Triangulated categories of rings

Grigory Garkusha

Manchester & St. Petersburg

1 September 2006
Motivation:
Motivation:

- new homology theories on rings. Well-known homology theories are \( KV_* \) of Karoubi-Villamayor and \( KH_* \) of Weibel
Motivation:

- new homology theories on rings. Well-known homology theories are $KV_\ast$ of Karoubi-Villamayor and $KH_\ast$ of Weibel
- if we look at rings (without unit) as spaces is there a homotopy theory for them?
Motivation:

- new homology theories on rings. Well-known homology theories are $KV_*$ of Karoubi-Villamayor and $KH_*$ of Weibel
- if we look at rings (without unit) as spaces is there a homotopy theory for them?
- another definition for the triangulated category $kk$ constructed very recently by Cortiñas and Thom.
Motivation:

- new homology theories on rings. Well-known homology theories are $KV_*$ of Karoubi-Villamayor and $KH_*$ of Weibel
- if we look at rings (without unit) as spaces is there a homotopy theory for them?
- another definition for the triangulated category $kk$ constructed very recently by Cortiñas and Thom. The latter category is an algebraic analog for triangulated structures on operator algebras used in the Kasparov $KK$-theory.
We shall work in the category $\textit{Rings}$ of associative rings (without unit) and ring homomorphisms.
Definitions

We shall work in the category $Rings$ of associative rings (without unit) and ring homomorphisms. Following Gersten a category of rings $\mathcal{R}$ is \textit{admissible} if it is a full subcategory of $Rings$ and
We shall work in the category *Rings* of associative rings (without unit) and ring homomorphisms.
Following Gersten a category of rings $\mathcal{R}$ is *admissible* if it is a full subcategory of *Rings* and

- $R$ in $\mathcal{R}$, $I$ a (two-sided) ideal of $R$ then $I$ and $R/I$ are in $\mathcal{R}$;
We shall work in the category $\text{Rings}$ of associative rings (without unit) and ring homomorphisms. Following Gersten a category of rings $\mathcal{R}$ is admissible if it is a full subcategory of $\text{Rings}$ and

- $R$ in $\mathcal{R}$, $I$ a (two-sided) ideal of $R$ then $I$ and $R/I$ are in $\mathcal{R}$;
- if $R$ is in $\mathcal{R}$, then so is $R[x]$, the polynomial ring in one variable;
We shall work in the category $\textit{Rings}$ of associative rings (without unit) and ring homomorphisms. Following Gersten a category of rings $\mathcal{R}$ is admissible if it is a full subcategory of $\textit{Rings}$ and

- $R$ in $\mathcal{R}$, $I$ a (two-sided) ideal of $R$ then $I$ and $R/I$ are in $\mathcal{R}$;
- if $R$ is in $\mathcal{R}$, then so is $R[x]$, the polynomial ring in one variable;
- given a cartesian square

$$
\begin{array}{ccc}
D & \xrightarrow{\rho} & A \\
\downarrow{\sigma} & & \downarrow{f} \\
B & \xrightarrow{g} & C
\end{array}
$$

in $\textit{Rings}$ with $A, B, C$ in $\mathcal{R}$, then $D$ is in $\mathcal{R}$.
We shall work in the category $\text{Rings}$ of associative rings (without unit) and ring homomorphisms. Following Gersten a category of rings $\mathcal{R}$ is \textit{admissible} if it is a full subcategory of $\text{Rings}$ and

- $R$ in $\mathcal{R}$, $I$ a (two-sided) ideal of $R$ then $I$ and $R/I$ are in $\mathcal{R}$;
- if $R$ is in $\mathcal{R}$, then so is $R[x]$, the polynomial ring in one variable;
- given a cartesian square

\[
\begin{array}{ccc}
D & \xrightarrow{\rho} & A \\
\sigma \downarrow & & \downarrow f \\
B & \xrightarrow{g} & C
\end{array}
\]

in $\text{Rings}$ with $A, B, C$ in $\mathcal{R}$, then $D$ is in $\mathcal{R}$.

We shall work in a fixed small admissible category $\mathcal{R}$. 
If $R$ is a ring then the polynomial ring $R[x]$ admits two homomorphisms onto $R$

\[ R[x] \xrightarrow{\partial_x^0} R \xleftarrow{\partial_x^1} \]

where $\partial_x^i|_R = 1_R$, $\partial_x^i(x) = i, i = 0, 1$. Of course, $\partial_x^1(x) = 1$ has to be understood in the sense that $\sum r_n x^n \mapsto \sum r_n$. 
If $R$ is a ring then the polynomial ring $R[x]$ admits two homomorphisms onto $R$

\[ R[x] \xrightarrow{\partial_x^0} R \xleftarrow{\partial_x^1} R \]

where $\partial_x^0|_R = 1_R$, $\partial_x^i(x) = i$, $i = 0, 1$. Of course, $\partial_x^1(x) = 1$ has to be understood in the sense that $\Sigma r_n x^n \mapsto \Sigma r_n$.

