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TESIS DOCTORAL:

Equivariant algebraic kk-theory and isomorphism conjectures

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Resumen

La kk-teoría algebraica fue introducida por G. Cortiñas y A. Thom en [**5**]. Esta teoría es una K-teoría bivariante en la categoría de ℓ-álgebras siendo ℓ un anillo conmutativo con unidad. Para cada par de álgebras (A, B) se define un grupo $\mathrm{kk}(A, B)$. Se obtiene una categoría $\mathfrak{KK}$ cuyos objetos son las álgebras y cuyos morfismos son los elementos de $\mathrm{kk}(A, B)$. La categoría $\mathfrak{KK}$ es una categoría triangulada y existe un funtor canónico $j : \mathrm{Alg}_\ell \to \mathfrak{KK}$ con ciertas propiedades universales. Estas propiedades son la de invarianza homotópica polinomial, invarianza por matrices y la propiedad de escisión. La definición de la kk-teoría algebraica fue motivada por los trabajos de J. Cuntz [**6**] y N. Higson [**12**] sobre las propiedades universales de la KK-teoría de C^*-álgebras definida por Kasparov en [**19**].

En esta tesis se continua la línea de trabajo de [**5**] verficando que la kk-teoría algebraica admite una versión equivariante. Sean ℓ un anillo conmutativo con unidad, G un grupo numerable y $\mathcal{H}$ un álgebra de Hopf sobre un cuerpo. En el capítulo 1 definimos una k-teoría bivariante para la categoría de G-álgebras, álgebras G-graduadas, $\mathcal{H}$-módulo álgebras y $\mathcal{H}$-comódulo álgebras. Denotamos por $\mathcal{X}$-Alg a cualquiera de estas categorías. Se introduce el concepto de $\mathcal{X}$-estabilidad que consiste en una noción equivariante de invarianza matricial. Posteriormente se establecen las propiedades universales que verifica el funtor canónico $j : \mathcal{X}\text{-Alg} \to \mathfrak{KK}^{\mathcal{X}}$: invarianza por homotopías polinómicas, $\mathcal{X}$-estabilidad y la propiedad de escisión.

En el capítulo 2 se estudian los teoremas de adjunción en la kk-teoría algebraica equivariante. Este estudio nos permite seguir completando el diccionario iniciado en [**5**] entre la kk-teoría algebraica y la KK-teoría de Kasparov. Si G es un grupo numerable, se definen funtores que extienden el producto cruzado por G y la acción trivial

$$\rtimes \mathrm{G} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK} \qquad \tau : \mathfrak{KK} \to \mathfrak{KK}^{\mathrm{G}}.$$

El primer resultado de adjunción es una versión algebraica del Teorema de Green-Julg y nos permite relacionar los kk^{G}-grupos con los KH-grupos de la K-teoría homotópica de C. Weibel definida en [**32**]. Si G es un grupo finito, $1/|\mathrm{G}| \in \ell$, B es un álgebra y A es una G-álgebra entonces existe un isomorfismo

$$\psi_{GJ} : \mathrm{kk}^{\mathrm{G}}(B^\tau, A) \to \mathrm{kk}(B, A \rtimes \mathrm{G})$$

Aquí B^τ indica la G-álgebra con la acción trivial de G. En particular tomando $B = \ell$,

$$\mathrm{kk}^{\mathrm{G}}(\ell, A) \simeq \mathrm{KH}(A \rtimes \mathrm{G}) \qquad \text{para toda } A \in \mathrm{G}\text{-Alg}.$$

Para cada subgrupo H de G se definen funtores de inducción y restricción

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}^{\mathrm{H}} \qquad \mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}^{\mathrm{H}}$$

los cuales son funtores adjuntos. En otras palabras,

$$\mathrm{kk}^{\mathrm{G}}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B), A) \simeq \mathrm{kk}^{\mathrm{H}}(B, \mathrm{Res}_{\mathrm{G}}^{\mathrm{H}}(A)) \quad \text{para } A \in \text{G-Alg y } B \in \text{H-Alg}.$$

En particular si A es una G-álgebra,

$$\mathrm{kk}^{\mathrm{G}}(\ell^{(\mathrm{G})}, A) \simeq \mathrm{KH}(A).$$

Aquí $\ell^{(\mathrm{G})} = \bigoplus_{g\in\mathrm{G}} \ell$ con la acción regular de G. Una versión algebraica del Teorema de imprimitividad de Green nos permite identificar en $\mathfrak{KK}$ al álgebra $A \rtimes \mathrm{H}$ con el álgebra $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} A \rtimes \mathrm{G}$. También se prueba que $\mathfrak{KK}^{\mathrm{G}}$ la categoría de G-algebras y $\mathfrak{KK}^{\hat{\mathrm{G}}}$ la categoría de álgebras G-graduadas son equivalentes. Esta equivalencia está dada por los funtores

$$\rtimes \mathrm{G} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}^{\hat{\mathrm{G}}} \qquad \hat{\mathrm{G}} \rtimes : \mathfrak{KK}^{\hat{\mathrm{G}}} \to \mathfrak{KK}^{\mathrm{G}}.$$

inducidos por los productos cruzados. Es el análogo algebraico a la dualidad de Baaj-Skandalis, [**1**]. Por último estudiamos el caso de $\mathfrak{KK}^{\mathcal{H}}$ cuando $\mathcal{H}$ es un álgebra de Hopf de dimensión finita sobre un cuerpo. Existen funtores

$$\tau : \mathfrak{KK} \to \mathfrak{KK}^{\mathcal{H}} \qquad \# : \mathfrak{KK}^{\mathcal{H}} \to \mathfrak{KK}$$

inducidos por la acción trivial y el producto *smash*. Cuando $\mathcal{H}$ es un álgebra semisimple obtenemos una versión del teorema de Green-Julg en este contexto. En particular si A es un $\mathcal{H}$-módulo álgebra entonces

$$\mathrm{kk}^{\mathcal{H}}(\ell, A) \simeq \mathrm{KH}(A\#\mathcal{H}).$$

En el último capítulo estudiamos conjeturas de isomorfismo siguiendo la línea de trabajo de [**8**]. Consideramos estructuras de modelo en la categoría de G-conjuntos simpliciales y en la categoría de G-espacios topológicos. Dichas estructuras estan definidas en función de una familia $\mathcal{F}$ de subconjuntos de G y son tales que las equivalencias débiles y las fibraciones son punto a punto. Los objetos cofibrantes son aquellos X tales que el estabilizador G_x pertenece a la familia $\mathcal{F}$ para todo $x \in X$. Al probar que el siguiente par de funtores es una equivalencia de Quillen

$$\mathbf{Top}^{\mathrm{G}} \underset{|\,|_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} \mathbb{S}^{\mathrm{G}}.$$

obtenemos que es equivalente trabajar con un modelo simplicial de espacio o con un modelo topológico.

Decimos que un funtor $\mathrm{H} : \mathbb{S}^{\mathrm{G}} \to \mathbf{Spt}$ de la categoría de G-conjuntos simpliciales en la categoría de espectros satisface la $(\mathrm{G}, \mathcal{F})$-conjetura de isomorfismo, si para el reemplazo cofibrante $\pi : \mathcal{E}(\mathrm{G}, \mathcal{F}) \to *$ en la categoría de modelos mencionada anteriormente el morfismo

$$\mathrm{H}(\pi) : \mathrm{H}(\mathcal{E}(\mathrm{G}, \mathcal{F})) \to \mathrm{H}(*)$$

es una equivalencia. Si $\mathrm{E} : \mathbb{Z}\text{-}\mathbf{Cat} \to \mathbf{Spt}$ es un funtor y R es un G-anillo con unidad, podemos construir, siguiendo la línea Davis-Luck [**8**], un funtor

$$H^{\mathrm{G}}(-, \mathrm{E}(R)) : \mathbb{S}^{\mathrm{G}} \to \mathbf{Spt}$$

tal que $H^{\mathrm{G}}(*, \mathrm{E}(R)) = \mathrm{E}(R \rtimes \mathrm{G})$. La $(\mathrm{G}, \mathcal{F}, \mathrm{E}, R)$-conjetura de isomorfismo es la $(\mathrm{G}, \mathcal{F})$-conjetura para el funtor $H^{\mathrm{G}}(-, \mathrm{E}(R))$. Pobamos que considerando unas condiciones mínimas sobre E, las STANDING ASSUMPTIONS 3.2.5 (condiciones que satisfacen K y KH, ver proposiciones 3.4.18 y 3.5.3), $H^{\mathrm{G}}(-, \mathrm{E}(A))$ está definido no sólo para los G-anillos con unidad, si no que está definido también para todo G-anillo A que satisface E-escisión. Más aún, probamos que si

$$0 \to A' \to A \to A'' \to 0 \tag{1}$$

es una sucesión exacta de anillos E-escisivos y X es un G-conjunto simplicial, entonces

$$H^{\mathrm{G}}(X, \mathrm{E}(A')) \to H^{\mathrm{G}}(X, \mathrm{E}(A)) \to H^{\mathrm{G}}(X, \mathrm{E}(A''))$$

es una fibración homotópica. Esta es una propiedad necesaria para establecer un análogo algebraico al médodo dual-Dirac que es un método usado en la prueba de la conjetura de Baum-Connes para algunos grupos. Otra propiedad básica que nos proporciona sucesiones (1) en las cuales al menos uno de los anillos satisface la conjetura de isomorfismos, es el teorema 3.7.3. En éste se muestra que si E satisface las standing assumptions y A es un anillo E-escisivo de la forma

$$A = \bigoplus_i \operatorname{Ind}_{\mathrm{K}_i}^{\mathrm{G}} B_i \tag{2}$$

en donde B_i es un K_i-anillo y $\mathrm{K}_i \in \mathcal{F}$ para todo i, entonces el funtor $H^{\mathrm{G}}(-, \mathrm{E}(A))$ lleva $(\mathrm{G}, \mathcal{F})$-equivalencias en equivalencias. En particular la $(\mathrm{G}, \mathcal{F}, \mathrm{E}, A)$-conjetura de isomorfismo se satisface. Usamos este hecho en la sección 8 para mostrar que bajo otros supuestos adicionales 3.8.1 (los cuales siguen siendo verificados por $\mathrm{E} = K$ y KH), para cada G-anillo E-escisivo A existe de manera funtorial una sucesión de G-anillos E-escisivos

$$\mathfrak{F}^0 B \to \mathfrak{F}^\infty B \to \mathfrak{F}^\infty B / \mathfrak{F}^0 B$$

y un morfismo natural $A \to \mathfrak{F}^0 B$ tal que

i) $H^{\mathrm{G}}(X, \mathrm{E}(A)) \to H^{\mathrm{G}}(X, \mathrm{E}(\mathfrak{F}^0 B))$ es una equivalencia para todo G-conjunto simplicial X.
ii) $H^{\mathrm{G}}(X, \mathrm{E}(\mathfrak{F}^\infty B)) \to *$ es una equivalencia si X es $(\mathrm{G}, \mathcal{F})$-cofibrante.
iii) $H^{\mathrm{G}}(-, \mathrm{E}(\mathfrak{F}^\infty B / \mathfrak{F}^0 B))$ lleva $(\mathrm{G}, \mathcal{F})$-equivalencias en equivalencias.

A partir de esta sucesión obtenemos que

$$H^{\mathrm{G}}(\mathcal{E}(\mathrm{G}, \mathcal{F}), \mathrm{E}(A)) \to \mathrm{E}(A \rtimes \mathrm{G})$$

es una equivalencia si y sólo si el morfismo de conexión

$$\Omega(\mathrm{E}(\mathfrak{F}^\infty B / \mathfrak{F}^0 B \rtimes \mathrm{G})) \to \mathrm{E}(A \rtimes \mathrm{G})$$

es una equivalencia. En particular lo anterior se aplica cuando $\mathrm{E} = K, \mathrm{KH}$. En el teorema 3.9.2 probamos que bajo hipótesis más fuertes en E, la cual la más importante es que E satisface escisión (e.g. KH satisface pero K no), entonces la $(\mathrm{G}, \mathcal{F}, \mathrm{E}, A)$-conjetura de isomorfismo es verdad para cuando A es un anillo $(\mathrm{G}, \mathcal{F})$-propio. Si X es un conjunto simplicial localmente finito con una acción de G entonces el G-anillo A es propio sobre X si este es un álgebra sobre el anillo $\mathbb{Z}^{(X)}$ de funciones polinomiales finitamente soportadas en X, la acción es compatible con la acción de G en A y en X y $\mathbb{Z}^{(X)} \cdot A = A$. Decimos que A es $(\mathrm{G}, \mathcal{F})$-propio si es propio sobre algún conjunto simplicial X localmente finito en el cual G actúa de manera que los estabilizadores pertenecen a la familia $\mathcal{F}$. Por ejemplo un álgebra es de la forma (2) si y solamente si es propia sobre algún G-conjunto simplicial cero dimensional de la forma $X = \coprod \mathrm{G}\,/\,\mathrm{K}_i$. La noción de anillo $(\mathrm{G}, \mathcal{F})$-propio usada aquí es la noción algebraica de la G-C^*-algebra propia, y el teorema 3.9.2 es una versión algebraica del conocido hecho de que la conjetura de Baum-Connes es cierta para las G-C^*-algebras [**11**].

Introduction

Some concepts in operator algebras and in non-commutative geometry can fit into algebraic contexts; this observation can lead to interesting results in algebraic settings. In this thesis we analyse some items in a diccionary between operator algebras and algebras without topological structure.

Algebraic kk-theory has been introduced by G. Cortiñas and A. Thom in [**5**]. This is a bivariant K-theory on the category of ℓ-algebras where ℓ is a commutative ring with unit. For each pair (A, B) of ℓ-algebras a group $\mathrm{kk}(A, B)$ is defined. A category $\mathfrak{KK}$ is obtained whose objects are the ℓ-algebras and whose morphisms are the elements of the group $\mathrm{kk}(A, B)$. The category $\mathfrak{KK}$ is triangulated and there is a canonical functor $j : \mathrm{Alg}_\ell \to \mathfrak{KK}$ with universal properties. These properties are algebraic homotopy invariance, matrix invariance and excision.

The definition of algebraic kk-theory was inspired by the work of J. Cuntz [**6**] and N. Higson [**12**] on the universal properties of Kasparov KK-theory [**19**]. The KK-theory of separable C^*-algebras is a common generalization both of topological K-homology and topological K-theory as an additive bivariant functor. Let A, B be separable C^*-algebras then

$$(3) \qquad KK_*(\mathbb{C}, B) \simeq K_*^{top}(B) \qquad KK^*(A, \mathbb{C}) = K^*_{hom}(A)$$

here $K_*^{top}(B)$ denotes the K-theory of B and $K^*_{hom}(A)$ the K-homology of A. J. Cuntz in [**6**] gave another equivalent definition of the original one given in [**19**]. This new approach allowed to put bivariant K-theory in algebraic context. Higson in [**12**] stated the universal property of KK whose algebraic analogoue is studied in [**5**], where also an analogue of (3) is proved. On the algebraic side, if A is an ℓ-algebra then

$$\mathrm{kk}(\ell, A) \simeq \mathrm{KH}(A)$$

here KH is Weibel's homotopy K-theory defined in [**32**]. We can start to build a dictionary between Kasparov's KK-theory and algebraic kk-theory in the following way

Kasparov's KK-theory	algebraic kk-theory
bivariant K-theory on separable C^*-algebras C^*-Alg	bivariant K-theory on ℓ-algebras Alg_ℓ
$k : C^*\text{-Alg} \to KK$	$j : \mathrm{Alg}_\ell \to \mathfrak{KK}$
k is stable with respect to compact operators $\mathcal{K}$	j is stable with respect to $M_\infty = \bigcup_{n\in\mathbb{N}} M_n$
k is continous homotopy invariant	j is polynomial homotopy invariant
k is split exact	j is excisive
k is universal for the properties described above	j is universal for the properties described above
$KK_*(\mathbb{C}, A) \simeq K_*(A)$	$\mathrm{kk}_*(\ell, A) \simeq \mathrm{KH}_*(A)$

In this work we obtain an equivariant version of this dictionary. Let G be a countable group and $\mathcal{H}$ be a Hopf algebra over a field. We introduce an algebraic kk-theory for the categories of G-algebras, G-graded algebras, $\mathcal{H}$-module algebras and $\mathcal{H}$-comodule algebras. We define an equivariant algebraic notion of matrix invariance. We study the diferent cases separately. In the category of G-algebras, every object A is stably isomorphic to the equivariant matrix algebra $M_{\mathrm{G}}(A)$. In the category of G-graded algebras, which we call $\hat{\mathrm{G}}$-algebras, stabilization is with respect to the graded matrix algebra $M_{\hat{\mathrm{G}}}$. The definition of G-stability was inspired by the definition of equivariant stability in G-C^*-algebras (see [**24**]). In the case of $\mathcal{H}$-algebras, we fix a basis of $\mathcal{H}$ as a ℓ-space and we define an $\mathcal{H}$-algebra called $\mathrm{End}_\ell^F(\mathcal{H})$. The $\mathcal{H}$-stability identifies A with $\mathrm{End}_\ell^F(\mathcal{H}) \otimes A$. This identification depends on a chosen basis of $\mathcal{H}$. We put a finiteness condition in $\mathrm{End}_\ell^F(\mathcal{H})$ and M_{G} but these condition are different if we take $\mathcal{H} = \ell\,\mathrm{G}$. The equivariant matrix invariance in the case of $\mathcal{H}$-comodule algebras is similar to that of $\mathcal{H}$-algebras. After that we introduce the appropiate brand of algebraic kk-theory and we establish its universal properties in each case. We consider several properties wich are valid for G-algebras, $\hat{\mathrm{G}}$-algebras, $\mathcal{H}$-algebras and $\mathcal{H}$-comodule algebras and we write $\mathcal{X}$-algebra to refer either of them. We can resume Chapter 1 in the following table

Equivariant Kasparov's KK-theory	Equivariant algebraic kk-theory
bivariant K-theory on separable G-C^*-algebras G-C^*-Alg	bivariant K-theory on $\mathfrak{X}$-algebras $\mathfrak{X}$-Alg
$k : \text{G-}C^*\text{-Alg} \to KK^{\mathrm{G}}$	$j : \mathfrak{X}\text{-Alg} \to \mathfrak{K}\mathfrak{K}^{\mathfrak{X}}$
k is stable with respect to $\mathcal{K}(\ell^2(\mathrm{G} \times \mathbb{N}))$	j is $\mathfrak{X}$-stable
k is continous homotopy invariant	j is polynomial homotopy invariant
k is split exact	j is excisive
k is universal for the properties described above	j is universal for the properties described above

In Chapter 2 we study adjointness theorems in equivariant kk-theory. We put in an algebraic context some of the adjointness theorems which appear in Kasparov KK-theory. Let G be a countable group and ℓ a commutative ring with unit. We define the functors of trivial action and crossed product between $\mathfrak{K}\mathfrak{K}$ and $\mathfrak{K}\mathfrak{K}^{\mathrm{G}}$. The first adjointness theorem is Theorem 2.1.4 which is an algebraic version of the Green-Julg Theorem. This result gives us the first computation related with homotopy K-theory. If G is a finite group, A is a G-algebra, B is an algebra and $\frac{1}{|\mathrm{G}|} \in \ell$ then there is an isomorphism

$$\psi_{GJ} : \mathrm{kk}^{\mathrm{G}}(B^{\tau}, A) \to \mathrm{kk}(B, A \rtimes \mathrm{G}).$$

In particular, if $B = \ell$ then

$$\mathrm{kk}^{\mathrm{G}}(\ell, A) \simeq \mathrm{KH}(A \rtimes \mathrm{G}).$$

We consider a subgroup H of G, define induction and restriction funtors between $\mathfrak{K}\mathfrak{K}^{\mathrm{G}}$ and $\mathfrak{K}\mathfrak{K}^{\mathrm{H}}$ and study the adjointness between them. If B is an H-algebra and A is a G-algebra then there is an isomorphism

$$\psi_{IR} : \mathrm{kk}^{\mathrm{G}}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} B, A) \to \mathrm{kk}^{\mathrm{H}}(B, \mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} A).$$

This result gives us another computation. Taking H the trivial group and $B = \ell$ we obtain that

$$\mathrm{kk}^{\mathrm{G}}(\ell^{(\mathrm{G})}, A) \simeq \mathrm{KH}(A) \qquad \forall A \in \mathrm{G} - \mathrm{Alg}\,.$$

Here $\ell^{(\mathrm{G})} = \bigoplus_{g \in \mathrm{G}} \ell$ with the regular action of G. More general, if H is a finite subgroup of G and $1/|\,\mathrm{H}\,| \in \ell$ we combine ψ_{GJ} and ψ_{IR} and obtain

$$\mathrm{kk}^{\mathrm{G}}(\ell^{(\mathrm{G}/\mathrm{H})}, A) \simeq \mathrm{KH}(A \rtimes \mathrm{H}) \qquad \forall A \in \mathrm{G} - \mathrm{Alg}\,.$$

We also prove an algebraic version of GREEN IMPRIMITIVITY THEOREM and obtain that

$$\mathrm{KH}(A \rtimes \mathrm{H}) \simeq \mathrm{KH}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} A \rtimes \mathrm{G}).$$

We also obtain an algebraic version of the Baaj-Skandalis theorem. We show that the funtors

$$\rtimes\,\mathrm{G} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}^{\hat{\mathrm{G}}} \qquad \hat{\mathrm{G}}\rtimes : \mathfrak{KK}^{\hat{\mathrm{G}}} \to \mathfrak{KK}^{\mathrm{G}}$$

are inverse category equivalences. Let $\mathcal{H}$ be a Hopf algebra with finite dimension. We define functors between $\mathfrak{KK}$ and $\mathfrak{KK}^{\mathcal{H}}$, the smash product and the trivial action. We study the adjointness between them in theorem 2.5.4. We obtain that if $\mathcal{H}$ is semisimple, B is an algebra and A is an $\mathcal{H}$-algebra then there is an isomorphism

$$\psi : \mathrm{kk}^{\mathcal{H}}(B^{\tau}, A) \to \mathrm{kk}(B, A\#\mathcal{H}).$$

In particular,

$$\mathrm{kk}^{\mathcal{H}}(\ell, A) \simeq \mathrm{KH}(A\#\mathcal{H}).$$

In the last chapter we study isomorphism conjecures in the sense of [**8**]. If $\mathcal{F}$ is a family of subgroups of G we consider model category structures on $\mathbb{S}^{\mathrm{G}}$ and $\mathbf{Top}^{\mathrm{G}}$. With this structure weak equivalences and fibration are object-wise. Cofibrant object are those X such that the stabilizer subgroup G_x is a subgroup of $\mathcal{F}$ for all $x \in X$. We prove that the following is a Quillen equivalence

$$\mathbf{Top}^{\mathrm{G}} \underset{|\,|_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} \mathbb{S}^{\mathrm{G}}.$$

We say that a functor $\mathrm{H} : \mathbb{S}^{\mathrm{G}} \to \mathbf{Spt}$ from the category of G-simplicial set to the category of spectra satisfies the $(\mathrm{G}, \mathcal{F})$-isomorphism conjecture if for the cofibrant replacement $\pi : \mathcal{E}(\mathrm{G}, \mathcal{F}) \to *$ in the $\mathcal{F}$-model category mentioned aboved, the map

$$\mathrm{H}(\pi) : \mathrm{H}(\mathcal{E}(\mathrm{G}, \mathcal{F})) \to \mathrm{H}(*)$$

is an equivalence. If $\mathrm{E} : \mathbb{Z}\text{-}\mathbf{Cat} \to \mathbf{Spt}$ is a functor and R is a unital G-ring, one constructs, following Davis-Luck [**8**], a functor

$$H^{\mathrm{G}}(-, \mathrm{E}(R)) : \mathbb{S}^{\mathrm{G}} \to \mathbf{Spt}$$

such that $H^{\mathrm{G}}(*, \mathrm{E}(R)) = \mathrm{E}(R \rtimes \mathrm{G})$. The $(\mathrm{G}, \mathcal{F}, \mathrm{E}, R)$-isomorphism conjecture is the $(\mathrm{G}, \mathcal{F})$-conjecture for the functor $H^{\mathrm{G}}(-, \mathrm{E}(R))$. We show that under very mild assumptions on E, the STANDING ASSUMPTIONS 3.2.5 (which are satisfied for example when E is either K or KH, see propositions 3.4.18 and 3.5.3), $H^{\mathrm{G}}(-, \mathrm{E}(A))$ is defined not only for unital G-rings, but also for all E-excisive G-rings A, that is all G-rings on which E-salisfies excision. Moreover whe show that if

$$0 \to A' \to A \to A'' \to 0 \tag{4}$$

is an exact sequence of E-excisive rings and X is a G-simplicial set, then

$$H^{\mathrm{G}}(X, \mathrm{E}(A')) \to H^{\mathrm{G}}(X, \mathrm{E}(A)) \to H^{\mathrm{G}}(X, \mathrm{E}(A''))$$

is a homotopy fibration. This is a basic property needed to establish an algebraic analogue of the Dirac-dual method which is used to prove the Baum-Connes conjecture for some groups. Another basic property, which provide us with enough sequences (4) in which at least one of the rings satisfies the isomorphism conjecture, is Theorem 3.7.3 which shows that if E satisfies the standing assumtions and A is an E-excisive G-ring of the form

$$A = \bigoplus_i \operatorname{Ind}_{\mathrm{K}_i}^{\mathrm{G}} B_i \tag{5}$$

with B_i a K_i-ring and $\mathrm{K}_i \in \mathcal{F}$ for all i, then the functor $H^{\mathrm{G}}(-, \mathrm{E}(A))$ maps $(\mathrm{G}, \mathcal{F})$-equivalences to equivalences. In particular the $(\mathrm{G}, \mathcal{F}, \mathrm{E}, A)$-isomorphism conjecture holds. We use this in Section 8 to show that under some additional assumptions (which are still satisfied if $\mathrm{E} = K$, KH, see 3.8.1), for each E-excisive G-ring A there is a functorial exact sequence of E-excisive G-rings

$$\mathfrak{F}^0 B \to \mathfrak{F}^\infty B \to \mathfrak{F}^\infty B / \mathfrak{F}^0 B$$

and a natural map $A \to \mathfrak{F}^0 B$ such that

i) $H^{\mathrm{G}}(X, \mathrm{E}(A)) \to H^{\mathrm{G}}(X, \mathrm{E}(\mathfrak{F}^0 B))$ is an equivalence for all G-simplicial set X.
ii) $H^{\mathrm{G}}(X, \mathrm{E}(\mathfrak{F}^\infty B)) \to *$ is an equivalence if X is $(\mathrm{G}, \mathcal{F})$-cofibrant.
iii) $H^{\mathrm{G}}(-, \mathrm{E}(\mathfrak{F}^\infty B / \mathfrak{F}^0 B))$ maps $(\mathrm{G}, \mathcal{F})$-equivalences to equivalences.

It follows that the assembly map

$$H^{\mathrm{G}}(\mathcal{E}(\mathrm{G}, \mathcal{F}), \mathrm{E}(A)) \to \mathrm{E}(A \rtimes \mathrm{G})$$

is an equivalence iff the conecting map

$$\Omega(\mathrm{E}(\mathfrak{F}^\infty B / \mathfrak{F}^0 B \rtimes \mathrm{G})) \to \mathrm{E}(A \rtimes \mathrm{G})$$

is an equivalence. In particular all this applies when $\mathrm{E} = K, \mathrm{KH}$. We also show in Theorem 3.9.2 that under stronger hypotesis on E, of which the main one is that E satisfies excision (e.g. KH satisfies this but K does not), then the $(\mathrm{G}, \mathcal{F}, \mathrm{E}, A)$-isomorphism conjecture is true whenever A is $(\mathrm{G}, \mathcal{F})$-proper. If X is a locally finite simplicial set with a G-action then a G-ring A is proper over X if it is an algebra over the ring $\mathbb{Z}^{(X)}$ of finitely supported polynomial maps on X, the algebra action is compatible with the actions of G on A and on X, and $\mathbb{Z}^{(X)} \cdot A = A$. We say that A is $(\mathrm{G}, \mathcal{F})$-proper if it is proper over a locally finite simplicial set X on which G acts with all stabilizers in $\mathcal{F}$. For example an algebra is of the form (5) if and only if it is proper over the zero-dimensional G-simplicial set $X = \coprod \mathrm{G} / \mathrm{K}_i$.

We remark that the notion of $(\mathrm{G}, \mathcal{F})$-proper ring used here is the algebraic analogue of the notion of proper G-C^*-algebra, and that Teorem 3.9.2 is an algebraic version of the known fact that Baum-Connes conjecture holds for proper G-C^*-algebras [**11**].

CHAPTER 1

Equivariant algebraic kk-theory

Algebraic kk-theory has been introduced by G. Cortiñas and A. Thom in [**5**]. This is a bivariant K-theory on the category of ℓ-algebras where ℓ is a commutative ring with unit. For each pair (A, B) of ℓ-algebras a group $\mathrm{kk}(A, B)$ is defined. A category $\mathfrak{KK}$ is obtained whose objects are the ℓ-algebras and whose morphisms are the elements of the group $\mathrm{kk}(A, B)$. The category $\mathfrak{KK}$ is triangulated and there is a canonical functor $j : \mathrm{Alg}_\ell \to \mathfrak{KK}$ with universal properties. These properties are algebraic homotopy invariance, matrix invariance and excision.

The definition of algebraic kk-theory was inspired by the work of J. Cuntz [**6**] and N. Higson [**12**] on the universal properties of Kasparov KK-theory [**19**]. The KK-theory of separable C^*-algebras is a common generalization both of topological K-homology and topological K-theory as an additive bivariant functor. Let A, B be separable C^*-algebras then

$$KK_*(\mathbb{C}, B) \simeq K_*^{top}(B) \qquad KK^*(A, \mathbb{C}) = K^*_{hom}(A) \tag{6}$$

here $K_*^{top}(B)$ denotes the K-theory of B and $K^*_{hom}(A)$ the K-homology of A. J. Cuntz in [**6**] gave another equivalent definition of the original one given in [**19**]. This new approach allowed to put bivariant K-theory in algebraic context. Higson in [**12**] stated the universal property of KK whose algebraic analogoue is studied in [**5**], where also an analogue of (6) is proved. On the algebraic side, if A is an ℓ-algebra then

$$\mathrm{kk}(\ell, A) \simeq \mathrm{KH}(A)$$

here KH is Weibel's homotopy K-theory defined in [**32**]. We can start to build a dictionary between Kasparov's KK-theory and algebraic kk-theory in the following way

Kasparov's KK-theory	algebraic kk-theory
bivariant K-theory on separable C^*-algebras C^*-Alg	bivariant K-theory on ℓ-algebras Alg_ℓ
$k : C^*\text{-Alg} \to KK$	$j : \mathrm{Alg}_\ell \to \mathfrak{KK}$
k is stable with respect to compact operators $\mathcal{K}$	j is stable with respect to $M_\infty = \bigcup_{n\in\mathbb{N}} M_n$
k is continous homotopy invariant	j is polynomial homotopy invariant
k is split exact	j is excisive
k is universal for the properties described above	j is universal for the properties described above
$KK_*(\mathbb{C}, A) \simeq K_*(A)$	$\mathrm{kk}_*(\ell, A) \simeq \mathrm{KH}_*(A)$

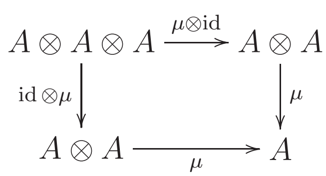
In this chapter we obtain an equivariant version of this dictionary. Let G be a countable group and $\mathcal{H}$ be a Hopf algebra over a field. We introduce an algebraic kk-theory for the categories of G-algebras, G-graded algebras, $\mathcal{H}$-module algebras and $\mathcal{H}$-comodule algebras. We define an equivariant algebraic notion of matrix invariance. We study the diferent cases separately. In the category of G-algebras, every object A is stably isomorphic to the equivariant matrix algebra $M_{\mathrm{G}}(A)$. In the category of G-graded algebras, which we call $\hat{\mathrm{G}}$-algebras, stabilization is with respect to the graded matrix algebra $M_{\hat{\mathrm{G}}}$. The definition of G-stability was inspired by the definition of equivariant stability in G-C^*-algebras (see [**24**]). In the case of $\mathcal{H}$-algebras, we fix a basis of $\mathcal{H}$ as a ℓ-space and we define an $\mathcal{H}$-algebra called $\mathrm{End}^F_\ell(\mathcal{H})$. The $\mathcal{H}$-stability identifies A with $\mathrm{End}^F_\ell(\mathcal{H}) \otimes A$. This identification depends on a chosen basis of $\mathcal{H}$. We put a finiteness condition in $\mathrm{End}^F_\ell(\mathcal{H})$ and M_{G} but these condition are different if we take $\mathcal{H} = \ell\,\mathrm{G}$. The equivariant matrix invariance in the case of $\mathcal{H}$-comodule algebras is similar to that of $\mathcal{H}$-algebras. After that we introduce the appropiate brand of algebraic kk-theory and we establish its universal properties in each case. We consider several properties wich are valid for G-algebras, $\hat{\mathrm{G}}$-algebras, $\mathcal{H}$-algebras and $\mathcal{H}$-comodule algebras and we write $\mathcal{X}$-algebra to refer either of them. We can resume this chapter in the following table

Equivariant Kasparov's KK-theory	Equivariant algebraic kk-theory
bivariant K-theory on separable G-C^*-algebras G-C^*-Alg	bivariant K-theory on $\mathcal{X}$-algebras $\mathcal{X}$-Alg
k : G-C^*-Alg $\to KK^{\mathrm{G}}$	$j : \mathcal{X}\text{-Alg} \to \mathfrak{KK}^{\mathcal{X}}$
k is stable with respect to $\mathcal{K}(\ell^2(\mathrm{G} \times \mathbb{N}))$	j is $\mathcal{X}$-stable
k is continous homotopy invariant	j is polynomial homotopy invariant
k is split exact	j is excisive
k is universal for the properties described above	j is universal for the properties described above

1. $\mathcal{H}$-algebras and $\hat{\mathcal{H}}$-algebras

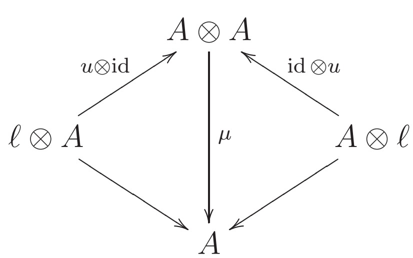
In this section we set up the notation and terminology related to Hopf algebras, module algebras and comodule algebras. We follow [**25**] for this part.

1.1. Hopf algebras. Let ℓ be a commutative ring with unit. In this section all tensor products are over ℓ; we write $\otimes = \otimes_\ell$. An ℓ-algebra is an ℓ-bimodule A together with an ℓ-linear map called multiplication $\mu : A \otimes A \to A$. It is associative if the following diagram is commutative:



(associativity)

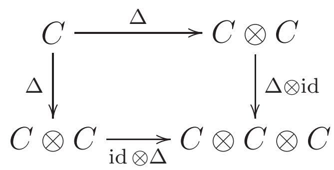
An ℓ-algebra A with unit is an associative ℓ-algebra A together a with an ℓ-linear map called unit $u : \ell \to A$ such that the following diagram is commutative:



(unit)

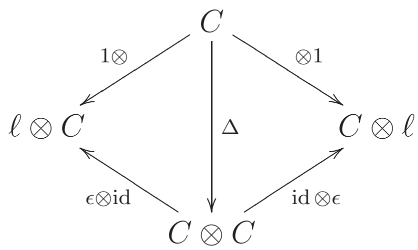
The two lower maps are given by scalar multiplication. We will denote by $1_A := u(1_\ell)$ and we will write ab for $\mu(a, b)$. Let V, W be ℓ-modules. Let us denote by $\tau : V \otimes W \to W \otimes V$ the twisting map, $\tau(v \otimes w) = w \otimes v$. An ℓ-algebra A is commutative if $\mu \circ \tau = \mu$.

