^T(C(G/T))$. This action does not come from an action of W on the C^*-algebra $C(G/T)$.

Equivariant Fredholm modules for the Podleś sphere

We want to describe a family $(\mathcal{E}_l, \phi_l, F_l)$ of even *G-equivariant Fredholm modules* over $C(G/T)$ for $l \in \mathbb{Z}$.

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- the Hilbert space $\mathcal{E}_l = \mathcal{E}_l^+ \oplus \mathcal{E}_l^-$ is the completion of

$$\Gamma(G \times_T \mathbb{C}_{l-1}) \oplus \Gamma(G \times_T \mathbb{C}_{l+1})$$

with respect to the scalar product induced by $L^2(G)$.

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  with respect to the scalar product induced by $L^2(G)$.
- The representation ϕ_l of $C(G/T)$ is induced by left multiplication on the induced bundles.

- According to *Frobenius reciprocity* one has

$$\Gamma(G \times_T \mathbb{C}_j) = \bigoplus_{m=0}^{\infty} V_{|j|/2+m}$$

where V_j denotes the irreducible representation of $SU_q(2)$ of dimension $2j+1$. The operator F_l is defined by the matrix

$$F_l = \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$$

where $S : \Gamma(G \times_T \mathbb{C}_{l+1}) \to \Gamma(G \times_T \mathbb{C}_{l-1})$ is the natural isometry.

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Proposition

This data defines an equivariant Fredholm module $F_l = (\mathcal{E}_l, \phi_l, F_l)$ over $C(G/T)$ for every $l \in \mathbb{Z}$.

Remark

These Fredholm modules represent *twisted Dirac operators* on G/T. For $l = 0$ the corresponding Dirac operator has been defined and studied by Dabrowski-Sitarz.

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Theorem

The Kasparov product $\Gamma(G \times_T \mathbb{C}_k) \circ F_l \in KK_0^G(\mathbb{C}, \mathbb{C}) = R(G)$ *is given by*

$$\Gamma(G \times_T \mathbb{C}_k) \circ F_l = \begin{cases} V_{(k+l-1)/2} & \text{for } k + l > 0 \\ 0 & \text{for } k + l = 0 \\ -V_{-(k+l-1)/2} & \text{for } k + l < 0 \end{cases}$$

for all $k, l \in \mathbb{Z}$.

... completing the proof of the main theorem

- Define elements $\alpha \in KK_0^G(C(G/T), \mathbb{C}^{|W(G)|})$ by

$$\alpha = F_0 \oplus F_{-1}$$

and $\beta \in KK_0^G(\mathbb{C}^{|W(G)|}, C(G/T))$ by

$$\beta = \Gamma(G \times_T \mathbb{C}_1) \oplus -\Gamma(G \times_T \mathbb{C}_0)$$

... completing the proof of the main theorem

- Define elements $\alpha \in KK_0^G(C(G/T), \mathbb{C}^{|W(G)|})$ by
$$\alpha = F_0 \oplus F_{-1}$$
and $\beta \in KK_0^G(\mathbb{C}^{|W(G)|}, C(G/T))$ by
$$\beta = \Gamma(G \times_T \mathbb{C}_1) \oplus -\Gamma(G \times_T \mathbb{C}_0)$$

- Compute $\beta \circ \alpha = 1$ using the previous theorem and $\alpha \circ \beta = 1$ using in addition induction, UCT for KK^T and the description of $K_*^T(C(G/T))$ obtained before. $\square$

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Theorem

Let $q \in (0,1]$. For $G = SU_q(2)$ the K-theory of $C(G)$ is given by

$$K_0(C(G)) = \mathbb{Z}, \qquad K_1(C(G)) = \mathbb{Z}.$$

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Proof. In $KK^{\mathbb{Z}}$ one has

$$\begin{tikzcd}
\Sigma C_0(\mathbb{Z}) \arrow[r] \arrow[d, equal] & C_0(\mathbb{R}) \arrow[r, "i"] \arrow[d, "D"] & C_0(\mathbb{Z}) \arrow[r, "1-S"] \arrow[d, equal] & C_0(\mathbb{Z}) \arrow[d, equal] \\
\Sigma C_0(\mathbb{Z}) \arrow[r] & \Sigma\mathbb{C} \arrow[r, "i"] & C_0(\mathbb{Z}) \arrow[r, "1-S"] & C_0(\mathbb{Z})
\end{tikzcd}$$

K-theory of $C(SU_q(2))$

Applying the crossed product functor yields

$$\begin{array}{ccccccc}
\Sigma C_0(\mathbb{Z}) \rtimes \mathbb{Z} & \longrightarrow & C_0(\mathbb{R}) \rtimes \mathbb{Z} & \longrightarrow & C_0(\mathbb{Z}) \rtimes \mathbb{Z} & \longrightarrow & C_0(\mathbb{Z}) \rtimes \mathbb{Z} \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\Sigma\mathbb{C} & \longrightarrow & \Sigma C(T) & \longrightarrow & \mathbb{C} & \xrightarrow{1-z} & \mathbb{C}
\end{array}$$

in KK^T.

We apply the induction functor ind_T^G and obtain an exact sequence

$$0 \longrightarrow K_1(C(G)) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow K_0(C(G)) \longrightarrow 0$$

Identify $K_0^T(C(G/T))$ with the free $R(T)$-module generated by $1 \otimes 1$ and $1 \otimes z$ in $R(T) \otimes_{R(G)} R(T)$. It follows that multiplication by z corresponds to the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & z^{-1} + z \end{pmatrix}$$

in $M_2(R(T))$. Hence

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

represents the map induced by $1 - z$ in $\mathrm{End}(\mathbb{Z}^2)$. One checks that $\ker(1-z) = \{(k,-k) | k \in \mathbb{Z}\} \cong \mathbb{Z}$ and $\mathrm{Coker}(1-z) \cong \mathbb{Z}$. $\square$

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- The method of the above proof should also work for q-deformations of other classical Lie groups.