Two ring homomorphisms $f_0, f_1 : S \to R$ are \textit{elementary homotopic}, written $f_0 \sim f_1$, if there exists a ring homomorphism

\[ f : S \to R[x] \]

such that $\partial_x^0 f = f_0$ and $\partial_x^1 f = f_1$. 
If $R$ is a ring then the polynomial ring $R[x]$ admits two homomorphisms onto $R$

$$R[x] \xrightarrow{\partial^0_x} R \xrightarrow{\partial^1_x} R$$

where $\partial^i_x|_R = 1_R$, $\partial^i_x(x) = i$, $i = 0, 1$. Of course, $\partial^1_x(x) = 1$ has to be understood in the sense that $\Sigma r_n x^n \mapsto \Sigma r_n$.

Two ring homomorphisms $f_0, f_1 : S \to R$ are elementary homotopic, written $f_0 \sim f_1$, if there exists a ring homomorphism

$$f : S \to R[x]$$

such that $\partial^0_x f = f_0$ and $\partial^1_x f = f_1$. A map $f : S \to R$ is called an elementary homotopy equivalence if there is a map $g : R \to S$ such that $fg$ and $gf$ are elementary homotopic to $\text{id}_R$ and $\text{id}_S$ respectively.
The relation “elementary homotopic” is reflexive and symmetric. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol “≃”).
The relation “elementary homotopic” is reflexive and symmetric. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol “$\simeq$”).

The set of equivalence classes of morphisms $R \to S$ is written $[R, S]$. 

Homotopy behaves well with respect to composition and we have category $H(\mathcal{R})$, the homotopy category of $\mathcal{R}$, whose objects are rings from $\mathcal{R}$ and such that $H(\mathcal{R})(R, S) = [R, S]$. 

Grigory Garkusha (Manchester & St. Petersburg)
The relation “elementary homotopic” is reflexive and symmetric. One may take the transitive closure of this relation to get an equivalence relation (denoted by the symbol “≈”).

The set of equivalence classes of morphisms $R \to S$ is written $[R, S]$. Homotopy behaves well with respect to composition and we have category $\mathcal{H}(\mathcal{R})$, the homotopy category of $\mathcal{R}$, whose objects are rings from $\mathcal{R}$ and such that $\mathcal{H}(\mathcal{R})(R, S) = [R, S]$. 
A ring $R$ is **contractible** if $0 \sim 1$; that is, if there is a ring homomorphism $f : R \to R[x]$ such that $\partial^0_x f = 0$ and $\partial^1_x f = 1_R$.
A ring $R$ is **contractible** if $0 \sim 1$; that is, if there is a ring homomorphism $f : R \to R[x]$ such that $\partial_x^0 f = 0$ and $\partial_x^1 f = 1_R$.

Following Karoubi and Villamayor we define $ER$, the **path ring** on $R$, as the kernel of $\partial_x^0 : R[x] \to R$. Also $\partial_x^1 : R[x] \to R$ induces a surjection

$$\partial_x^1 : ER \to R$$

and we define the **loop ring** $\Omega R$ of $R$ to be its kernel, so we have a short exact sequence in $\text{Rings}$

$$\Omega R \longrightarrow ER \overset{\partial_x^1}{\longrightarrow} R.$$ 

$ER$ is contractible for any ring $R$. 
A functor $\mathcal{X}$ from rings to simplicial sets is **homotopy invariant** if for every ring $R$ the natural map $R \to R[t]$ induces a weak equivalence of simplicial sets $\mathcal{X}(R) \simeq \mathcal{X}(R[t])$. 
A functor $\mathcal{X}$ from rings to simplicial sets is **homotopy invariant** if for every ring $R$ the natural map $R \to R[t]$ induces a weak equivalence of simplicial sets $\mathcal{X}(R) \simeq \mathcal{X}(R[t])$.