An ℓ-coalgebra is an ℓ-module C together with an ℓ-linear map called comultiplication $\Delta : C \to C \otimes C$ such that the following diagram is commutative



(coassociativity)

An ℓ-coalgebra with counit is a coalgebra C together with an ℓ-linear map called counit $\epsilon : C \to \ell$ such that the following diagram is commutative:



(counit)

We say that C is cocommutative if $\tau \circ \Delta = \Delta$. An ℓ-module $(B, \mu, u, \Delta, \epsilon)$ is a bialgebra if (B, μ, u) is an algebra with unit and (B, Δ, ϵ) is a coalgebra with counit such that Δ and ϵ are algebra morphisms. We will use the Sweedler notation or sigma notation for Δ. If $c \in C$ we write

$$\Delta c = \sum c_1 \otimes c_2$$

The subscripts 1 and 2 are symbolic, and do not indicate particular elements of C. The power of the notation becomes apparent when Δ must be applied more than once. If we apply Δ $(n-1)$-times to c we can write

$$\Delta_{n-1}(c) = \sum c_1 \otimes c_2 \otimes \ldots \otimes c_n$$

because of the coassociativity.

An ℓ-module $(\mathcal{H}, \mu, u, \Delta, \epsilon, S)$ is a Hopf algebra if $(\mathcal{H}, \mu, u, \Delta, \epsilon)$ is a bialgebra and $S : \mathcal{H} \to \mathcal{H}$ is a bijective ℓ-linear map such that

$$\sum (Sh_1)h_2 = \epsilon(h)1_{\mathcal{H}} = \sum h_1(Sh_2) \qquad \forall h \in \mathcal{H}$$

The map S is called an antipode of $\mathcal{H}$. We will write $\overline{S}$ for the inverse of S. A map $f : \mathcal{H} \to \mathcal{K}$ is a Hopf morphism if it is a bialgebra morphism and $f(S_{\mathcal{H}}h) = S_{\mathcal{K}} f(h)$.

Example 1.1.1. Let G be a group and ℓ a commutative ring with unit. The group algebra $\ell\,\mathrm{G}$ of G is a Hopf algebra with the following structure:

$$\ell\,\mathrm{G} := \{\textstyle\sum_{g \in \mathrm{G}} a_g \delta_g : a_g \in \ell \text{ and } \{g \in \mathrm{G} : a_g \neq 0\} \text{ is a finite set}\}$$

$$\mu : \ell\,\mathrm{G} \otimes \ell\,\mathrm{G} \to \ell\,\mathrm{G} \qquad \delta_g \otimes \delta_h \mapsto \delta_{gh} \qquad u : \ell \to \ell\,\mathrm{G} \quad u(1) = \delta_e$$

$$\Delta : \ell\,\mathrm{G} \to \ell\,\mathrm{G} \otimes \ell\,\mathrm{G} \qquad \delta_g \mapsto \delta_g \otimes \delta_g \qquad \epsilon : \ell\,\mathrm{G} \to \ell \quad \epsilon(\delta_g) = 1$$

$$S : \ell\,\mathrm{G} \to \ell G \qquad \delta_g \mapsto \delta_{g^{-1}}$$

1.2. $\mathcal{H}$-algebras. Let $\mathcal{H}$ be a Hopf algebra. A left $\mathcal{H}$-module A is an ℓ-module with an ℓ-linear map $\gamma : \mathcal{H} \otimes A \to A$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{H}\otimes\mathcal{H}\otimes A & \xrightarrow{\mu\otimes \mathrm{id}} & \mathcal{H}\otimes A \\ {\scriptstyle \mathrm{id}\otimes\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathcal{H}\otimes A & \xrightarrow[\gamma]{} & A \end{array} \qquad \begin{array}{ccc} \ell\otimes A & \xrightarrow{u\otimes \mathrm{id}} & \mathcal{H}\otimes A \\ & \searrow{\scriptstyle \text{scalar mult.}} & \downarrow{\scriptstyle\gamma} \\ & & A \end{array}$$

We will write $h \cdot a$ for $\gamma(h \otimes a)$. A left $\mathcal{H}$-module algebra is an ℓ-algebra A and a left $\mathcal{H}$-module structure on A such that

- $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$, for all $h \in \mathcal{H}$, $a, b \in A$.
- $h \cdot 1_A = \epsilon(h)1_A$, for all $h \in \mathcal{H}$.

We call A an $\mathcal{H}$-algebra for short. A map $f : A \to B$ is an $\mathcal{H}$-algebra morphism if

- $f(ab) = f(a)f(b)$ for all $a,b \in A$.
- $f(h \cdot a) = h \cdot f(a)$ for all $a \in A$, $h \in \mathcal{H}$.

Example 1.1.2. For any algebra A we can consider the trivial action $h \cdot a = \epsilon(h)a$. In this case we say A is a trivial $\mathcal{H}$-algebra.

Example 1.1.3. Let M be a left $\mathcal{H}$-module. Put

$$\mathrm{End}_\ell(M) = \{\varphi : M \to M : \varphi \text{ is } \ell\text{-linear}\},$$

$\mathrm{End}_\ell(M)$ is an ℓ-algebra with the composition and is an $\mathcal{H}$-algebra with the following action

$$(h \cdot \varphi)(m) = \sum h_1 \cdot \varphi(S(h_2) \cdot m) \qquad\qquad \varphi \in \mathrm{End}_\ell(M), h \in \mathcal{H}$$

More generally we have the following example.

Example 1.1.4. Let M be an $\mathcal{H}$-module and let A be an $\mathcal{H}$-algebra. If $\varphi \in \mathrm{End}_\ell(M)$ and h, $k \in \mathcal{H}$ we put

$$\varphi_{h,k} \in \mathrm{End}_\ell(M) \quad \varphi_{h,k}(m) = h \cdot \varphi(S(k) \cdot m) \tag{7}$$

It is easy to check that $(\varphi_{h,k})_{\tilde{h},\tilde{k}} = \varphi_{\tilde{h}h,\tilde{k}k}$. Define the following action in $\mathrm{End}_\ell(M) \otimes A$

$$h \cdot (\varphi \otimes a) = \sum \varphi_{h_1,h_3} \otimes h_2 \cdot a$$

Let us see that $\mathrm{End}_\ell(M) \otimes A$ is an $\mathcal{H}$-algebra,

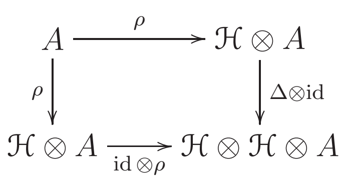
$$\begin{aligned} k \cdot (h \cdot (\varphi \otimes a)) &= k \cdot (\textstyle\sum \varphi_{h_1,h_3} \otimes h_2 \cdot a) \\ &= \textstyle\sum (\varphi_{h_1,h_3})_{k_1,k_3} \otimes k_2 \cdot (h_2 \cdot a) \\ &= \textstyle\sum \varphi_{k_1h_1,k_3h_3} \otimes (k_2h_2) \cdot a \\ &= (kh) \cdot (\varphi \otimes a) \end{aligned}$$

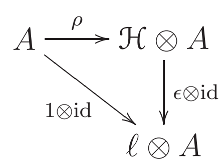
$$\begin{aligned}
h\cdot((\varphi\otimes a)(\tilde\varphi\otimes\tilde a)) &= \sum(\varphi\tilde\varphi)_{h_1,h_3}\otimes h_2\cdot(a\tilde a)\\
&= \sum(\varphi\tilde\varphi)_{h_1,h_4}\otimes(h_2\cdot a)(h_3\cdot\tilde a)\\
&= \sum h_1\cdot(\varphi(\tilde\varphi(S(h_5)\cdot\quad))\otimes(h_2\cdot a)(\epsilon(h_3)h_4\cdot\tilde a)\\
&= \sum h_1\cdot(\varphi(S(h_3)h_4\cdot\tilde\varphi(S(h_6)\cdot\quad))\otimes(h_2\cdot a)(h_5\cdot\tilde a)\\
&= \sum\varphi_{h_1,h_3}\tilde\varphi_{h_4,h_6}\otimes(h_2\cdot a)(h_5\cdot\tilde a)\\
&= \sum h_1\cdot(\varphi\otimes a)h_2\cdot(\tilde\varphi\otimes\tilde a)
\end{aligned}$$

$$1\cdot(\varphi\otimes a)=\varphi_{1,1}\otimes 1\cdot a=\varphi\otimes a$$

$$h\cdot(\mathrm{id}\otimes 1_A)=\sum\mathrm{id}_{h_1,h_3}\otimes h_2\cdot 1_A=\sum\mathrm{id}_{h_1,h_3}\otimes\epsilon(h_2)1_A=\sum\mathrm{id}_{h_1,h_2}\otimes 1_A=\mathrm{id}\otimes 1_A$$

1.3. $\hat{\mathcal{H}}$-algebras. A left $\mathcal{H}$-comodule A is an ℓ-module with an ℓ-linear map $\rho: A\to\mathcal{H}\otimes A$ such that the following diagrams commute:





Following the Sweedler notation we write $\rho(a)=\sum a_{-1}\otimes a_0$. A left $\mathcal{H}$-comodule algebra A is an ℓ-algebra and a left $\mathcal{H}$-comodule such that

- μ_A is a morphism of left $\mathcal{H}$-comodules: $\rho(ab)=\sum a_{-1}b_{-1}\otimes a_0b_0 \qquad \forall a,b\in A$
- u_A is a morphism of left $\mathcal{H}$-comodules: $\rho(1_A)=1\otimes 1_A$

We call A an $\hat{\mathcal{H}}$-algebra for short; note that $\hat{\mathcal{H}}$ does not denote any object. A map $f: A\to B$ is an $\hat{\mathcal{H}}$-algebra morphism if it is an algebra morphism and a comodule morphism, that is:

- $f(ab)=f(a)f(b)$ for all $a,b\in A$.
- $\rho_B\circ f=(\mathrm{id}\otimes f)\circ\rho_A$.

Example 1.1.5. Suppose ℓ is a field. Let A be an $\hat{\mathcal{H}}$-algebra and M an $\mathcal{H}$-comodule with finite dimension over ℓ. Then $\mathrm{End}_\ell(M)\otimes A$ is an $\hat{\mathcal{H}}$-algebra with the following structure

$$\rho:\mathrm{End}_\ell(M)\otimes A\to\mathcal{H}\otimes\mathrm{End}_\ell(M)\otimes A \qquad \rho(\varphi\otimes a)(m)=\sum(\varphi(m_0))_{-1}a_{-1}\overline{S}(m_{-1})\otimes(\varphi(m_0))_0\otimes a_0$$

1.4. Dual of Hopf algebras. Let $(\mathcal{H},\mu,u,\Delta,\epsilon,S)$ be a Hopf algebra. The dual of $\mathcal{H}$ is

$$\mathcal{H}^*=\{\varphi:\mathcal{H}\to\ell:\varphi\text{ is }\ell\text{-linear}\}.$$

We consider the dual of the multiplication Δ^*, the dual unit ϵ^*, the dual counit u^* and the dual antipode S^*. We also want to consider the dual comultiplication but some difficulties arise. The image of $\mu^* : \mathcal{H}^* \to (\mathcal{H} \otimes \mathcal{H})^*$ may not lie in $\mathcal{H}^* \otimes \mathcal{H}^*$. The finite dual of $\mathcal{H}$ is defined as

$$\mathcal{H}^o = \{\varphi \in \mathcal{H}^* : \qquad \mu^*(\varphi) \in \mathcal{H}^* \otimes \mathcal{H}^*\}$$

There are many other equivalent conditions which define $\mathcal{H}^o$ (see for instance [**25**], Chapter 9). One of them is

$$\mu^*(\varphi) \in \mathcal{H}^* \otimes \mathcal{H}^* \Longleftrightarrow \dim(\varphi \leftharpoondown \mathcal{H}) < \infty \qquad \varphi \leftharpoondown \mathcal{H} := \{\varphi \leftharpoondown h \in \mathcal{H}^* : \quad (\varphi \leftharpoondown h)(k) = \varphi(hk)\}$$

If $\mathcal{H}$ is finite dimensional with basis $B = \{g_1, \cdots, g_n\}$, the set $B^* = \{g_1^*, \cdots, g_n^*\}$ is a basis of $\mathcal{H}^* = \mathcal{H}^o$. In this case we can write the dual comultiplication in the following way

$$\delta = \mu^* : \mathcal{H}^* \to \mathcal{H}^* \otimes \mathcal{H}^* \qquad \delta(\varphi) = \sum_i g_i^* \otimes (\varphi \leftharpoondown g_i)$$

Proposition 1.1.6. Let $(\mathcal{H}, \mu, u, \Delta, \epsilon, S)$ be a Hopf algebra. Then $(\mathcal{H}^o, \mu^*, u^*, \Delta^*, \epsilon^*, S^*)$ is also a Hopf algebra.

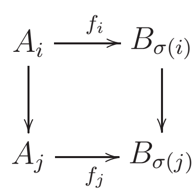
Proof: See [**25**] Theorem 9.1.3. □

1.5. ind**-categories.** In Chapter 1 we are going to work with directed diagrams of $\mathcal{H}$-algebras and $\hat{\mathcal{H}}$-algebras.

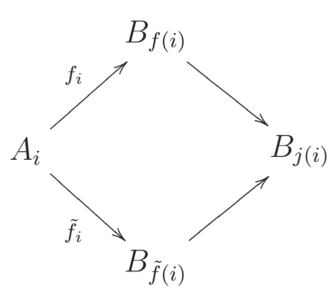
Let $\mathcal{C}$ be a category. The category of ind-objects of $\mathcal{C}$ is the category ind-$\mathcal{C}$ of directed diagrams in $\mathcal{C}$. An object in ind-$\mathcal{C}$ is described by a filtering partially ordered set $(I, \leq)$ and a functor $A : I \to \mathcal{C}$. The set of homomorphisms in ind-$\mathcal{C}$ is defined by

$$\hom_{\text{ind-}\mathcal{C}}((A, I), (B, J)) := \lim_{i \in I} \operatorname{colim}_{j \in J} \hom_{\mathcal{C}}(A_i, B_j).$$

Note each homomorphism in ind-$\mathcal{C}$ is represented by a natural transformation $\sigma : I \to J$ that is cofinal (i.e. $\sigma(I)$ is cofinal in J) and a family of homomorphisms $\{f_i\}_{i \in I}$ in $\hom_{\mathcal{C}}(A_i, B_{\sigma(i)})$ such that if $i \leq j$ the following diagram is commutative



We write $f(i)$ for $\sigma(i)$. Two families of homomorphisms in $\mathcal{C}$, $\{f_i\}_{i \in I}$ and $\{\tilde{f}_i\}_{i \in I}$ represent the same homomorphism in ind-$\mathcal{C}$ if for all $i \in I$ there exists $j(i) \in J$ such that $f(i), \tilde{f}(i) \leq j(i)$ and the following diagram is commutative



Note that there is a natural functor ind-(ind-$\mathcal{C}$) $\to$ ind-$\mathcal{C}$ mapping

$$((A_i, J_i), I) \mapsto (A_{ij}, \bigcup_{i\in I} J_i \times \{i\})$$

We shall use this functor to collapse any ind-ind-object to an ind-object. Any functor $F : \mathcal{C} \to \mathcal{D}$ extends to ind-F : ind-$\mathcal{C} \to$ ind-D by

$$F(A_i, i \in I) = (F(A_i), i \in I) \tag{8}$$

We shall identify objects of $\mathcal{C}$ with the full subcategory of ind-$\mathcal{C}$ consisting of the constant ind-objects.

2. Homotopy invariance

In this section we recall the notion of algebraic homotopy and give a brief exposition of some of its properties. In particular we discuss some properties of homotopy invariant functors. All algebras are over ℓ. $\mathcal{H}$ is a fixed Hopf algebra. We consider several properties which are valid for $\mathcal{H}$-algebras and $\hat{\mathcal{H}}$-algebras, we write $\mathcal{X}$-algebra to refer either of them. In this section tensor products are over $\mathbb{Z}$.

2.1. Algebraic homotopies. Let A be an $\mathcal{X}$-algebra. Put

$$A^{\mathbf{\Delta}^1} := A[t] = A \otimes_{\mathbb{Z}} \mathbb{Z}[t].$$

Note that $A[t]$ is an $\mathcal{X}$-algebra. Let write by $c_A : A \to A[t]$ for the inclusion of A as constant polynomials in $A[t]$ and $\mathrm{ev}_i : A[t] \to A$ for the evaluation of t at i ($i = 0, 1$). Note these morphisms are morphisms of $\mathcal{X}$-algebras and that c_A is a section of ev_i.

Let $f_0, f_1 : A \to B$ be morphisms in $\mathcal{X}$-Alg. We call f_0 and f_1 elementarily homotopic if there exists a morphism $H : A \to B[t]$ such that $\mathrm{ev}_i H = f_i$, $i = 0, 1$. We denote it by $f_0 \sim_e f_1$. A morphism $f : A \to B$ is an elementary homotopy equivalence if there exists a morphism $g : B \to A$ such that $f \circ g \sim_e \mathrm{id}_B$ and $g \circ f \sim_e \mathrm{id}_A$.

An $\mathcal{X}$-algebra A is elementarily contractible if the null morphism and the identity morphism are elementarily homotopic. In other words, A is elementarily contractible if there exists a morphism $f : A \to A[t]$ such that $\mathrm{ev}_0 \circ f = 0$ and $\mathrm{ev}_1 \circ f = \mathrm{id}_A$.

EXAMPLE 1.2.1. Let A be an $\mathcal{X}-$algebra. The path algebra $PA := \{p \in A[u] : p(0) = 0\}$ is elementarily contractible and the homotopy is given by

$$PA \to PA[t] \qquad \sum_{i\geq 1} a_i u^i \mapsto \sum_{i\geq 1} a_i (tu)^i.$$

EXAMPLE 1.2.2. Let $A = \oplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}_{\geq 0}$-graded $\mathcal{X}$-algebra. The inclusion $\iota : A_0 \to A$ is an elementary homotopy equivalence. The projection $p : A \to A_0$ is the homotopy inverse of ι because $p \circ \iota = \mathrm{id}_{A_0}$ and $\iota \circ p \sim_e \mathrm{id}_A$ where the homotopy is given by

$$H : A \to A[t] \qquad H(\sum_{i\in\mathbb{N}} a_i) = a_0 + \sum_{i\geq 1} a_i t^i.$$

Note PA is an $\mathbb{N}_{\geq 0}$-graded $\mathcal{X}$-algebra.

It is easy to check that elementary homotopy is a reflexive and symmetric relation. In general, it is not transitive.

Definition 1.2.3 (algebraic homotopy). Let $f, g : A \to B$ be morphisms of $\mathcal{X}$-algebras. We call f and g **homotopic**, and write $f \sim g$, if they can be connected by a chain of elementary homotopies,

$$f \sim_e h_0 \sim_e \ldots \sim_e h_n \sim_e g.$$

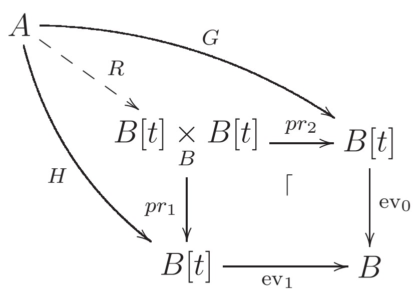
We denote the set of homotopy clases by $[A, B]_{\mathcal{X}}$. A morphism $f : A \to B$ is an **homotopy equivalence** if there exists a morphism $g : B \to A$ such that $f \circ g \sim \mathrm{id}_B$ and $g \circ f \sim \mathrm{id}_A$.

Definition 1.2.4. An $\mathcal{X}$-algebra A is **contractible** if the identity morphism and the null morphism are homotopic.

Remark 1.2.5. Let A, B be $\mathcal{X}$-algebras. We suppose $f, h, g : A \to B$ are morphisms of $\mathcal{X}$-algebras such that $f \sim_e h \sim_e g$. We can consider morphisms $H, G : A \to B[t]$ such that

$$\mathrm{ev}_0 \circ H = f \quad \mathrm{ev}_1 \circ H = h \quad \mathrm{ev}_1 \circ G = g \quad \mathrm{ev}_0 \circ G = h$$

Then we have a morphism $R : A \to B[t] \underset{B}{\times} B[t]$ such that the following diagram is commutative



$$\mathrm{ev}_0 \circ pr_1 \circ R = f \qquad \mathrm{ev}_1 \circ pr_2 \circ R = g$$

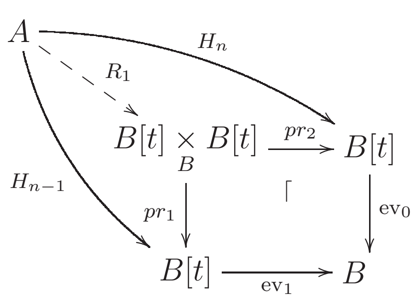
Suppose there exists a chain of elementary homotopies

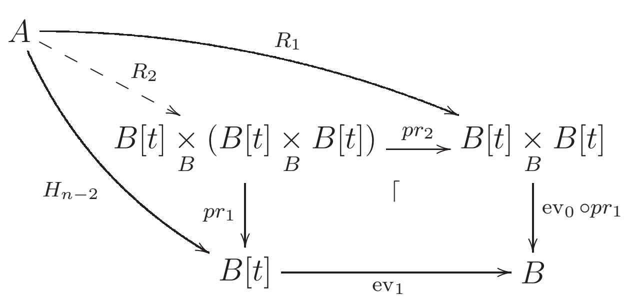
$$f \sim_e h_1 \sim_e \ldots \sim_e h_{n-1} \sim_e g.$$

We denote this chain in the following way

(9) $$\underset{f}{\bullet} \xrightarrow{H_1} \underset{h_1}{\bullet} \xrightarrow{H_2} \underset{h_2}{\bullet} \xrightarrow{H_3} \ldots \xrightarrow{H_{n-1}} \underset{h_{n-1}}{\bullet} \xrightarrow{H_n} \underset{g}{\bullet}$$

where the beginning of the arrow denotes de value at 0 and the end of the arrow denotes the value at 1. There exists a morphism $R : A \to B[t] \underset{B}{\times} (B[t] \underset{B}{\times} \ldots \underset{B}{\times} (B[t] \underset{B}{\times} B[t]))$ such that the following pull back diagrams commute





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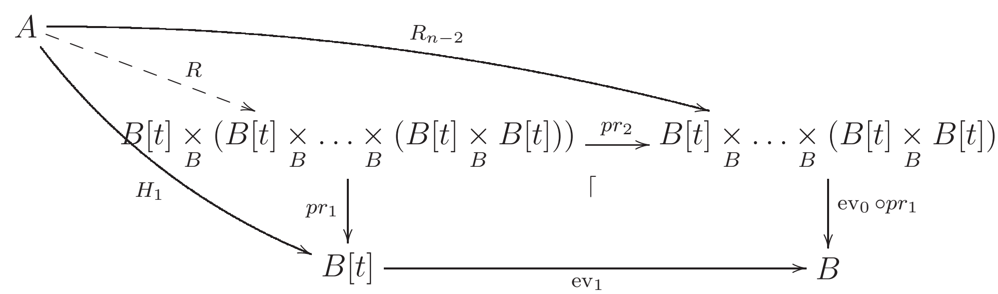


Note that

(10)
$$B[t] \underset{B}{\times} (B[t] \underset{B}{\times} \ldots \underset{B}{\times} (B[t] \underset{B}{\times} B[t])) = \{(p_1(t), \ldots, p_n(t)) : \quad p_i(t) \in B[t] \quad p_i(1) = p_{i+1}(0)\}$$

We have $\mathrm{ev}_0 \circ pr_1 \circ R = f$ and $\mathrm{ev}_1 \circ pr_2 \circ R = g$ where pr_2 is the projection to the last coordinate. It is equivalent to have the chain of homotopies (9) and to have the morphism R.

2.2. Homotopies of ind-$\mathfrak{X}$-algebras. Let $A = (A, I)$ and $B = (B, J)$ be ind-$\mathfrak{X}$-algebras, we define

$$[A, B]_{\mathfrak{X}} = \lim_{i \in I} \operatorname*{colim}_{j \in J} [A_i, B_j]_{\mathfrak{X}}$$

Note that for each $i \in I$ and $j \in J$ there is a natural map

$$\hom_{\mathfrak{X}\text{-Alg}}(A_i, B_j) \mapsto [A_i, B_j]_{\mathfrak{X}}$$

which sends each f to its homotopy class $[f]$. We also have a map

(11)
$$\hom_{\text{ind-}\mathfrak{X}\text{-Alg}}(A, B) \mapsto [A, B]_{\mathfrak{X}}.$$

We say two morphisms in ind-$\mathfrak{X}$-Alg are homotopic if their images by (11) are equal. An ind-$\mathfrak{X}$-algebra A is contractible if the null morphism and the identity morphism are homotopic.

2.3. Homotopy invariant functors. A functor $F : \mathfrak{X}\text{-Alg} \to \mathfrak{C}$ is homotopy invariant if it maps the inclusion $c_A : A \to A[t]$ to an isomorphism. The following Lemma shows an equivalent definition.

LEMMA 1.2.6. Let $F : \mathfrak{X}\text{-Alg} \to \mathfrak{C}$ be a functor. F is a homotopy invariant functor if and only if $F(f) = F(g)$ when $f \sim g$ (or equivalenty when $f \sim_e g$).

□

2.4. The simplicial algebra A^{Δ}. Define the following simplicial ring

(12)
$$\mathbb{Z}^{\boldsymbol{\Delta}} : [n] \mapsto \mathbb{Z}^{\boldsymbol{\Delta}^n} \qquad \mathbb{Z}^{\boldsymbol{\Delta}^n} := \mathbb{Z}[t_0, \ldots, t_n] / < 1 - \sum_i t_i >$$

$$\Theta : [n] \to [m] \quad \mapsto \quad \Theta^* : \mathbb{Z}^{\Delta^m} \to \mathbb{Z}^{\Delta^n}$$
$$\Theta^*(t_i) = \begin{cases} 0 & \text{si } \Theta^{-1}(i) = \emptyset \\ \sum_{j \in \Theta^{-1}(i)} t_j & \text{si } \Theta^{-1}(i) \neq \emptyset \end{cases}$$

Let A be an $\mathfrak{X}$-algebra. Define

$$A^{\boldsymbol{\Delta}} : [n] \mapsto A^{\boldsymbol{\Delta}^n} \qquad A^{\boldsymbol{\Delta}^n} := A \otimes_{\mathbb{Z}} \mathbb{Z}^{\boldsymbol{\Delta}^n}$$

Note $A^{\mathbf{\Delta}}$ is a simplicial $\mathfrak{X}$-algebra because $A^{\mathbf{\Delta}^n}$ is an $\mathfrak{X}$-algebra with $\gamma\otimes\mathrm{id}$ or $\rho\otimes\mathrm{id}$. The face maps and degeneracy maps are given by

$$d_i^n : A^{\mathbf{\Delta}^n} \to A^{\mathbf{\Delta}^{n-1}} \quad d_i^n(p)(t_0,\dots,t_{n-1}) = p(t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1}) \qquad 0\le i\le n \quad n\neq 0$$

$$s_j^n : A^{\mathbf{\Delta}^n} \to A^{\mathbf{\Delta}^{n+1}} \quad s_j^n(p)(t_0,\dots,t_{n+1}) = p(t_0,\dots,t_j+t_{j+1},t_{j+2},\dots,t_{n+1}) \quad 0\le j\le n$$

If A has unit, it is easy to check that

$$A^{\Delta^1} = A[t_0,t_1]/ < 1-t_0-t_1 > \cong A[t] \qquad {d_0}^* = \mathrm{ev}_0 \qquad {d_1}^* = \mathrm{ev}_1$$

We can enrich the category $\mathfrak{X}$-Alg over simplicial sets, as follows. We have a mapping space functor

$$\hom^{\bullet}_{\mathfrak{X}\text{-Alg}} : \mathfrak{X}\text{-Alg}^{\mathrm{op}} \times \mathfrak{X}\text{-Alg} \to \mathbb{S} \qquad (A,B) \mapsto ([n] \mapsto \hom_{\mathfrak{X}\text{-Alg}}(A, B^{\mathbf{\Delta}^n}))$$

Let $\mu : \mathbb{Z}^{\mathbf{\Delta}} \otimes \mathbb{Z}^{\mathbf{\Delta}} \to \mathbb{Z}^{\mathbf{\Delta}}$ be the multiplication map. For $A,B,C \in \mathfrak{X}$-Alg we define

$$\circledcirc : \hom^{\bullet}_{\mathfrak{X}\text{-Alg}}(B,C) \times \hom^{\bullet}_{\mathfrak{X}\text{-Alg}}(A,B) \to \hom^{\bullet}_{\mathfrak{X}\text{-Alg}}(A,C) \qquad (g\circledcirc f)_n := (\mathrm{id}_C \otimes \mu)(g^{\mathbf{\Delta}^n} \otimes f)$$

2.5. Subdivision of simplicial sets. We will recall some concepts from [**10**], Chapter 3, Section 4. Recall that the nondegenerate simplices of the standard n-simplex

$$\Delta^n = \hom_{\mathbf{\Delta}}(,[n])$$

are the monic ordinal number maps $[m] \to [n]$. There is exactly one such monomorphism for each subset of $[n]$ of cardinality $m+1$. It follows that the nondegenerate simplices of Δ^n form a poset $P\Delta^n$, ordered by the face relation, and this poset is isomorphic to that of the nonempty subsets of the ordinal number $[n]$, ordered by inclusion. The subdivision of Δ^n, is the nerve of the poset $P\Delta^n$. We write it as $\mathrm{sd}\,\Delta^n$. If X is a simplicial set, the subdivision of X, is

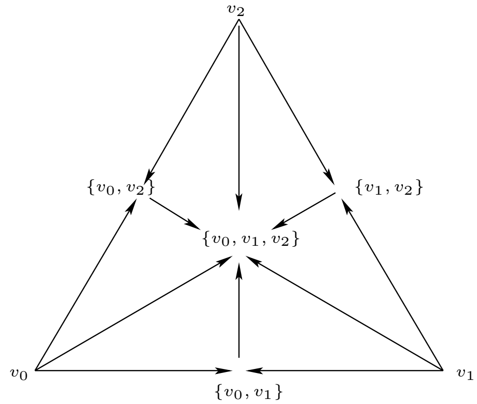
$$\mathrm{sd}\, X = \mathrm{colim}_{\sigma:\Delta^n\to X}\, \mathrm{sd}\,\Delta^n.$$

The map of posets $P\Delta^n \to [n]$ given by $[v_0,v_1,\dots,v_k] \mapsto v_k$, induces a natural map

$$h : \mathrm{sd}\,\Delta^n \to \Delta^n \tag{13}$$

which it is called last vertex map.

EXAMPLE 1.2.7. Consider Δ^2 and its vertices v_0, v_1, v_2. The vertices of $\mathrm{sd}\,\Delta^2$ are v_0, v_1, v_2, $\{v_0,v_1\}$, $\{v_0,v_2\}$, $\{v_1,v_2\}$, $\{v_0,v_1,v_2\}$, the 1-simplices are $[\{v_0\},\{v_0,v_1\}]$, $[\{v_1\},\{v_0,v_1\}]$, $[\{v_1\},\{v_1,v_2\}]$, $[\{v_2\},\{v_1,v_2\}]$, $[\{v_0\},\{v_0,v_2\}]$, $[\{v_2\},\{v_0,v_2\}]$, $[\{v_0,v_1\},\{v_0,v_1,v_2\}]$, $[\{v_1,v_2\},\{v_0,v_1,v_2\}]$, $[\{v_0,v_2\},\{v_0,v_1,v_2\}]$, $[\{v_2\},\{v_0,v_1,v_2\}]$, $[\{v_0\},\{v_0,v_1,v_2\}]$, $[\{v_2\},\{v_0,v_1,v_2\}]$, etc.



2.6. The algebra of polynomial functions on a simplicial set. Let A be an ℓ-algebra and X be a simplicial set. We define

$$A^X := \lim_{\mathbf{\Delta}^n \to X} A^{\mathbf{\Delta}^n} = \int_n \prod_{x \in X_n} A^{\mathbf{\Delta}^n} = \operatorname{map}_{\mathbb{S}}(X, A^{\mathbf{\Delta}})$$

If $(K, \star)$ is a pointed simplicial set, put

$$A^{(K,\star)} := \operatorname{map}_{\mathbb{S}_*}((K,\star), A^{\mathbf{\Delta}}) = \ker(\operatorname{map}_{\mathbb{S}}(K, A^{\mathbf{\Delta}}) \to \operatorname{map}_{\mathbb{S}}(\star, A^{\mathbf{\Delta}})) = \ker(A^K \to A)$$

From [**5**] we have the following lemmas.

Lemma 1.2.8. Let $j : K \to L$ in $\mathbb{S}$, $\star \in K$ and $A \in \mathrm{Alg}$. If j is a cofibration, then

- The map $A^L \to A^K$ is surjective.
- The sequence $0 \to A^{(L/K,\star)} \to A^{(L,\star)} \to A^{(K,\star)} \to 0$ is exact.

Proof: See Lemma 3.1.2 in [**5**] □

Lemma 1.2.9. Let K be a finite simplicial set, $\star$ a vertex of K, and A an algebra. Then $\mathbb{Z}^K$ and $\mathbb{Z}^{(K,\star)}$ are free abelian groups and there are natural isomorphisms

$$A \otimes \mathbb{Z}^K \xrightarrow{\sim} A^K \qquad\qquad A \otimes \mathbb{Z}^{(K,\star)} \xrightarrow{\sim} A^{(K,\star)}$$

Proof: See Lemma 3.1.3 in [**5**] □

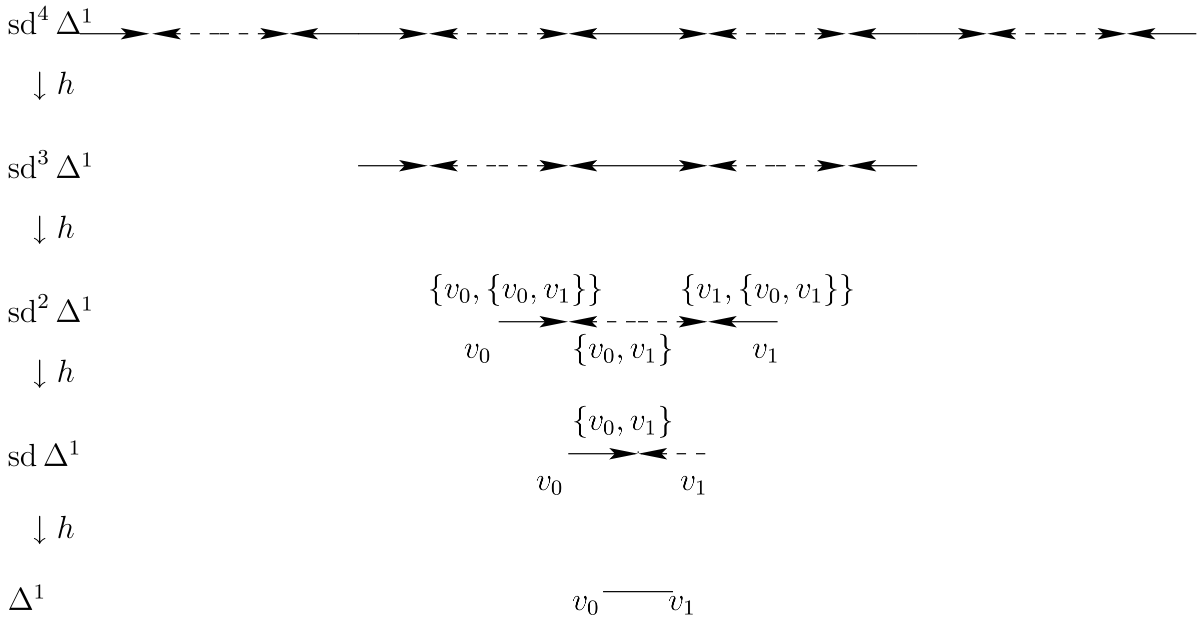
Let A be an $\mathfrak{X}$-algebra and K a finite simplicial set, by Lemma 1.2.9 A^K and $A^{(K,\star)}$ are $\mathfrak{X}$-algebras with $\gamma \otimes \mathrm{id}$ or $\rho \otimes \mathrm{id}$. We will denote by $\mathrm{sd}^\bullet X$ the following pro-simplicial set

$$\mathrm{sd}^\bullet X : \ldots \xrightarrow{h} \mathrm{sd}^n X \xrightarrow{h} \mathrm{sd}^{n-1} X \xrightarrow{h} \ldots \xrightarrow{h} \mathrm{sd}\, X \xrightarrow{h} X$$

where h is the morphism defined in (13). If A is an $\mathfrak{X}$-algebra, we consider the following ind-$\mathfrak{X}$-algebra

$$\mathrm{sd}^\bullet X^* : A^X \to A^{\mathrm{sd}\, X} \to \ldots \to A^{\mathrm{sd}^{n-1} X} \to A^{\mathrm{sd}^n X} \to \ldots$$

The next picture shows $\mathrm{sd}^\bullet \Delta^1$. The map h contracts the dashed lines to a point.



