Relative cyclic homology of square zero extensions

Jorge Alberto Guccione   Juan José Guccione

Departamento de Matemática
F.C.E. y N., Universidad de Buenos Aires
Argentina

Relative cyclic homology

In this talk

- k is a characteristic zero field,
- All the algebras are over k,
- For an algebra C, we let $\overline{C}$ denote C/k.

let M be a two sided ideal of C and $A = C/M$. The

Hochschild $\mathrm{HH}(C, M)$,
cyclic $\mathrm{HC}(C, M)$,
negative $\mathrm{HN}(C, M)$ and
periodic $\mathrm{HP}(C, M)$

homologies of C relative to M, are the respective homologies of the mixed complex $\ker \pi_*$, where

$$(C \otimes \overline{C}^{\otimes *}, b, B) \xrightarrow{\pi_*} (A \otimes \overline{A}^{\otimes *}, b, B),$$

is the morphism of mixed complexes induced by the canonical surjection $\pi \colon C \to A$.

From now on $M^2 = 0$. Then M is an A-bimodule and there is a normalized Hochschild cocycle

$$f\colon A \otimes A \to M$$

so that $C \simeq A \ltimes_f M$, where

$$A \ltimes_f M = A \oplus M$$

endowed with the multiplication

$$(a, m)(a', m') = (aa', am' + ma' + f(a \otimes a')).$$

Definition

The algebra $E = A \ltimes_f M$ is called the square zero extension of A by M associated with f.

In this case the Hochschild complex of $\ker \pi_*$ is the total complex of the double complex

$$
(\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}) \quad = \quad
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\Big\downarrow \hat{\mathfrak{b}} & & \Big\downarrow \hat{\mathfrak{b}} & & \Big\downarrow \hat{\mathfrak{b}} \\
\hat{\mathfrak{X}}^2_4 & \xleftarrow{\ \hat{\mathfrak{d}}\ } & \hat{\mathfrak{X}}^1_4 & \xleftarrow{\ \hat{\mathfrak{d}}\ } & \hat{\mathfrak{X}}^0_4 \\
 & & \Big\downarrow \hat{\mathfrak{b}} & & \Big\downarrow \hat{\mathfrak{b}} \\
 & & \hat{\mathfrak{X}}^1_3 & \xleftarrow{\ \hat{\mathfrak{d}}\ } & \hat{\mathfrak{X}}^0_3 \\
 & & \Big\downarrow \hat{\mathfrak{b}} & & \Big\downarrow \hat{\mathfrak{b}} \\
 & & \hat{\mathfrak{X}}^1_2 & \xleftarrow{\ \hat{\mathfrak{d}}\ } & \hat{\mathfrak{X}}^0_2 \\
 & & & & \Big\downarrow \hat{\mathfrak{b}} \\
 & & & & \hat{\mathfrak{X}}^0_1 \\
 & & & & \Big\downarrow \hat{\mathfrak{b}} \\
 & & & & \hat{\mathfrak{X}}^0_0
\end{array}
$$

in which

- $\hat{\mathfrak{X}}_v^w$ is the subspace of

$$E \otimes \overline{E}^{\otimes n} \quad (n = v - w)$$

  spanned by the simple tensors $x_0 \otimes \cdots \otimes x_n$ with $w + 1$ factors in M and the other ones in $A \cup \overline{A}$,
- the maps $\hat{\mathfrak{b}}$ are given by the same formula as the Hochschild boundary map of an algebra,
- the horizontal boundary maps $\hat{\mathfrak{d}}$ are defined by

$$\hat{\mathfrak{d}}(x_0 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-1} (-1)^i x_0 \otimes \cdots \otimes f(x_i, x_{i+1}) \otimes \cdots \otimes x_n.$$

The column $(\hat{\mathfrak{X}}^w, \hat{\mathfrak{b}})$ of the previous complex is the total complex of the double complex

$$
(\mathfrak{X}^w, \mathfrak{b}^w, \alpha^w) \quad = \quad
\begin{array}{ccc}
\vdots & & \vdots \\
\Big\downarrow{\scriptstyle \mathfrak{b}_0} & & \Big\downarrow{\scriptstyle \mathfrak{b}_1} \\
\mathfrak{X}^w_{0,2w+2} & \xleftarrow{\ \alpha\ } & \mathfrak{X}^w_{1,2w+2} \\
\Big\downarrow{\scriptstyle \mathfrak{b}_0} & & \Big\downarrow{\scriptstyle \mathfrak{b}_1} \\
\mathfrak{X}^w_{0,2w+1} & \xleftarrow{\ \alpha\ } & \mathfrak{X}^w_{1,2w+1} \\
\Big\downarrow{\scriptstyle \mathfrak{b}_0} & & \Big\downarrow{\scriptstyle \mathfrak{b}_0} \\
\mathfrak{X}^w_{0,2w} & \xleftarrow{\ \alpha\ } & \mathfrak{X}^w_{1,2w}
\end{array}
$$