Let $\mathcal{R}$ be an admissible category of rings and let $\mathcal{F}$ be a family of fibrations. A simplicial functor $\mathcal{X} \in U\mathcal{R} := (\mathcal{R}, \text{SSets})$ is called **excisive** with respect to $\mathcal{F}$ if for any distinguished square in $\mathcal{R}$

\[
\begin{array}{ccc}
D & \rightarrow & A \\
\downarrow & & \downarrow \\
B & \rightarrow & C
\end{array}
\]

the square of simplicial sets

\[
\begin{array}{ccc}
\mathcal{X}(D) & \rightarrow & \mathcal{X}(A) \\
\downarrow & & \downarrow \\
\mathcal{X}(B) & \rightarrow & \mathcal{X}(C)
\end{array}
\]

is a homotopy pullback square.
A ring homomorphism $A \to B$ in $\mathcal{R}$ is a **quasi-isomorphism** if for every excisive and homotopy invariant simplicial functor $\mathcal{X} \in \text{U}\mathcal{R}$ the map $\mathcal{X}(A) \to \mathcal{X}(B)$ is a weak equivalence of simplicial sets.
A ring homomorphism $A \to B$ in $\mathcal{R}$ is a quasi-isomorphism if for every excisive and homotopy invariant simplicial functor $\mathcal{X} \in U\mathcal{R}$ the map $\mathcal{X}(A) \to \mathcal{X}(B)$ is a weak equivalence of simplicial sets.

Let $\mathcal{R}$ be an admissible category of rings and let $\mathcal{F}$ be a family of fibrations. The left derived category $D^-(\mathcal{R}, \mathcal{F})$ of $\mathcal{R}$ with respect to $\mathcal{F}$ is the category obtained from $\mathcal{R}$ by inverting quasi-isomorphisms.
Given a fibration $g : A \to B$ with fibre $F$, consider the commutative diagram as follows:

\[
\begin{array}{c}
\Omega B \ar[r] \ar[d]_j & \Omega B \ar[d] \\\nF \ar[r]^i \ar[d]_\iota & P(g) \ar[r] \ar[d]_{g_1} & EB \ar[d]_{\partial^1_x} \\
F \ar[r]_\iota & A \ar[r]_g & B
\end{array}
\]

Since $E_B$ is contractible, it follows that $i$ is a quasi-isomorphism.
Given a fibration $g : A \to B$ with fibre $F$, consider the commutative diagram as follows:

Since $EB$ is contractible, it follows that $i$ is a quasi-isomorphism.
We deduce the sequence in $D^-(\mathcal{R}, \mathcal{F})$

$$\Omega B \xrightarrow{i^{-1}_0 j} F \xrightarrow{\iota} A \xrightarrow{g} B.$$
We deduce the sequence in $D^{-}(R, F)$

$$
\Omega B \xrightarrow{i^{-1} \circ j} F \xrightarrow{i} A \xrightarrow{g} B.
$$

We shall refer to such sequences as **standard left triangles**.
We deduce the sequence in $D^{-}(R, F)$

$$\Omega B \xrightarrow{i^{-1} \circ j} F \xrightarrow{\iota} A \xrightarrow{g} B.$$ 

We shall refer to such sequences as standard left triangles.

Any diagram in $D^{-}(R, F)$ which is isomorphic to the latter sequence will be called a left triangle.
We deduce the sequence in $D^{-}(\mathcal{R}, \mathcal{F})$

$$\Omega B \xrightarrow{i^{-1} \circ j} F \xrightarrow{\iota} A \xrightarrow{g} B.$$ 

We shall refer to such sequences as **standard left triangles**.

Any diagram in $D^{-}(\mathcal{R}, \mathcal{F})$ which is isomorphic to the latter sequence will be called a **left triangle**.

Let $\mathcal{L}tr(\mathcal{R}, \mathcal{F})$ denote the category of left triangles having the usual set of morphisms from $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ to $\Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$. 
Theorem (GG 2005)

$Ltr(\mathcal{R}, \mathcal{F})$ is a left triangulation of $D^-(\mathcal{R}, \mathcal{F})$, i.e. it is closed under isomorphisms and enjoys the following four axioms:
Theorem (GG 2005)

$L_{tr}(\mathcal{R}, \mathcal{F})$ is a left triangulation of $D^{-}(\mathcal{R}, \mathcal{F})$, i.e. it is closed under isomorphisms and enjoys the following four axioms:

- for any ring $A \in \mathcal{R}$ the sequence $0 \to A \to A \to 0 \in L_{tr}(\mathcal{R}, \mathcal{F})$
  and for any map $h : B \to C$ there is a left triangle
  \[ \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C ; \]
Theorem (GG 2005)

$Ltr(\mathcal{R}, \mathcal{F})$ is a left triangulation of $D^-(\mathcal{R}, \mathcal{F})$, i.e. it is closed under isomorphisms and enjoys the following four axioms:

- for any ring $A \in \mathcal{R}$ the sequence $0 \to A \to A \to 0 \in Ltr(\mathcal{R}, \mathcal{F})$ and for any map $h : B \to C$ there is a left triangle
  \[ \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C ; \]
- for any left triangle $\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \in Ltr(\mathcal{R}, \mathcal{F})$, the diagram $\Omega B \xrightarrow{-\Omega h} \Omega C \xrightarrow{f} A \xrightarrow{g} B$ is also in $Ltr(\mathcal{R}, \mathcal{F})$;
Theorem (GG 2005)