Note that

$$A^{\mathrm{sd}^n \Delta^1} = \{(p_1, \dots, p_{2^n}) : p_{2i}(0) = p_{2i+1}(0) \quad p_{2j-1}(1) = p_{2j}(1) \quad i = 1, \dots 2^{n-1}-1 \quad j = 1, \dots 2^{n-1}\}$$

and

$$h^* : A^{\mathrm{sd}^n \Delta^1} \to A^{\mathrm{sd}^{n+1} \Delta^1}$$

$$(p_1, p_2, p_3, \dots, p_{2^n-1}, p_{2^n}) \mapsto (p_1, p_1(1), p_2(1), p_2, p_3, p_3(1), p_4(1), \dots, p_{2^n-1}, p_{2^n-1}(1), p_{2^n-1}(1), p_{2^n})$$

Let us see that the algebra defined in (10) is isomorphic to $A^{\mathrm{sd}^n \Delta^1}$. In fact, put $\sigma : A[t] \to A[t]$, $\sigma(p)(t) = p(1-t)$ then the isomorphism is the following

$$\underbrace{A[t] \underset{A}{\times} A[t] \underset{A}{\times} \dots \underset{A}{\times} A[t]}_{2^n \text{ times}} \mapsto A^{\mathrm{sd}^n \Delta^1} \quad (p_1, p_2, p_3, \dots, p_{2^n}) \mapsto (p_1, \sigma(p_2), p_3, \dots, \sigma(p_{2^n}))$$

For every $n \in \mathbb{N}$ define

$$\mathrm{ev}_i : A^{\mathrm{sd}^n \Delta^1} \to A \quad i = 0, 1 \quad \mathrm{ev}_0(p_1, \dots, p_{2^n}) = \mathrm{ev}_0(p_1) \quad \mathrm{ev}_1(p_1, \dots, p_{2^n}) = \mathrm{ev}_1(p_{2^n})$$

Remark 1.2.10. Two morphisms $f, g : A \to B$ of $\mathfrak{X}$-algebras are homotopic if and only if for some n there exists a morphism $H : A \to B^{\mathrm{sd}^n \Delta^1}$ such that $\mathrm{ev}_0 \circ H = f$ and $\mathrm{ev}_1 \circ H = g$. In other words, $f \sim g$ if and only if we have an morphism $H : A \to B^{\mathrm{sd}^\bullet \Delta^1}$ in ind-$\mathfrak{X}$-Alg such that $\mathrm{ev}_0 \circ H = f$ and $\mathrm{ev}_1 \circ H = g$.

Lemma 1.2.11. The $\mathfrak{X}$-algebra $A^{(\mathrm{sd}^k \Delta^1, \star)}$ is contractible.

Proof: Put

$$B = \{(p_1, \dots, p_{2^k}) : p_i \in A[u], p_1(0) = 0, p_i(1) = p_{i+1}(0)\} \simeq A^{(\mathrm{sd}^k \Delta^1, \star)}$$

Let $H : B \to B[t] \underset{B}{\times} \dots \underset{B}{\times} B[t]$ such that

$$H(p_1, \dots, p_{2^k}) = (Q_1(p_1, \dots, p_{2^k}), \dots, Q_{2^k}(p_1, \dots, p_{2^k}))$$

$$Q_i(p_1, \dots, p_{2^k}) = (p_1(u), \dots, p_{i-1}(u), p_i(ut), p_i(t), \dots, p_i(t))$$

Note H is well-defined

$$\begin{aligned} \mathrm{ev}_1 \circ Q_i(p_1, \dots, p_{2^k}) &= (p_1(u), \dots, p_{i-1}(u), p_i(u), p_i(1), \dots, p_i(1)) \\ &= (p_1(u), \dots, p_{i-1}(u), p_i(u), p_{i+1}(0), \dots, p_{i+1}(0)) \\ &= \mathrm{ev}_0 \circ Q_{i+1}(p_1, \dots, p_{2^k}) \end{aligned}$$

and

$$\mathrm{ev}_0 \circ pr_1 \circ H(p_1, \dots, p_{2^k}) = \mathrm{ev}_0(p_1(ut), p_1(t) \dots, p_1(t)) = (p_1(0), p_1(0) \dots, p_1(0)) = 0$$
$$\mathrm{ev}_1 \circ pr_2 \circ H(p_1, \dots, p_{2^k}) = \mathrm{ev}_1(p_1(u), \dots, p_{2^k-1}(u), p_{2^k}(ut)) = (p_1(u), \dots, p_{2^k-1}(u), p_{2^k}(u))$$

□

COROLLARY 1.2.12. The ind-$\mathfrak{X}$-algebra $A^{(\mathrm{sd}^{\bullet} \Delta^1, \star)}$ is contractible. □

3. Matrix invariance

3.1. Matrix algebra. Let Λ be a infinite set and let $\mathcal{P}_F(\Lambda)$ be the set of its finite sets. We consider the inclusion relation in $\mathcal{P}_F(\Lambda)$ and we obtain a poset. Let $X \in \mathcal{P}_F(\Lambda)$ and define

$$M_X := \{\varphi : X \times X \to \mathbb{Z}\}.$$

For each $(x, y) \in X \times X$ we consider

$$e_{x,y} : X \times X \to \mathbb{Z} \quad e_{x,y}(s,t) = \begin{cases} 1 & (s,t) = (x,y) \\ 0 & (s,t) \neq (x,y) \end{cases}$$

Every matrix in M_X is a $\mathbb{Z}$-linear combination of elements $e_{x,y}$, $x, y \in X$. If $X \subset Y$ we define $\iota : M_X \to M_Y$ in the obvious way

$$\iota : M_X \to M_Y \qquad \iota(\varphi)(x,y) = \begin{cases} \varphi(x,y) & \text{if } (x,y) \in X \times X \\ 0 & \text{otherwise} \end{cases}$$

Put

$$M_\bullet := \{M_X\}_{X \in \mathcal{P}_F(\Lambda)} \qquad M_{|\Lambda|} := \underset{X}{\mathrm{colim}} M_X \qquad \mathcal{M}_{|\Lambda|} = M_\bullet \otimes M_{|\Lambda|}$$

Note $M_\bullet$ and $\mathcal{M}_{|\Lambda|}$ are ind-algebras and $M_{|\Lambda|}$ is an algebra with the multiplication given by

$$e_{x,y} \cdot e_{z,w} = \begin{cases} e_{x,w} & \text{if } y = z \\ 0 & \text{if } y \neq z \end{cases}$$

The set

$$\Gamma^l_\Lambda := \{m \in \mathbb{Z}^{\Lambda \times \Lambda} : m \cdot M_{|\Lambda|} \subseteq M_{|\Lambda|} \supseteq M_{|\Lambda|} \cdot m\}.$$

consists of those matrices in $\mathbb{Z}^{\Lambda \times \Lambda}$ having finitely many nonzero elements in each row and column. Every element T in Γ^l_Λ can be written as a formal sum

$$T = \sum_{i,j \in \Lambda} a_{i,j} e_{i,j} \quad \text{with } a_{i,j} \in \mathbb{Z} \text{ such that } \forall i \in \Lambda \text{ the sets } \{j : a_{i,j} \neq 0\} \text{ and } \{j : a_{j,i} \neq 0\} \text{ are finite sets}$$

LEMMA 1.3.1. Let $V, W \in \Gamma^l_\Lambda$ such that $VW = 1$. Define

$$\psi : M_{|\Lambda|} \to M_{|\Lambda|} \quad \psi(A) = WAV \quad \hat{\psi} : M_\bullet \to M_\bullet \quad \hat{\psi}(A) = WAV \quad 1 \otimes \psi : \mathcal{M}_{|\Lambda|} \to \mathcal{M}_{|\Lambda|}$$

ψ is an morphism of algebras, $\hat{\psi}$ and $1 \otimes \psi$ are morphisms of ind-algebras. If ι denotes the inclusion in the first entry of M_2, then $\iota\psi$ is homotopic to ι. The morphisms $\hat{\psi}$ and $1 \otimes \psi$ are homotopic to the identity.

PROOF: See Lemma 4.1.1 in [**5**]. □

As Λ is a infinite set, we can choose a subset of Λ biyectable to $\mathbb{N}$. Suppose $\Lambda = \mathbb{N} \times \lambda$. In order to define a direct sum of matrices in $M_{|\Lambda|}$ consider, $n, m \in \mathbb{N}$, $g, h \in \lambda$ and the elements in Γ^l_Λ

$$V(n,g,m,h) = \begin{cases} 1 & if \quad m = 2n \quad g = h \\ 0 & \text{otherwise} \end{cases} \qquad V^*(n,g,m,h) = \begin{cases} 1 & if \quad 2m = n \quad g = h \\ 0 & \text{otherwise} \end{cases}$$

$$W(n,g,m,h) = \begin{cases} 1 & if \quad m = 2n+1 \quad g = h \\ 0 & \text{otherwise} \end{cases} \qquad W(n,g,m,h) = \begin{cases} 1 & if \quad 2m+1 = n \quad g = h \\ 0 & \text{otherwise} \end{cases}$$

Define $\oplus : M_{|\Lambda|} \times M_{|\Lambda|} \to M_{|\Lambda|}$ as follows

$$\varphi \oplus \tilde{\varphi} = V\varphi V^* + W\tilde{\varphi}W^*.$$

There is also a tensor product defined in $M_{|\Lambda|}$. Let $\eta : (\mathbb{N} \times \lambda) \to (\mathbb{N} \times \lambda) \times (\mathbb{N} \times \lambda)$ a biyection. Define $\star : M_{|\Lambda|} \times M_{|\Lambda|} \to M_{|\Lambda|}$ as

$$(\varphi \star \tilde{\varphi})(x,y) := \varphi(\eta(x))\tilde{\varphi}(\eta(y))$$

We obtain that $(\mathcal{M}_{|\Lambda|}, \oplus, \star)$ is a homotopy semiring, see Section 4.1 of [**5**]. If $\Lambda = \mathbb{N}$ we will denote $M_{|\mathbb{N}|} = M_\infty$ and $\mathcal{M}_{|\mathbb{N}|} = \mathcal{M}_\infty$.

3.2. Matrix invariant functors. Let A be an $\mathcal{X}$-algebra. We define

$$M_n A = M_n \otimes_{\mathbb{Z}} A \qquad M_\infty A = M_\infty \otimes_{\mathbb{Z}} A$$

which are $\mathcal{X}$-algebras in the obvious way. Denote by $\iota_n : \mathbb{Z} \to M_n$ and $\iota_\infty : \mathbb{Z} \to M_\infty$ the inclusions at the upper left corner. A functor $F : \mathcal{X}\text{-Alg} \to \mathcal{C}$ is M_n-stable (M_∞-stable) if $F(\iota_n \otimes \mathrm{id}_A)$ $(F(\iota_\infty \otimes \mathrm{id}_A))$ is an isomorphism for all $A \in \mathcal{X}\text{-Alg}$.

4. Extensions and classifying maps

4.1. Extension. Following [**5**], a sequence of morphisms in ind-$\mathcal{X}$-Alg

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{14}$$

is called an extension if f is a kernel of g and g is a cokernel of f.

Let $\mathcal{X}$-mod be the category of $\mathcal{X}$-modules with linear and equivariant maps. Let $F : \mathcal{X}\text{-Alg} \to \mathcal{X}\text{-mod}$ be the forgetful functor. This functor can be extended to F : ind-$\mathcal{X}$-Alg $\to$ ind-$\mathcal{X}$-mod as is shown in (8). We will call an extension (14) weakly split if $F(g)$ has a section in $\mathcal{X}$-mod.

4.2. Path extension. Let S^1 be the simplicial circle $\mathbf{\Delta}^1/\partial\mathbf{\Delta}^1$, we define

$$\Omega := \mathbb{Z}^{(S^1,\star)}$$

The path extension of an $\mathcal{X}$-algebra A is the extension iduced by the cofibration $\partial\mathbf{\Delta}^1 \subset \mathbf{\Delta}^1$, see 1.2.8.

$$\Omega A \longrightarrow A^{\mathbf{\Delta}^1} \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} A \oplus A \tag{15}$$

The extension (15) is weakly split because we have a linear and $\mathcal{X}$-equivariant section of $(\mathrm{ev}_0, \mathrm{ev}_1)$

$$(a, b) \mapsto (1 - t)a + tb. \tag{16}$$

4.3. Universal extension. Let M be an $\mathcal{H}$-module. Consider in

$$\tilde{T}(M) = \bigoplus_{n\geq 1} M^{\otimes^n} \quad M^{\otimes^n} = \underbrace{M \otimes \ldots \otimes M}_{n\text{-times}}$$

the usual structure of $\mathcal{H}$-module. The action in $M^{\otimes^n}$ is given by

$$h \cdot (m_1 \otimes m_2 \otimes \ldots \otimes m_n) = \sum h_1 \cdot m_1 \otimes h_2 \cdot m_2 \otimes \ldots \otimes h_n \cdot m_n$$

It is easy to check that $\tilde{T}(M)$ is an $\mathcal{H}$-module algebra with this action. Similarily, let R be an $\mathcal{H}$-comodule. Consider in

$$\tilde{T}(R) = \bigoplus_{n\geq 1} R^{\otimes^n}$$

the usual structure of $\mathcal{H}$-comodule. The coaction in $R^{\otimes^n}$ is given by

$$\rho(r^1 \otimes r^2 \otimes \ldots \otimes r^n) = \sum r^1_{-1} r^2_{-1} \ldots r^n_{-1} \otimes r^1_0 \otimes \ldots \otimes r^n_0$$

It is easy to check that $\tilde{T}(R)$ is an $\mathcal{H}$-comodule algebra with this coaction. Both constructions are functorial hence we consider a functor $\tilde{T} : \mathcal{X}\text{-mod} \to \mathcal{X}\text{-Alg}$. Put

$$T := \tilde{T} \circ F : \mathcal{X}\text{-Alg} \to \mathcal{X}\text{-Alg}$$

If A is an $\mathcal{X}$-algebra there exists an $\mathcal{X}$-algebra morphism

$$\eta_A : T(A) \to A \qquad \eta_A(a_1 \otimes \ldots \otimes a_n) = a_1 \ldots a_n$$

and an $\mathcal{X}$-module morphism $\mu_A : A \to T(A)$ which is the inclusion at the first summand of $T(A)$.

Remark 1.4.1. Let A, B be $\mathcal{X}$-algebras, it is easy to check that

$$\hom_{\mathcal{X}\text{-Alg}}(T(A), B) \simeq \hom_{\mathcal{X}\text{-mod}}(F(A), F(B)).$$

Hence if we have a morphism $A \to B$ in $\mathcal{X}$-mod, we can extend it to a morphism $T(A) \to B$ of $\mathcal{X}$-algebras. It shows that T is the left adjoint of F.

The counit of the adjuntion is $\eta_A : T(A) \to A$. It is surjective (see [**22**], IV.3 Thm 1). We define

$$J(A) := \ker \eta_A.$$

The universal extension of A is

$$J(A) \xrightarrow{\imath_A} T(A) \xrightarrow{\eta_A} A.$$

Proposition 1.4.2. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an weakly split extension. There exists a commutative diagram of extensions as follows

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \uparrow{\scriptstyle \xi} & & \uparrow{\scriptstyle \hat{\xi}} & & \uparrow{\scriptstyle \mathrm{id}_C} \\ J(C) & \xrightarrow[\imath_C]{} & T(C) & \xrightarrow[\eta_C]{} & C \end{array}$$

where the map ξ is unique up to elementary homotopy.

Proof: Let s be a section of $F(g)$ and define $\hat{\xi} = \eta_B \circ \tilde{T}(s)$, then

$$\eta_C = \eta_C \circ T(g) \circ \tilde{T}(s) = g \circ \eta_B \circ \tilde{T}(s) = g \circ \hat{\xi}.$$

Define $\xi : J(C) \to A$ as the restriction of $\hat{\xi}$ to $J(C)$. This construction of ξ depends on which s is chosen. If $\xi_1, \xi_2 : J(C) \to A$ are morphisms constructed taking different sections of $F(g)$ we will show ξ_1, ξ_2 are homotopic. Define a linear equivariant map

$$H : C \to A[t] \quad H(c) = (1-t)\xi_1(c) + t\xi_2(c).$$

By 1.4.1 it extends to a homomorphism and there exists a morphism $H : T(C) \to A[t]$ in $\mathfrak{X}$-Alg such that

$$\mathrm{ev}_0 \circ H_{|J(C)} = \xi_1 \qquad \mathrm{ev}_1 \circ H_{|J(C)} = \xi_2.$$

□

We call ξ the classifying map of the extension $A \xrightarrow{f} B \xrightarrow{g} C$. We abuse notation because we shall work with maps up to homotopy.

Proposition 1.4.3. ([**5**] 4.4.2) Let

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

be a commutative diagram of weakly split extensions. Then there is a diagram

$$\begin{array}{ccc} J(C) & \longrightarrow & A \\ \downarrow{\scriptstyle J(g)} & & \downarrow{\scriptstyle f} \\ J(C') & \longrightarrow & A' \end{array}$$

of classifying maps, which is commutative up to elementary homotopy.

Proposition 1.4.4. ([**5**] 4.4.3) Let L be a ring and A an $\mathfrak{X}$-algebra. Then the extension

$$J(A)\otimes_{\mathbb{Z}} L \to T(A)\otimes_{\mathbb{Z}} L \to A\otimes_{\mathbb{Z}} L \tag{17}$$

is weakly split, and there is a chioce for classifying map $\phi_{A,L} : J(A\otimes_{\mathbb{Z}} L) \to J(A)\otimes_{\mathbb{Z}} L$ of this extension, which is natural in both variables. □

Corollary 1.4.5. ([**5**] 4.4.4) Let K be a finite pointed simplicial set. There is a homotopy class of maps

$$\phi_{A,K} : J(A^K) \to J(A)^K \tag{18}$$

natural with respect to K, which is represented by a classifying map of following the extension

$$J(A)^K \xrightarrow{\iota_A^*} T(A)^K \xrightarrow{\eta_A^*} A^K$$

Naturality means that for any map of finite pointed simplicial sets $f : K \to L$, the following diagram commutes

$$\begin{array}{ccc} J(A^K) & \longrightarrow & J(A)^K \\ {\scriptstyle J(f^*)}\downarrow & & \downarrow{\scriptstyle f^*} \\ J(A^L) & \longrightarrow & J(A)^L \end{array}$$

□

4.4. Loop extension. Let A be an $\mathfrak{X}$-algebra. The path algebra of A is $PA := P\otimes_{\mathbb{Z}} A$ with $P := \mathbb{Z}^{(\boldsymbol{\Delta}^1,\star)}$. The loop extension of A is

$$\Omega A \to PA \xrightarrow{\mathrm{ev}_1} A \tag{19}$$

Note it is naturally weakly split because $a \mapsto at$ is a natural section of $F(\mathrm{ev}_1)$. Thus we can pick a natural choice for the classifying map of (19). We call it

$$\rho_A : J(A) \to \Omega A$$

Let

$$\mathcal{P} := \mathbb{Z}^{(\mathrm{sd}^\bullet\, \boldsymbol{\Delta}^1,\star)}$$

we have an extension of ind-$\mathfrak{X}$-algebras

$$A^{\mathbb{S}^1} := A^{(\mathrm{sd}^\bullet\, S^1,\star)} \qquad A^{\mathbb{S}^1} \to \mathcal{P}A \xrightarrow{\mathrm{ev}_1} A$$

which is naturally weakly split. The classifying map $J(A) \to A^{\mathbb{S}^1}$ is the following composition

$$J(A) \xrightarrow{\rho_A} \Omega A \xrightarrow{h} A^{\mathbb{S}^1} \tag{20}$$

where h is induced by the last vertex map. We will sometimes abuse notation and write ρ_A for the map (20).

4.5. Mapping path extension. Let $f : A \to B$ be a morphism of $\mathcal{X}$-algebras. The mapping path extension of f is the extension obtained from the loop extension of B by pulling it back to A

$$P_f := PB \times_B A \qquad \begin{array}{ccccc} \Omega B & \xrightarrow{\iota} & PB \times_B A & \xrightarrow{\pi_f} & A \\ \downarrow & & \downarrow & & \downarrow f \\ \Omega B & \xrightarrow[\iota]{} & PB & \xrightarrow[\mathrm{ev}_1]{} & B \end{array} \qquad \begin{array}{c} (\mathrm{E}') \\ \\ (\mathrm{E}) \end{array}$$

We call P_f the path algebra of f. Note the extension (E′) is naturally weakly split because $\tilde{s}(a) = (s \circ f(a), a)$ is a natural section of π_f where s is the natural section of (E) given in the Section 4.4. We define

$$\mathcal{P}_f := \mathcal{P}B \times_B A \qquad B^{\mathcal{S}^1} \to \mathcal{P}B \times_B A \xrightarrow{\pi_f} A. \tag{21}$$

Note $\rho_f := \rho_B J(f)$ is the classifying map of the extension (21).

5. Excision

5.1. Triangulated categories. We recall the definition of a triangulated category [**21**], [**5**]. We will express the definition in terms of a loop functor Ω. A triangulated category $(\mathcal{T}, \Omega, \mathcal{Q})$ is an additive category $\mathcal{T}$ with an equivalence $\Omega : \mathcal{T} \to \mathcal{T}$ and a class $\mathcal{Q}$ of sequences in $\mathcal{T}$ called distinguished triangles

$$(T) \qquad \Omega C \to A \to B \to C$$

satisfying the following axioms:

TR0: Any sequence isomorphic to a distinguished triangle is a distinguished triangle. The sequence

$$\Omega A \to 0 \to A \xrightarrow{\mathrm{id}_A} A$$

is a distinguished triangle for every object A in $\mathcal{T}$.

TR1: For any morphism $\alpha : A \to B$ in $\mathcal{T}$, there exists a distinguished triangle of the form

$$\Omega B \to C \to A \xrightarrow{\alpha} B$$

TR2: Consider the two triangles

$$\Omega C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \qquad\qquad \Omega B \xrightarrow{-\Omega(h)} \Omega C \xrightarrow{-f} A \xrightarrow{-g} B$$

If one is distinguished, then so is the other.

TR3: For any solid arrow commutative diagram

$$\begin{array}{ccccccc} \Omega C & \xrightarrow{f} & A & \xrightarrow{g} & B & \xrightarrow{h} & C \\ \downarrow \Omega k & & \downarrow j & & \downarrow l & & \downarrow k \\ \Omega C' & \xrightarrow[f']{} & A' & \xrightarrow[g']{} & B' & \xrightarrow[h']{} & C' \end{array}$$

there exists a filler $j : A \to A'$ which makes the whole diagram commutative.

TR4: Let $\alpha : A \to B$ and $\beta : B \to C$ be morphism in $\mathcal{T}$. We set $\gamma = \beta\alpha$. There exists a commutative diagram

$$\begin{array}{ccccccc}
\Omega^2 C & \longrightarrow & \Omega D' & \longrightarrow & \Omega B & \xrightarrow{\Omega\beta} & \Omega C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & D''' & \xrightarrow{1} & D''' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow \\
\Omega C & \xrightarrow{j} & D'' & \longrightarrow & A & \xrightarrow{\gamma} & C \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1} \\
\Omega C & \longrightarrow & D' & \longrightarrow & B & \xrightarrow{\beta} & C
\end{array}$$

in which each row and column is a distinghished triangle. Furthermore, the square

$$\begin{array}{ccc}
\Omega B & \xrightarrow{\Omega\beta} & \Omega C \\
\downarrow & & \downarrow{\scriptstyle j} \\
D''' & \xrightarrow[h]{} & D''
\end{array}$$

commutes in $\mathcal{T}$.

A triangle functor from $(\mathcal{T}_1, \Omega_1, \mathcal{Q}_1)$ to $(\mathcal{T}_2, \Omega_2, \mathcal{Q}_2)$ is a pair consisting of an additive functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ and a natural transformation $\alpha : \Omega_2 F \to F\Omega_1$ such that

$$\Omega_2 F(C) \xrightarrow{F(f)\circ\alpha_C} F(A) \xrightarrow{F(g)} F(B) \xrightarrow{F(h)} F(C)$$

is a distinguished triangle in $\mathcal{T}_2$ for each triangle $\Omega_1 C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$.

5.2. Excisive homology theories. Let $\mathcal{E}$ be a class of extensions

$$\text{(E)} \qquad A \xrightarrow{f} B \xrightarrow{g} C \tag{22}$$

in $\mathfrak{X}$-Alg, and let $(\mathcal{T}, \Omega, \mathcal{Q})$ be a triangulated category. An $\mathcal{E}$-excisive homology theory for $\mathfrak{X}$-algebras with values in $\mathcal{T}$ consists of a functor $H : \mathfrak{X}\text{-Alg} \to \mathcal{T}$, together with a collection $\{\partial_{\mathrm{E}} : \mathrm{E} \in \mathcal{E}\}$ of maps

$$\partial_{\mathrm{E}}^H = \partial_{\mathrm{E}} \in \hom_{\mathcal{T}}(\Omega H(C), H(A)).$$

The maps ∂_{E} are to satisfy the following requirements:

- For all E as in (22)
$$\Omega H(C) \xrightarrow{\partial_{\mathrm{E}}} H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C)$$
is a distinguished triangle in $\mathcal{T}$.
- If
$$\begin{array}{rccccc}
(\mathrm{E}): & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow \\
(\mathrm{E}'): & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C'
\end{array}$$

is a map of extensions, then the following diagram commutes:

$$\begin{array}{ccc} \Omega H(C) & \xrightarrow{\partial_{\mathrm{E}}} & H(A) \\ {\scriptstyle \Omega H(\gamma)}\downarrow & & \downarrow{\scriptstyle H(\alpha)} \\ \Omega H(C') & \xrightarrow[\partial_{\mathrm{E}'}]{} & H(A) \end{array}$$

6. Algebraic kk-theory

6.1. Universal functors. Let P be a property for functors defined on $\mathfrak{X}$-Alg to some triangulated category. A functor $u : \mathfrak{X}\text{-Alg} \to \mathbb{U}$ is a universal functor with P if it has the property P and if $F : \mathfrak{X}\text{-Alg} \to \mathfrak{T}$ is another functor with P there exists a unique triangle functor $G : \mathbb{U} \to \mathfrak{T}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{X}\text{-Alg} & \xrightarrow{u} & \mathbb{U} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle G} \\ & & \mathfrak{T} \end{array}$$

In the next section we shall see the definition of algebraic kk-theory and a construction of a universal functor with $P = \{P_1, P_2, P_3\}$ where the properties are: excision (see 5.2), homotopy invariance (see 2.3), and matrix invariance (see 3.2). In Section 8 we define equivariant kk-theory and we construct a functor which is universal for the properties: excision, homotopy invariance and equivariant matrix invariance (defined in Section 7).

6.2. kk-groups. Let A, B be $\mathfrak{X}$-algebras. Define

$$E_n(A, B) := [J^n(A), \mathcal{M}_\infty B^{\mathcal{S}^n}]_{\mathfrak{X}}.$$

Consider the following morphism $\imath_n : E_n(A, B) \to E_{n+1}(A, B)$ such that

$$J^n(A) \xrightarrow{f} \mathcal{M}_\infty B^{\mathcal{S}^n} \quad \mapsto \quad J^{n+1}(A) \xrightarrow{J(f)} J(\mathcal{M}_\infty B^{\mathcal{S}^n}) \xrightarrow{\phi_{\mathcal{M}_\infty, B^{\mathcal{S}^n}}} \mathcal{M}_\infty J(B^{\mathcal{S}^n}) \xrightarrow{\rho_{B^{\mathcal{S}^n}}} \mathcal{M}_\infty B^{\mathcal{S}^{n+1}}$$

Define

$$\mathrm{kk}^{|\mathfrak{X}|}(A, B) = \mathrm{colim}_{n \in \mathbb{N}} E_n(A, B)$$

6.3. Composition product. A morphism of $\mathfrak{X}$-algebras $f : A \to B$ induces a morphism of weakly split extensions

$$\begin{array}{ccccc} A^{\mathcal{S}^1} & \longrightarrow & \mathcal{P}A & \longrightarrow & A \\ {\scriptstyle f^{\mathcal{S}^1}}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ B^{\mathcal{S}^1} & \longrightarrow & \mathcal{P}B & \longrightarrow & B \end{array}$$

By Proposition 1.4.3 the following diagram is commutative

$$\begin{array}{ccc} J(A) & \xrightarrow{\rho_A} & A^{\mathcal{S}^1} \\ {\scriptstyle J(f)}\downarrow & & \downarrow{\scriptstyle f^{\mathcal{S}^1}} \\ J(B) & \xrightarrow[\rho_B]{} & B^{\mathcal{S}^1} \end{array}$$

Then $f^{\mathcal{S}^1}\rho_A$ and $\rho_f = \rho_B J(f)$ represent the same element in $\mathrm{kk}^{|\mathfrak{X}|}(A,B)$. Denote by $\gamma_A : J(A^{\mathcal{S}^1}) \to J(A)^{\mathcal{S}^1}$ the morphism (18) and note

$$\rho_{J(A)} \simeq \gamma_A J(\rho_A) : J^2(A) \to J(A)^{\mathcal{S}^1}$$

THEOREM 1.6.1. ([**5**]) Let A, B and C be $\mathfrak{X}$-algebras. There exists an associative product

$$\circ : \mathrm{kk}^{|\mathfrak{X}|}(B,C) \times \mathrm{kk}^{|\mathfrak{X}|}(A,B) \to \mathrm{kk}^{|\mathfrak{X}|}(A,C)$$

which extends the composition of algebra homomorphisms.

If $[\alpha] \in \mathrm{kk}^{|\mathfrak{X}|}(B,C)$ is an element represented by $\alpha : J^n(B) \to C^{\mathcal{S}^n}$ and $[\beta] \in \mathrm{kk}^{|\mathfrak{X}|}(A,B)$ is an element represented by $\beta : J^m(A) \to B^{\mathcal{S}^m}$ then $[\alpha]\circ[\beta]$ is an element $\mathrm{kk}^{|\mathfrak{X}|}(A,C)$ represented by

$$J^{n+m}(A) \xrightarrow{J^n(\beta)} J^n(B^{\mathcal{S}^m}) \to J^n(B)^{\mathcal{S}^m} \xrightarrow{\alpha^{\mathcal{S}^m}} C^{\mathcal{S}^{n+m}}$$

6.4. The category $\mathfrak{KK}^{|\mathfrak{X}|}$. The product given in theorem 1.6.1 allows us to define a composition in the following category $\mathfrak{KK}^{|\mathfrak{X}|}$:

- The objects are the $\mathfrak{X}$-algebras: $\mathrm{ob}(\mathfrak{KK}^{|\mathfrak{X}|}) = \mathrm{ob}(\mathfrak{X}\text{-Alg})$.
- The morphisms are the $\mathrm{kk}^{|\mathfrak{X}|}$-groups: $\hom_{\mathfrak{KK}^{|\mathfrak{X}|}}(A,B) = \mathrm{kk}^{|\mathfrak{X}|}(A,B)$.

Denote

$$j : \mathfrak{X}\text{-Alg} \to \mathfrak{KK}^{|\mathfrak{X}|}$$

to the functor which at the level of objects is the identity and at level of morphism sends $f : A \to B$ to $[f] \in \mathrm{kk}^{|\mathfrak{X}|}(A,B)$.

REMARK 1.6.2. Let C be an $\mathfrak{X}$-algebra. A morphism of $\mathfrak{X}$-algebras $f : C \to \mathcal{M}_\infty C$ represents an element $j(f)$ in $\mathrm{kk}^{|\mathfrak{X}|}(C, \mathcal{M}_\infty C)$. But also represents an element in $\mathrm{kk}^{|\mathfrak{X}|}(C,C)$ because

$$\mathrm{kk}^{|\mathfrak{X}|}(C,C) = \operatorname{colim} E_n(C,C) \quad \text{and} \quad E_0(C,C) = [C, \mathcal{M}_\infty C].$$

A priori, kk-theory is only defined for $\mathfrak{X}$-algebras. However, if (A,J) is an ind-$\mathfrak{X}$-algebra for which all structure maps are $kk^{|\mathfrak{X}|}$-equivalences, we can equally well speak about its $kk^{|\mathfrak{X}|}$-groups. The $kk^{|\mathfrak{X}|}$-groups with an ind-$\mathfrak{X}$-algebra in the right argument are defined via the colimit of the induces diagram of $kk^{|\mathfrak{X}|}$-groups, i.e.

$$\mathrm{kk}^{|\mathfrak{X}|}(A,(B,J)) := \operatorname{colim}_{j\in J} \mathrm{kk}^{|\mathfrak{X}|}(A,B_j)$$

The statement that two ind-$\mathfrak{X}$-algebras B and C are $kk^{|\mathfrak{X}|}$-equivalent, has the rather strict meaning, that all structure maps of B and C are $kk^{|\mathfrak{X}|}$-equivalences and all morphisms that constitute the morphism of ind-$\mathfrak{X}$-algebras are $kk^{|\mathfrak{X}|}$-equivaleneces as well.

6.5. The triangulated structure of $\mathfrak{KK}^{|\mathfrak{X}|}$. In this section we describe the triangulated structure (see 5.1) in $\mathfrak{KK}^{|\mathfrak{X}|}$. We shall define the endofunctor $\Omega : \mathfrak{KK}^{|\mathfrak{X}|} \to \mathfrak{KK}^{|\mathfrak{X}|}$ and the class $\mathcal{Q}$ of distinghished triangles.