where

- the vector spaces

$$\mathfrak{X}_{0v}^{w} \quad \text{and} \quad \mathfrak{X}_{1,v-1}^{w}$$

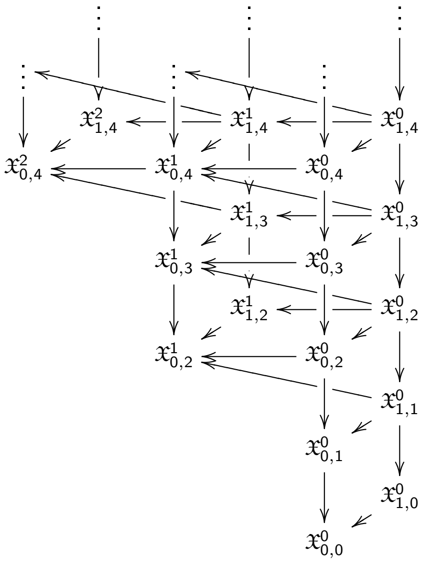
  are the subspaces of $\hat{\mathfrak{X}}_{v}^{w}$ spanned by the simple tensors

$$x_0 \otimes \cdots \otimes x_n \quad (n = v - w)$$

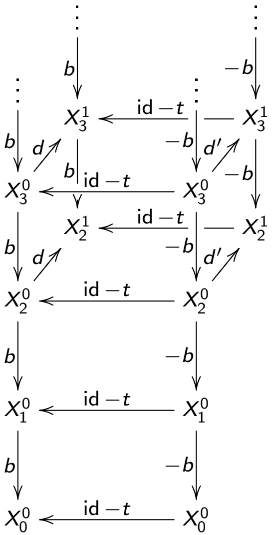
  so that $x_0 \in M$ and $x_0 \in A$, respectively,
- the boundary maps $\mathfrak{b}_0$ and $\mathfrak{b}_1$ are similar to the Hochschild boundary maps,
- the maps α are defined by

$$\alpha_v^w(\mathbf{x}_0^{n+1}) = x_0\pi_M(x_1)\otimes\mathbf{x}_2^{n+1}+(-1)^{n+1}\pi_M(x_{n+1})x_0\otimes\mathbf{x}_1^{n}.$$

$$\mathrm{Tot}(\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}) \quad =$$



The following complex (in which $X_v^w = \mathfrak{X}_{0v}^w$) also gives $\mathrm{HH}(E, M)$



Let $\mathbf{x}_0^n = x_0 \otimes \cdots \otimes x_n$. The boundary maps are given by

$$b(\mathbf{x}_0^n) = \sum_{j=0}^{n-1} (-1)^j \mathbf{x}_0^{j-1} \otimes x_j x_{j+1} \otimes \mathbf{x}_{j+1}^n + (-1)^n x_n x_0 \otimes \mathbf{x}_1^{n-1},$$

$$d(\mathbf{x}_0^n) = \sum_{j=1}^{n-1} (-1)^j \mathbf{x}_0^{j-1} \otimes f(x_j \otimes x_{j+1}) \otimes \mathbf{x}_{j+2}^n,$$

$$d'(\mathbf{x}_0^n) = -d(\mathbf{x}_0^n) - \sum_{j=i+1}^{n-1} (-1)^{jn} f(x_j \otimes x_{j+1}) \otimes \mathbf{x}_{j+2}^n \otimes \mathbf{x}_0^{j-1},$$

$$t(\mathbf{x}_0^n) = (-1)^{in} \mathbf{x}_i^n \otimes \mathbf{x}_0^{i-1},$$

where i denotes the last index so that $x_i \in M$.

Let $\hat{X}_v^w = X_v^w \oplus X_{v-1}^w$. The double complex

$$
(\hat{X}, \hat{b}, \hat{d}) =
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle \hat{b}} & & \downarrow{\scriptstyle \hat{b}} & & \downarrow{\scriptstyle \hat{b}} \\
\hat{X}_4^2 & \xleftarrow{\hat{d}} & \hat{X}_4^1 & \xleftarrow{\hat{d}} & \hat{X}_4^0 \\
 & & \downarrow{\scriptstyle \hat{b}} & & \downarrow{\scriptstyle \hat{b}} \\
 & & \hat{X}_3^1 & \xleftarrow{\hat{d}} & \hat{X}_3^0 \\
 & & \downarrow{\scriptstyle \hat{b}} & & \downarrow{\scriptstyle \hat{b}} \\
 & & \hat{X}_2^1 & \xleftarrow{\hat{d}} & \hat{X}_2^0 \\
 & & & & \downarrow{\scriptstyle \hat{b}} \\
 & & & & \hat{X}_1^0 \\
 & & & & \downarrow{\scriptstyle \hat{b}} \\
 & & & & \hat{X}_0^0
\end{array}
$$

where

$$\hat{b}_v^w = \begin{pmatrix} b & \operatorname{id} - t \\ 0 & -b \end{pmatrix} \qquad \text{and} \qquad \hat{d}_v^w = \begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix},$$

obtained taking one of the partial total complexes of the two faces triple complex considered before, is a double mixed complex via the Connes operator

$$\hat{B}_v^w \colon \hat{X}_v^w \to \hat{X}_{v+1}^w,$$

defined by $\hat{B}(\mathbf{x}, \mathbf{y}) = (0, N(\mathbf{x}))$, where

$$N(\mathbf{x}) = \operatorname{id} + t + \cdots + t^w.$$

Theorem

The cyclic homology of $(\hat{X}, \hat{b}, \hat{d})$ is the cyclic homology of E relative to M.