\( Ltr(\mathcal{R}, \mathcal{F}) \) is a left triangulation of \( D^{-}(\mathcal{R}, \mathcal{F}) \), i.e. it is closed under isomorphisms and enjoys the following four axioms:

- for any ring \( A \in \mathcal{R} \) the sequence \( 0 \rightarrow A \rightarrow A \rightarrow 0 \in Ltr(\mathcal{R}, \mathcal{F}) \) and for any map \( h : B \rightarrow C \) there is a left triangle
  \[
  \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C ;
  \]

- for any left triangle \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \in Ltr(\mathcal{R}, \mathcal{F}) \), the diagram \( \Omega B \xrightarrow{-\Omega h} \Omega C \xrightarrow{f} A \xrightarrow{g} B \) is also in \( Ltr(\mathcal{R}, \mathcal{F}) \);

- for any two left triangles \( \Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \), \( \Omega C' \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C' \) and any two maps \( \beta : B \rightarrow B' \), \( \gamma : C \rightarrow C' \) of \( D^{-}(\mathcal{R}, \mathcal{F}) \) with \( \gamma h = h' \beta \), there is a map \( \alpha : A \rightarrow A' \) of \( D^{-}(\mathcal{R}, \mathcal{F}) \) such that the triple \( (\alpha, \beta, \gamma) \) gives a map from the first triangle to the second;
(Octahedron) any two morphisms $B \xrightarrow{h} C \xrightarrow{k} D$ of $D^-(\mathcal{A}, \mathcal{F})$ can be fitted into a commutative diagram

\[
\begin{array}{ccccccc}
\Omega E & \xrightarrow{f} & \Omega C & \xrightarrow{f} & \Omega D & \xrightarrow{i} & \Omega D \\
& & \downarrow \Omega k & & \downarrow 1_{\Omega D} & & \\
& \Omega A & \xrightarrow{g} & \Omega B & \xrightarrow{1_B} & \Omega F & \xrightarrow{m} \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \Omega D \\
& \Omega B & \xrightarrow{h} & \Omega C & \xrightarrow{k} & \Omega D & \xrightarrow{1_D} \\
\end{array}
\]

in which the rows and the second column from the left are left triangles in $\mathcal{Ltr}(\mathcal{A}, \mathcal{F})$. 
There is a general method of stabilizing the loop functor $\Omega$ and producing a triangulated category $D(\mathcal{R}, \mathcal{F})$ from the left triangulated structure on $D^-(\mathcal{R}, \mathcal{F})$. 
There is a general method of stabilizing the loop functor $\Omega$ and producing a triangulated category $D(\mathcal{R}, \mathcal{F})$ from the left triangulated structure on $D^-(\mathcal{R}, \mathcal{F})$.

We use stabilization to define a $\mathbb{Z}$-graded bivariant homology theory $k_*(A, B)$ on $\mathcal{R}$, i.e. it is contravariant in the first variable and covariant in the second and produces long exact sequences in each variable out of $\mathcal{F}$-fibre sequences.
The construction of the derived category $D(\mathcal{R}, \mathcal{F})$ consists of formally inverting the endofunctor $\Omega$. 
The construction of the derived category $D(\mathcal{R}, \mathcal{F})$ consists of formally inverting the endofunctor $\Omega$.

An object of $D(\mathcal{R}, \mathcal{F})$ is a pair $(A, m)$ with $A \in D^-(\mathcal{R}, \mathcal{F})$ and $m \in \mathbb{Z}$.
The construction of the derived category $D(\mathcal{R}, \mathcal{F})$ consists of formally inverting the endofunctor $\Omega$.

An object of $D(\mathcal{R}, \mathcal{F})$ is a pair $(A, m)$ with $A \in D^{-}(\mathcal{R}, \mathcal{F})$ and $m \in \mathbb{Z}$. If $m, n \in \mathbb{Z}$ then we consider the directed set $I_{m,n} = \{k \in \mathbb{Z} \mid m, n \leq k\}$. 

Morphisms of $D(\mathcal{R}, \mathcal{F})$ are composed in the obvious fashion. We define the loop automorphism on $D(\mathcal{R}, \mathcal{F})$ by $\Omega(A, m) := (A, m - 1)$. 

There is a natural functor $S : D^{-}(\mathcal{R}, \mathcal{F}) \to D(\mathcal{R}, \mathcal{F})$ defined by $A \mapsto (-\to, A, 0)$. 

Grigory Garkusha (Manchester & St. Petersburg) 
Triangulated categories of rings 
1 September 2006
The construction of the derived category $D(\mathcal{A}, \mathcal{F})$ consists of formally inverting the endofunctor $\Omega$.