6.5.1. *The endofunctor* Ω. The functor $\Omega : \mathfrak{KK}^{|\mathfrak{X}|} \to \mathfrak{KK}^{|\mathfrak{X}|}$ sends any $\mathfrak{X}$-algebra A to the path $\mathfrak{X}$-algebra ΩA. Let $[\alpha]$ be an element of $\mathrm{kk}^{|\mathfrak{X}|}(A, B)$ represented by $\alpha : J^n(A) \to B^{\mathbb{S}^n}$. The class of $\Omega[\alpha]$ is represented by

$$J^n(A^{\mathbb{S}^1}) \to J^n(A)^{\mathbb{S}^1} \xrightarrow{\alpha^{\mathbb{S}^1}} B^{\mathbb{S}^{n+1}} \tag{23}$$

Note (23) represents an element of $\mathrm{kk}^{|\mathfrak{X}|}(A^{\mathbb{S}^1}, B^{\mathbb{S}^1})$ (see Lemma 6.3.8 [**5**] to check it is well defined). As $\imath : \Omega A \to A^{\mathbb{S}^1}$ is a $\mathrm{kk}^{|\mathfrak{X}|}$-equivalence (see corollary of Lemma 6.3.2 [**5**]), we have the natural morphism

$$\mathrm{kk}^{|\mathfrak{X}|}(A^{\mathbb{S}^1}, B^{\mathbb{S}^1}) \xrightarrow{\sim} \mathrm{kk}^{|\mathfrak{X}|}(\Omega A, \Omega B)$$

6.5.2. *Distinguished Triangles.* A diagram

$$\Omega C \to A \to B \to C$$

of morphisms in $\mathfrak{KK}^{|\mathfrak{X}|}$ is called a distinguished triangle if it is isomorphic in $\mathfrak{KK}^{|\mathfrak{X}|}$ to the path sequence

$$\Omega B' \xrightarrow{j(\iota)} P_f \xrightarrow{j(\pi_f)} A' \xrightarrow{j(f)} B'$$

associated with a homomorphism $f : A' \to B'$ of $\mathfrak{X}$-algebras. Denote the class of distinghished triangles by $\mathcal{Q}$.

THEOREM 1.6.3 ([**5**]). *The category* $\mathfrak{KK}^{|\mathfrak{X}|}$ *is triangulated with respect to the endofunctor* $\Omega : \mathfrak{KK}^{|\mathfrak{X}|} \to \mathfrak{KK}^{|\mathfrak{X}|}$ *and the class* $\mathcal{Q}$ *of distinguished triangles.*

6.6. Universal properties of $\mathfrak{KK}^{|\mathfrak{X}|}$. Let $\mathrm{E} : A \xrightarrow{f} B \xrightarrow{g} C$ be an weakly split extension and let $c_{\mathrm{E}} \in \mathrm{kk}^{|\mathfrak{X}|}(J(C), A)$ be the classifying map of E. As the natural map $\rho_A : J(A) \to \Omega A$ induces a kk-equivalence (see Lemma 6.3.10, [**5**]) we can consider the following morphisms in $\mathrm{kk}^{|\mathfrak{X}|}(\Omega C, A)$

$$\partial_{\mathrm{E}} := c_{\mathrm{E}} \circ \rho_C^{-1}. \tag{24}$$

The functor $j : \mathfrak{X}\text{-Alg} \to \mathfrak{KK}^{|\mathfrak{X}|}$ with the morphisms $\{\partial_{\mathrm{E}} : \mathrm{E} \in \mathcal{E}\}$ is an excisive homology theory, homotopy invariant and M_∞-stable. In fact, let E be an extension as in (22). Take the path sequence asociated to g and the following diagram in $\mathfrak{KK}^{|\mathfrak{X}|}$

$$\begin{array}{ccccccccc}
\Omega C & \xrightarrow{j(\iota)} & P_g & \xrightarrow{j(\pi_g)} & B & \xrightarrow{j(g)} & C & & (T) \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \iota_f} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
\Omega C & \xrightarrow[\partial_{\mathrm{E}}]{} & A & \xrightarrow[j(f)]{} & B & \xrightarrow[j(g)]{} & C & & (T')
\end{array}$$

The first square commutes beacuse $\iota_f \circ c_{\mathrm{E}}$ is elementarily homotopic to $\iota \circ \rho_C$. By Lemma 6.3.2 [**5**], the morphism ι_f is a $\mathrm{kk}^{|\mathfrak{X}|}$-equivalence. Finally (T) and (T') are isomorphic in $\mathfrak{KK}^{|\mathfrak{X}|}$ and (T') is a distinguished triangle.

THEOREM 1.6.4. ([**5**]) The functor $j : \mathcal{X}\text{-Alg} \to \mathfrak{KK}^{|\mathcal{X}|}$ is universal with the properties defined above. In other words, if $\mathcal{C}$ is a triangulated category and $G : \mathcal{X}\text{-Alg} \to \mathcal{C}$ together a class of morphisms $\{\overline{\partial}_{\mathrm{E}} : \mathrm{E} \in \mathcal{E}\}$ is an excisive, homotopy invariant and M_∞-stable functor, then there exists a unique triangle functor $\overline{G} : \mathfrak{KK}^{|\mathcal{X}|} \to \mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{X}\text{-Alg} & \xrightarrow{\;j\;} & \mathfrak{KK}^{|\mathcal{X}|} \\ & {\scriptstyle G}\searrow & \downarrow {\scriptstyle \overline{G}} \\ & & \mathcal{C} \end{array}$$

THEOREM 1.6.5 ([**5**]). Consider $\mathcal{X} = \ell$, then

$$\mathrm{kk}_*^{|\ell|}(\ell, A) = \mathrm{kk}_*(\ell, A) \simeq \mathrm{KH}_*(A)$$

Here KH_* is the homotopy K-theory of Wiebel, see [**32**]. We recall this definition in a general context in Section 5 of Chapter 3.

7. Equivariant matrix invariance

In the equivariant setting we will replace the property of M_∞-stability by an $\mathcal{X}$-stability condition. Let consider the different cases of $\mathcal{X}$ separately.

7.1. G-stability. Let G be a group and ℓ a commutative ring with unit. In this section we define a G-algebra called M_{G}. Let $M_{|\,\mathrm{G}\,|}$ be the matrix algebra defined in 3.1. The translation action in G defines an action in $M_{|\,\mathrm{G}\,|}$

$$g \cdot e_{s,t} = e_{gs,gt}.$$

Recall a functor is homotopy invariant if it sends the inclusion $A \to A[t]$ to an isomorphism. A $M_{|\,\mathrm{G}\,|}$-stable functor on G-Alg sends the morphism $a \to a \otimes e_{1_{\mathrm{G}},1_{\mathrm{G}}}$ to an isomorphism. We want a definition of G-stability such that a G-stable functor on G-Alg, identifies an G-algebra A with the algebra $M_{\mathrm{G}} \otimes A$. In this case we may not have an equivariant map between A and $M_{\mathrm{G}} \otimes A$. Note that the map $a \to a \otimes e_{1_{\mathrm{G}},1_{\mathrm{G}}}$ is not equivariant. For this reason we will define G-stability in terms of G-modules.

DEFINITION 1.7.1 (G-module with basis).

- A pair $(\mathcal{W}, B)$ is a G-**module with basis** if $\mathcal{W}$ is a G-module, free as an ℓ-module and B is a basis of $\mathcal{W}$.
- A pair $(\mathcal{W}', B')$ is a **submodule with basis** of $(\mathcal{W}, B)$ if $\mathcal{W}'$ is a submodule of $\mathcal{W}$ and $B' \subset B$.

Note that if $(\mathcal{W}_1, B_1)$ and $(\mathcal{W}_2, B_2)$ are G-modules with basis then $(\mathcal{W}_1 \oplus \mathcal{W}_2, B_1 \sqcup B_2)$ is a G-module with basis.

DEFINITION 1.7.2. Let $(\mathcal{W}, B)$ be a G-module with basis $B = \{v_i : i \in \Lambda\}$. We define

$$\mathcal{L}(\mathcal{W}, B) := \{\psi : \Lambda \times \Lambda \to \ell : \{i : \psi(i,j) \neq 0\} \text{ is finite for all } j\}$$

Let $p_i : \mathcal{W} \to \ell$ be the projection to the submodule of $\mathcal{W}$ generated by v_i,

$$p_i(\sum_{j\in\Lambda} a_j v_j) = a_i.$$

Note that $\mathcal{L}(\mathcal{W}, B)$ and $\mathrm{End}_\ell(\mathcal{W}) = \{f : \mathcal{W} \to \mathcal{W} : \ f \text{ is } \ell\text{-linear}\}$ are isomorphic; indeed we have inverse isomorphisms

$$\mathrm{End}_\ell(\mathcal{W}) \to \mathcal{L}(\mathcal{W}, B) \qquad f \mapsto \psi_f \qquad \psi_f(i,j) = p_i(f(v_j)) \tag{25}$$

$$\mathcal{L}(\mathcal{W}, B) \to \mathrm{End}_\ell(\mathcal{W}) \qquad \psi \mapsto f_\psi \qquad f_\psi(v_k) = \sum_{i\in\Lambda} \psi(i,k)v_i \tag{26}$$

Define

$$\mathcal{C}(\mathcal{W}, B) := \{\psi \in \mathcal{L}(\mathcal{W}, B) : \{j : \psi(i,j) \neq 0\} \text{ is finite for all } i\}$$

$$\mathcal{F}(\mathcal{W}, B) := \{\psi \in \mathcal{L}(\mathcal{W}, B) : \{(i,j) : \psi(i,j) \neq 0\} \text{ is finite }\}$$

and

$$\mathrm{End}^F_\ell(\mathcal{W}, B) := \{f \in \mathrm{End}_\ell(\mathcal{W}) : \psi_f \in \mathcal{F}(\mathcal{W}, B)\} \quad \mathrm{End}^C_\ell(\mathcal{W}, B) := \{f \in \mathrm{End}_\ell(\mathcal{W}) : \psi_f \in \mathcal{C}(\mathcal{W}, B)\}$$

Note that $\mathcal{C}(\mathcal{W}, B)$ is a ring with the matrix product and $\mathrm{End}^C_\ell(\mathcal{W}, B)$ is a ring with the composition. By (25) and (26), $\mathrm{End}^C_\ell(\mathcal{W}, B)$ and $\mathcal{C}(\mathcal{W}, B)$ are isomorphic rings.

Definition 1.7.3 (finiteness conditions for G-modules). Let $(\mathcal{W}, B)$ be a G-module with basis. Consider the representation

$$\rho : \mathrm{G} \to \mathrm{End}_\ell(\mathcal{W}) \qquad g \mapsto \rho_g \qquad \rho_g(w) = g \cdot w$$

We say that $(\mathcal{W}, B)$ is a G-**module by finite automorphisms** if $\rho(\mathrm{G}) \subset \mathrm{End}^F_\ell(\mathcal{W}, B)$. We say that $(\mathcal{W}, B)$ is a G-**module by almost finite automorphisms** if $\rho(\mathrm{G}) \subset \mathrm{End}^C_\ell(\mathcal{W}, B)$. If $(\mathcal{W}, B)$ is a G-module by almost finite automorphisms, $\mathrm{End}^F_\ell(\mathcal{W}, B)$ is a G-algebra with the following action

$$g \cdot f = \rho(g) f(\rho(g))^{-1}$$

Note that $\mathrm{End}^F_\ell(\mathcal{W}, B)$ is an ideal of $\mathrm{End}^C_\ell(\mathcal{W}, B)$.

Example 1.7.4. Let $\mathcal{W} = \ell\,\mathrm{G}$ be the group algebra considered as a G-module via the regular representation with basis $B = \{\delta_g : g \in \mathrm{G}\}$,

$$g \cdot (\sum_{h\in\mathrm{G}} a_h \delta_h) = \sum_{h\in\mathrm{G}} a_h \delta_{gh}. \tag{27}$$

Note

$$\rho : \mathrm{G} \to \mathrm{End}_\ell(\ell\,\mathrm{G}) \simeq \mathcal{L}(\ell\,\mathrm{G}, B) \quad g \mapsto M_g = \sum_{t\in\mathrm{G}} e_{gt,t} \quad (M_g)(s,t) = \begin{cases} 1 & s = gt \\ 0 & s \neq gt \end{cases}$$

As $M_g \in \mathcal{C}(\ell\,\mathrm{G}, B)$ for all $g \in \mathrm{G}$, $(\ell\,\mathrm{G}, B)$ is a G-module by almost finite automorphisms. Moreover we have

$$(M_g)^{-1} = M_{g^{-1}} = (M_g)^t.$$

Observe that

$$M_{\mathrm{G}} = \mathcal{F}(\ell\,\mathrm{G}, B).$$

DEFINITION 1.7.5. Let A be a G-algebra. Consider the tensor product

$$M_{\mathrm{G}}A = M_{\mathrm{G}} \otimes A$$

with the diagonal action of G

$$(M \otimes a)(N \otimes b) = MN \otimes ab \qquad g \cdot (e_{s,t} \otimes a) = e_{gs,gt} \otimes g \cdot a.$$

PROPOSITION 1.7.6. Let $\mathcal{W}$ be a G-module. Let $\mathcal{W}^\tau$ be $\mathcal{W}$ considered as a G-module with trivial action, then

$$\ell\,\mathrm{G} \otimes \mathcal{W} \simeq \ell\,\mathrm{G} \otimes \mathcal{W}^\tau$$

PROOF: The isomorphisms are given by

$$T : \ell\,\mathrm{G} \otimes \mathcal{W}^\tau \to \ell\,\mathrm{G} \otimes \mathcal{W} \qquad S : \ell\,\mathrm{G} \otimes \mathcal{W} \to \ell\,\mathrm{G} \otimes \mathcal{W}^\tau$$

$$T(\delta_g \otimes h) = \delta_g \otimes g(h) \qquad S(\delta_g \otimes h) = \delta_g \otimes g^{-1}(h)$$

It is clear that each is the inverse of the other and that they are equivariant morphisms. □

If $(\mathcal{W}, B)$ is a G-module with basis we will write $\mathrm{End}^C_\ell(\mathcal{W})$ and $\mathrm{End}^F_\ell(\mathcal{W})$ ommiting the basis when there is no confusion.

PROPOSITION 1.7.7. Let $(\mathcal{W}, B)$ be a G-module with basis. Then

$$\mathrm{End}^F_\ell(\ell\,\mathrm{G} \otimes \mathcal{W}) \simeq \mathrm{End}^F_\ell(\ell\,\mathrm{G}) \otimes \mathrm{End}^F_\ell(\mathcal{W})$$

PROOF: Define $T : \mathrm{End}^F_\ell(\ell\,\mathrm{G}) \otimes \mathrm{End}^F_\ell(\mathcal{W}) \to \mathrm{End}^F_\ell(\ell\,\mathrm{G} \otimes \mathcal{W})$ by

$$T(e_{g,h} \otimes e_{v,w}) = e_{g,v,h,w} \qquad v, w \in B \quad g, h \in \mathrm{G}$$

As T is a bijection between the basis, T is an isomorphism. □

DEFINITION 1.7.8 (G-stability). Let $(\mathcal{W}_1, B_1)$ and $(\mathcal{W}_2, B_2)$ be G-modules by almost finite automorphisms such that $\mathrm{card}(B_i) \leq \mathrm{card}(\mathrm{G}) \times \mathrm{card}\,\mathbb{N}$, $i = 1, 2$. The inclusion $\iota : \mathcal{W}_1 \to \mathcal{W}_1 \oplus \mathcal{W}_2$ induces a morphism of G-algebras

$$\tilde{\iota} : \mathrm{End}^F_\ell(\mathcal{W}_1) \to \mathrm{End}^F_\ell(\mathcal{W}_1 \oplus \mathcal{W}_2) \qquad f \mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \tag{28}$$

Let A be a G-algebra and consider

$$\tilde{\iota} \otimes 1 : \mathrm{End}^F_\ell(\mathcal{W}_1) \otimes A \to \mathrm{End}^F_\ell(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A.$$

A functor $F : \text{G-Alg} \to \mathcal{C}$ is G-stable if for $(\mathcal{W}_1, B_1)$, $(\mathcal{W}_2, B_2)$ and A as above $F(\tilde{\iota} \otimes 1)$ is an isomorphism in $\mathcal{C}$.

PROPOSITION 1.7.9. If $F : \text{G-Alg} \to \mathcal{C}$ is a G-stable functor then F is M_∞-stable.

PROOF: Consider $(\mathcal{W}_1, B_1) = (\ell, \{1\})$ and $(\mathcal{W}_2, B_2) = (\ell^{(\mathbb{N})}, \{e_i : i \in \mathbb{N}\})$ with $\ell^{(\mathbb{N})} = \oplus_{i=1}^\infty \ell$, $\{e_i : i \in \mathbb{N}\}$ is the canonical basis and both modules have the trivial action of G. Note

$$\mathrm{End}^F_\ell(\ell) = \mathrm{End}_\ell(\ell) = \ell \quad \mathrm{End}^F_\ell(\ell \oplus \ell^{(\mathbb{N})}) = \mathrm{End}^F_\ell(\ell^{(\mathbb{N})}) = M_\infty$$

and $\tilde{\iota} : \ell \to M_\infty$ is the inclusion at the upper left corner. Then $\tilde{\iota} \otimes 1 = \iota : A \to M_\infty(A)$ and $F(\iota)$ is an isomorphism. □

COROLLARY 1.7.10. If $\mathrm{G} = \{e\}$, F is G-stable if and only if F is M_∞-stable.

DEFINITION 1.7.11 (zig-zag). Let A, B be G-algebras and $F : \text{G-Alg} \to \mathcal{C}$ be a functor. A zig-zag between A and B by F is a diagram in G-Alg

$$A \xrightarrow{f_1} C_1 \xleftarrow{g_1} \dots \xrightarrow{f_n} C_n \xleftarrow{g_n} B$$

such that $F(g_i)$, $i = 1, \dots, n$, are isomorphisms in $\mathcal{C}$.

REMARK 1.7.12. Let A be a G-algebra and F be a G-stable functor. There exists a zig-zag between A and $M_{\mathrm{G}}A$ by F. Consider $\mathcal{W}_1 = (\ell\,\mathrm{G}, B)$ as in the example 1.7.4 and consider $\mathcal{W}_2 = (\ell, \{1\})$ with the trivial action of G. Put $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ and $C = \mathrm{End}_\ell^F(\mathcal{W})$ with the induced action, then

$$\iota : A = A \otimes \ell = A \otimes \mathrm{End}_\ell^F(\ell) \to A \otimes C \leftarrow A \otimes M_{\mathrm{G}} : \iota'$$

is a zig-zag between A and $M_{\mathrm{G}}A$.

PROPOSITION 1.7.13. Suppose G is countable. Let $F : \text{G-Alg} \to \mathcal{C}$ be an M_∞-stable functor. Define

$$\hat{F} : \text{G-Alg} \to \mathcal{C} \qquad A \mapsto F(M_{\mathrm{G}}A)$$

where $M_{\mathrm{G}}A$ was defined in 1.7.5. Then $\hat{F}$ is G-stable.

PROOF: Let $(\mathcal{W}_1, B_1)$, $(\mathcal{W}_2, B_2)$ be G-modules by almost finite automorphisms and A be a G-algebra. Consider

$$\tilde{\iota} \otimes 1 : \mathrm{End}_\ell^F(\mathcal{W}_1) \otimes A \to \mathrm{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A.$$

Let us check that $\hat{F}(\tilde{\iota} \otimes 1)$ is an isomorphism:

$$\begin{array}{llll} F(M_{\mathrm{G}} \otimes \mathrm{End}_\ell^F(\mathcal{W}_1) \otimes A) & \simeq & F(\mathrm{End}_\ell^F(\ell\,\mathrm{G} \otimes \mathcal{W}_1) \otimes A) & \text{by 1.7.7} \\ & \simeq & F(\mathrm{End}_\ell^F(\ell\,\mathrm{G} \otimes \mathcal{W}_1^\tau) \otimes A) & \text{by 1.7.6} \\ & \simeq & F(M_{\mathrm{G}} \otimes \mathrm{End}_\ell^F(\mathcal{W}_1^\tau) \otimes A) & \text{by 1.7.7} \\ F(M_{\mathrm{G}} \otimes \mathrm{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes A) & \simeq & F(\mathrm{End}_\ell^F((\mathcal{W}_1 \oplus \mathcal{W}_2)^\tau) \otimes M_{\mathrm{G}} \otimes A) & \end{array}$$

$\hat{F}(\tilde{\iota} \otimes 1) = F(M_{\mathrm{G}} \otimes \tilde{\iota} \otimes 1)$ is an isomorphism because F is M_∞-stable. □

EXAMPLE 1.7.14. Suppose G is a finite group of order n such that $1/n \in \ell$. Define $\xi = (1/n)\sum_{g\in\mathrm{G}} \delta_g$, then ξ is idempotent. The map $s : \ell \to \ell\,\mathrm{G}$, $s(1) = \xi$, is a G-equivariant section of the canonical argumentation $\varphi : \ell\,\mathrm{G} \to \ell$. Thus the sequence of G-modules

$$0 \longrightarrow I \longrightarrow \ell\,\mathrm{G} \xrightarrow{\varphi} \ell \longrightarrow 0 \tag{29}$$

splits. Then

$$\ell\,\mathrm{G} = \ell\xi \oplus I$$

I is a G-module with basis $\{\delta_e - \delta_g : g \neq e\}$. Define

$$\lambda_g = \begin{cases} \xi & g = e \\ \delta_e - \delta_g & g \neq e \end{cases}$$

The set $\Lambda = \{\lambda_g : g \in \mathrm{G}\}$ is a basis of $\ell\,\mathrm{G}$ and the relations with the elements of $B = \{\delta_g : g \in \mathrm{G}\}$ are the following

$$\begin{array}{lll} \lambda_e = \frac{1}{n}\sum_{g\in\mathrm{G}} \delta_g & \lambda_h = \delta_e - \delta_h & \\ \delta_e = \lambda_e + \frac{1}{n}\sum_{g\neq e} \lambda_g & \delta_h = \lambda_e - \lambda_h + \frac{1}{n}\sum_{g\neq e}\lambda_g & h \neq e \end{array}$$

Consider $\mathcal{W}_1 = \ell = (\ell\xi, \{\xi\})$ and $\mathcal{W}_2 = (I, \{\lambda_g\}_{g\neq e})$, in this case the morphism (28) is

$$\iota : \ell \to M_{\mathrm{G}} \simeq \mathrm{End}(\ell\,\mathrm{G}, \Lambda) \quad 1 \mapsto \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

If we write it in the canonical basis we have

$$\overline{\iota} : \ell \to M_{\mathrm{G}} = \mathrm{End}(\ell\,\mathrm{G}, B) \quad 1 \mapsto \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix}$$

If $F : \mathrm{G}\text{-Alg} \to \mathcal{C}$ is a G-stable functor then $F(\overline{\iota})$ is an isomorphism in $\mathcal{C}$.

Remark 1.7.15. Let I be a set with a distinguished element ι_0. Let A be an ℓ-algebra and $\{A_i : i \in I\}$ subalgebras such that $A = \oplus_{i\in I} A_i$. For each $i \in I$ we consider the projection $\pi_{A_i} : A \to A_i$. The following matrices with coeficients in $\mathrm{End}_\ell^F(A)$ and with indices in I are conjugated

$$M(i,j) = \begin{cases} \mathrm{id} & i = j = \iota_0 \\ 0 & \text{otherwise} \end{cases} \qquad N(i,j) = \begin{cases} \pi_{A_i} & i = j \\ 0 & i \neq j \end{cases}$$

In fact, if we choose T such that

$$T(i,j) = \begin{cases} \pi_{A_j} & i = \iota_0 \\ \pi_{A_i} & j = \iota_0 \\ 1 - \pi_{A_i} & i = j \neq \iota_0 \\ 0 & \text{otherwise} \end{cases}$$

it is easy to see that $T^2 = Id$ and $TMT = N$.

Remark 1.7.16. Let H be a subgroup of G. Define

$$\ell(\mathrm{G}/\mathrm{H}) = \bigoplus_{g\,\mathrm{H} \in \mathrm{G}/\mathrm{H}} \ell$$

Consider $\mathcal{W}_1 = (\ell(\mathrm{G}/\mathrm{H}), \{g\,\mathrm{H} : g \in \mathrm{G}\})$ with the regular action and $\mathcal{W}_2 = (\ell(\mathrm{G}/\mathrm{H}), \{g\,\mathrm{H} : g \in \mathrm{G}\})$ with the trivial action. Put

$$M_{\mathrm{G}/\mathrm{H}} = \mathcal{F}(\mathcal{W}_1) \qquad M_{|\mathrm{G}/\mathrm{H}|} = \mathcal{F}(\mathcal{W}_2)$$

There is an isomorphism

$$\begin{array}{ccc} M_{\mathrm{G}/\mathrm{H}} \otimes M_{\mathrm{G}} & \xrightarrow{\simeq} & M_{|\mathrm{G}/\mathrm{H}|} \otimes M_{\mathrm{G}} \\ e_{s\,\mathrm{H}, g\,\mathrm{H}} \otimes e_{t,r} & \mapsto & e_{t^{-1}s\,\mathrm{H}, r^{-1}g\,\mathrm{H}} \otimes e_{t,r} \end{array}$$

7.2. $\hat{\mathrm{G}}$-stability. In this section we consider the dual of the notion of G-stability. We want to identify a G-graded algebra A with the G-graded matrix algebra $M_{\hat{\mathrm{G}}}A$. The definition of $\hat{\mathrm{G}}$-stability is easier than that of G-stability beacause the morphism $A \to M_{\hat{\mathrm{G}}}A$ is homogeneous.

DEFINITION 1.7.17 ($\hat{\mathrm{G}}$-algebra). A $\hat{\mathrm{G}}$-algebra A is an associative algebra with a familiy of submodules $\{A_s\}_{s\in\mathrm{G}}$ such that

$$A = \bigoplus_{s\in\mathrm{G}} A_s \qquad A_sA_t \subseteq A_{st} \qquad s,t\in\mathrm{G}$$

We write $|a| = s$ if $a \in A_s$. A morphism $f : A \to B$ of $\hat{\mathrm{G}}$-algebras is an algebra morphism such that $f(A_s) \subset B_s$, $s \in \mathrm{G}$. Write $\hat{\mathrm{G}}$-Alg for the category of $\hat{\mathrm{G}}$-algebras.

EXAMPLE 1.7.18. An associative algebra A is a $\hat{\mathrm{G}}$-algebra with the trivial grading:

$$A = \bigoplus_{s\in\mathrm{G}} A_s \qquad A_s = \begin{cases} A & s = e \\ 0 & s \neq e \end{cases}$$

In particular ℓ is a $\hat{\mathrm{G}}$-algebra.

EXAMPLE 1.7.19. Let A be a $\hat{\mathrm{G}}$-algebra. Define the following grading in $M_{\mathrm{G}}A$

$$(M_{\mathrm{G}}A)_g := < e_{s,t} \otimes a : g = s|a|t^{-1} > \tag{30}$$

As

$$(e_{s,t} \otimes a)(e_{\tilde{s},\tilde{t}} \otimes b) = \begin{cases} e_{s,\tilde{t}} \otimes ab & t = \tilde{s} \\ 0 & t \neq \tilde{s} \end{cases}$$

we have

$$\begin{aligned} |(e_{s,t} \otimes a)(e_{\tilde{s},\tilde{t}} \otimes b)| &= s|a||b|\tilde{t}^{-1} \qquad \text{if } t = \tilde{s} \\ &= s|a|t^{-1}\tilde{s}|b|\tilde{t}^{-1} \\ &= |e_{s,t} \otimes a||e_{\tilde{s},\tilde{t}} \otimes b| \end{aligned}$$

and $M_{\mathrm{G}}A$ is a $\hat{\mathrm{G}}$-algebra. We will write $M_{\hat{\mathrm{G}}}A$ for $M_{\mathrm{G}}A$ with this structure.

DEFINITION 1.7.20 ($\hat{\mathrm{G}}$-stability). Let A be a $\hat{\mathrm{G}}$-algebra. Consider $M_{\hat{\mathrm{G}}}A$ as in the example 1.7.19. Observe the map

$$\iota_A : A \to M_{\hat{\mathrm{G}}}A \qquad a \mapsto e_{1_{\mathrm{G}},1_{\mathrm{G}}} \otimes a$$

is homogeneous. A functor $F : \hat{\mathrm{G}}\text{-Alg} \to \mathcal{C}$ is a $\hat{\mathrm{G}}$-stable if $F(\iota_A)$ is an isomorphism in $\mathcal{C}$ for all $A \in \hat{\mathrm{G}}$-Alg.

7.3. $\mathcal{H}$-stability. In this section we suppose $\mathcal{H}$ is a Hopf algebra over a field ℓ. Recall that a left Hopf module over $\mathcal{H}$ is an ℓ-module M such that

- M is a left $\mathcal{H}$-module.
- M is a left $\mathcal{H}$-comodule, via $\rho : M \to \mathcal{H} \otimes M$.
- ρ is a left $\mathcal{H}$-module map, where $\mathcal{H} \otimes M$ is a left module via

$$h \cdot (k \otimes m) = \sum h_1k \otimes h_2 \cdot m.$$

We may also write

$$\rho(h\cdot m) = h\cdot\rho(m) \quad \sum (h\cdot m)_{-1}\otimes(h\cdot m)_0 = \sum h_1\cdot m_{-1}\otimes h_2\cdot m_0 \qquad \forall m\in M \qquad \forall h\in\mathcal{H}$$

PROPOSITION 1.7.21. Let M be a left $\mathcal{H}$-module, $\mathcal{H}\otimes M$ is a Hopf module over $\mathcal{H}$ via

$$h\cdot(k\otimes m) = \sum h_1k\otimes h_2\cdot m \qquad \rho(k\otimes m) = \Delta(k)\otimes m$$

Write M^τ for the trivial $\mathcal{H}$-module, then

$$\mathcal{H}\otimes M \simeq \mathcal{H}\otimes M^\tau$$

PROOF: The isomorphisms are given by

$$\alpha : \mathcal{H}\otimes M^\tau \to \mathcal{H}\otimes M \quad \alpha(h\otimes m) = \sum h_1\otimes h_2\cdot m$$

$$\beta : \mathcal{H}\otimes M \to \mathcal{H}\otimes M^\tau \quad \beta(h\otimes m) = \sum h_1\otimes S(h_2)\cdot m$$

One checks that one is the inverse of the each other and that they are Hopf modules morphisms. □

Let M be an $\mathcal{H}$-module. In the Example 1.1.3 we have seen an $\mathcal{H}$-algebra structure on $\mathrm{End}_\ell(M)$. Define

$$\mathrm{End}_\ell^F(M) := \{\varphi\in\mathrm{End}_\ell(M) : \varphi(M) \text{ is a finite dimensional subspace of } M\}. \tag{31}$$

Let $\varphi\in\mathrm{End}_\ell^F(M)$ and $h\in\mathcal{H}$. Consider C_h the subcoalgebra of $\mathcal{H}$ generated by h. By Fundamental theorem of coalgebras we know C_h has finite dimension over ℓ. Because $(h\cdot\varphi)(M)\subset C_h\cdot\varphi(M)$ we conclude $(h\cdot\varphi)(M)$ is finite dimensional and $h\cdot\varphi\in\mathrm{End}_\ell^F(M)$. Hence $\mathrm{End}_\ell^F(M)$ is an $\mathcal{H}$-subalgebra of $\mathrm{End}_\ell(M)$. It is easy to check that $\mathrm{End}_\ell^F(M)$ is an ideal of $\mathrm{End}_\ell(M)$.

REMARK 1.7.22. Let M be an $\mathcal{H}$-module, $\varphi\in\mathrm{End}_\ell^F(M)$ and $h,k\in\mathcal{H}$. Recall from (7) the definition of $\varphi_{h,k}$, then $\varphi_{h,k}(M)\subset h\cdot\varphi(M)$. The space $\varphi(M)$ has finite dimension and h is fixed. Thus $h\cdot\varphi(M)$ and $\varphi_{h,k}(M)$ has also finite dimension. We conclude that $\varphi_{h,k}\in\mathrm{End}_\ell^F(M)$.

DEFINITION 1.7.23. Let M_1 and M_2 be $\mathcal{H}$-modules and let A be an $\mathcal{H}$-algebra. Let us consider the following homomorphism of $\mathcal{H}$-algebras

$$\begin{array}{rlcl} \iota : & \mathrm{End}_\ell^F(M_1)\otimes A & \to & \mathrm{End}_\ell^F(M_1\oplus M_2)\otimes A \\ & f\otimes a & \mapsto & \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\otimes a \end{array}$$

A functor $F : \mathcal{H}\text{-Alg}\to\mathcal{C}$ is $\mathcal{H}$-stable if it is M_∞-stable and $F(\iota)$ is an isomorphism for every M_1, M_2 and A as above and with

$$\dim_\ell(M_i)\le\dim_\ell(\mathcal{H}) \qquad i=1,2.$$

REMARK 1.7.24. Let A be an $\mathcal{H}$-algebra and $F : \mathcal{H}\text{-Alg}\to\mathcal{C}$ an $\mathcal{H}$-stable functor. There exists a zig-zag between A and $\mathrm{End}_\ell^F(\mathcal{H})\otimes A$ by F. Consider $\mathcal{H}$ as a $\mathcal{H}$-module with the regular action and ℓ with the trivial action. Then

$$A\cong\ell\otimes A\cong\mathrm{End}_\ell^F(\ell)\otimes A\xrightarrow{\iota}\mathrm{End}_\ell^F(\ell\oplus\mathcal{H})\otimes A\xleftarrow{\iota'}\mathrm{End}_\ell^F(\mathcal{H})\otimes A$$

is a zig-zag by F.

PROPOSITION 1.7.25. Let $\mathcal{H}$ be a finite dimensional Hopf algebra and M an $\mathcal{H}$-module with finite dimension. Then

$$\mathrm{End}_\ell(\mathcal{H} \otimes M) \simeq \mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M)$$

as a $\mathcal{H}$-algebras. The structures of $\mathrm{End}_\ell(\mathcal{H}\otimes M)$ and $\mathrm{End}_\ell(\mathcal{H})\otimes\mathrm{End}_\ell(M)$ are those given in the examples 1.1.3 and 1.1.4 respectively.

PROOF: Define

$$\Gamma : \mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M) \to \mathrm{End}_\ell(\mathcal{H} \otimes M) \qquad \Gamma(\varphi \otimes \psi)(h \otimes m) = \varphi(h) \otimes \psi(m)$$

One checks Γ is an $\mathcal{H}$-algebra monomorphism. By a dimension argument we conclude that it is an isomorphism. □

PROPOSITION 1.7.26. Suppose $\dim_\ell \mathcal{H}$ is finite. Let $F : \mathcal{H}\text{-Alg} \to \mathcal{C}$ be a homotopy invariant and M_∞-stable functor. The following functor

$$\hat{F} : \mathcal{H}\text{-Alg} \to \mathcal{C} \qquad A \mapsto F(\mathrm{End}_\ell(\mathcal{H}) \otimes A)$$

is homotopy invariant and $\mathcal{H}$-stable.

PROOF: Let M_1 and M_2 be $\mathcal{H}$-modules with finite dimension and A be an $\mathcal{H}$-algebra. Put

$$\tilde{\iota} \otimes 1 : \mathrm{End}_\ell(M_1) \otimes A \to \mathrm{End}_\ell(M_1 \oplus M_2) \otimes A.$$

Let $\hat{F} : \mathcal{H}\text{-Alg} \to \mathcal{C}$, $\hat{F}(A) := F(\mathrm{End}_\ell(\mathcal{H}) \otimes A)$. We have to prove $\hat{F}(\tilde{\iota} \otimes 1)$ is an isomorphism. As

$$\begin{array}{rcll}
F(\mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M_1) \otimes A) & \simeq & F(\mathrm{End}_\ell(\mathcal{H} \otimes M_1) \otimes A) & \text{see 1.7.25} \\
 & \simeq & F(\mathrm{End}_\ell(\mathcal{H} \otimes M_1^\tau) \otimes A) & \text{by 1.7.21} \\
 & \simeq & F(\mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M_1^\tau) \otimes A) & \text{see 1.7.25} \\
 & \simeq & F(\mathrm{End}_\ell(M_1^\tau) \otimes \mathrm{End}_\ell(\mathcal{H}) \otimes A) & \\
F(\mathrm{End}_\ell(\mathcal{H}) \otimes \mathrm{End}_\ell(M_1 \oplus M_2) \otimes A) & \simeq & F(\mathrm{End}_\ell((M_1 \oplus M_2)^\tau) \otimes \mathrm{End}_\ell(\mathcal{H}) \otimes A) &
\end{array}$$

$\hat{F}(\tilde{\iota} \otimes 1)$ is an isomorphism by the M_∞-stability of F. □

In order to obtain an $\mathcal{H}$-stable functor when $\mathcal{H}$ is not finite dimensional, we have tried to apply the argument used in Proposition 1.7.26. We could not prove an equivalent assumption of Proposition 1.7.25. We are not sure if the right finiteness condition on $\mathrm{End}_\ell(M)$ is that written on (31). If it is the correct one, we have to redefine the $\mathrm{kk}^{|\mathcal{H}|}$-groups stabilizing by the $\mathbb{N} \times \mathbb{N}$-matrices with finite rank instead.