Let

$$
(\overline{X}, \overline{b}, \overline{d}) = \begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle \overline{b}} & & \downarrow{\scriptstyle \overline{b}} & & \downarrow{\scriptstyle \overline{b}} \\
\overline{X}_4^2 & \xleftarrow{\overline{d}} & \overline{X}_4^1 & \xleftarrow{\overline{d}} & \overline{X}_4^0 \\
 & & \downarrow{\scriptstyle \overline{b}} & & \downarrow{\scriptstyle \overline{b}} \\
 & & \overline{X}_3^1 & \xleftarrow{\overline{d}} & \overline{X}_3^0 \\
 & & \downarrow{\scriptstyle \overline{b}} & & \downarrow{\scriptstyle \overline{b}} \\
 & & \overline{X}_2^1 & \xleftarrow{\overline{d}} & \overline{X}_2^0 \\
 & & & & \downarrow{\scriptstyle \overline{b}} \\
 & & & & \overline{X}_1^0 \\
 & & & & \downarrow{\scriptstyle \overline{b}} \\
 & & & & \overline{X}_0^0
\end{array}
$$

be the cokernel of the map

$$\mathrm{id} - t\colon (X, -b, d') \to (X, b, d).$$

Theorem

The cyclic homology of E relative to M is the homology of $(\overline{X}, \overline{b}, \overline{d})$.

Aplications

We get a new proof of the following celebrated theorem.

Theorem (Goodwillie)

If M is a nilpotent two sided ideal of an algebra C, then

$$\mathrm{HP}(C/M) = \mathrm{HP}(C).$$

Goodwillie also proved that if $M^{m+1} = 0$, then

$$S^{m(n+1)}\colon \mathrm{HC}_{n+2m(n+1)}(C,M) \to \mathrm{HC}_n(C,M)$$

is the zero map. We obtained the following improvement.

Theorem

$$S^{m([n/2]+1)}\colon \mathrm{HC}_{n+2m([n/2]+1)}(C,M) \to \mathrm{HC}_n(C,M)$$

vanish, if $M^{2^m} = 0$.

Let A be an arbitrary algebra,

$$\Omega^2 A = \Omega^2 = A \otimes \overline{A}^{\otimes^2} \qquad \text{and} \qquad E = A \ltimes_f \Omega^2 A,$$

where $f\colon A \otimes A \to \Omega^2$ is the cocycle $f(a \otimes b) = dadb$.

Let

$$\overline{\mathrm{HD}}_n(A) = \operatorname{Im}\big(S\colon \overline{\mathrm{HC}}_{n+2}(A) \to \overline{\mathrm{HC}}_n(A)\big)$$

be the reduced de Rham homology of A, and let

$$L_n = \frac{\overline{A}^{\otimes^{n+1}}}{\operatorname{Im}(\mathrm{id} - \lambda) + b^{-1}((\mathrm{id} - \lambda)(\overline{A}^{\otimes^n}))} \oplus \frac{\overline{A}^{\otimes^{n+2}}}{\operatorname{Im}(\mathrm{id} - \lambda) + \operatorname{Im}(b)},$$

where $\lambda(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$.

Theorem

We have:

$$\mathrm{HC}_0(E;\Omega^2) = \mathrm{HH}_2(A) \oplus \overline{\mathrm{HD}}_0(A) \oplus L_1 \oplus P_{-1}(\Omega^2),$$

$$\mathrm{HC}_{4n}(E;\Omega^2) = \left(\bigoplus_{j=1}^{2n} \mathrm{HH}_{4n+2j}(A)\right) \oplus \overline{\mathrm{HD}}_{8n-1}(A) \oplus L_{8n+1} \oplus P_{-1}(\Omega^{4n+2}),$$

$$\mathrm{HC}_{4n+2}(E;\Omega^2) = \left(\bigoplus_{j=1}^{2n+1} \mathrm{HH}_{4n+2+2j}(A)\right) \oplus \overline{\mathrm{HD}}_{8n+3}(A) \oplus L_{8n+5} \oplus P_{\pm i}(\Omega^{4n+4}),$$

$$\mathrm{HC}_1(E;\Omega^2) = \mathrm{HH}_3(A),$$

$$\mathrm{HC}_{2n+1}(E;\Omega^2) = \left(\bigoplus_{\substack{j=1 \\ j\neq n}}^{n+1} \mathrm{HH}_{2n+2j+1}(A)\right) \oplus \overline{\mathrm{HD}}_{4n+1}(A) \oplus \frac{\overline{\mathrm{HC}}_{4n}(A)}{\overline{\mathrm{HD}}_{4n}(A)},$$

where $P_{\pm i}(\Omega^{4n+4}) = P_i(\Omega^{4n+4}) \oplus P_{-i}(\Omega^{4n+4})$.