An object of $D(\mathcal{A}, \mathcal{F})$ is a pair $(A, m)$ with $A \in D^{-}(\mathcal{A}, \mathcal{F})$ and $m \in \mathbb{Z}$. If $m, n \in \mathbb{Z}$ then we consider the directed set

$$I_{m, n} = \{ k \in \mathbb{Z} \mid m, n \leq k \}.$$ 

The set of morphisms between $(A, m)$ and $(B, n) \in D(\mathcal{A}, \mathcal{F})$ is defined by

$$D(\mathcal{A}, \mathcal{F})[(A, m), (B, n)] := \lim_{k \in I_{m, n}} D^{-}(\mathcal{A}, \mathcal{F})(\Omega^{k-m}(A), \Omega^{k-n}(B)).$$
The construction of the derived category $D(\mathcal{R}, \mathcal{F})$ consists of formally inverting the endofunctor $\Omega$.

An object of $D(\mathcal{R}, \mathcal{F})$ is a pair $(A, m)$ with $A \in D^-(\mathcal{R}, \mathcal{F})$ and $m \in \mathbb{Z}$. If $m, n \in \mathbb{Z}$ then we consider the directed set

$$l_{m,n} = \{ k \in \mathbb{Z} | m, n \leq k \}.$$  

The set of morphisms between $(A, m)$ and $(B, n) \in D(\mathcal{R}, \mathcal{F})$ is defined by

$$D(\mathcal{R}, \mathcal{F})[(A, m), (B, n)] := \lim_{k \in l_{m,n}} D^-(\mathcal{R}, \mathcal{F})(\Omega^{k-m}(A), \Omega^{k-n}(B)).$$

Morphisms of $D(\mathcal{R}, \mathcal{F})$ are composed in the obvious fashion. We define the loop automorphism on $D(\mathcal{R}, \mathcal{F})$ by $\Omega(A, m) := (A, m - 1)$. There is a natural functor $S : D^-(\mathcal{R}, \mathcal{F}) \to D(\mathcal{R}, \mathcal{F})$ defined by

$$A \mapsto (A, 0).$$
We define a triangulation $\mathcal{T}r(\mathcal{R}, \mathcal{F})$ of the pair $(D(\mathcal{R}, \mathcal{F}), \Omega)$. 

**Theorem (GG 2006)**

$D(\mathcal{R}, \mathcal{F})$ is additive and $\mathcal{T}r(\mathcal{R}, \mathcal{F})$ is a triangulation of $D(\mathcal{R}, \mathcal{F})$ in the classical sense of Verdier.
We define a triangulation $\mathcal{T}r(\mathcal{R}, \mathcal{F})$ of the pair $(D(\mathcal{R}, \mathcal{F}), \Omega)$.

A sequence
\[
\Omega(A, l) \to (C, n) \to (B, m) \to (A, l)
\]
belongs to $\mathcal{T}r(\mathcal{R}, \mathcal{F})$ if there is an even integer $k$ and a left triangle of representatives $\Omega(\Omega^{k-l}(A)) \to \Omega^{k-n}(C) \to \Omega^{k-m}(B) \to \Omega^{k-l}(A)$ in $D^-(\mathcal{R}, \mathcal{F})$. 
We define a triangulation $\mathcal{T}_r(\mathcal{R}, \mathcal{F})$ of the pair $(D(\mathcal{R}, \mathcal{F}), \Omega)$.

A sequence

$$\Omega(A, l) \rightarrow (C, n) \rightarrow (B, m) \rightarrow (A, l)$$

belongs to $\mathcal{T}_r(\mathcal{R}, \mathcal{F})$ if there is an even integer $k$ and a left triangle of representatives $\Omega(\Omega^{k-l}(A)) \rightarrow \Omega^{k-n}(C) \rightarrow \Omega^{k-m}(B) \rightarrow \Omega^{k-l}(A)$ in $D^-(\mathcal{R}, \mathcal{F})$.

The functor $S$ takes left triangles in $D^-(\mathcal{R}, \mathcal{F})$ to triangles in $D(\mathcal{R}, \mathcal{F})$. 

Theorem (GG 2006)

$D(\mathcal{R}, \mathcal{F})$ is additive and $\mathcal{T}_r(\mathcal{R}, \mathcal{F})$ is a triangulation of $D(\mathcal{R}, \mathcal{F})$ in the classical sense of Verdier.
We define a triangulation $Tr(\mathcal{R}, \mathcal{F})$ of the pair $(D(\mathcal{R}, \mathcal{F}), \Omega)$.

A sequence
$$\Omega(A, l) \rightarrow (C, n) \rightarrow (B, m) \rightarrow (A, l)$$

belongs to $Tr(\mathcal{R}, \mathcal{F})$ if there is an even integer $k$ and a left triangle of representatives $\Omega(\Omega^{k-l}(A)) \rightarrow \Omega^{k-n}(C) \rightarrow \Omega^{k-m}(B) \rightarrow \Omega^{k-l}(A)$ in $D^-((\mathcal{R}, \mathcal{F}))$.