7.4. $\hat{\mathcal{H}}$-stability. Suppose $\mathcal{H}$ is a Hopf algebra over a field ℓ with finite dimension. Let A be an $\hat{\mathcal{H}}$-algebra.

PROPOSITION 1.7.27. Let N be a $\mathcal{H}$-comodule and N^τ the trivial comodule. There is a isomorphism

$$\mathcal{H} \otimes N \simeq \mathcal{H} \otimes N^\tau$$

□

Let N be a $\mathcal{H}$-comodule with finite dimension. Recall from Example 1.1.5 the structure of $\mathcal{H}$-algebra on $\mathrm{End}_\ell(N) \otimes A$.

DEFINITION 1.7.28 ($\hat{\mathcal{H}}$-stability). Let N_1 and N_2 be $\mathcal{H}$-comodules with finite dimension and let A be an $\hat{\mathcal{H}}$-algebra. Let us consider the following homomorphism of $\hat{\mathcal{H}}$-algebras

$$\begin{array}{rlcl} \iota: & \mathrm{End}_\ell(N_1)\otimes A & \to & \mathrm{End}_\ell(N_1\oplus N_2)\otimes A \\ & f\otimes a & \mapsto & \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\otimes a \end{array}$$

A functor $F:\hat{\mathcal{H}}\text{-Alg}\to\mathcal{C}$ is $\hat{\mathcal{H}}$-stable if it is M_∞-stable and $F(\iota)$ is an isomorphism for every N_1, N_2 and A as above.

PROPOSITION 1.7.29. Let $F:\hat{\mathcal{H}}\text{-Alg}\to\mathcal{C}$ be a homotopy invariant and M_∞-stable functor. The following functor

$$\hat{F}:\hat{\mathcal{H}}\text{-Alg}\to\mathcal{C}\qquad A\mapsto F(\mathrm{End}_\ell(\mathcal{H})\otimes A)$$

is homotopy invariant and $\hat{\mathcal{H}}$-stable.

□

8. Equivariant algebraic kk-theory

Suppose G is a countable group and ℓ is a commutative ring with unit. Let A, B be G-algebras, we define

$$\mathrm{kk}^{\mathrm{G}}(A,B):=\mathrm{kk}^{|\mathrm{G}|}(M_{\mathrm{G}}\otimes A,M_{\mathrm{G}}\otimes B).$$

Consider the category $\mathfrak{KK}^{\mathrm{G}}$ whose objects are the G-algebras and the morphisms between A and B are the elements of $\mathrm{kk}^{\mathrm{G}}(A,B)$. In other words,

$$\hom_{\mathfrak{KK}^{\mathrm{G}}}(A,B)=\hom_{\mathfrak{KK}^{|\mathrm{G}|}}(M_{\mathrm{G}}\otimes A,M_{\mathrm{G}}\otimes B)$$

Let $j^{\mathrm{G}}:\mathrm{G}\text{-Alg}\to\mathfrak{KK}^{\mathrm{G}}$ be the functor such that is the identity on objects and sends each morphism of G-algebras $f:A\to B$ to its class $[f]\in\mathrm{kk}^{\mathrm{G}}(A,B)$.

THEOREM 1.8.1. The functor $j^{\mathrm{G}}:\mathrm{G}\text{-Alg}\to\mathfrak{KK}^{\mathrm{G}}$ is an excisive, equivariantly homotopy invariant, and G-stable functor. Moreover, it is the universal functor for these properties.

PROOF: Let E be an extension as in (22), define

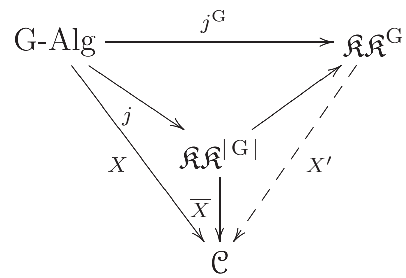
$$\partial^{\mathrm{G}}_{\mathrm{E}}\in\hom_{\mathfrak{KK}^{\mathrm{G}}}(\Omega C,A)=\hom_{\mathfrak{KK}^{|\mathrm{G}|}}(M_{\mathrm{G}}\otimes\Omega C,M_{\mathrm{G}}\otimes A)=\hom_{\mathfrak{KK}^{|\mathrm{G}|}}(\Omega M_{\mathrm{G}}\otimes C,M_{\mathrm{G}}\otimes A)$$

as the morphism $\partial_{\mathrm{E}'}$ defined in (24) asociated to the following weakly split extension

$$M_{\mathrm{G}}\otimes A\to M_{\mathrm{G}}\otimes B\to M_{\mathrm{G}}\otimes C\qquad(\mathrm{E}')$$

By theorem 1.6.4 and proposition 1.7.13 the functor $j:\mathrm{G}\text{-Alg}\to\mathfrak{KK}^{\mathrm{G}}$ with the family $\{\partial^{\mathrm{G}}_{\mathrm{E}}:\mathrm{E}\in\mathcal{E}\}$ is an excisive, homotopy invariant and G-stable functor. Let us check it is universal for these properties. Let $X:\mathrm{G}\text{-Alg}\to\mathcal{C}$ be a functor which has the mentioned properties with a family $\{\overline{\partial}_{\mathrm{E}} : \mathrm{E} \in \mathcal{E}\}$. By theorem 1.6.4 there exists a unique triangle functor $\overline{X} : \mathfrak{KK}^{|\,\mathrm{G}\,|} \to \mathcal{C}$ such that the following diagram commutes

(32)



We will define $X' : \mathfrak{KK}^{\mathrm{G}} \to \mathcal{C}$. We know that $X' = X$ on objects. As X is G-stable the following morphisms are a zig-zag between A and $M_{\mathrm{G}} \otimes A$ by X, (see Remark 1.7.12)

$$(33) \qquad A \xrightarrow{\iota_A} M_{\mathrm{G}\,\sqcup\{*\}} \otimes A \xleftarrow{\iota'_A} M_{\mathrm{G}} \otimes A$$

Let $\alpha \in \mathrm{kk}^{\mathrm{G}}(A, B)$ and define

$$X'(\alpha) := X(\iota_B)^{-1} X(\iota'_B)\overline{X}(\alpha) X(\iota'_A)^{-1} X(\iota_A).$$

Note this definition is the unique possibility to make the diagram (32) commutative. □

Let A, B be $\hat{\mathrm{G}}$-algebras. We define

$$\mathrm{kk}^{\hat{\mathrm{G}}}(A, B) := \mathrm{kk}^{|\hat{\mathrm{G}}|}(A, B).$$

Consider the category $\mathfrak{KK}^{\hat{\mathrm{G}}}$ whose objects are the $\hat{\mathrm{G}}$-algebras and the morphisms between A and B are the elements of $\mathrm{kk}^{\hat{\mathrm{G}}}(A, B)$. Let $j^{\hat{\mathrm{G}}} : \hat{\mathrm{G}}\text{-Alg} \to \mathfrak{KK}^{\hat{\mathrm{G}}}$ be the functor that is the identity on objects and which sends each morphism of G-algebras $f : A \to B$ to its class $[f] \in \mathrm{kk}^{\hat{\mathrm{G}}}(A, B)$.

Theorem 1.8.2. The functor $j^{\hat{\mathrm{G}}} : \hat{\mathrm{G}}\text{-Alg} \to \mathfrak{KK}^{\hat{\mathrm{G}}}$ is an excisive, graded homotopy invariant, and $\hat{\mathrm{G}}$-stable functor. Moreover, it is the universal functor for these properties.

Proof: : It follows from Theorem 1.6.4 and the fact that $\hat{\mathrm{G}}$-stability property holds by the M_∞-stability property. □

Let $\mathcal{H}$ be a Hopf algebra over a field ℓ with finite dimension. Similary, if A and B are $\mathcal{H}$-algebras, we define

$$\mathrm{kk}^{\mathcal{H}}(A, B) := \mathrm{kk}^{|\mathcal{H}|}(\mathrm{End}_\ell(\mathcal{H}) \otimes A, \mathrm{End}_\ell(\mathcal{H}) \otimes B).$$

Consider the category $\mathfrak{KK}^{\mathcal{H}}$ whose objects are the $\mathcal{H}$-algebras and the morphisms between A and B are the elements of $\mathrm{kk}^{\mathcal{H}}(A, B)$. In other words,

$$\hom_{\mathfrak{KK}^{\mathcal{H}}}(A, B) = \hom_{\mathfrak{KK}^{|\mathcal{H}|}}(\mathrm{End}_\ell(\mathcal{H}) \otimes A, \mathrm{End}_\ell(\mathcal{H}) \otimes B)$$

Let $j^{\mathcal{H}} : \mathcal{H}\text{-Alg} \to \mathfrak{KK}^{\mathcal{H}}$ be the functor which is the identity on objects and sends each morphism of $\mathcal{H}$-algebras $f : A \to B$ to its class $[f] \in \mathrm{kk}^{\mathcal{H}}(A, B)$.

THEOREM 1.8.3. The functor $j^{\mathcal{H}} : \mathcal{H}\text{-Alg} \to \mathfrak{KK}^{\mathcal{H}}$ is universal for the properties: excisive, equivariantly homotopy invariant, and $\mathcal{H}$-stable.

PROOF: The proof is similar to the proof of Theorem 1.8.1. Just we have to replace M_{G} by $\mathrm{End}_\ell^F(\mathcal{H})$. ☐

Similary, if A and B are $\hat{\mathcal{H}}$-algebras, we define

$$\mathrm{kk}^{\hat{\mathcal{H}}}(A,B) := \mathrm{kk}^{|\hat{\mathcal{H}}|}(\mathrm{End}_\ell(\mathcal{H}) \otimes A, \mathrm{End}_\ell(\mathcal{H}) \otimes B).$$

Consider the category $\mathfrak{KK}^{\hat{\mathcal{H}}}$ whose objects are the $\hat{\mathcal{H}}$-algebras and the morphisms between A and B are the elements of $\mathrm{kk}^{\hat{\mathcal{H}}}(A,B)$. Let $j^{\hat{\mathcal{H}}} : \hat{\mathcal{H}}\text{-Alg} \to \mathfrak{KK}^{\hat{\mathcal{H}}}$ be the functor which is the identity on objects and sends each morphism of $\hat{\mathcal{H}}$-algebras $f : A \to B$ to its class $[f] \in \mathrm{kk}^{\hat{\mathcal{H}}}(A,B)$.

THEOREM 1.8.4. The functor $j^{\hat{\mathcal{H}}} : \hat{\mathcal{H}}\text{-Alg} \to \mathfrak{KK}^{\hat{\mathcal{H}}}$ is universal for the properties: excisive, equivariantly homotopy invariant, and $\hat{\mathcal{H}}$-stable.

☐

CHAPTER 2

Adjointness theorems in kk-theory.

In this chapter we study adjointness theorems in equivariant kk-theory. We put in an algebraic context some of the adjointness theorems which appear in Kasparov KK-theory. Let G be a countable group and ℓ a commutative ring with unit. We define the functors of trivial action and crossed product between $\mathfrak{KK}$ and $\mathfrak{KK}^{\mathrm{G}}$. The first adjointness theorem is Theorem 2.1.4 which is an algebraic version of the Green-Julg Theorem. This result gives us the first computation related with homotopy K-theory. If G is a finite group, A is a G-algebra, B is an algebra and $\frac{1}{|\mathrm{G}|} \in \ell$ then there is an isomorphism

$$\psi_{GJ} : \mathrm{kk}^{\mathrm{G}}(B^{\tau}, A) \to \mathrm{kk}(B, A \rtimes \mathrm{G}).$$

In particular, if $B = \ell$ then

$$\mathrm{kk}^{\mathrm{G}}(\ell, A) \simeq \mathrm{KH}(A \rtimes \mathrm{G}).$$

We consider a subgroup H of G, define induction and restriction funtors between $\mathfrak{KK}^{\mathrm{G}}$ and $\mathfrak{KK}^{\mathrm{H}}$ and study the adjointness between them. If B is an H-algebra and A is a G-algebra then there is an isomorphism

$$\psi_{IR} : \mathrm{kk}^{\mathrm{G}}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} B, A) \to \mathrm{kk}^{\mathrm{H}}(B, \mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} A).$$

This result gives us another computation. Taking H the trivial group and $B = \ell$ we obtain that

$$\mathrm{kk}^{\mathrm{G}}(\ell^{(\mathrm{G})}, A) \simeq \mathrm{KH}(A) \qquad \forall A \in \mathrm{G} - \mathrm{Alg}\,.$$

Here $\ell^{(\mathrm{G})} = \bigoplus_{g \in \mathrm{G}} \ell$ with the regular action of G. More general, if H is a finite subgroup of G and $1/|\mathrm{H}| \in \ell$ we combine ψ_{GJ} and ψ_{IR} and obtain

$$\mathrm{kk}^{\mathrm{G}}(\ell^{(\mathrm{G}/\mathrm{H})}, A) \simeq \mathrm{KH}(A \rtimes \mathrm{H}) \qquad \forall A \in \mathrm{G} - \mathrm{Alg}\,.$$

We also prove an algebraic version of Green imprimitivity Theorem and obtain that

$$\mathrm{KH}(A \rtimes \mathrm{H}) \simeq \mathrm{KH}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} A \rtimes \mathrm{G}).$$

We also obtain an algebraic version of the Baaj-Skandalis theorem. We show that the funtors

$$\rtimes\,\mathrm{G} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}^{\hat{\mathrm{G}}} \qquad\qquad \hat{\mathrm{G}} \rtimes : \mathfrak{KK}^{\hat{\mathrm{G}}} \to \mathfrak{KK}^{\mathrm{G}}$$

are inverse category equivalences. Let $\mathcal{H}$ be a Hopf algebra with finite dimension. We define functors between $\mathfrak{KK}$ and $\mathfrak{KK}^{\mathcal{H}}$, the smash product and the trivial action. We study the adjointness between them in theorem 2.5.4. We obtain that if $\mathcal{H}$ is semisimple, B is an algebra and A is an $\mathcal{H}$-algebra then there is an isomorphism

$$\psi : \mathrm{kk}^{\mathcal{H}}(B^{\tau}, A) \to \mathrm{kk}(B, A\#\mathcal{H}).$$

In particular,

$$\mathrm{kk}^{\mathcal{H}}(\ell, A) \simeq \mathrm{KH}(A\#\mathcal{H}).$$

1. Crossed product and trivial action

1.1. Trivial action. Let A be an ℓ-algebra. Recall A^τ is A with the trivial action of G. This gives us a functor $\tau : \mathrm{Alg} \to \text{G-Alg}$. It is easy to check that $j^{\mathrm{G}} \circ \tau$ satisfies excision and is homotopy invariant and M_∞-stable. By theorem 1.6.4 there exists a unique functor $\hat{\tau} : \mathfrak{KK} \to \mathfrak{KK}^{\mathrm{G}}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Alg} & \xrightarrow{\tau} & \text{G-Alg} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j^{\mathrm{G}}} \\ \mathfrak{KK} & \xrightarrow[\hat{\tau}]{} & \mathfrak{KK}^{\mathrm{G}} \end{array}$$

We will write τ for $\hat{\tau}$.

1.2. Crossed product. Let A be a G-algebra. The crossed product algebra $A \rtimes \mathrm{G}$ is the ℓ-module $A \otimes \ell\,\mathrm{G}$ with the following multiplication

$$(a \rtimes g)(b \rtimes h) = a(g \cdot b) \rtimes gh \qquad a, b \in A\ g, h \in \mathrm{G}\,.$$

Proposition 2.1.1. Let A be a G-algebra and $\mathcal{W}$ be a G-module by almost finite automorphisms. The following algebras are naturally isomorphic

$$(A \rtimes \mathrm{G}) \otimes \mathrm{End}_\ell^F(\mathcal{W}) \simeq (A \otimes \mathrm{End}_\ell^F(\mathcal{W})) \rtimes \mathrm{G}\,.$$

Proof: Let $\rho : \mathrm{G} \to (\mathrm{End}_\ell^K(\mathcal{W}))^\times$ be the structure map. Define

$$\phi : (A \rtimes \mathrm{G}) \otimes \mathrm{End}_\ell^F(\mathcal{W}) \to (A \otimes \mathrm{End}_\ell^F(\mathcal{W})) \rtimes \mathrm{G} \qquad \phi(a \rtimes g \otimes \varphi) = a \otimes \varphi\rho(g^{-1}) \rtimes g$$

It is an algebra morphism:

$$\begin{aligned} \phi((a \rtimes g \otimes \varphi)(\tilde{a} \rtimes \tilde{g} \otimes \tilde{\varphi})) &= \phi(ag(\tilde{a}) \rtimes g\tilde{g} \otimes \varphi\tilde{\varphi}) \\ &= ag(\tilde{a}) \otimes \varphi\tilde{\varphi}\rho(\tilde{g}^{-1}g^{-1}) \rtimes g\tilde{g} \\ &= (a \otimes \varphi\rho(g^{-1}))(g(\tilde{a}) \otimes \rho(g)\tilde{\varphi}\rho(\tilde{g}^{-1})\rho(g^{-1})) \rtimes g\tilde{g} \\ &= (a \otimes \varphi\rho(g^{-1}) \rtimes g)(\tilde{a} \otimes \tilde{\varphi}\rho(\tilde{g}^{-1}) \rtimes \tilde{g}) \\ &= \phi(a \rtimes g \otimes \varphi)\phi(\tilde{a} \rtimes \tilde{g} \otimes \tilde{\varphi}) \end{aligned}$$

On the other hand, define

$$\psi : (A \otimes \mathrm{End}_\ell^F(\mathcal{W})) \rtimes \mathrm{G} \to (A \rtimes \mathrm{G}) \otimes \mathrm{End}_\ell^F(\mathcal{W}) \qquad \psi(a \otimes \varphi \rtimes g) = a \rtimes g \otimes \varphi\rho(g).$$

It is also an algebra morphism:

$$\begin{aligned}
\psi(a\otimes\varphi\rtimes g)(\tilde a\otimes\tilde\varphi\rtimes\tilde g) &= \psi([a\otimes\varphi][g(\tilde a)\otimes\rho(g)\tilde\varphi\rho(g^{-1})]\rtimes g\tilde g)\\
&= \psi(ag(\tilde a)\otimes\varphi\rho(g)\tilde\varphi\rho(g^{-1})\rtimes g\tilde g)\\
&= ag(\tilde a)\rtimes g\tilde g\otimes\varphi\rho(g)\tilde\varphi\rho(\tilde g)\\
&= (a\rtimes g\otimes\varphi\rho(g))(\tilde a\rtimes\tilde g\otimes\tilde\varphi\rho(\tilde g))\\
&= \psi(a\otimes\varphi\rtimes g)\psi(\tilde a\otimes\tilde\varphi\rtimes\tilde g)
\end{aligned}$$

It is clear ϕ and ψ are inverse of each other. □

Proposition 2.1.2. There exists a unique functor $\rtimes\mathrm{G} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}$ such that the following diagram is commutative

$$\begin{array}{ccc}
\text{G-Alg} & \xrightarrow{\rtimes \mathrm{G}} & \text{Alg}\\
{\scriptstyle j^{\mathrm{G}}}\downarrow & & \downarrow{\scriptstyle j}\\
\mathfrak{KK}^{\mathrm{G}} & \xrightarrow[\rtimes \mathrm{G}]{} & \mathfrak{KK}
\end{array}$$

Proof: We shall show $j(-\rtimes\mathrm{G})$ is excisive, homotopy invariant and G-stable. Because $\rtimes\mathrm{G}$ maps split sequences to split sequences and j is excisive, then $j(-\rtimes\mathrm{G})$ is excisive. That $j(-\rtimes\mathrm{G})$ is homotopy invariant follows from the fact that

$$A[t]\rtimes\mathrm{G} = (A\rtimes\mathrm{G})[t].$$

Let $(\mathcal{W}_1, B_1)$, $(\mathcal{W}_2, B_2)$ be G-modules by almost finite automorphisms and A a G-algebra. Consider the isomorphism ψ defined in Proposition 2.1.1. Note that the following diagram is commutative

$$\begin{array}{ccc}
(A\otimes\mathrm{End}^F_\ell(\mathcal{W}_1))\rtimes\mathrm{G} & \xrightarrow{(1\otimes\tilde\imath)\rtimes\mathrm{G}} & (A\otimes\mathrm{End}^F_\ell(\mathcal{W}_1\oplus\mathcal{W}_2))\rtimes\mathrm{G}\\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi}\\
(A\rtimes\mathrm{G})\otimes\mathrm{End}^F_\ell(\mathcal{W}_1) & \xrightarrow[(1\rtimes\mathrm{G})\otimes\tilde\imath]{} & (A\rtimes\mathrm{G})\otimes\mathrm{End}^F_\ell(\mathcal{W}_1\oplus\mathcal{W}_2)
\end{array}$$

Because j is M_∞-stable, $j((1\rtimes\mathrm{G})\otimes\tilde\imath)$ is an isomorphism. Hence $j(-\rtimes\mathrm{G})(1\otimes\tilde\imath)$ is an isomorphism by the diagram above. □

Remark 2.1.3. Let $[\alpha]\in\mathrm{kk}^{\mathrm{G}}(A,B)$ be an element represented by $\alpha : J^n(M_{\mathrm{G}}A)\to(M_{\mathrm{G}}B)^{\mathrm{sd}^p\,\mathcal{S}^n}$ which is a morphism in $[J^n(M_{\mathrm{G}}A), \mathcal{M}_\infty(M_{\mathrm{G}}B)^{\mathrm{sd}^p\,\mathcal{S}^n}]$. Let us see who $[\alpha]\rtimes\mathrm{G}$ is. Consider the classifying map

$$J^n(M_{\mathrm{G}}A\rtimes\mathrm{G})\to J^n(M_{\mathrm{G}}A)\rtimes\mathrm{G}$$

The element $[\alpha]\rtimes\mathrm{G}$ is represented by the following composition

$$J^n(M_{\mathrm{G}}A\rtimes\mathrm{G})\to J^n(M_{\mathrm{G}}A)\rtimes\mathrm{G}\xrightarrow{\alpha\rtimes\mathrm{G}}(M_{\mathrm{G}}B)^{\mathrm{sd}^p\,\mathcal{S}^n}\rtimes\mathrm{G}\,.$$

1.3. Adjointness between τ and $\rtimes$G. In this section we shall see an algebraic version of the GREEN-JULG THEOREM. In [**24**] there is a version of this result in the context of Kasparov KK-theory. In [**11**] there is a version of this theorem in the context of E-theory.

THEOREM 2.1.4. Let G be a finite group of n elements and $1/n \in \ell$. The functors $\tau : \mathfrak{KK} \to \mathfrak{KK}^{\mathrm{G}}$ and $\rtimes \mathrm{G} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}$ are adjoint functors. Hence

$$\mathrm{kk}^{\mathrm{G}}(A^{\tau}, B) \simeq \mathrm{kk}(A, B \rtimes \mathrm{G}) \qquad A \in \mathrm{Alg} \qquad B \in \mathrm{G}\text{-}\mathrm{Alg}.$$

PROOF: By [[**22**] Theorem 2, pag 81], it is enough to prove that there exist natural transformations

$$\overline{\alpha}_A \in \mathrm{kk}(A, A^{\tau} \rtimes \mathrm{G}) \qquad \text{and} \qquad \overline{\beta}_B \in \mathrm{kk}^{\mathrm{G}}((B \rtimes \mathrm{G})^{\tau}, B)$$

such that

$$A^{\tau} \xrightarrow{\tau(\overline{\alpha}_A)} (A^{\tau} \rtimes \mathrm{G})^{\tau} \xrightarrow{\overline{\beta}_{\tau(A)}} A^{\tau} \qquad B \rtimes \mathrm{G} \xrightarrow{\overline{\alpha}_{B \rtimes \mathrm{G}}} (B \rtimes \mathrm{G})^{\tau} \rtimes \mathrm{G} \xrightarrow{\overline{\beta}_B \rtimes \mathrm{G}} B \rtimes \mathrm{G}$$

are the identities in $\mathrm{kk}^{\mathrm{G}}(A^{\tau}, A^{\tau})$ and $\mathrm{kk}(B \rtimes \mathrm{G}, B \rtimes \mathrm{G})$ respectively.

Put $\epsilon = 1/n \sum_{g \in \mathrm{G}} g$ in $\ell\,\mathrm{G}$ and define

$$\alpha_A : A \to A^{\tau} \rtimes \mathrm{G} \qquad \alpha(a) = a \otimes \epsilon \tag{34}$$

Note $A^{\tau} \rtimes \mathrm{G} = A \otimes \ell\,\mathrm{G}$ and α_A is an algebra morphism since ϵ is idempotent. Consider the element $\overline{\alpha}_A \in \mathrm{kk}(A, A^{\tau} \rtimes \mathrm{G})$ represented by α_A. Let

$$\beta_B : (B \rtimes \mathrm{G})^{\tau} \to M_{\mathrm{G}} B \qquad \beta_B(b \rtimes g) = \sum_{s \in \mathrm{G}} s(b) e_{s,sg}$$

Let us check that β_B is an algebra morphism:

$$\begin{aligned}
\beta_B((b \rtimes g)(a \rtimes h)) &= \beta_B(bg(a) \rtimes gh) \\
&= \textstyle\sum_{s \in \mathrm{G}} s(bg(a)) e_{s,sgh} \\
&= \textstyle\sum_{s,t \in \mathrm{G}} s(b) t(a) e_{s,sg} e_{t,th} \\
&= (\textstyle\sum_{s \in \mathrm{G}} s(b) e_{s,sg})(\textstyle\sum_{t \in \mathrm{G}} t(a) e_{t,th}) \\
&= \beta_B(b \rtimes g) \beta_B(a \rtimes h)
\end{aligned}$$

Let us check that β_B is equivariant:

$$\begin{aligned}
\beta_B(h(b \rtimes g)) &= \beta_B(b \rtimes g) \\
&= \textstyle\sum_{s \in \mathrm{G}} s(b) e_{s,sg} \\
&= \textstyle\sum_{s \in \mathrm{G}} hs(b) e_{hs,hsg} \\
&= h(\textstyle\sum_{s \in \mathrm{G}} s(b) e_{s,sg}) \\
&= h(\beta_B(b \rtimes g))
\end{aligned}$$

Let $\overline{\beta}_B \in \mathrm{kk}^{\mathrm{G}}((B \rtimes \mathrm{G})^\tau, B)$ be represented by β_B. The composite $\beta_{\tau(A)}\tau(\alpha_A)$ is

$$A^\tau \xrightarrow{\tau(\alpha_A)} (A^\tau \rtimes \mathrm{G})^\tau \xrightarrow{\beta_{\tau(A)}} M_{\mathrm{G}}A^\tau \qquad a \mapsto \frac{1}{n}\sum_{s,t\in G} ae_{s,t}. \tag{35}$$

Thus $\beta_{\tau(A)}\tau(\alpha_A) = \mathrm{id}_{A^\tau} \otimes \overline{\iota}$ where $\overline{\iota}$ is the map defined in 1.7.14. As j^{G} is G-stable, $j^{\mathrm{G}}(\mathrm{id}_{A^\tau} \otimes \overline{\iota})$ is the identity in $\mathrm{kk}^{\mathrm{G}}(A^\tau, A^\tau)$.

On the other hand take the morphism ψ defined in the proof of the Proposition 2.1.1. By 2.1.3, $\overline{\beta}_B \rtimes \mathrm{G}$ is represented by $\psi \circ (\beta_B \rtimes \mathrm{G})$. We want to prove that the identity in $\mathrm{kk}(B \rtimes \mathrm{G}, B \rtimes \mathrm{G})$ is represented by the following composition

$$B \rtimes \mathrm{G} \xrightarrow{\alpha_{B\rtimes \mathrm{G}}} (B \rtimes \mathrm{G})^\tau \rtimes \mathrm{G} \xrightarrow{\beta_B \rtimes \mathrm{G}} (B \otimes M_{\mathrm{G}}) \rtimes \mathrm{G} \xrightarrow{\psi} (B \rtimes \mathrm{G}) \otimes M_{\mathrm{G}}$$

$$b \rtimes g \mapsto \tfrac{1}{n}\textstyle\sum_{h,s\in \mathrm{G}} (s(b) \rtimes h)e_{s,h^{-1}sg}$$

Put $t = h^{-1}sg$ and note

$$\frac{1}{n}\sum_{h,s\in \mathrm{G}} (s(b) \rtimes h)e_{s,h^{-1}sg} = \frac{1}{n}\sum_{t,s\in \mathrm{G}} (s(b) \rtimes sgt^{-1})e_{s,t} \tag{36}$$

and

$$s(b) \rtimes sgt^{-1} = (\mathbf{1} \rtimes s)(b \rtimes g)(\mathbf{1} \rtimes t^{-1}) \qquad \text{in } \tilde{B} \rtimes \mathrm{G}$$

We can write (36) as $TA_{b\rtimes g}T^{-1}$, where

$$A_{b\rtimes g} = \frac{1}{n}\sum_{t,s\in \mathrm{G}} (b \rtimes g)e_{s,t} \quad T = \sum_{t\in \mathrm{G}} (\mathbf{1} \rtimes t)e_{t,t}$$

Because $b \rtimes g \mapsto A_{b\rtimes g}$ represents the identity, the same is true of $b \rtimes g \mapsto TA_{b\rtimes g}T^{-1}$. $\square$

Example 2.1.5. We give an example to show that the adjointness between of τ and $\rtimes\mathrm{G}$ of Theorem 2.1.4 fails to hold at the algebra level. Let $\mathrm{G} = \mathbb{Z}_2 = \{1, \sigma\}$, $A = \ell$ and $B = (\ell\,\mathrm{G})^*$ the dual algebra of $\ell\,\mathrm{G}$ with the regular action. Note $\hom_{\text{G-Alg}}(A^\tau, B)$ has two elements only:

$$\varphi_i : \ell \to (\ell\,\mathrm{G})^* \qquad \varphi_0(1) = 0 \qquad \varphi_1(1) = \chi_1 + \chi_\sigma$$

One the other hand $\hom_{\mathrm{Alg}}(A, B \rtimes \mathrm{G}) = \hom_{\mathrm{Alg}}(\ell, (\ell\,\mathrm{G})^* \rtimes \mathrm{G})$ has at least as many elements as ℓ. For each $\lambda \in \ell$ we can define

$$\varphi_\lambda : \ell \to (\ell\,\mathrm{G})^* \rtimes \mathrm{G} \quad \varphi_\lambda(1) = \chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma) \quad \lambda \in \ell$$

Note φ_λ is an algebra morphism because $\chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma)$ is an idempotent element:

$$(\chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma))^2 = \chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma) + \lambda(\chi_1\chi_\sigma \rtimes \sigma) + \lambda^2(\chi_1\chi_\sigma \rtimes 1) = \chi_1 \rtimes 1 + \lambda(\chi_1 \rtimes \sigma)$$

Write ψ_{GJ} for the isomorphism of the Theorem 2.1.4

$$\psi_{GJ} : \mathrm{kk}^{\mathrm{G}}(B^\tau, A) \to \mathrm{kk}(B, A \rtimes \mathrm{G}) \qquad \psi_{GJ} = \alpha^* \circ \rtimes\mathrm{G} \tag{37}$$

where α is the morphism defined in (34).

Corollary 2.1.6. Let G be a finite group such that $1/|\,\mathrm{G}\,| \in \ell$. Let A be a G-algebra, then

$$\mathrm{kk}^{\mathrm{G}}(\ell, A) \simeq \mathrm{kk}(\ell, A \rtimes \mathrm{G}) \simeq \mathrm{KH}(A \rtimes \mathrm{G})$$

□

2. Induction and Restriction

Let H be a subgroup of G. In this section we study the functors of induction and restriction between the categories $\mathfrak{KK}^{\mathrm{H}}$ and $\mathfrak{KK}^{\mathrm{G}}$

2.1. Finitely supported polynomial maps. Let A be an ℓ-algebra and X a simplicial set. We write $\sigma \in X$ when σ is an n-simplex for some n, in other words when $\sigma \in X_n$. Denote by $<\sigma>\subset X$ the subsimplicial set of X generated by σ:

$$<\sigma>_i = \{\tau \in X_i : \exists\ \alpha : [i] \to [n] \text{ such that } \alpha^*(\sigma) = \tau\}$$

Let $\sigma \in X$ be a simplex. The star of σ in X is the following set of simplices of X

$$\mathrm{St}(\sigma) := \mathrm{St}_X(\sigma) = \{\tau \in X : <\tau> \cap <\sigma> \neq \emptyset\}.$$

The closed star of σ is the sub simplicial set $\overline{\mathrm{St}}(\sigma)$ generated by $\mathrm{St}(\sigma)$,

$$\overline{\mathrm{St}}(\sigma) = <\mathrm{St}(\sigma)> .$$

Let M be the set of simplices of X. We define

$$\mathrm{St}_X(M) := \bigcup_{\sigma \in M} \mathrm{St}_X(\sigma) \qquad \overline{\mathrm{St}}_X(M) = <\mathrm{St}_X(M)> = \bigcup_{\sigma \in M} \overline{\mathrm{St}}(\sigma).$$

We define the link of M as

$$\mathrm{Link}(M) := \overline{\mathrm{St}}_X(M) \setminus \mathrm{St}_X(M).$$

LEMMA 2.2.1. Let X be a simplicial set; write NX for the set of nondegenerate simplices. The following are equivalent.

i) $(\forall \sigma \in X)\{\tau \in NX : <\tau> \supset <\sigma>\}$ is a finite set.

ii) For every $\sigma \in X$, $\overline{\mathrm{St}}_X(\sigma)$ is a finite simplicial set.

PROOF. If $\sigma \in X$, then $<\sigma>$ has finitely many nondegenerate simplices, and thus the set $\{<\tau> \cap <\sigma> : \tau \in X\}$ is finite. Hence if i) holds, there are finitely many $\tau \in NX$ such that $<\tau> \cap <\sigma> \neq \emptyset$; in other words, $NX \cap \mathrm{St}_X(\sigma)$ is a finite set, and therefore $\overline{\mathrm{St}}_X(\sigma)$ is a finite simplicial set. Thus i)⇒ii). Next note that $<\tau> \supset <\sigma>$ implies $\tau \in \mathrm{St}_X(\sigma)$, whence ii)⇒i). □

DEFINITION 2.2.2. A simplicial set X is locally finite if for all $\sigma \in X$, $\overline{\mathrm{St}}_X(\sigma)$ is a finite simplicial set.