The functor $S$ takes left triangles in $D^-((\mathcal{R}, \mathcal{F}))$ to triangles in $D((\mathcal{R}, \mathcal{F}))$.

**Theorem (GG 2006)**

$D((\mathcal{R}, \mathcal{F}))$ is additive and $Tr((\mathcal{R}, \mathcal{F}))$ is a triangulation of $D((\mathcal{R}, \mathcal{F}))$ in the classical sense of Verdier.
We use the triangulated category $D(\mathcal{R}, \mathcal{F})$ to define a $\mathbb{Z}$-graded bivariant homology theory on $\mathcal{R}$ as follows:

$$k_n(A, B) := D(\mathcal{R}, \mathcal{F})((A, 0), (B, n)), \quad n \in \mathbb{Z}.$$
We use the triangulated category $D(\mathcal{R}, \mathcal{F})$ to define a $\mathbb{Z}$-graded bivariant homology theory on $\mathcal{R}$ as follows:

$$k_n(A, B) := D(\mathcal{R}, \mathcal{F})((A, 0), (B, n)), \quad n \in \mathbb{Z}.$$  

**Corollary**

For any $\mathcal{F}$-fibre sequence $A \to B \to C$ and any $D \in \mathcal{R}$, we have long exact sequences of abelian groups

$$\cdots \to k_{n+1}(D, C) \to k_n(D, A) \to k_n(D, B) \to k_n(D, C) \to \cdots$$

and

$$\cdots \to k_{n+1}(A, D) \to k_n(C, D) \to k_n(B, D) \to k_n(A, D) \to \cdots$$
Motivated by ideas and work of J. Cuntz on bivariant $K$-theory of locally convex algebras, Cortiñas and Thom construct a bivariant homology theory $kk_*(A, B)$ on the category $Alg_H$ of algebras over a unital ground ring $H$. 
Motivated by ideas and work of J. Cuntz on bivariant $K$-theory of locally convex algebras, Cortiñas and Thom construct a bivariant homology theory $kk_{*}(A, B)$ on the category $Alg_H$ of algebras over a unital ground ring $H$. It is Morita invariant, homotopy invariant, excisive $K$-theory of algebras, which is universal in the sense that it maps uniquely to any other such theory.
Motivated by ideas and work of J. Cuntz on bivariant $K$-theory of locally convex algebras, Cortiñas and Thom construct a bivariant homology theory $kk_*(A, B)$ on the category $Alg_H$ of algebras over a unital ground ring $H$. It is Morita invariant, homotopy invariant, excisive $K$-theory of algebras, which is universal in the sense that it maps uniquely to any other such theory. This bivariant $K$-theory is defined in a triangulated category $kk$ whose objects are the $H$-algebras without unit and $kk_*(A, B) = kk(A, \Omega^*B)$. 
Motivated by ideas and work of J. Cuntz on bivariant $K$-theory of locally convex algebras, Cortiñas and Thom construct a bivariant homology theory $kk_*(A, B)$ on the category $\text{Alg}_H$ of algebras over a unital ground ring $H$. It is Morita invariant, homotopy invariant, excisive $K$-theory of algebras, which is universal in the sense that it maps uniquely to any other such theory. This bivariant $K$-theory is defined in a triangulated category $kk$ whose objects are the $H$-algebras without unit and $kk_*(A, B) = kk(A, \Omega^*B)$.

We make use of our machinery to study various triangulated structures on admissible categories of rings which are not necessarily small. As an application, we give another, but equivalent, description of the triangulated category $kk$. 
Theorem (GG 2006)

Let $\mathcal{R}$ be an arbitrary admissible category of rings and let $\mathcal{W}$ be any subcategory of homomorphisms containing $A \to A[x]$ such that the triple $(\mathcal{R}, \mathcal{W}, \mathcal{F} = \{\text{surjective maps}\})$ is a Brown category. There is a triangulated category $D(\mathcal{R}, \mathcal{W})$ whose objects and morphisms are defined similar to those of $D(\mathcal{R}, \mathcal{F})$. If $\mathcal{R} = \text{Alg}_H$ and $\mathcal{W}_{CT}$ is the class of weak equivalences generated by Morita invariant, homotopy invariant, excisive homology theories, then there is a natural triangulated equivalence of the triangulated categories $D(\text{Alg}_H, \mathcal{W}_{CT})$ and $kk$. 
A homology theory $H_*$ on $\mathcal{R}$ relative to $\mathcal{F}$ consists of
A homology theory $H_\ast$ on $\mathcal{K}$ relative to $\mathcal{F}$ consists of