DEFINITION 2.2.3. The support of an element $\phi \in A^X$ is generated by the simplices σ such that $\phi(\sigma) \neq 0$,

$$\mathrm{supp}(\phi) := <\sigma \in X : \varphi(\sigma) \neq 0>$$

Let $\phi, \psi \in A^X$ and let $f : Y \to X$ be a simplicial map then

$$\mathrm{supp}(\phi \cdot \psi) \subset \mathrm{supp}(\phi) \cap \mathrm{supp}(\psi) \qquad \mathrm{supp}(f^*(\phi)) \subset f^{-1}(\mathrm{supp}(\phi)). \tag{38}$$

We say ϕ has finite support if $\mathrm{supp}(\phi) \subset X$ is a finite simplicial set. We define the algebra of polinomial maps with finite support in X as

$$A^{(X)} := \{\phi \in A^X : \mathrm{supp}(\phi) \text{ is finite }\}$$

If X is finite then $A^X = A^{(X)}$. In general, by (38) we have that $A^{(X)} \lhd A^X$ is an ideal. Note that if $f : X \to Y$ is a morphism of simplicial sets, the image of $A^{(Y)}$ by $f^* : A^Y \to A^X$ it not necessarily in $A^{(X)}$. By this reason $X \mapsto A^{(X)}$ can not be extended to a functor in $\mathbb{S}$. But if f es proper (i.e. $f^{-1}(K)$ is finite for all finite $K \subset Y$) then $f^*(A^{(Y)}) \subset A^{(X)}$ by (38). We conclude that $A^{(-)}$ is a functor from the category of simplicial sets with proper morphisms. Next we consider the behaviour of this functor with respect to colimits. First of all, if $\{X_i\}$ is a family of simplicial sets, then we have

$$A^{(\coprod X_i)} = \bigoplus_i A^{(X_i)} \tag{39}$$

Here $\bigoplus$ indicates the direct sum of abelian groups, equipped with coordinatewise multiplication. Second, $A^{(-)}$ maps coequalizers of proper maps to equalizers; if $\{f_j : X \to Y\}$ is a family of proper maps, then

$$A^{(\mathrm{coeq}_j\{f_j:X\to Y\})} = \mathrm{eq}_j\{f_j^* : A^{(Y)} \to A^{(X)}\} \tag{40}$$

Next recall that if I is a small category and $X : I \to \mathbb{S}$ is a functor, then the colimit of X can be computed as a coequalizer:

$$\operatorname*{colim}_i X_i = \mathrm{coeq}(\coprod_{\alpha\in \mathrm{Ar}(I)} X_{s(\alpha)} \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \coprod_{i\in \mathrm{Ob}(I)} X_i)$$

Here Ob and Ar are respectively the sets of objects and of arrows of I, and if $\alpha \in \mathrm{Ar}(I)$ then $s(\alpha) \in \mathrm{Ob}(I)$ is its source; we also write $r(\alpha)$ for the range of α. The maps ∂_0 and ∂_1 are defined as follows. The restriction of ∂_i to the copy of $X_{s(\alpha)}$ indexed by α is the inclusion $X_{s(\alpha)} \subset \coprod_j X_j$ if $i = 0$ and the composite of $X(\alpha)$ followed by the inclusion $X_{r(\alpha)} \subset \coprod_j X_j$ if $i = 1$. The conditions that ∂_0 and ∂_1 be proper are equivalent to the following

- ∂_0) Each object of I is the source of finitely many arrows.
- ∂_1) Each object of I is the range of finitely many arrows, and X sends each map of I to a proper map.

Example 2.2.4. For example the functor $\sigma \mapsto <\sigma>$ from the set of nondegenerate simplices of X, ordered by $\sigma \le \tau$ if $<\sigma> \subset <\tau>$, always satisfies ∂_1; condition ∂_0 is precisely condition i) of Lemma 2.2.1. Hence ∂_0 is satisfied if and only if X is locally finite, and in that case we have

$$A^{(X)} = \mathrm{eq}(\bigoplus_{\sigma\in NX} A^{<\sigma>} \overset{\partial_0^*}{\underset{\partial_1^*}{\rightrightarrows}} \bigoplus_{\substack{<\tau>\subset<\sigma>,\\ \sigma,\tau\in NX}} A^{<\tau>})$$

2.2. Restriction. Let A be a G-algebra and $\mathrm{H} \subset \mathrm{G}$ a subgroup. If we restrict the action to H we obtain an H-algebra $\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}}(A)$. It is clear this construction defines a functor $\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} : \text{G-Alg} \to \text{H-Alg}$. It is easily seen that we can extend $\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} : \text{G-Alg} \to \text{H-Alg}$ to a triangle functor $\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}^{\mathrm{H}}$ so that the following diagram commutes

$$\begin{array}{ccc} \text{G-Alg} & \xrightarrow{\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}}} & \text{H-Alg} \\ {\scriptstyle j^{\mathrm{G}}}\downarrow & & \downarrow{\scriptstyle j^{\mathrm{H}}} \\ \mathfrak{KK}^{\mathrm{G}} & \xrightarrow[\mathrm{Res}_{\mathrm{G}}^{\mathrm{H}}]{} & \mathfrak{KK}^{\mathrm{H}} \end{array}$$

Note that for $\mathrm{H} = \{1\}$ we get $j^{\{1\}} = j : \mathrm{Alg} \to \mathfrak{KK}$ and $\mathrm{Res}_{\mathrm{G}}^{\{1\}} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}$ is the induced by the forgetful functor G-Alg $\to$ Alg.

2.3. Induction. We shall define an induction functor $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}$: H-Alg $\to$ G-Alg and extend it to a functor $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} : \mathfrak{KK}^{\mathrm{H}} \to \mathfrak{KK}^{\mathrm{G}}$ such the following diagram is commutative

$$\begin{array}{ccc} \text{H-Alg} & \xrightarrow{\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}} & \text{G-Alg} \\ {\scriptstyle j^{\mathrm{H}}}\downarrow & & \downarrow{\scriptstyle j^{\mathrm{G}}} \\ \mathfrak{KK}^{\mathrm{H}} & \xrightarrow[\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}]{} & \mathfrak{KK}^{\mathrm{G}} \end{array}$$

Definition 2.2.5 ($\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$). Let $\mathrm{H} \subset \mathrm{G}$ be a subgroup, $\pi : \mathrm{G} \to \mathrm{G}/\mathrm{H}$ the projection and A an H-algebra. Consider

$$A^{(\mathrm{G},\mathrm{H})} := \{f : \mathrm{G} \to A : \#\pi(\mathrm{supp}(f)) < \infty\}$$

Define

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) = \{f \in A^{(\mathrm{G},\mathrm{H})} : \quad f(s) = h(f(sh)) \quad \forall s \in \mathrm{G}, \quad h \in \mathrm{H}\}.$$

One checks that $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ is a G-algebra with the pointwise multiplication and the following action

$$(g \cdot f)(s) = f(g^{-1}s) \quad f \in \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \quad g, s \in \mathrm{G}\,. \tag{41}$$

Definition 2.2.6 ($\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\varphi)$). Let $\varphi : A \to B$ be a morphism of H-algebras. Define

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\varphi) : \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \to \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B) \qquad \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\varphi)(f) = \varphi \circ f$$

As φ is equivariant $\varphi \circ f$ lies in $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B)$.

Let A be a G-algebra and $\mathrm{H} \subset \mathrm{G}$ a subgroup. The following identities are easy to check:

$$\mathrm{Ind}_{\mathrm{G}}^{\mathrm{G}}(A) \simeq A \qquad \mathrm{Ind}_{\{e\}}^{\mathrm{G}}(A) = A^{(\mathrm{G})} \qquad \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\ell) = \ell^{(\mathrm{G}/\mathrm{H})}$$

If H is finite, then

$$A^{(\mathrm{G},H)} = A^{(\mathrm{G})}.$$

Let A be an H-algebra. Consider

$$\mathrm{Big\,Ind}_{\mathrm{H}}^{\mathrm{G}}(A) := \{f : \mathrm{G} \to A : f(s) = h(f(sh)), \quad \forall s \in \mathrm{G}, \quad h \in \mathrm{H}\}$$

Note $\mathrm{Big\,Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ is a G-algebra with operations defined pointwise, and where G acts like in (41). If $f \in \mathrm{Big\,Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ and $x = s\,\mathrm{H} \in \mathrm{G}/\mathrm{H}$, then the value of f at any $g \in x$ determines f on the whole x; in particular,

$$\mathrm{supp}(f) \cap s\,\mathrm{H} \neq \emptyset \Rightarrow s\,\mathrm{H} \subset \mathrm{supp}(f) \quad (s\,\mathrm{H} \in \mathrm{G}/\mathrm{H})$$

Hence

$$\operatorname{supp}(f) = \coprod_{s\,\mathrm{H} \cap \operatorname{supp}(f) \neq \emptyset} s\,\mathrm{H}$$

Consider the projection $\pi : \mathrm{G} \to \mathrm{G}\,/\,\mathrm{H}$. We obtain

$$\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) = \{f \in \operatorname{Big}\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) : \#\pi(\operatorname{supp}(f)) < \infty\}$$

One checks that $\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \subset \operatorname{Big}\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ is a subalgebra; we shall presently introduce some of its typical elements. If $g \in \mathrm{G}$, write $\chi_g : \mathrm{G} \to \mathbb{Z}$ for the characteristic function. If $a \in A$ and $g \in \mathrm{G}$, then

$$\xi_{\mathrm{H}}(g,a) = \sum_{h \in \mathrm{H}} \chi_{gh} h^{-1}(a) \qquad \xi_{\mathrm{H}}(g,a)(s) = \begin{cases} h^{-1}(a) & s = gh \\ 0 & s \notin g\,\mathrm{H} \end{cases}$$

These elements are in $\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ because

$$\operatorname{supp}(\xi_{\mathrm{H}}(g,a)) = g\,\mathrm{H} \qquad \xi_{\mathrm{H}}(g,a)(s) = t \cdot \xi_{\mathrm{H}}(g,a)(st) \quad \forall s \in \mathrm{G}, t \in \mathrm{H}$$

Let $r : \mathrm{G}\,/\,\mathrm{H} \to \mathrm{G}$ be a pointed section and $\mathcal{R} = r(\mathrm{G}\,/\,\mathrm{H})$. Every element $\phi \in \operatorname{Big}\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ can be written as a formal sum

$$\phi = \sum_{g \in \mathcal{R}} \xi_{\mathrm{H}}(g, \phi(g)) \tag{42}$$

Note that $\phi \in \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ if and only if the sum above is finite. In particular

$$\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) = \sum_{g \in \mathrm{G}, a \in A} \ell \xi_{\mathrm{H}}(g,a) \subset \operatorname{Big}\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$$

It is easy to see that if $\varphi : A \to B$ is an H-equivariant map then

$$\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(\varphi)(\xi_{\mathrm{H}}(g,a)) = \xi_{\mathrm{H}}(g, \varphi(a)) \tag{43}$$

Observe that for each $s \in \mathrm{G}$, the map

$$\xi_{\mathrm{H}}(s, -) : A \to \operatorname{Big}\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$$

is an algebra homomorphism. Moreover, we have the following relations

$$s \cdot \xi_{\mathrm{H}}(g,a) = \xi_{\mathrm{H}}(sg, a) \tag{44}$$

$$\xi_{\mathrm{H}}(g,a)\xi_{\mathrm{H}}(\tilde{g},\tilde{a}) = \begin{cases} \xi_{\mathrm{H}}(\tilde{g}, \tilde{g}^{-1}g(a)\tilde{a}) & \tilde{g}^{-1}g \in \mathrm{H} \\ 0 & \tilde{g}^{-1}g \notin \mathrm{H} \end{cases} \tag{45}$$

$$\xi_{\mathrm{H}}(g,a) = \xi_{\mathrm{H}}(gh, h^{-1} \cdot a) \quad h \in \mathrm{H} \tag{46}$$

It follows that $(g,a) \mapsto \xi_{\mathrm{H}}(g,a)$ gives a G-equivariant map

$$\mathrm{G} \times_{\mathrm{H}} A \to \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A).$$

Here $\mathrm{G} \times_{\mathrm{H}} A = \mathrm{G} \times A / \sim$, where

$$(g_1, a_1) \sim (g_2, a_2) \iff h = g_1^{-1} g_2 \in \mathrm{H} \quad \textit{and} \quad a_1 = h \cdot a_2.$$

Extending by linearity we obtain an isomorphism of left G-modules

$$\ell[\mathrm{G}] \otimes_{\ell[\mathrm{H}]} A \to \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$$

Thus we may think of $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ as the G-module induced from the H-module A, equipped with an algebra structure compatible with that of A. In fact (45) implies that if $r : \mathrm{G}/\mathrm{H} \to \mathrm{G}$ is a section, then

$$\ell^{(\mathrm{G}/\mathrm{H})} \otimes A \to \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \qquad \chi_x \otimes a \mapsto \xi_{\mathrm{H}}(r(x), a) \tag{47}$$

is a *nonequivariant* algebra isomorphism.

LEMMA 2.2.7. Let X be an H-simplicial set; put

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X) = \mathrm{G} \times_{\mathrm{H}} X$$

There is a natural, G-equivariant isomorphism $\ell^{(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X))} \simeq \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\ell^{(X)})$.

PROOF: Let $\pi : \mathrm{G} \times X \to \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X)$ be the projection. We have a G-algebra isomorphism

$$\Theta : \mathrm{Big}\,\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\ell^{X}) \to \ell^{\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X)}, \quad \Theta(f)(\pi(g,x)) = f(g)(x)$$

For $s \in \mathrm{G}$ and $\phi \in \ell^{X}$,

$$\Theta(\xi_{\mathrm{H}}(s,\phi))(\pi(g,x)) = \begin{cases} \phi(s^{-1}g \cdot x) & if\ g \in s\,\mathrm{H} \\ 0 & if\ g \notin s\,\mathrm{H} \end{cases}$$

In particular, for $\Theta(\xi_{\mathrm{H}}(s,\phi))$ not to vanish on $\pi(g,x)$, we must have $g = sh$ and $x \in h^{-1} \cdot \{\phi \neq 0\}$ for some $h \in \mathrm{H}$. Hence $\mathrm{supp}(\Theta(\xi_{\mathrm{H}}(s,\phi))) \subset \pi(\{s\} \times \mathrm{supp}(\phi))$ which is a finite simplicial set if $\phi \in \ell^{(X)}$. Therefore Θ maps $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\ell^{(X)})$ inside $\ell^{(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X))}$. It remains to show that $\Theta^{-1}(\ell^{(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X))}) \subset \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\ell^{(X)})$. Let $\{g_i\} \subset \mathrm{G}$ be a full set of representatives of G/H. Every element of $\mathrm{G} \times_{\mathrm{H}} X$ can be written uniquely as $\pi(g_i, x)$ for some i and some $x \in X$. Hence as a simplicial set, $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X)$ is the disjoint union of the $Y_i = \pi(\{g_i \times X)$. In particular if $\phi \in \ell^{(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X))}$, then its support meets finitely many of the Y_i, and $\mathrm{supp}(\phi) \cap Y_i$ is a finite simplicial set. Thus there is a finite number of i such that $\psi = \Theta^{-1}(\phi)$ is nonzero on $g_i\,\mathrm{H}$, and its restriction to each of these subsets takes values in $\ell^{(X)}$. By (42), this implies that $\psi \in \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\ell^{(X)})$, as we had to prove. □

PROPOSITION 2.2.8. The functor $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} : \mathrm{H}\text{-Alg} \to \mathrm{G}\text{-Alg}$ is exact.

PROOF: The exactness only depends on the group structure of the objects involved. It follows from the fact that (47) is an isomorphism. □

PROPOSITION 2.2.9. Let A be a G-algebra and B be an H-algebra, then

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B \otimes \mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} A) \simeq \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B) \otimes A$$

PROOF: The isomorphisms are given by

$$\begin{array}{llll} S: & \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B) \otimes A & \to & \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B \otimes \mathrm{Res}_{\mathrm{G}}^{\mathrm{H}}(A)) \\ & \xi_{\mathrm{H}}(g,b) \otimes a & \mapsto & \xi_{\mathrm{H}}(g, b \otimes g^{-1} \cdot a) \end{array} \qquad \begin{array}{llll} T: & \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B \otimes \mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} A) & \to & \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B) \otimes A \\ & \xi_{\mathrm{H}}(g, b \otimes a) & \mapsto & \xi_{\mathrm{H}}(g,b) \otimes g \cdot a \end{array}$$

It easy to check that they are mutually inverse equivariant maps. □

COROLLARY 2.2.10. Let A be a G-algebra. Then

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \mathrm{Res}_{\mathrm{G}}^{\mathrm{H}} A \to \ell^{(\mathrm{G}/\mathrm{H})} \otimes A \qquad \xi_{\mathrm{H}}(s,b) \mapsto \chi_{s\,\mathrm{H}} \otimes s \cdot b$$

is an isomorphism of G-algebras.

$\square$

Let $x \subset \mathrm{G}$ and define

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)[x] := \{f \in \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) : \mathrm{supp}(f) \subset x\}.$$

Note that if $\mathrm{G} = \bigsqcup_i x_i$ is a disjoint union with $x_i = g_i\,\mathrm{H}$ then

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) = \bigoplus_i \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)[x_i] \tag{48}$$

Suppose K is another subgroup of G. Let $x = \mathrm{K}\,\theta\,\mathrm{H}$ for some $\theta \in \mathrm{G}$. Consider the subgroup of K

$$\mathrm{K}_\theta = \mathrm{K} \cap \theta\,\mathrm{H}\,\theta^{-1} \tag{49}$$

Put $\mathrm{H}_{\theta^{-1}} = \theta^{-1}\,\mathrm{K}\,\theta \cap \mathrm{H}$. Conjugation by θ^{-1} defines an isomorphism

$$c_{\theta^{-1}} : \mathrm{K}_\theta \to \mathrm{H}_{\theta^{-1}} \qquad c_{\theta^{-1}}(k) = \theta^{-1}k\theta \tag{50}$$

Hence we may view an $\mathrm{H}_{\theta^{-1}}$-algebra A as a K_θ-algebra via $c_{\theta^{-1}}$. We denote it by $c^*_{\theta^{-1}}(A)$.

Proposition 2.2.11. The map

$$\alpha : \mathrm{Res}_{\mathrm{G}}^{\mathrm{K}} \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)[\mathrm{K}\,\theta\,\mathrm{H}] \to \mathrm{Ind}_{\mathrm{K}_\theta}^{\mathrm{K}}(c^*_{\theta^{-1}}(\mathrm{Res}_{\mathrm{H}}^{\mathrm{H}_{\theta^{-1}}}(A))) \qquad \alpha(f)(k) = f(k\theta)$$

is an isomorphism of K-algebras.

Proof: We can check $\alpha(\xi_{\mathrm{H}}(s,a)) = \xi_{\mathrm{K}_\theta}(s\theta^{-1},a)$. The map α is equivariant:

$$\alpha(k\cdot f)(x) = (k\cdot f)(x\theta) = f(k^{-1}x\theta) = \alpha(f)(k^{-1}x) = (k\cdot\alpha(f))(x).$$

It is an isomorphism because $\alpha(\xi_{\mathrm{H}}(k\theta,a)) = \xi_{\mathrm{K}_\theta}(k,a)$. $\square$

Proposition 2.2.12. Let T be a simplicial set regarded as an H-simplicial set with the trivial action, then

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(T) = \coprod_{\mathrm{H}\,\theta\,\mathrm{K}} \mathrm{Ind}_{\mathrm{K}_{\theta^{-1}}}^{\mathrm{K}}(T) \tag{51}$$

Proof: It is enough to prove that $\mathrm{G}\,/\,\mathrm{H} \cong \coprod_{\mathrm{H}\,\theta\,\mathrm{K}} \mathrm{K}\,/\,\mathrm{K}_{\theta^{-1}}$, as K-sets with the right regular action. Consider de decomposition

$$\mathrm{G} = \coprod_{\theta\in\mathcal{R}} \mathrm{H}\,\theta\,\mathrm{K}\,.$$

Here θ runs among a full set of representatives $\mathcal{R}$ of the double coclasses $\mathrm{H}\setminus\mathrm{G}\,/\,\mathrm{K}$. We have

$$\mathrm{G}\,/\,\mathrm{H} = \coprod_{\mathrm{H}\,\theta\,\mathrm{K}} \mathrm{H}\,\theta\,\mathrm{K}\,/\,\mathrm{H}\,\theta\,\mathrm{K} \cap \mathrm{H}$$

Observe the function $\mathrm{K} \to \mathrm{H}\,\theta\,\mathrm{K}$, $k \mapsto \theta k$ is equivariant and if $k, \tilde{k} \in \mathrm{K}$ then

$$\theta k \equiv \theta\tilde{k}_{(\mathrm{H})} \iff \theta k\tilde{k}^{-1}\theta^{-1} \in H \iff k\tilde{k}^{-1} \in \theta^{-1}H\theta \iff k\tilde{k}^{-1} \in K_{\theta^{-1}} \iff k \equiv \tilde{k}_{(\mathrm{K}_{\theta^{-1}})}$$

Then

$$\frac{\mathrm{H}\,\theta\,\mathrm{K}}{\mathrm{H}\,\theta\,\mathrm{K}\cap\mathrm{H}} \cong \frac{\mathrm{K}}{\mathrm{K}_{\theta^{-1}}}$$

$\square$

PROPOSITION 2.2.13. Let Ind : H-Alg $\to$ G-Alg be the functor such that $\operatorname{Ind}(A) = \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(M_{\mathrm{H}} \otimes A)$. There exists a functor $\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} : \mathfrak{KK}^{\mathrm{H}} \to \mathfrak{KK}^{\mathrm{G}}$ such that the following diagram is commutative

$$\begin{array}{ccc} \text{H-Alg} & \xrightarrow{\operatorname{Ind}} & \text{G-Alg} \\ {\scriptstyle j^{\mathrm{H}}}\downarrow & & \downarrow{\scriptstyle j^{\mathrm{G}}} \\ \mathfrak{KK}^{\mathrm{H}} & \xrightarrow[\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}]{} & \mathfrak{KK}^{\mathrm{G}} \end{array} \tag{52}$$

PROOF: The functor $j^{\mathrm{G}} \circ \operatorname{Ind}$ is excisive because Ind maps split extensions to split extensions and j^{G} is excisive. Let us see the functor $j^{\mathrm{G}} \circ \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}$ is homotopy invariant and M_∞-stable. Let $f, g : A \to B$ be elementary homotopic morphisms of H-algebras. Then there exists $H : A \to B[t]$ such that $f = \mathrm{ev}_0 \circ H$ $g = \mathrm{ev}_1 \circ H$. Hence $\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(f)$ and $\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(g)$ are elementary homotopic morphisms of G-algebras and the homotopy is given by

$$\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} H : \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \to \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(B[t]) \simeq \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(B)[t] \qquad \text{by 2.2.9.}$$

As j^{G} is homotopy invariant then $j^{\mathrm{G}} \circ \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(f) = j^{\mathrm{G}} \circ \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(g)$. Let $\iota : A \to M_\infty A$, then $j^{\mathrm{G}} \circ \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(\iota)$ is an isomorphism because by Proposition 2.2.9 we have

$$\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(M_\infty A) \simeq M_\infty \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$$

and j^{G} is M_∞-stable. By Proposition 1.7.13 the functor $j^{\mathrm{G}} \circ \operatorname{Ind}$ is homotopy invariant and H-stable. As j^{H} : H-Alg $\to \mathfrak{KK}^{\mathrm{H}}$ is universal for this properties there exists $\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}$ such that (52) conmmutes. □

2.4. Adjointness between $\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}$ and $\operatorname{Res}_{\mathrm{G}}^{\mathrm{H}}$.

THEOREM 2.2.14. Let G be a group and H be a subgroup of G. The functors

$$\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} : \mathfrak{KK}^{\mathrm{H}} \to \mathfrak{KK}^{\mathrm{G}} \qquad \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} : \mathfrak{KK}^{\mathrm{G}} \to \mathfrak{KK}^{\mathrm{H}}$$

are adjoint. Hence

$$\mathrm{kk}^{\mathrm{G}}(\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(B), A) \simeq \mathrm{kk}^{\mathrm{H}}(B, \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}}(A)) \qquad \forall B \in \text{H-Alg} \quad A \in \text{G-Alg}$$

PROOF: Let $A \in$ G-Alg and $B \in$ H-Alg. We need to have natural transformations

$$\alpha_A \in \mathrm{kk}^{\mathrm{G}}(\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} A, A) \qquad \beta_B \in \mathrm{kk}^{\mathrm{H}}(B, \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} B)$$

which verify the unit and counit condition.

Define $\varphi_A : \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(\operatorname{Res}_{\mathrm{G}}^{\mathrm{H}}(A)) \to M_{\mathrm{G}/\mathrm{H}} \otimes A$ such that

$$\varphi_A(f) = \sum_{g\,\mathrm{H} \in \mathrm{G}/\mathrm{H}} e_{g\,\mathrm{H}, g\,\mathrm{H}} \otimes g(f(g)) \quad \varphi_A(\xi_{\mathrm{H}}(s, b)) = e_{s\,\mathrm{H}, s\,\mathrm{H}} \otimes s \cdot b$$

Put

$$\psi_B : B \to \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(B) \quad \psi_B(b) = \xi_{\mathrm{H}}(e, b) \tag{53}$$

It is easy to check that ψ_B is well-defined and is a map of H-algebras. Let $[\varphi_A] \in \mathrm{kk}^{\mathrm{G}}(\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} A, A)$ the element represented by φ_A and $\psi_B \in \mathrm{kk}^{\mathrm{H}}(B, \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} B)$ the element represented by ψ_B. The composite

$$\delta : \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} A \xrightarrow{\beta_{\operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} A}} \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}} \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} A \xrightarrow{\operatorname{Res}_{\mathrm{G}}^{\mathrm{H}}(\alpha_A)} \operatorname{Res}_{\mathrm{G}}^{\mathrm{H}} A$$

is represented by

$$\begin{array}{ccccc} \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} A & \xrightarrow{\psi_{\mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} A}} & \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} A & \xrightarrow{\mathrm{Res}^{\mathrm{H}}_{\mathrm{G}}(\varphi_A)} & \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}}(M_{\mathrm{G}\,/\,\mathrm{H}} \otimes A) = M_{|\,\mathrm{G}\,/\,\mathrm{H}\,|} \otimes \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} A \\ a & \longmapsto & \xi_{\mathrm{H}}(e,a) & \longmapsto & \varphi_A(\xi_{\mathrm{H}}(e,a)) = e_{\mathrm{H},\mathrm{H}} \otimes a \end{array}$$

which is kk^{H}-equivalent to the identity in the sense of remark 1.6.2. The morphism $\chi \in \mathrm{kk}^{\mathrm{G}}(\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B, \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B)$

$$\chi : \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B \xrightarrow{\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}(\beta_B)} \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B \xrightarrow{\alpha_{\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B}} \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B$$

is represented by

$$\begin{array}{ccccc} \gamma : \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B & \xrightarrow{\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}(\psi_B)} & \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B & \xrightarrow{\varphi_{\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B}} & M_{\mathrm{G}\,/\,\mathrm{H}} \otimes \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B \\ \xi_{\mathrm{H}}(g,b) & \longmapsto & \xi_{\mathrm{H}}(g, \xi_{\mathrm{H}}(e,b)) & \longmapsto & e_{g\,\mathrm{H}, g\,\mathrm{H}} \otimes \xi_{\mathrm{H}}(g,b) \end{array}$$

The following morphism of H-algebras

$$\theta : C \to M_{\mathrm{G}\,/\,\mathrm{H}} \otimes C \qquad \theta(c) = e_{\mathrm{H},\mathrm{H}} \otimes c$$

represents to the identity in the sense of remark 1.6.2. Then $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}(\theta)$ is kk^{G}-equivalent to the identity. It is easy to check $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}(\theta) = \gamma$ with $C = B$. □

Write ψ_{IR} for the isomorphism

$$\psi_{IR} : \mathrm{kk}^{\mathrm{G}}(\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}} B, A) \to \mathrm{kk}^{\mathrm{H}}(B, \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} A) \qquad \psi_{IR} = {\psi_B}^* \circ \mathrm{Res}^{\mathrm{H}}_{\mathrm{G}} \tag{54}$$

where ψ_B is the morphism defined (53).

Corollary 2.2.15. Let G be a group, H be a finite subgroup of G and A be a G-algebra then

$$\mathrm{kk}^{\mathrm{G}}(\ell^{(\mathrm{G}\,/\,\mathrm{H})}, A) \simeq \mathrm{kk}(\ell, A \rtimes \mathrm{H}) \simeq \mathrm{KH}(A \rtimes \mathrm{H})$$

Proof: The isomorphism is the composition of ψ_{GJ} and ψ_{IR} defined in (37) and in (54). □

3. A discrete variant of Green's imprimitivity

Let G be a group, H a subgroup of G and A an H-algebra. We consider the following left action of $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}(A) \rtimes \mathrm{G}$ in $A^{(\mathrm{G})}$

$$((f \rtimes g) \cdot \varphi)(t) = f(t)\varphi(g^{-1}t) \qquad f \in \mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}(A) \quad \varphi \in A^{(\mathrm{G})} \quad g, t \in \mathrm{G} \tag{55}$$

Observe that $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}(A)$ acts on $A^{(\mathrm{G})}$ by multiplication because $\mathrm{Ind}^{\mathrm{G}}_{\mathrm{H}}(A)$ is a subalgebra of A^{G} and $A^{(\mathrm{G})}$ is an ideal of A^{G}. The action of $\ell\,\mathrm{G}$ over $A^{(\mathrm{G})}$ is $(g \cdot \varphi)(t) = \varphi(g^{-1}t)$. Taking the morphism obtained by the universal property of the crossed product we obtain the following commutative diagram:

$$
(56)\qquad
\begin{array}{ccccc}
\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) & \hookrightarrow & A^{\mathrm{G}} & & \\
\downarrow & & \downarrow & \searrow^{\eta} & \\
\Upsilon : \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G} & \hookrightarrow & A^{\mathrm{G}} \rtimes \mathrm{G} & \dashrightarrow & \operatorname{End}_{\ell}(A^{(\mathrm{G})}) \\
 & & \uparrow & \nearrow_{\mu} & \\
 & & \ell\,\mathrm{G} & &
\end{array}
\qquad
\begin{array}{l}
\eta(f)(\varphi)(t) = f(t)\varphi(t) \\[2em]
\mu(g)(\varphi)(t) = \varphi(g^{-1}t)
\end{array}
$$

Here $\Upsilon : \operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G} \to \operatorname{End}_{\ell}(A^{(\mathrm{G})})$ is the map asociated to the action defined in (55).

We also define on $A^{(\mathrm{G})}$ a right action of $A \rtimes \mathrm{H}$ as follows

$$
(57)\qquad (\varphi \cdot (a \rtimes h))(t) = h^{-1} \cdot (\varphi(th^{-1})a) \qquad \varphi \in A^{(\mathrm{G})} \quad a \in A \quad h \in \mathrm{H} \quad t \in \mathrm{G}
$$

It is easy to prove (55) and (57) are in fact a left action and a right action respectively. One checks that these two actions satisfy

$$
(f \rtimes g) \cdot [\varphi \cdot (a \rtimes h)] = [(f \rtimes g) \cdot \varphi] \cdot (a \rtimes h)
$$

Hence they make $A^{(\mathrm{G})}$ into an $(\operatorname{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G}, A \rtimes \mathrm{H})$-bimodule. In particular $\operatorname{Im}\Upsilon \subseteq \operatorname{End}_{A \rtimes \mathrm{H}}(A^{(\mathrm{G})})$. The decomposition

$$
\mathrm{G} = \coprod_{x \in \mathrm{G}\,/\,\mathrm{H}} x
$$

induces

$$
(58)\qquad A^{(\mathrm{G})} = \bigoplus_{x \in \mathrm{G}\,/\,\mathrm{H}} A^{(x)}
$$

and $A^{(x)} \cdot (A \rtimes \mathrm{H}) \subseteq A^{(x)}$. Hence (58) is a direct sum of $A \rtimes \mathrm{H}$-right modules.

Let $T_{x,y} : A^{(y)} \to A^{(x)}$ be a morphism of $A \rtimes \mathrm{H}$-modules such that for each $v \in A^{(y)}$, $T_{x,y}(v) = 0$ for all but a finite number of x. Note the matrix $T = \{T_{x,y}\}_{x,y \in \mathrm{G}\,/\,\mathrm{H}}$ represents an element of $\operatorname{End}_{A \rtimes H}(A^{(\mathrm{G})})$. Moreover, every element $T \in \operatorname{End}_{A \rtimes H}(A^{(\mathrm{G})})$ can be represented in that way. If $x = g\,\mathrm{H}$,

$$
A \rtimes \mathrm{H} \to A^{(g\,\mathrm{H})} \qquad a \rtimes h \mapsto \chi_g \cdot (a \rtimes h) = \chi_{gh} h^{-1}(a)
$$

is an isomorphism of right $A \rtimes \mathrm{H}$-modules. Fix a full set of representatives $\mathcal{R}$ of $\mathrm{G}\,/\,\mathrm{H}$. Write $M_{\mathcal{R}}$ for the algebra of $\mathcal{R} \times \mathcal{R}$-matrices with finitely many nonzero coefficients in ℓ and put

$$
M_{\mathcal{R}}(A \rtimes \mathrm{H}) = M_{\mathcal{R}} \otimes (A \rtimes \mathrm{H})
$$

We have an homomorphism

$$
(59)\qquad
\begin{array}{rcl}
M_{\mathcal{R}}(A \rtimes \mathrm{H}) & \longrightarrow & \operatorname{End}_{A \rtimes \mathrm{H}}(A^{(\mathrm{G})}) \\
M = \{m_{x,y}\}_{x,y \in \mathcal{R}} & \longmapsto & \sum_{x \in X} \chi_x \cdot \alpha_x \mapsto \sum_{y \in \mathcal{R}} \chi_y \cdot \sum_{x \in \mathcal{R}} m_{x,y} \alpha_x
\end{array}
$$

Furthermore, we have a map $\mathrm{G} \to \mathcal{R}$ which sends each $s \in \mathrm{G}$ to the representative $\hat{s} \in \mathcal{R}$ of $s\,\mathrm{H}$. Observe $e_{s\,\mathrm{H},t\,\mathrm{H}} \mapsto e_{\hat{s},\hat{t}}$ is an isomorphism between $M_{|\,\mathrm{G}\,/\,\mathrm{H}\,|}$ and $M_{\mathcal{R}}$. By composition, we obtain an algebra homomorphism

$$(60) \qquad \begin{array}{rlcl} \Gamma : & M_{|\,\mathrm{G}\,/\,\mathrm{H}\,|}(A \rtimes \mathrm{H}) & \to & \mathrm{End}_{A\rtimes\mathrm{H}}(A^{(\mathrm{G})}) \\ & e_{s\,\mathrm{H},t\,\mathrm{H}}(a \rtimes h) & \mapsto & \sum_{y\in\mathcal{R}} \chi_y \cdot \alpha_y \mapsto \chi_{\hat{s}} \cdot (a \rtimes h)\alpha_{\hat{t}} \end{array}$$

Theorem 2.3.1. Let G be a group, $\mathrm{H} \subset \mathrm{G}$ a subgroup, and A an H-algebra. Then there is an isomorphism

$$\alpha : \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G} \to M_{|\,\mathrm{G}\,/\,\mathrm{H}\,|}(A \rtimes \mathrm{H})$$

such that the following diagrams commute

$$\begin{array}{ccc} \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G} & \xrightarrow{\quad\Upsilon\quad} & \mathrm{End}_{A\rtimes\mathrm{H}}(A^{(\mathrm{G})}) \\ & {\scriptstyle\alpha}\searrow \qquad \nearrow{\scriptstyle\Gamma} & \\ & M_{|\,\mathrm{G}\,/\,\mathrm{H}\,|}(A \rtimes \mathrm{H}) & \end{array}$$

$$\begin{array}{ccc} \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G} & \xrightarrow{\quad\alpha\quad} & M_{|\,\mathrm{G}\,/\,\mathrm{H}\,|}(A \rtimes \mathrm{H}) \\ & {\scriptstyle\xi_{\mathrm{H}}(e,-)\rtimes\mathrm{id}}\nwarrow \qquad \nearrow{\scriptstyle e_{\mathrm{H},\mathrm{H}}\otimes -} & \\ & A \rtimes \mathrm{H} & \end{array}$$

The morphism Υ is defined in (56) and the morphism Γ in (60).