- a family $\{H_n, n \geq 0\}$ of functors $H_n : \mathcal{K} \to \text{Sets}$ with $H_{n \geq 1}(A)$ a group,
A homology theory $H_\ast$ on $\mathcal{R}$ relative to $\mathcal{F}$ consists of

- a family $\{H_n, n \geq 0\}$ of functors $H_n : \mathcal{R} \to \text{Sets}_\bullet$ with $H_{n \geq 1}(A)$ a group,
- for every $\mathcal{F}$-fibre sequence

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

with $g \in \mathcal{F}$, morphisms

$$H_{n+1}(C) \xrightarrow{\partial_{n+1}(g)} H_n(A), \quad n \geq 0,$$

satisfying axioms
A homology theory $H_*$ on $\mathcal{R}$ relative to $\mathcal{F}$ consists of

- a family \( \{H_n, n \geq 0\} \) of functors $H_n : \mathcal{R} \to \text{Sets}_{\bullet}$ with $H_{n \geq 1}(A)$ a group,
- for every $\mathcal{F}$-fibre sequence

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C,
\end{array}
\]

with $g \in \mathcal{F}$, morphisms

\[
H_{n+1}(C) \xrightarrow{\partial_{n+1}(g)} H_n(A), \quad n \geq 0,
\]

satisfying axioms

Ax 1) $H_n(u) = H_n(v)$ for any homotopic homomorphisms $u, v$,
Ax 2) given a commutative diagram in $\mathcal{R}$ with rows $\mathcal{F}$-fibre sequences

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
\end{array}
$$

and with $g, g' \in \mathcal{F}$, then the diagram is commutative for $n \geq 0$

$$
\begin{align*}
H_{n+1}(C) & \xrightarrow{\partial_{n+1}(g)} H_n(A) \\
H_{n+1}(c) & \downarrow \quad \quad \quad \quad \downarrow H_n(a) \\
H_{n+1}(C') & \xrightarrow{\partial_{n+1}(g')} H_n(A');
\end{align*}
$$
Ax 2) given a commutative diagram in $\mathcal{K}$ with rows $\mathcal{F}$-fibre sequences

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \xrightarrow{g} C \\
\downarrow{a} & & \downarrow{b} \\
A' & \xrightarrow{f'} & B' \xrightarrow{g'} C'
\end{array}
\]

and with $g, g' \in \mathcal{F}$, then the diagram is commutative for $n \geq 0$

\[
\begin{array}{c}
H_{n+1}(C) \xrightarrow{\partial_{n+1}(g)} H_n(A) \\
\downarrow{H_{n+1}(c)} & \downarrow{H_n(a)} \\
H_{n+1}(C') \xrightarrow{\partial_{n+1}(g')} H_n(A')
\end{array}
\]

Ax 3) if $A \xrightarrow{f} B \xrightarrow{g} C$, is an $\mathcal{F}$-fibre sequence then we have a long exact sequence of pointed sets

\[
\cdots \rightarrow H_{n+1}(A) \xrightarrow{H_{n+1}(f)} H_{n+1}(B) \xrightarrow{H_{n+1}(g)} H_{n+1}(C) \\
\downarrow{\partial_{n+1}(g)} \rightarrow H_n(A) \rightarrow \cdots \rightarrow H_0(B) \xrightarrow{} H_0(C)
\]
Theorem (GG 2005)

To any functor $\mathcal{X}$ from $\mathcal{R}$ to pointed simplicial sets a homology theory $\{k^\mathcal{X}_i, i \geq 0\}$ is associated. Such a homology theory is defined by means of an explicitly constructed functor $\text{Ex}_I,J(\mathcal{X})$ from $\mathcal{R}$ to pointed simplicial sets and, by definition,

$$k^\mathcal{X}_i(A) := \pi_i(\text{Ex}_I,J(\mathcal{X})(A))$$

for any $A \in \mathcal{R}$ and $i \geq 0$. Moreover, there is a natural transformation $\theta_{\mathcal{X}} : \mathcal{X} \rightarrow \text{Ex}_I,J(\mathcal{X})$, functorial in $\mathcal{X}$. 

**Remark:** Roughly speaking, we turn any pointed simplicial functor into a homology theory. The important simplicial functors $GL$ and $K$ give rise to the homology theories $\{K^V_i | F = \text{GL-fibrations}\}$ and $\{K^H_i | F = \text{surjective maps}\}$ respectively. Here $KH$ stands for the (non-negative) homotopy $K$-theory in the sense of Weibel.
Theorem (GG 2005)

To any functor $\mathcal{X}$ from $\mathcal{R}$ to pointed simplicial sets a homology theory $\{k^\mathcal{X}_i, i \geq 0\}$ is associated. Such a homology theory is defined by means of an explicitly constructed functor $Ex^I_J(\mathcal{X})$ from $\mathcal{R}$ to pointed simplicial sets and, by definition,

$$k^\mathcal{X}_i(A) := \pi_i(Ex^I_J(\mathcal{X})(A))$$

for any $A \in \mathcal{R}$ and $i \geq 0$. Moreover, there is a natural transformation $\theta_\mathcal{X} : \mathcal{X} \to Ex^I_J(\mathcal{X})$, functorial in $\mathcal{X}$.