Proof: For simplicity suppose $1_{\mathrm{G}} \in \mathcal{R}$. For each $s \in \mathrm{G}$ denote $\phi(s) = \hat{s}^{-1}s \in \mathrm{H}$. Observe $\phi(sh) = \phi(s)h$ for all $s \in \mathrm{G}$, $h \in \mathrm{H}$. Define

$$(61) \qquad \begin{array}{rlcl} \alpha : & \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G} & \to & M_{|\,\mathrm{G}\,/\,\mathrm{H}\,|}(A \rtimes \mathrm{H}) \\ & \xi_{\mathrm{H}}(s,a) \rtimes g & \mapsto & e_{s\,\mathrm{H},g^{-1}s\,\mathrm{H}} \otimes \phi(s) \cdot a \rtimes \phi(s)\phi(g^{-1}s)^{-1} \end{array}$$

The map (61) is well-defined:

$$\begin{aligned} \alpha(\xi_{\mathrm{H}}(sh, h^{-1} \cdot a) \rtimes g) &= e_{sh\,\mathrm{H},g^{-1}sh\,\mathrm{H}} \otimes \phi(sh) \cdot h^{-1} \cdot a \rtimes \phi(sh)\phi(g^{-1}sh)^{-1} \\ &= e_{s\,\mathrm{H},g^{-1}s\,\mathrm{H}} \otimes \phi(s) \cdot a \rtimes \phi(s)\phi(g^{-1}s)^{-1} \\ &= \alpha(\xi_{\mathrm{H}}(s,a) \rtimes g) \end{aligned}$$

Note that if $g, t, s \in \mathrm{G}$ are elements such that $t^{-1}g^{-1}s \in \mathrm{H}$ then $\phi(gt) = \phi(s)\phi(g^{-1}s)^{-1}\phi(t)$. The map (61) is an algebra homomorphism:

$$\begin{aligned}
\alpha((\xi_{\mathrm{H}}(s,a) \rtimes g)(\xi_{\mathrm{H}}(t,b) \rtimes r)) &= \alpha(\xi_{\mathrm{H}}(s,a)\xi_{\mathrm{H}}(gt,b) \rtimes gr) \\
&= \begin{cases} \alpha(\xi_{\mathrm{H}}(gt,[(t^{-1}g^{-1}s)\cdot a]b) \rtimes gr) & t^{-1}g^{-1}s \in \mathrm{H} \\ 0 & t^{-1}g^{-1}s \notin \mathrm{H} \end{cases} \\
&= \begin{cases} e_{gt\,\mathrm{H}, r^{-1}g^{-1}gt\,\mathrm{H}} \otimes \phi(gt)\cdot([(t^{-1}g^{-1}s)\cdot a]b) \rtimes \phi(gt)\phi(r^{-1}t)^{-1} & t^{-1}g^{-1}s \in \mathrm{H} \\ 0 & t^{-1}g^{-1}s \notin \mathrm{H} \end{cases} \\
&= \begin{cases} e_{s\,\mathrm{H}, r^{-1}t\,\mathrm{H}} \otimes [\phi(s)\cdot a][\phi(gt)\cdot b] \rtimes \phi(gt)\phi(r^{-1}t)^{-1} & t^{-1}g^{-1}s \in \mathrm{H} \\ 0 & t^{-1}g^{-1}s \notin \mathrm{H} \end{cases} \\
&= \alpha(\xi_{\mathrm{H}}(s,a) \rtimes g)\alpha(\xi_{\mathrm{H}}(t,b) \rtimes r)
\end{aligned}$$

The map (61) is bijective and its inverse is the following

$$\beta : M_{|\mathrm{G}/\mathrm{H}|}(A \times \mathrm{H}) \to \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G} \qquad e_{s\,\mathrm{H}, g\,\mathrm{H}}(a \rtimes h) \mapsto \xi_{\mathrm{H}}(\hat{s}, a) \rtimes \hat{s}h\hat{g}^{-1}$$

Let us check that the diagrams of the theorem are commutative

$$\begin{aligned}
\Upsilon(\xi_{\mathrm{H}}(s,a) \rtimes g)(\textstyle\sum_{y\in\mathcal{R}} \chi_y \cdot \alpha_y) &= \chi_s \cdot [(a \rtimes \phi(g^{-1}s)^{-1})\alpha_{\widehat{g^{-1}s}}] \\
&= \chi_{\hat{s}} \cdot [(\phi(s)\cdot a \rtimes \phi(s)\phi(g^{-1}s)^{-1})\alpha_{\widehat{g^{-1}s}}] \\
&= (\Gamma \circ \alpha)(\xi_{\mathrm{H}}(s,a) \rtimes g)(\textstyle\sum_{y\in\mathcal{R}} \chi_y \cdot \alpha_y)
\end{aligned}$$

$$\begin{aligned}
(\alpha \circ \xi_{\mathrm{H}}(1_{\mathrm{G}}, -) \rtimes \mathrm{id})(a \rtimes h) &= \alpha(\xi_{\mathrm{H}}(1_{\mathrm{G}}, a) \rtimes h) \\
&= e_{\mathrm{H},\mathrm{H}} \otimes (a \rtimes h)
\end{aligned}$$

□

REMARK 2.3.2. The isomorphism α of the Theorem 2.3.1 is natural in A but not in the pair (G, H), as it depends on a choice of a full set of representatives $\mathcal{R}$ of G/H.

COROLLARY 2.3.3. Let G be a countable group, $\mathrm{H} \subset \mathrm{G}$ a subgroup, and A an H-algebra. Let $\mathrm{E} : \mathrm{Alg} \to \mathcal{C}$ be a M_∞-stable functor. The following map is an isomorphism

$$\mathrm{E}(\xi_{\mathrm{H}}(1_{\mathrm{G}}, -) \rtimes \mathrm{id}) : \mathrm{E}(A \rtimes \mathrm{H}) \to \mathrm{E}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) \rtimes \mathrm{G}).$$

□

4. Duality

Let $\hat{\mathrm{G}}$-Alg be the category of $\hat{\mathrm{G}}$-algebras with homogeneous homomorphisms. In this section we define functors between the categories G-Alg and $\hat{\mathrm{G}}$-Alg. We prove that they extend to equivalences between $\mathfrak{KK}^{\mathrm{G}}$ and $\mathfrak{KK}^{\hat{\mathrm{G}}}$. In this way we obtain an algebraic duality theorem similar to the duality given by Baaj-Skandalis in [**1**].

Let A be a G-algebra. Then

$$A \rtimes \mathrm{G} = \bigoplus_{s\in \mathrm{G}} A \rtimes s \qquad \text{and } (A \rtimes s)(A \rtimes t) \subset A \rtimes st$$

thus $A \rtimes \mathrm{G}$ is a $\hat{\mathrm{G}}$-algebra. If $f : A \to B$ is a homomorphism of $\hat{\mathrm{G}}$-algebras then $f \rtimes \mathrm{G} : A \rtimes \mathrm{G} \to B \rtimes \mathrm{G}$ is a graded homomorphism. Hence we have a functor

$$\rtimes \mathrm{G} : \mathrm{G}\text{-Alg} \to \hat{\mathrm{G}}\text{-Alg}$$

We can also define a functor

$$\rtimes \hat{\mathrm{G}} : \hat{\mathrm{G}}\text{-Alg} \to \mathrm{G}\text{-Alg}$$

as follows. Let B be a $\hat{\mathrm{G}}$-algebra. Let $\hat{\mathrm{G}} \rtimes B$ be the algebra which as a module is $\ell^{(\mathrm{G})} \otimes B$ and the product is the following

$$(\chi_g \rtimes a)(\chi_h \rtimes b) := \chi_g \rtimes a_{g^{-1}h} b. \tag{62}$$

Recall b_g is the homogeneous element associated to g in the decomposition

$$b = \sum_{g\in \mathrm{G}} b_g.$$

One check that the product (62) is associative and the following action of G makes it into a G-algebra

$$s \cdot (\chi_g \rtimes a) = \chi_{sg} \rtimes a$$

If $f : A \to B$ is a homogeneous homomorphism define

$$\hat{\mathrm{G}} \rtimes f : \hat{\mathrm{G}} \rtimes A \to \hat{\mathrm{G}} \rtimes B \qquad (\hat{\mathrm{G}} \rtimes f)(\chi_g \rtimes a) := \chi_g \rtimes f(a)$$

It is a G-algebra homomorphism. Thus we have a functor

$$\rtimes \hat{\mathrm{G}} : \hat{\mathrm{G}}\text{-Alg} \to \mathrm{G}\text{-Alg} \tag{63}$$

Proposition 2.4.1. Let A be a G-algebra and let B be a $\hat{\mathrm{G}}$-algebra.

a) There are natural isomorphisms of G-algebras

$$\hat{\mathrm{G}} \rtimes (A \rtimes \mathrm{G}) \simeq M_{\mathrm{G}} \otimes A$$

b) There are natural isomorphisms of $\hat{\mathrm{G}}$-algebras

$$(\hat{\mathrm{G}} \rtimes B) \rtimes \mathrm{G} \simeq M_{\hat{\mathrm{G}}} \otimes B$$

Proof:

a) Define $T : \hat{\mathrm{G}} \rtimes (A \rtimes \mathrm{G}) \to M_{\mathrm{G}} \otimes A$ as

$$T(\chi_g \rtimes a \rtimes s) = g \cdot a \otimes e_{g,gs}.$$

It is easy to check T is an equivariant algebra isomorphism with inverse given by

$$S(a \otimes e_{r,t}) := \chi_r \rtimes r^{-1} \cdot a \rtimes r^{-1} t$$

b) Define $T : (\hat{G} \ltimes B) \rtimes G \to M_{\hat{G}} \otimes B$ as

$$T(\chi_h \ltimes b \rtimes s) = \sum_{r \in G} e_{h,s^{-1}hr} \otimes b_r.$$

It is easy to check T is a graded algebra isomorphism with inverse given by

$$S(e_{r,s} \otimes b_q) = \chi_r \ltimes b_q \rtimes rqs^{-1} \tag{64}$$

□

THEOREM 2.4.2. The functors $\rtimes G$ and $\hat{G} \ltimes$ extend to inverse equivalences

$$- \rtimes G : \mathfrak{KK}^G \longrightarrow \mathfrak{KK}^{\hat{G}} \qquad \hat{G} \ltimes - : \mathfrak{KK}^{\hat{G}} \longrightarrow \mathfrak{KK}^G$$

Hence if A and B are G-algebras and C and D are $\hat{G}$-algebras then

$$\mathrm{kk}^G(A, B) \simeq \mathrm{kk}^{\hat{G}}(A \rtimes G, B \rtimes G) \quad \mathrm{kk}^{\hat{G}}(C, D) \simeq \mathrm{kk}^G(\hat{G} \ltimes C, \hat{G} \ltimes D)$$

PROOF: As $\rtimes G$ maps split sequences to split sequences, $j^{\hat{G}}(- \rtimes G)$ is excisive. By Proposition 2.1.2 $j^{\hat{G}}(- \rtimes G)$ is G-stable and homotopy invariant, whence it extends to $- \rtimes G : \mathfrak{KK}^G \to \mathfrak{KK}^{\hat{G}}$ by universality. Similary, as $\ltimes_{\hat{G}}$ maps split sequences to split sequences then $j^G(\hat{G} \ltimes -)$ is excisive. Because $\hat{G} \ltimes$ maps graded homotopies to equivariant homotopies and $j^G(\hat{G} \ltimes -)$ is M_∞-stable, $j^G(\hat{G} \ltimes -)$ extends to $\hat{G} \ltimes - : \mathfrak{KK}^{\hat{G}} \to \mathfrak{KK}^G$ by universality. To finish we must show that the maps

$$\mathrm{kk}^G(A, B) \to \mathrm{kk}^{\hat{G}}(A \rtimes G, B \rtimes G) \quad \text{and} \quad \mathrm{kk}^{\hat{G}}(C, D) \to \mathrm{kk}^G(\hat{G} \ltimes C, \hat{G} \ltimes D)$$

are isomorphisms. This is true by Proposition 2.4.1. □

5. Green-Julg theorem for $\mathrm{kk}^{\mathcal{H}}$

In this section we consider a finite dimensional semisimple Hopf algebra $\mathcal{H}$ and prove a version of the Green-Julg theorem for $\mathrm{kk}^{\mathcal{H}}$.

5.1. Smash product. We shall recall the smash product for $\mathcal{H}$-algebras. It is a generalization for $\mathcal{H}$-algebras of the crossed product $A \rtimes G$.

DEFINITION 2.5.1. Let A be an $\mathcal{H}$-algebra. The smash product algebra $A\#\mathcal{H}$ is the ℓ-module $A \otimes \mathcal{H}$ with the following product

$$(a\#h)(b\#k) = \sum a(h_1 \cdot b)\#h_2k \qquad a, b \in A \quad h, k \in \mathcal{H}$$

If $f : A \to B$ is a morphism of $\mathcal{H}$-algebras, we put

$$f\#\mathcal{H} : A\#\mathcal{H} \to B\#\mathcal{H} \quad f\#\mathcal{H}(a\#h) = f(a)\#h$$

which is a morphism of algebras. Hence, we have a functor $\# \mathcal{H} : \mathcal{H}\text{-Alg} \to \text{Alg}$.

PROPOSITION 2.5.2. Let M be an $\mathcal{H}$-module and A be an $\mathcal{H}$-algebra. The following is an isomorphism of algebras

$$\phi : \mathrm{End}_\ell(M) \otimes (A\#\mathcal{H}) \to (\mathrm{End}_\ell(M) \otimes A)\#\mathcal{H} \quad \phi(\varphi \otimes a\#h) = \sum \varphi_{1,h_1} \otimes a\#h_2 \tag{65}$$

Moreover, the following restriction of ϕ is also an ismorphism

$$\phi : \mathrm{End}^F_\ell(M) \otimes (A\#\mathcal{H}) \to (\mathrm{End}^F_\ell(M) \otimes A)\#\mathcal{H}$$

$\square$

PROOF: Let us check (65) is an algebra morphism:

$$\begin{aligned}
\phi((\varphi\otimes a\#h)(\tilde{\varphi}\otimes\tilde{a}\#\tilde{h})) &= \sum\phi(\varphi\tilde{\varphi}\otimes a(h_1\cdot\tilde{a})\#h_2\tilde{h}) \\
&= \sum(\varphi\tilde{\varphi})_{1,h_2\tilde{h}_1}\otimes a(h_1\cdot\tilde{a})\#h_3\tilde{h}_2 \\
&= \sum(\varphi(\tilde{\varphi}(S(h_3\tilde{h}_1)\cdot\quad))\otimes a(\epsilon(h_1)h_2\cdot\tilde{a})\#h_4\tilde{h}_2 \\
&= \sum\varphi(S(h_1)h_2\tilde{\varphi}(S(h_4\tilde{h}_1)\cdot\quad))\otimes a(h_3\cdot\tilde{a})\#h_5\tilde{h}_2 \\
&= \sum\varphi_{1,h_1}\tilde{\varphi}_{h_2,h_4\tilde{h}_1}\otimes a(h_3\cdot\tilde{a})\#h_5\tilde{h}_2 \\
&= \sum(\varphi_{1,h_1}\otimes a)(\tilde{\varphi}_{h_2,h_4\tilde{h}_1}\otimes h_3\cdot\tilde{a})\#h_5\tilde{h}_2 \\
&= \sum(\varphi_{1,h_1}\otimes a)((\tilde{\varphi}_{1,\tilde{h}_1})_{h_2,h_4}\otimes h_3\cdot\tilde{a})\#h_5\tilde{h}_2 \\
&= \sum(\varphi_{1,h_1}\otimes a)h_2\cdot(\varphi_{1,\tilde{h}_1}\otimes\tilde{a})\#h_3\tilde{h}_2 \\
&= \phi(\varphi\otimes a\#h)\phi(\tilde{\varphi}\otimes\tilde{a}\#\tilde{h})
\end{aligned}$$

It is easy to check that the following map is also an algebra morphism and is the inverse of (65).

$$\psi:(\mathrm{End}_\ell(M)\otimes A)\#\mathcal{H}\to\mathrm{End}_\ell(M)\otimes(A\#\mathcal{H})\quad \psi(\varphi\otimes a\#h)=\sum\varphi_{1,\overline{S}(h_1)}\otimes a\#h_2$$

By remark 1.7.22, we can restrict the homomorphisms defined above and obtain the following isomorphisms

$$\psi:(\mathrm{End}_\ell^F(M)\otimes A)\#\mathcal{H}\to\mathrm{End}_\ell^F(M)\otimes(A\#\mathcal{H})\quad \phi:\mathrm{End}_\ell^F(M)\otimes(A\#\mathcal{H})\to(\mathrm{End}_\ell^F(M)\otimes A)\#\mathcal{H}.\quad\square$$

PROPOSITION 2.5.3. Let $\mathcal{H}$ a Hopf algebra with finite dimension. There exists a unique functor $\#\,\mathcal{H}:\mathfrak{KK}^{\mathcal{H}}\to\mathfrak{KK}$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{H}\text{-Alg} & \xrightarrow{\#\mathcal{H}} & \text{Alg} \\
{\scriptstyle j^{\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle j} \\
\mathfrak{KK}^{\mathcal{H}} & \xrightarrow[\#\mathcal{H}]{} & \mathfrak{KK}
\end{array}$$

PROOF: By 1.8.3 it is enough to prove $j(-\#\mathcal{H})$ is excisive, homotopy invariant and $\mathcal{H}$-stable. The two first properties are straightforward. Let M_1, M_2 be $\mathcal{H}$ modules with countable dimension and A be an $\mathcal{H}$-algebra. Consider the isomorphism ψ defined in the

Proposition 2.5.2. Note the following diagram is commutative

$$\begin{array}{ccc} (\mathrm{End}_\ell(M_1)\otimes A)\#\mathcal{H} & \xrightarrow{(\tilde{\iota}\otimes 1)\#\mathcal{H}} & (\mathrm{End}_\ell(M_1\oplus M_2)\otimes A)\#\mathcal{H} \\ \psi\downarrow & & \downarrow\psi \\ \mathrm{End}_\ell(M_1)\otimes(A\#\mathcal{H}) & \xrightarrow[\tilde{\iota}\otimes 1\#\mathcal{H}]{} & \mathrm{End}_\ell(M_1\oplus M_2)\otimes(A\#\mathcal{H}) \end{array}$$

As j is M_∞-stable we know $j(\tilde{\iota}\otimes 1\#\mathcal{H})$ is an isomorphism. Hence $j(-\#\mathcal{H})(\tilde{\iota}\otimes 1)$ is an isomorphism and $j(-\#\mathcal{H})$ is $\mathcal{H}$-stable. □

Let A be an $\mathcal{H}$-algebra. Consider $\mathcal{H}\otimes A$ as a left $\mathcal{H}$-module with the diagonal action. Also consider $\mathcal{H}\otimes A$ as a right A-module with the regular action. In other words

$$h\cdot(k\otimes a)=\sum h_1k\otimes h_2\cdot a \qquad (k\otimes a)\cdot c=k\otimes ac$$

It is easy to check that

$$t\cdot((h\otimes a)\cdot c)=\sum(t_1\cdot(h\otimes a))\cdot(t_2\cdot c) \tag{66}$$

We define

$$\mathrm{End}_A(\mathcal{H}\otimes A):=\{\varphi\in\mathrm{End}_\ell(\mathcal{H}\otimes A)\text{ such that }\varphi(k\otimes ac)=\varphi(k\otimes a)\cdot c\}$$

The structure of $\mathcal{H}$-module in $\mathcal{H}\otimes A$ gives an $\mathcal{H}$-algebra structure in $\mathrm{End}_\ell(\mathcal{H}\otimes A)$, see Example 1.1.3. It is easy to check that $\mathrm{End}_A(\mathcal{H}\otimes A)$ is a sub-$\mathcal{H}$-algebra of $\mathrm{End}_\ell(\mathcal{H}\otimes A)$. Consider $\mathrm{End}_\ell(\mathcal{H})\otimes A$ as the $\mathcal{H}$-algebra defined in Example 1.1.4 We have the following homomorphism of $\mathcal{H}$-algebras

$$T:\mathrm{End}_\ell(\mathcal{H})\otimes A\to\mathrm{End}_A(\mathcal{H}\otimes A) \qquad T(\varphi\otimes a)(h\otimes b)=\varphi(h)\otimes ab \tag{67}$$

If $\mathcal{H}$ is finite dimensional, then (67) is an isomorphism.

Theorem 2.5.4. Let $\mathcal{H}$ be a semisimple Hopf ℓ-algebra. The functor given by the trivial action $\tau:\mathfrak{KK}\to\mathfrak{KK}^{\mathcal{H}}$ is left adjoint to the functor given by the smash product $\#\mathcal{H}:\mathfrak{KK}^{\mathcal{H}}\to\mathfrak{KK}$. In particular there is a natural isomorphism

$$\mathrm{kk}^{\mathcal{H}}(A^\tau,B)\simeq\mathrm{kk}(A,B\#\mathcal{H}) \qquad A\in\mathrm{Alg} \quad B\in\mathcal{H}\text{-Alg}$$

Proof: It is enough to prove that there exist

$$\overline{\alpha}_A\in\mathrm{kk}(A,A^\tau\#\mathcal{H}) \qquad\text{and}\qquad \overline{\beta}_B\in\mathrm{kk}^{\mathcal{H}}((B\#\mathcal{H})^\tau,B)$$

such that

$$A^\tau\xrightarrow{\tau(\overline{\alpha}_A)}(A^\tau\#\mathcal{H})^\tau\xrightarrow{\overline{\beta}_{A^\tau}}A^\tau \quad\text{and}\quad B\#\mathcal{H}\xrightarrow{\overline{\alpha}_{B\#\mathcal{H}}}(B\#\mathcal{H})^\tau\#\mathcal{H}\xrightarrow{\#(\overline{\beta}_B)}B\#\mathcal{H}$$

are the identities in $\mathrm{kk}^{\mathcal{H}}(A^\tau,A^\tau)$ and $\mathrm{kk}(B\#\mathcal{H},B\#\mathcal{H})$ respectively. As $\mathcal{H}$ is semisimple there exists an element $t\in\mathcal{H}$ such that

$$\epsilon(t)=1 \qquad th=\epsilon(h)t \qquad \forall h\in\mathcal{H}$$

Define

$$\alpha_A:A\to A^\tau\#\mathcal{H}=A\otimes\mathcal{H} \qquad \alpha_A(a)=a\otimes t$$

It is an algebra morphism because t is idempotent. Let

$$\begin{array}{rccl} \beta_B: & (B\#\mathcal{H})^\tau & \to & \mathrm{End}_B(\mathcal{H}\otimes B) \\ & b\#h & \mapsto & \varphi_{b\#h} \end{array}$$

$$\varphi_{b\#h}(k\otimes a)=\sum k_1\overline{S}(h_2)\otimes(k_2\overline{S}(h_1)\cdot b)a$$

One checks that β_B is an equivariant algebra homomorphism. Consider

$$\begin{array}{ccccc} A^\tau & \xrightarrow{\tau(\alpha_A)} & (A^\tau\#\mathcal{H})^\tau & \xrightarrow{\beta_{A^\tau}} & \mathrm{End}_{A^\tau}(\mathcal{H}\otimes A^\tau) \\ a & \mapsto & a\otimes t & \mapsto & \varphi_{a\otimes t} \end{array} \tag{68}$$

Note

$$\varphi_{a\otimes t}(k\otimes b)=\sum k_1\overline{S}(t_2)\otimes(k_2\overline{S}(t_1)\cdot a)b=\sum k_1\overline{S}(t_2)\otimes\epsilon(k_2)\epsilon(t_1)\cdot ab=k\overline{S}(t)\otimes ab=\epsilon(k)t\otimes ab$$

As $\mathcal{H}$ is unimodular we have $t=\overline{S}(t)$. We write $<t>$ for the subspace of $\mathcal{H}$ generated by t. Let $I=\ker\epsilon$ and note $\mathcal{H}=<t>\oplus I$ as $\mathcal{H}$-modules. Let $\varphi:\mathcal{H}\to\mathcal{H}$ be the projection over $<t>$, $\varphi(h)=\epsilon(h)t$. Because the following diagram commutes the map (68) represents the identity in $\mathrm{kk}^{\mathcal{H}}(A^\tau,A^\tau)$ in the sense of remark 1.6.2.

$$\begin{array}{ccc} A^\tau & \xrightarrow{\iota} & \mathrm{End}_\ell(\mathcal{H})\otimes A^\tau \\ & \searrow^{(68)} & \downarrow{\scriptstyle (67)} \\ & & \mathrm{End}_{A^\tau}(\mathcal{H}\otimes A^\tau) \end{array} \qquad \begin{array}{ccc} a & \longrightarrow & \varphi\otimes a \\ & \searrow & \downarrow \\ & & \varphi_{a\otimes t} \end{array}$$

It remains to prove that the following morphism represents the identity in $\mathrm{kk}(B\#\mathcal{H},B\#\mathcal{H})$

$$\begin{array}{ccccc} B\#\mathcal{H} & \xrightarrow{\alpha_{B\#\mathcal{H}}} & (B\#\mathcal{H})^\tau\otimes\mathcal{H} & \xrightarrow{\#\beta_B} & \mathrm{End}_B(\mathcal{H}\otimes B)\#\mathcal{H} \\ b\#h & \mapsto & b\#h\otimes t & \mapsto & \varphi_{b\#h}\#t \end{array} \tag{69}$$

The following morphism

$$\Omega:\mathrm{End}_B(\mathcal{H}\otimes B)\#\mathcal{H}\to\mathrm{End}_{B\#\mathcal{H}}(\mathcal{H}\otimes B\#\mathcal{H}) \qquad \Omega(\eta\#l)(x\otimes c\#y)=\sum\eta(l_1x\otimes l_2\cdot c)\#l_3y$$

makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{End}_B(\mathcal{H}\otimes B)\#\mathcal{H} & \xleftarrow{T_B\#\,\mathrm{id}} & (\mathrm{End}_\ell(\mathcal{H})\otimes B)\#\mathcal{H} \\ {\scriptstyle\Omega}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathrm{End}_{B\#\mathcal{H}}(\mathcal{H}\otimes B\#\mathcal{H}) & \xleftarrow[T_{B\#\mathcal{H}}]{} & \mathrm{End}_\ell(\mathcal{H})\otimes(B\#\mathcal{H}) \end{array}$$

here ψ is the isomorphism defined in Proposition 2.5.2 and T_B and $T_{\mathcal{H}\#B}$ are the morphisms defined in (67). As $\mathcal{H}$ is finite dimensional, T_B and $T_{\mathcal{H}\#B}$ are isomorphisms. Hence Ω is an isomorphism too. Write by $\Lambda_{b\#h}=\Omega(\varphi_{b\#h}\#t)$. We shall prove that the map $b\#h\mapsto\Lambda_{b\#h}$ represents to the identity. Define

$$\delta:\mathcal{H}\otimes B\#\mathcal{H}\to\mathcal{H}\otimes B\#\mathcal{H} \quad \gamma:\mathcal{H}\otimes B\#\mathcal{H}\to\mathcal{H}\otimes B\#\mathcal{H}$$

$$\delta(x\otimes a\#y)=\sum x_1\otimes x_2\cdot a\#x_3y \qquad \gamma(x\otimes a\#y)=\sum x_1\otimes S(x_3)\cdot a\#S(x_2)y$$

It is easy to check that they are mutually inverse. By M_∞-stability the following morphism represents the identity

$$B\#\mathcal{H} \to \mathrm{End}_\ell(\mathcal{H}) \otimes (B\#\mathcal{H}) \qquad b\#h \mapsto \varphi \otimes b\#h$$

Hence

$$B\#\mathcal{H} \to \mathrm{End}_{B\#\mathcal{H}}(\mathcal{H} \otimes B\#\mathcal{H}) \qquad b\#h \mapsto \delta \circ T_{B\#\mathcal{H}}(\varphi \otimes b\#h) \circ \gamma$$

also represents the identity. We finish the proof verifying that

$$\begin{aligned}
(\delta \circ T_{B\#\mathcal{H}}(\varphi \otimes b\#h) \circ \gamma)(x \otimes a\#y) &= \textstyle\sum (\delta \circ \Gamma_{b\#h})(x_1 \otimes S(x_3) \cdot a\#S(x_2)y) \\
&= \textstyle\sum \delta(\epsilon(x_1)t \otimes b(h_1S(x_3) \cdot a)\#h_2S(x_2)y) \\
&= \textstyle\sum \delta(t \otimes b(h_1S(x_2) \cdot a)\#h_2S(x_1)y) \\
&= \textstyle\sum t_1 \otimes t_2(b(h_1S(x_2)) \cdot a)\#t_3h_2S(x_1)y \\
&= \textstyle\sum t \cdot (1 \otimes b(h_1S(x_2)) \cdot a\#h_2S(x_1)y) \\
&= \textstyle\sum t\epsilon(x_1)\epsilon(h_3) \cdot (1 \otimes b(h_1S(x_3)) \cdot a\#h_2S(x_2)y) \\
&= \textstyle\sum tx_1\overline{S}(h_3) \cdot (1 \otimes b(h_1S(x_3)) \cdot a\#h_2S(x_2)y) \\
&= \textstyle\sum t_1x_1\overline{S}(h_6) \otimes (t_2x_2\overline{S}(h_5) \cdot b)(t_3x_3\overline{S}(h_4)h_1S(x_6) \cdot a)\#t_4x_4\overline{S}(h_3)h_2S(x_5)y \\
&= \textstyle\sum t_1x_1\overline{S}(h_4) \otimes (t_2x_2\overline{S}(h_3) \cdot b)(t_3x_3\overline{S}(h_2)h_1S(x_4) \cdot a)\#t_4y \\
&= \textstyle\sum t_1x_1\overline{S}(h_2) \otimes (t_2x_2\overline{S}(h_1) \cdot b)(t_3 \cdot a)\#t_4y \\
&= \Lambda_{b\#h}(x \otimes a\#y)
\end{aligned}$$

□

CHAPTER 3

Isomorphism conjectures with proper coefficients

In this chapter we study isomorphism conjecures in the sense of [**8**]. We consider model category structures on G-simplicial sets and G-topological spaces. If $\mathcal{F}$ is a family of subgroups of G we consider model category structures on $\mathbb{S}^{\mathrm{G}}$ and $\mathbf{Top}^{\mathrm{G}}$. With this structure weak equivalences and fibration are object-wise. Cofibrant object are those X such that the stabilizer subgroup G_x is a subgroup in $\mathcal{F}$ for all $x \in X$. We prove that the following is a Quillen equivalence

$$\mathbf{Top}^{\mathrm{G}} \underset{||_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} \mathbb{S}^{\mathrm{G}}.$$

We say that a functor $\mathrm{H} : \mathbb{S}^{\mathrm{G}} \to \mathbf{Spt}$ from the category of G-simplicial set to the category of spectra satisfies the $(\mathrm{G}, \mathcal{F})$-isomorphism conjecture if for the cofibrant replacement $\pi : \mathcal{E}(\mathrm{G}, \mathcal{F}) \to *$ in the $\mathcal{F}$-model category mentioned aboved, the map

$$\mathrm{H}(\pi) : \mathrm{H}(\mathcal{E}(\mathrm{G}, \mathcal{F})) \to \mathrm{H}(*)$$

is an equivalence. If $\mathrm{E} : \mathbb{Z}\text{-}\mathbf{Cat} \to \mathbf{Spt}$ is a functor and R is a unital G-ring, one constructs, following Davis-Luck [**8**], a functor

$$H^{\mathrm{G}}(-, \mathrm{E}(R)) : \mathbb{S}^{\mathrm{G}} \to \mathbf{Spt}$$

such that $H^{\mathrm{G}}(*, \mathrm{E}(R)) = \mathrm{E}(R \rtimes \mathrm{G})$. The $(\mathrm{G}, \mathcal{F}, \mathrm{E}, R)$-isomorphism conjecture is the $(\mathrm{G}, \mathcal{F})$-conjecture for the functor $H^{\mathrm{G}}(-, \mathrm{E}(R))$. We show that under very mild assumptions on E, the Standing Assumptions 3.2.5 (which are satisfied for example when E is either K or KH, see propositions 3.4.18 and 3.5.3), $H^{\mathrm{G}}(-, \mathrm{E}(A))$ is defined not only for unital G-rings, but also for all E-excisive G-rings A, that is all G-rings on which E-salisfies excision. Moreover we show that if

$$0 \to A' \to A \to A'' \to 0 \tag{70}$$

is an exact sequence of E-excisive rings and X is a G-simplicial set, then

$$H^{\mathrm{G}}(X, \mathrm{E}(A')) \to H^{\mathrm{G}}(X, \mathrm{E}(A)) \to H^{\mathrm{G}}(X, \mathrm{E}(A''))$$

is a homotopy fibration. This is a basic property needed to establish an algebraic analogue of the Dirac-dual method which is used to prove the Baum-Connes conjecture for some groups. Another basic property, which provide us with enough sequences (70) in which at least one of the rings satisfies the isomorphism conjecture, is Theorem 3.7.3 which shows that if E satisfies the standing assumptions and A is an E-excisive G-ring of the form

$$A = \bigoplus_i \mathrm{Ind}_{\mathrm{K}_i}^{\mathrm{G}} B_i \tag{71}$$

with B_i a K_i-ring and $\mathrm{K}_i \in \mathcal{F}$ for all i, then the functor $H^{\mathrm{G}}(-, \mathrm{E}(A))$ maps $(\mathrm{G}, \mathcal{F})$-equivalences to equivalences. In particular the $(\mathrm{G}, \mathcal{F}, \mathrm{E}, A)$-isomorphism conjecture holds. We use this in Section 8 to show that under some additional assumptions (which are still satisfied if $\mathrm{E} = K$, KH, see 3.8.1), for each E-excisive G-ring A there is a functorial exact sequence of E-excisive G-rings

$$\mathfrak{F}^0 B \to \mathfrak{F}^\infty B \to \mathfrak{F}^\infty B / \mathfrak{F}^0 B$$

and a natural map $A \to \mathfrak{F}^0 B$ such that

i) $H^{\mathrm{G}}(X, \mathrm{E}(A)) \to H^{\mathrm{G}}(X, \mathrm{E}(\mathfrak{F}^0 B))$ is an equivalence for all G-simplicial set X.
ii) $H^{\mathrm{G}}(X, \mathrm{E}(\mathfrak{F}^\infty B)) \to *$ is an equivalence if X is $(\mathrm{G}, \mathcal{F})$-cofibrant.
iii) $H^{\mathrm{G}}(-, \mathrm{E}(\mathfrak{F}^\infty B / \mathfrak{F}^0 B))$ maps $(\mathrm{G}, \mathcal{F})$-equivalences to equivalences.

It follows that the assembly map

$$H^{\mathrm{G}}(\mathcal{E}(\mathrm{G}, \mathcal{F}), \mathrm{E}(A)) \to \mathrm{E}(A \rtimes \mathrm{G})$$

is an equivalence iff the conecting map

$$\Omega(\mathrm{E}(\mathfrak{F}^\infty B / \mathfrak{F}^0 B \rtimes \mathrm{G})) \to \mathrm{E}(A \rtimes \mathrm{G})$$

is an equivalence. In particular all this applies when $\mathrm{E} = K, \mathrm{KH}$. We also show in Theorem 3.9.2 that under stronger hypotesis on E, of which the main one is that E satisfies excision (e.g. KH satisfies this but K does not), then the $(\mathrm{G}, \mathcal{F}, \mathrm{E}, A)$-isomorphism conjecture is true whenever A is $(\mathrm{G}, \mathcal{F})$-proper. If X is a locally finite simplicial set with a G-action then a G-ring A is proper over X if it is an algebra over the ring $\mathbb{Z}^{(X)}$ of finitely supported polynomial maps on X, the algebra action is compatible with the actions of G on A and on X, and $\mathbb{Z}^{(X)} \cdot A = A$. We say that A is $(\mathrm{G}, \mathcal{F})$-proper if it is proper over a locally finite simplicial set X on which G acts with all stabilizers in $\mathcal{F}$. For example an algebra is of the form (71) if and only if it is proper over the zero-dimensional G-simplicial set $X = \coprod \mathrm{G} / \mathrm{K}_i$.