Remark: Roughly speaking, we turn any pointed simplicial functor into a homology theory. The important simplicial functors $GL$ and $K$ give rise to the homology theories $\{KV_i \mid \mathcal{F} = GL$-fibrations$\}$ and $\{KH_i \mid \mathcal{F} = surjective$ maps$\}$ respectively. Here $KH$ stands for the (non-negative) homotopy $K$-theory in the sense of Weibel.
Theorem (GG 2005)

The functor

\[[A, -] = \text{Hom}_{D^- (\mathcal{R}, \mathcal{F})} (A, -)\]

gives rise to a homology theory $H_*$ on $\mathcal{R}$ with $H_n(B) = [A, \Omega^n B]$, $n \geq 0$, and

\[H_n(f) = \begin{cases} 
[A, (-1)^n \Omega(f)], & n \geq 1 \\
[A, f], & n = 0 
\end{cases}\]
The main tools are coming from modern homotopical algebra.
The main tools are coming from modern homotopical algebra.

To develop homotopy theory of rings we consider the model category $U\mathcal{R}$ of simplicial functors on $\mathcal{R}$, i.e. simplicial presheaves on $\mathcal{R}^{op}$ instead of simplicial presheaves on $\mathcal{R}$. 
Concluding remarks

The main tools are coming from modern homotopical algebra.

To develop homotopy theory of rings we consider the model category \( UR \) of simplicial functors on \( R \), i.e. simplicial presheaves on \( R^{op} \) instead of simplicial presheaves on \( R \).

The model structure is given by injective maps (cofibrations) and objectwise weak equivalences of simplicial sets (Quillen equivalences). There is a contravariant embedding \( r \) of \( R \) into \( UR \) as representable functors.
We need to localize this model structure to take into account the pullback squares

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow^f \\
C & \longrightarrow & D \\
\end{array}
$$

(1)

with $f$ a fibration in $\mathcal{F}$ and the fact that $rA[x] \to rA$ should be a Quillen equivalence.
We need to localize this model structure to take into account the pullback squares

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow f \\
C & \longrightarrow & D
\end{array}
\]  \hspace{1cm} (1)

with \( f \) a fibration in \( \mathcal{F} \) and the fact that \( rA[x] \to rA \) should be a Quillen equivalence.

To do so, we define a set \( S \) to consist of the maps \( rA[x] \to rA \) for any ring \( A \) and maps \( rB \bigsqcup_{rD} rC \to rA \) for every pullback square (1) in \( \mathcal{R} \) with \( f \) a fibration.
We need to localize this model structure to take into account the pullback squares

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \quad f \\
C & \quad \longrightarrow & D \\
\end{array}
\]

with \( f \) a fibration in \( \mathcal{F} \) and the fact that \( rA[x] \to rA \) should be a Quillen equivalence.

To do so, we define a set \( S \) to consist of the maps \( rA[x] \to rA \) for any ring \( A \) and maps \( rB \bigsqcup_{rD} rC \to rA \) for every pullback square (1) in \( \mathcal{K} \) with \( f \) a fibration.

Then one localizes \( UR \) at \( S \).
This procedure is a reminiscence of an unstable motivic model category.
This procedure is a reminiscence of an unstable motivic model category.

The latter model structure is obtained from simplicial presheaves $\mathcal{E}$ on smooth schemes by localizing $\mathcal{E}$ at the set $S$ of the maps $X \times \mathbb{A}^1 \to X$ for any smooth scheme $X$ and maps $P \to D$ for every pullback square (1) of smooth schemes with $f$ etale, $g$ an open embedding, and $f^{-1}(D - C) \to D - C$ an isomorphism.
This procedure is a reminiscence of an unstable motivic model category.

The latter model structure is obtained from simplicial presheaves $\mathcal{E}$ on smooth schemes by localizing $\mathcal{E}$ at the set $\mathcal{S}$ of the maps $X \times \mathbb{A}^1 \to X$ for any smooth scheme $X$ and maps $P \to D$ for every pullback square (1) of smooth schemes with $f$ etale, $g$ an open embedding, and $f^{-1}(D - C) \to D - C$ an isomorphism.

There is then some work involving properties of the Nisnevich topology to show that this model category is equivalent to the Morel-Voevodsky motivic model category.
THANK YOU FOR YOUR ATTENTION!