We remark that the notion of $(\mathrm{G}, \mathcal{F})$-proper ring used here is the algebraic analogue of the notion of proper G-C^*-algebra, and that Teorem 3.9.2 is an algebraic version of the known fact that Baum-Connes conjecture holds for proper G-C^*-algebras [**11**].

1. G-simplicial sets and model category structures

1.1. G-simplicial sets. A G-simplicial set or a G-complex X is a simplicial set with a simplicial action of G. Let X be a G-complex; consider the category $(\Delta \downarrow X)_{\mathrm{G}}$ defined as follows. Its objects are the triples $(\mathrm{G}/\mathrm{H}, [n], \sigma : \mathrm{G}/\mathrm{H} \times \Delta^n \to X)$ such that H is a subgroup of G and $\sigma : \mathrm{G}/\mathrm{H} \times \Delta^n \to X$ is a morphism such that $\sigma(g\mathrm{H}, x) = g \cdot \sigma(\mathrm{H}, x)$. An arrow is a pair (s, θ), $s : \mathrm{G}/\mathrm{H} \to \mathrm{G}/\mathrm{K}$, $\theta : [n] \to [m]$, such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{G}/\mathrm{H} \times \Delta^n & \xrightarrow{s \times \hom_\Delta(-, \theta)} & \mathrm{G}/\mathrm{K} \times \Delta^m \\ & {\scriptstyle\sigma} \searrow \quad \swarrow {\scriptstyle\tau} & \\ & X & \end{array} .$$

As in the nonequivariant case (see [**10**], I.2), any G-simplicial set is the colimit of its cells. Let recall this in the following lemma.

Lemma 3.1.1. Consider

$$T_X : (\Delta \downarrow X)_{\mathrm{G}} \to \mathbb{S}^{\mathrm{G}} \qquad (\mathrm{G}/\mathrm{H}, [n], \sigma : \mathrm{G}/\mathrm{H} \times \Delta^n \to X) \mapsto \sigma : \mathrm{G}/\mathrm{H} \times \Delta^n \to X$$

Then

$$X = \operatorname{colim}_{\mathrm{G}/\mathrm{H} \times \Delta^n \to X} \mathrm{G}/\mathrm{H} \times \Delta^n = \operatorname{colim} T_X$$

□

The n-skeleton X_n of a G-complex X can be obtained by attaching equivariant cells to X_{n-1} as the following pushout diagram shows

$$\begin{array}{ccc} \coprod_{i \in I_n} \mathrm{G}/\mathrm{H}_i \times \partial\Delta^n & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} \mathrm{G}/\mathrm{H}_i \times \Delta^n & \xrightarrow[\coprod_{i \in I_n} Q_i^n]{} & X_n \end{array}$$

A family $\mathcal{F}$ of subgroups of G is a nonempty family closed under conjugation and under taking subgroups. A G-complex X is a $(\mathrm{G}, \mathcal{F})$-complex if $\mathrm{H}_i \in \mathcal{F}$ for all $i \in I$, $n \in \mathbb{N}$.

1.2. Orbit category. Let $\mathcal{F}$ be a family of subgroups of G. We consider the orbit category of G relative to $\mathcal{F}$ and we write it $\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}$. Its objects are the G-sets G/H with $\mathrm{H} \in \mathcal{F}$ and its maps are the G-equivariant maps. We write $\mathbf{Or}\,\mathrm{G}$ for $\mathbf{Or}_{\mathbf{All}}\,\mathrm{G}$. If $\mathrm{H}, \mathrm{K} \in \mathcal{F}$ and $g \in \mathrm{G}$ is such that $g^{-1}\mathrm{H}\,g \subset \mathrm{K}$ then

$$r_g : \mathrm{G}/\mathrm{H} \to \mathrm{G}/\mathrm{K} \qquad r_g(t\,\mathrm{H}) = tg\,\mathrm{K}$$

is a morphism of $\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}$. Moreover, every morphism in $\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}$ is r_g for some $g \in \mathrm{G}$.

1.3. Quillen equivalences. We consider **Top** and $\mathbb{S}$ with their usual cofibrantly generated closed model category structures (see Section 2.4, Section 3.2 and Definition 2.1.17 of [**14**]). The sets of generating cofibrations $I_{\mathbf{Top}}$, $I_{\mathbb{S}}$ and the sets of generating trivial cofibration $J_{\mathbf{Top}}$, $J_{\mathbb{S}}$ are the following

$$I_{\mathbf{Top}} = \{f : S^{n-1} \hookrightarrow D^n : \quad n \geq 0\} \quad I_{\mathbb{S}} = \{f : \partial\Delta^n \hookrightarrow \Delta^n : \quad n \geq 0\}$$

$$J_{\mathbf{Top}} = \{f : D^n \hookrightarrow D^n \times I : \quad f(x) = (x, 0) \quad n \geq 0\} \quad J_{\mathbb{S}} = \{f : \Lambda^n_k \hookrightarrow \Delta^n : \quad n > 0, \quad 0 \leq k \leq n\}$$

By Theorem 3.6.7 of [**14**] we have that the geometric realization functor $|| : \mathbb{S} \to \mathbf{Top}$ and its right adjoint $\mathrm{Sing} : \mathbf{Top} \to \mathbb{S}$ form a Quillen equivalence (see Definition 1.3.12 in [**14**])

$$\mathbf{Top} \underset{||}{\overset{\mathrm{Sing}}{\rightleftarrows}} \mathbb{S}$$

If $\mathcal{C} = \mathbf{Top}$, $\mathbb{S}$, and I is any small category, then, by [**13**, Thm. 11.6.1], $\mathcal{C}^I$ is again a cofibrantly generated closed model category, with object-wise fibrations and weak equivalences, and where generating (trivial) cofibrations are of the form

$$\coprod_{\hom_I(\alpha,-)} f : \coprod_{\hom_I(\alpha,-)} \mathrm{dom} f \to \coprod_{\hom_I(\alpha,-)} \mathrm{cod} f$$

with $\alpha \in I$ and $f : \mathrm{dom} f \to \mathrm{cod} f$ a generating (trivial) cofibration in $\mathcal{C}$. By [**13**, Thm. 11.6.5], the induced functors

$$\mathbf{Top}^I \underset{|\,|_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} \mathbb{S}^I$$

are also Quillen equivalences.

Next fix a group G and a family $\mathcal{F}$ of subgroups of G. By the previous discussion applied to the orbit category $\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}$, we have a Quillen equivalence

$$\mathbf{Top}^{\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}} \underset{|\ \ |_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} \mathbb{S}^{\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}} \tag{72}$$

Let H be a subgroup of G and X an object in $\mathcal{C}^{\mathrm{G}}$. Consider

$$X^{\mathrm{H}} := \{x \in X : h \cdot x = x \quad \forall h \in \mathrm{H}\}$$

Note $X^{\mathrm{H}} = \mathrm{map}_{\mathrm{G}}(\mathrm{G}\,/\,\mathrm{H}, X)$.

For $\mathcal{C} = \mathbf{Top}, \mathbb{S}$, consider the functor

$$R : \mathcal{C}^{\mathrm{G}} \to \mathcal{C}^{\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}}, \quad R(X)(\mathrm{G}\,/\,\mathrm{H}) = \mathrm{map}_{\mathrm{G}}(\mathrm{G}\,/\,\mathrm{H}, X) = X^{\mathrm{H}}$$

and its left adjoint, the coend

$$L : \mathcal{C}^{\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}} \to \mathcal{C}^{\mathrm{G}}, \quad L(Y) = \int^{\mathrm{G}\,/\,\mathrm{H}} Y(\mathrm{G}\,/\,\mathrm{H}) \times \mathrm{G}\,/\,\mathrm{H}$$

The Quillen equivalence (72) fits into a diagram

$$\begin{array}{ccc}
\mathbf{Top}^{\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}} & \underset{|\ \ |_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} & \mathbb{S}^{\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}} \\
L \downarrow \uparrow R & & L \downarrow \uparrow R \\
\mathbf{Top}^{\mathrm{G}} & \underset{|\ \ |_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} & \mathbb{S}^{\mathrm{G}}
\end{array} \tag{73}$$

PROPOSITION 3.1.2. Let H be a subgroup of G and X an object in $\mathcal{C}^{\mathrm{G}}$.

(1) Let B an object in $\mathcal{C}$, B is also an object in $\mathcal{C}^{\mathrm{G}}$ with the trivial action, and

$$\mathrm{hom}_{\mathcal{C}^{\mathrm{G}}}(B \times \mathrm{G}\,/\,\mathrm{H}, X) \simeq \mathrm{hom}_{\mathcal{C}}(B, X^{\mathrm{H}})$$

(2) Let the following be a cocartesian diagram in $\mathcal{C}^{\mathrm{G}}$ with g injective,

$$\begin{array}{ccc}
A & \xrightarrow{\jmath} & D \\
g \uparrow & & \uparrow \imath \\
B & \xrightarrow[f]{} & C
\end{array}$$

Then

$$\begin{array}{ccc} A^{\mathrm{H}} & \xrightarrow{\jmath^{\mathrm{H}}} & D^{\mathrm{H}} \\ {\scriptstyle g^{\mathrm{H}}}\uparrow & & \uparrow{\scriptstyle \imath^{\mathrm{H}}} \\ B^{\mathrm{H}} & \xrightarrow[f^{\mathrm{H}}]{} & C^{\mathrm{H}} \end{array}$$

is cocartesian in $\mathcal{C}$.

PROOF: Straightforward. □

PROPOSITION 3.1.3. Let $\mathcal{C} = \mathbf{Top}, \mathbb{S}$.

i) $\mathcal{C}^{\mathrm{G}}$ is a closed model category where a map f is a fibration (resp. a weak equivalence) if and only if $R(f)$ is. Moreover $\mathcal{C}^{\mathrm{G}}$ is cofibrantly generated, where the generating (trivial) cofibrations are the maps $f \times \mathrm{id} : \mathrm{dom} f \times \mathrm{G}/\mathrm{H} \to \mathrm{cod} f \times \mathrm{G}/\mathrm{H}$, with f a generating (trivial) cofibration and $\mathrm{H} \in \mathcal{F}$.

ii) Each of the pairs of functors of diagram (73) is a Quillen equivalence .

PROOF. One can give conditions on two sets of maps and a subcategory of a category $\mathcal{D}$ to be respectively the generating cofibrations, generating trivial cofibrations and weak equivalences in a closed model structure of $\mathcal{D}$; see M. Hovey's book [**14**, Thm. 2.1.19]. It is straightforward that those conditions are satisfied in our case, for $\mathcal{D} = \mathcal{C}^{\mathrm{G}}$. This proves i). The top pair of functors in diagram (73) is a Quillen equivalence by the discussion above the proposition. By definition of fibrations and weak equivalences in $\mathcal{C}^{\mathrm{G}}$, these are both preserved and reflected by R. In particular (L, R) is a Quillen pair. To show that it is an equivalence, it suffices, by [**14**, Cor. 1.3.16], to show that if $X \in \mathcal{C}^{\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}}$ is cofibrant, then the unit map

$$X \to RLX \tag{74}$$

is a weak equivalence; in fact we shall see that it is an isomorphism. Because every cofibrant object is a retract of a cofibrant cell complex, it suffices to check that (74) is an isomorphism on cell complexes. By definition, the generating cofibrant cells in $\mathcal{C}^{\mathbf{Or}_{\mathcal{F}}\,\mathrm{G}^{op}}$ are of the form $\coprod_{\mathrm{map}_{\mathrm{G}}(-,\mathrm{G}/\mathrm{H})} \Delta^n$. But for every $T \in \mathbb{S}$, we have:

$$\begin{aligned} RL(\coprod_{\mathrm{map}_{\mathrm{G}}(-,\mathrm{G}/\mathrm{H})} T)(\mathrm{G}/\mathrm{K}) &= R(\mathrm{G}/\mathrm{H} \times T)(\mathrm{G}/\mathrm{K}) \\ &= (\mathrm{G}/\mathrm{H} \times T)^{\mathrm{K}} \\ &= \mathrm{map}_{\mathbf{Or}\,\mathrm{G}}(\mathrm{G}/\mathrm{K}, \mathrm{G}/\mathrm{H}) \times T = \coprod_{\mathrm{map}_{\mathrm{G}}(\mathrm{G}/\mathrm{K},\mathrm{G}/\mathrm{H})} T \end{aligned}$$

Thus the unit map is an isomorphism on cells, and therefore on coproducts of cells, since taking fixed points under a subgroup preserves coproducts of G-simplicial sets. In particular (74) is an isomorphism on the zero skeleton of X. Assume by induction that

(74) is an isomorphism on the n-skeleton. The $n+1$-skeleton is a pushout

$$\begin{array}{ccc} \coprod_{\mathrm{H}\in I_n}\coprod_{\mathrm{map}_{\mathrm{G}}(-,\mathrm{G}/\mathrm{H})}\Delta^n & \longrightarrow & X_{n+1}(-) \\ \uparrow & & \uparrow \\ \coprod_{\mathrm{H}\in I_n}\coprod_{\mathrm{map}_{\mathrm{G}}(-,\mathrm{G}/\mathrm{H})}\partial\Delta^n & \longrightarrow & X_n(-) \end{array}$$

By Lemma 3.1.2 and the inductive hypothesis, the diagram

$$\begin{array}{ccc} \coprod_{\mathrm{H}\in I_n}(\mathrm{map}_{\mathrm{G}}(-,\mathrm{G}/\mathrm{H}))\times\Delta^n & \longrightarrow & RLX_{n+1}(-) \\ \uparrow & & \uparrow \\ \coprod_{\mathrm{H}\in I_n}(\mathrm{map}_{\mathrm{G}}(-,\mathrm{G}/\mathrm{H}))\times\partial\Delta^n & \longrightarrow & X_n(-) \end{array}$$

is again a pushout. It follows that $RLX_{n+1}\cong X_{n+1}$ and thus (74) is an isomorphism on all cell complexes, as we had to prove. We have shown that the top horizontal and both vertical pairs of functors are Quillen equivalences; by [**14**, Cor. 1.3.15], this implies that also the bottom pair is a Quillen equivalence. □

1.4. Assembly map. For the model structures of Proposition 3.1.3, the functorial cofibrant replacement in $\mathbf{Top}^{\mathrm{G}}$ of the point space $*$ is a model for the classifying space of G with respect to $\mathcal{F}$ and the cofibrant replacement of $*$ in $\mathbb{S}^{\mathrm{G}}$ is a simplicial version. Moreover because $|-|_*:\mathbb{S}^{\mathrm{G}}\to\mathbf{Top}^{\mathrm{G}}$ is a Quillen equivalence, it takes the simplicial version to the topological one. In particular if E is a functor from $\mathbf{Top}^{\mathrm{G}}$ to spectra and $\pi:\mathcal{E}(\mathrm{G},\mathcal{F})\to *$ is the cofibrant replacement in $\mathbb{S}^{\mathrm{G}}$, then we have a map

$$\mathrm{E}(\pi):\mathrm{E}(|\mathcal{E}(\mathrm{G},\mathcal{F})|)\to\mathrm{E}(*) \tag{75}$$

If

$$\mathrm{E}(X)=F_{\%}(X)=R(X)\otimes_{\mathbf{Or}\,\mathrm{G}}F:=\int^{\mathbf{Or}\,\mathrm{G}}X_+^{\mathrm{H}}\wedge F(\mathrm{G}/\mathrm{H})$$

for some functor $F:\mathbf{Or}\,\mathrm{G}\to\mathbf{Spt}$, (75) is the Davis-Lück assembly map of [**8**, Section 5.1]. In case $F=|F'|$ is the geometric realization of a functorial spectrum in the simplicial set sense, we have further

$$|F'|_{\%}(|X|)=|\int^{\mathbf{Or}\,\mathrm{G}}X_+^{\mathrm{H}}\wedge F'(\mathrm{G}/\mathrm{H})|=|F'_{\%}(X)|$$

and the assembly map for F is the geometric realization of that of F'. Hence we can equivalently work with assembly maps in the topological or the simplicial setting; we choose to do the latter. In particular all spectra considered henceforth are simplicial.

2. Equivariant homology

2.1. Crossed products and equivariant homology. A groupoid is a small category where all arrows are isomorphisms. Let $\mathcal{G}$ be a groupoid, and let R be a unital ring. An action of $\mathcal{G}$ on R is a functor $\rho:\mathcal{G}\to\mathbf{Ring}_1$ such that $\rho(x)=R$ for all $x\in\mathrm{ob}\mathcal{G}$. For example we may take $\rho(g) = \mathrm{id}_R$ for all arrows $g \in \mathrm{ar}\mathcal{G}$; this is called the trivial action. Whenever ρ is fixed, we omit it from our notation, and write

$$g(r) = \rho(g)(r)$$

for $g \in \mathrm{ar}\mathcal{G}$ and $r \in R$. Given a triple $(\mathcal{G}, \rho, R)$, we consider a small $\mathbb{Z}$-linear category $R \rtimes \mathcal{G}$. The objects of $R \rtimes \mathcal{G}$ are those of $\mathcal{G}$, and

$$\hom_{R\rtimes\mathcal{G}}(x, y) = R \otimes \mathbb{Z}[\hom_{\mathcal{G}}(x, y)]$$

If $s \in R$ and $g \in \hom_{\mathcal{G}}(x, y)$, we write $s \rtimes g$ for $s \otimes g$. Composition is defined by the rule

$$(r \rtimes f) \cdot (s \rtimes g) = rf(s) \rtimes fg \tag{76}$$

here $r, s \in R$, and f and g are composable arrows in $\mathcal{G}$. In case the action of $\mathcal{G}$ on R is trivial, we also write $R[\mathcal{G}]$ for $R \rtimes \mathcal{G}$.

Let G be a group; consider the functor $\mathcal{G}^{\mathrm{G}} : \mathrm{G}-\mathbf{Set} \to \mathbf{Gpd}$ which sends a G-set S to its transport groupoid. By definition $\mathrm{ob}\mathcal{G}^{\mathrm{G}}(S) = S$, and $\hom_{\mathcal{G}^G(S)}(s, t) = \{g \in G : g{\cdot}s = t\}$.

Notation 3.2.1. If E is a functor from $\mathbb{Z}$-linear categories to spectra, R a unital G-ring, and X a G-space, we put

$$H^{\mathrm{G}}(X, \mathrm{E}(R)) := \mathrm{E}(R \rtimes \mathcal{G}^{\mathrm{G}}(?))_{\%}(X)$$

2.2. The ring $\mathcal{A}(\mathcal{C})$. Let $\mathcal{C}$ be a small $\mathbb{Z}$-linear category. Put

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{a,b\in\mathrm{ob}\mathcal{C}} \hom_{\mathcal{C}}(a, b) \tag{77}$$

The following multiplication law

$$(fg)_{a,b} = \sum_{c\in\mathrm{ob}\mathcal{C}} f_{c,b} \circ g_{a,c} \tag{78}$$

makes $\mathcal{A}(\mathcal{C})$ into an associative ring, which is unital if and only if $\mathrm{ob}\mathcal{C}$ is finite. Whatever the cardinal of $\mathrm{ob}\mathcal{C}$ is, $\mathcal{A}(\mathcal{C})$ is always a ring with local units, i.e. a filtering colimit of unital rings.

$\mathcal{A}(?)$ and tensor products. The tensor product of two $\mathbb{Z}$-linear categories $\mathcal{C}$ and $\mathcal{D}$ is the $\mathbb{Z}$-linear category $\mathcal{C} \otimes \mathcal{D}$ with $\mathrm{ob}(\mathcal{C} \otimes \mathcal{D}) = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{D})$ and

$$\hom_{\mathcal{C}\otimes\mathcal{D}}((c_1, d_1), (c_2, d_2)) = \hom_{\mathcal{C}}(c_1, c_2) \otimes \hom_{\mathcal{D}}(d_1, d_2)$$

We have

$$\mathcal{A}(\mathcal{C} \otimes \mathcal{D}) = \mathcal{A}(\mathcal{C}) \otimes \mathcal{A}(\mathcal{D})$$

Example 3.2.2. If $\mathcal{G}$ is a groupoid acting trivially on a unital ring R, then

$$\mathcal{A}(R[\mathcal{G}]) = \mathcal{A}(R \otimes \mathbb{Z}[\mathcal{G}]) = R \otimes \mathcal{A}(\mathbb{Z}[\mathcal{G}])$$

$\mathcal{A}(?)$ *and crossed products.* If A is any, not necessarily unital ring, and $\mathcal{G}$ is a groupoid acting on A, we put

$$\mathcal{A}(A \rtimes \mathcal{G}) = \bigoplus_{x,y\in \mathrm{ob}\,\mathcal{G}} A \otimes \mathbb{Z}[\hom_{\mathcal{G}}(x,y)]$$

The rules (76) and (78) make $\mathcal{A}(A \rtimes \mathcal{G})$ into a ring, which in general is nonunital and does not have local units. The ring $\mathcal{A}(A \rtimes \mathcal{G})$ may also be described in terms of the unitalization $\tilde{A}$ of A. By definition, $\tilde{A} = A \oplus \mathbb{Z}$ equipped with the trivial $\mathcal{G}$-action on the $\mathbb{Z}$-summand and the following multiplication

$$(a,\lambda)(b,\mu) = (ab + \lambda b + a\mu, \lambda\mu) \tag{79}$$

We have

$$\mathcal{A}(A \rtimes \mathcal{G}) = \ker(\mathcal{A}(\tilde{A} \rtimes \mathcal{G}) \to \mathcal{A}(\mathbb{Z}[\mathcal{G}])) \tag{80}$$

Note that $\mathcal{A}(A \rtimes \mathcal{G})$ is defined, even though $A \rtimes \mathcal{G}$ is not. One can actually define $A \rtimes \mathcal{G}$ as a nonunital category, i.e. a category without identity morphisms, but we do not go into that here.

Next we fix a group G and a subgroup $\mathrm{H} \subset \mathrm{G}$ and consider the ring $\mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$ associated to the crossed product by the transport groupoid. Note that

$$\hom_{\mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})}(\mathrm{H},\mathrm{H}) = \mathrm{H} = \hom_{\mathcal{G}^{\mathrm{H}}(\mathrm{H}/\mathrm{H})}(\mathrm{H},\mathrm{H})$$

thus there is a fully faithful functor $\mathcal{G}^{\mathrm{H}}(\mathrm{H}/\mathrm{H}) \to \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})$. This functor induces a ring homomorphism

$$\jmath : A \rtimes \mathrm{H} = \mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{H}}(\mathrm{H}/\mathrm{H})) \subset \mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$$

The next lemma compares the map $\jmath$ with the canonical inclusion

$$\iota : A \rtimes \mathrm{H} \to M_{\mathrm{G}/\mathrm{H}}(A \rtimes \mathrm{H}), \quad x \mapsto e_{\mathrm{H},\mathrm{H}} \otimes x$$

In the following lemma and elsewhere, we make use of a section $s : \mathrm{G}/\mathrm{H} \to \mathrm{G}$ of the canonical projection onto the quotient by a sugroup $\mathrm{H} \subset \mathrm{G}$. We say that the section s is pointed if it is a map of pointed sets, that is, if it maps the class of H to the element $1 \in \mathrm{G}$.

LEMMA 3.2.3. Let A be a ring, G a group acting on A, and $\mathrm{H} \subset \mathrm{G}$ a subgroup. Then there is an isomorphism $\alpha : \mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})) \xrightarrow{\cong} M_{\mathrm{G}/\mathrm{H}}(A \rtimes \mathrm{H})$ making the following diagram commute:

$$\begin{array}{ccc} A \rtimes \mathrm{H} & \xrightarrow{\jmath} & \mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})) \\ & {\scriptstyle\iota}\searrow & \downarrow{\scriptstyle\wr\alpha} \\ & & M_{\mathrm{G}/\mathrm{H}}(A \rtimes \mathrm{H}) \end{array}$$

The isomorphism α is natural in A but not in the pair (G,H), as it depends on a choice of pointed section $s : \mathrm{G}/\mathrm{H} \to \mathrm{G}$ of the projection $\pi : \mathrm{G} \to \mathrm{G}/\mathrm{H}$.

PROOF. Let s be as in the lemma; put $\hat{g} = s(\pi(g))$ $(g \in G)$. The isomorphism $\alpha : \mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})) \xrightarrow{\cong} M_{\mathrm{G}/\mathrm{H}}(A \rtimes \mathrm{H})$ is defined as follows. For $b \in A$, $s, t \in \mathrm{G}$, and $g \in \hom_{\mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})}(s\mathrm{H}, t\mathrm{H})$, put

$$\alpha(b \rtimes g) = e_{t\mathrm{H}, s\mathrm{H}} \otimes \hat{t}^{-1}(b) \rtimes (\hat{t}^{-1} g \hat{s})$$

It is straightforward to check that α is an isomorphism and that $\alpha \mathrm{J} = \iota$. □

Functoriality of $\mathcal{A}(?)$. If $F : \mathcal{C} \mapsto \mathcal{D}$ is a $\mathbb{Z}$-linear functor which is injective on objects, then it defines a homomorphism $\mathcal{A}(F) : \mathcal{A}(\mathcal{C}) \to \mathcal{A}(\mathcal{D})$ by the rule $\alpha \mapsto F(\alpha)$. Hence we may regard $\mathcal{A}$ as a functor

$$\mathcal{A} : \text{inj-}\mathbb{Z}\text{-}\mathbf{Cat} \to \mathbf{Ring} \tag{81}$$

from the category of $\mathbb{Z}$-linear categories and functors which are injective on objects, to the category of rings. However $\mathcal{A}(F)$ is not defined for general $\mathbb{Z}$-linear F.

REMARK 3.2.4. The use of the prefix inj here differs from that in [**8**]. Indeed, here inj indicates that functors are injective on objects, whereas in [**8**], it refers to functors which are injective on arrows.

2.3. The nonunital case. A Milnor square is a pullback square of rings

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow f \\ S' & \xrightarrow[g]{} & S \end{array} \tag{82}$$

such that either f or g is surjective. Below we shall assume f is surjective. Let $\mathrm{E} : \mathbb{Z} - \mathbf{Cat} \to \mathbf{Spt}$ be a functor. If A is a not necessarily unital ring, embedded as an ideal in a unital ring R, we write $\mathrm{E}(R : A) = \mathrm{hofiber}(\mathrm{E}(R) \to \mathrm{E}(R/A))$. The functor E is said to satisfy excision for the Milnor square (82) if

$$\begin{array}{ccc} \mathrm{E}(R') & \longrightarrow & \mathrm{E}(R) \\ \downarrow & & \downarrow \mathrm{E}(f) \\ \mathrm{E}(S') & \longrightarrow & \mathrm{E}(S) \end{array}$$

is homotopy cartesian. If $\ker f \cong A$, then E satisfies excision on (82) if and only if the spectrum

$$\mathrm{E}(R', R : A) = \mathrm{hofiber}(\mathrm{E}(R' : A) \to \mathrm{E}(R : A))$$

is weakly contractible. We say that the ring A is E-excisive if E satisfies excision on every Milnor square (82) with $\ker f \cong A$. Assume unital rings are E-excisive; if A is any, not necessarily E-excisive ring, we consider its unitalization $\tilde{A}$, defined in (79) above. Put

$$\mathrm{E}(A) = \mathrm{hofiber}(\mathrm{E}(\tilde{A}) \to \mathrm{E}(\mathbb{Z}))$$

Because of our assumption that unital rings are E-excisive, if A happens to be unital, the two definitions of $\mathrm{E}(A)$ are naturally homotopy equivalent. Note that if

$$0 \to A' \to A \to A" \to 0$$

is an exact sequence of rings and A' is E-excisive, then

$$\mathrm{E}(A') \to \mathrm{E}(A) \to \mathrm{E}(A")$$

is a homotopy fibration.

We have already considered rings with local units; we shall need an even weaker form of unitality, called s-unitality. A ring A is called s-unital if for every finite collection $a_1, \dots, a_n \in A$ there exists an element $e \in A$ such that $a_i e = e a_i = a_i$. Note that if we add the requirement that e be idempotent we recover the notion of ring with local units.

Standing Assumptions 3.2.5. From now on, we shall be primarily concerned with functors $\mathrm{E} : \mathbb{Z}\text{-}\mathbf{Cat} \to \mathbf{Spt}$ that satisfy the following:

i) Every s-unital ring is E-excisive.
ii) If H is a group and A an E-excisive H-ring, then $A \rtimes \mathrm{H}$ is E-excisive.
iii) If A is E-excisive, X a set and $x \in X$, then $M_X A$ is E-excisive, and E sends the map $A \to M_X A$, $a \mapsto e_{x,x} a$ to a weak equivalence.
iv) There is a natural weak equivalence $\mathrm{E}(\mathcal{A}(\mathcal{C})) \xrightarrow{\sim} \mathrm{E}(\mathcal{C})$ of functors inj-$\mathbb{Z}$-$\mathbf{Cat} \to \mathbf{Spt}$.
v) Let $\{A_i : i \in I\}$ be a family of rings, and let $A = \bigoplus_{i \in I} A_i$ be their direct sum, with coordinate-wise multiplication. Then A is E-excisive if and only if each A_i is. Moreover if these equivalent conditions are satisfied, then the map $\bigoplus_i \mathrm{E}(A_i) \to \mathrm{E}(A)$ is an equivalence.

Let G be a group. Assume E satisfies the standing assumptions above. For A an E-excisive G-ring, consider the **Or** G-spectrum

$$\mathrm{G}/\mathrm{H} \mapsto \mathrm{E}(A \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})) = \mathrm{hofiber}(\mathrm{E}(\tilde{A} \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})) \to \mathrm{E}(\mathbb{Z}[\mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})])) \tag{83}$$

Applying $(?)_{\%}$ to (83) defines an equivariant homology theory of G-simplicial sets, which we denote $H^{\mathrm{G}}(-, \mathrm{E}(A))$. Moreover, for each fixed G-simplicial set X, $H^{\mathrm{G}}(X, \mathrm{E}(?))$ is a functor of E-excisive rings. Observe that, for unital A, we have two definitions of $\mathrm{E}(A \rtimes \mathcal{G}^{\mathrm{G}}(-))$ and two definitions of $H^{\mathrm{G}}(-, \mathrm{E}(A))$; the next proposition says that the two definitions are equivalent.

Proposition 3.2.6. Let $\mathrm{E} : \mathbb{Z}\text{-}\mathbf{Cat} \to \mathbf{Spt}$ be a functor and G a group. Assume that E satisfies the standing assumptions 3.2.5 above.

a) If R is a unital G-ring, then the two definitions of $\mathrm{E}(R \rtimes \mathcal{G}^{\mathrm{G}}(-))$ and of $H^{\mathrm{G}}(-, \mathrm{E}(R))$ are equivalent.

b) If

$$0 \to A' \to A \to A" \to 0$$

is an exact sequence of E-excisive G-rings, and X is a G-simplicial set, then

$$\mathrm{E}(A' \rtimes \mathcal{G}^{\mathrm{G}}(-)) \to \mathrm{E}(A \rtimes \mathcal{G}^{\mathrm{G}}(-)) \to \mathrm{E}(A" \rtimes \mathcal{G}^{\mathrm{G}}(-))$$

and

$$H^{\mathrm{G}}(X, \mathrm{E}(A')) \to H^{\mathrm{G}}(X, \mathrm{E}(A)) \to H^{\mathrm{G}}(X, \mathrm{E}(A"))$$

are homotopy fibrations.

PROOF. If A is E-excisive and $\mathrm{H} \subset \mathrm{G}$ is a subgroup, then conditions ii) and iii) together with Lemma 3.2.3 imply that $\mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$ is E-excisive. Hence, by condition iv), the spectrum in (83) is equivalent to $\mathrm{E}(\mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$. In particular, by i), $\mathcal{A}(R \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$ is E-excisive for R unital, and the map

$$\mathrm{hofiber}(\mathrm{E}(\tilde{R} \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})) \to \mathrm{E}(\mathbb{Z}[\mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})])) \to \mathrm{E}(R \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$$

induced by the projection $\tilde{R} \cong R \times \mathbb{Z} \to R$ is an equivalence. This proves a). Moreover, because $\mathcal{A}(? \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$ preserves exact sequences, then applying (83) to the exact sequence of part b) yields an object-wise homotopy fibration of **Or** G-spectra, which is the first homotopy fibration of b). Applying $(?)_{\%}$ we obtain the second one. □

REMARK 3.2.7. Let $\mathrm{E} : \mathbb{Z}\text{-}\mathbf{Cat} \to \mathbf{Spt}$ and let A be any, not necessarily E-excisive G-ring, equivariantly embedded as an ideal in a unital G-ring R. Consider the **Or** G-spectrum

$$\mathrm{E}(R \rtimes \mathcal{G}^{\mathrm{G}}(-) : A \rtimes \mathcal{G}^{\mathrm{G}}(-)) = \mathrm{hofiber}(\mathrm{E}(R \rtimes \mathcal{G}^{\mathrm{G}}(-)) \to \mathrm{E}((R/A) \rtimes \mathcal{G}^{\mathrm{G}}(-)))$$

Put

$$H^{\mathrm{G}}(X, \mathrm{E}(R : A)) = \mathrm{E}(R \rtimes \mathcal{G}^{\mathrm{G}}(-) : A \rtimes \mathcal{G}^{\mathrm{G}}(-))_{\%}(X).$$

Assembly gives a map of homotopy fibrations

$$\begin{array}{ccccc} H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}), \mathrm{E}(R:A)) & \longrightarrow & H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}), \mathrm{E}(R)) & \longrightarrow & H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}), \mathrm{E}(R/A)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{E}(R \rtimes \mathrm{G} : A \rtimes \mathrm{G}) & \longrightarrow & \mathrm{E}(R \rtimes \mathrm{G}) & \longrightarrow & \mathrm{E}((R/A) \rtimes \mathrm{G}) \end{array}$$

Hence if the $(\mathrm{G}, \mathcal{F})$ assembly map for the functorial spectrum $\mathrm{E}(-)$ is an equivalence on unital rings, then both the middle and right hand side vertical maps are equivalences; it follows that the same is true of the map on the left. We record a particular case of this in the following corollary.

COROLLARY 3.2.8. Let $\mathrm{E} : \mathbb{Z}\text{-}\mathbf{Cat} \to \mathbf{Spt}$ be a functor; assume E satisfies the Standing assumptions 3.2.5. Further let G be a group and $\mathcal{F}$ a family of subgroups, and assume that the assembly map $H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}), \mathrm{E}(R)) \to \mathrm{E}(R \rtimes \mathrm{G})$ is an equivalence for every unital ring R. Then $H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}), \mathrm{E}(A)) \to \mathrm{E}(A \rtimes \mathrm{G})$ is an equivalence for every E-excisive ring A.

PROPOSITION 3.2.9. Let $A \triangleleft R$ be an ideal in a unital G-ring, closed under the action of G. Let $\mathrm{E} : \mathbf{Ring} \to \mathbf{Spt}$ be a functor satisfying the standing assumptions. If A is E-excisive then

$$\mathrm{E}(A \rtimes \mathcal{G}^{\mathrm{G}}(-)) \to \mathrm{E}(R \rtimes \mathcal{G}^{\mathrm{G}}(-) : A \rtimes \mathcal{G}^{\mathrm{G}}(-))$$

is an object-wise weak equivalence of **Or** $\mathrm{G}^{\mathbf{op}}$-spectra.

PROOF. Let H be a subgroup of G. By Standing Assumption ii)

$$\mathrm{E}(A \rtimes \mathrm{H}) \to \mathrm{E}(R \rtimes \mathrm{H} : A \rtimes \mathrm{H})$$

is an equivalence. The proof follows from Lemma 3.2.3, using assumptions iii) and iv). □

2.4. The ring $\mathcal{R}(\mathcal{C})$. Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category. Imitating a construction used by M. Joachim ([**15**]) in the C^*-algebra context, we shall associate to $\mathcal{C}$ a ring $\mathcal{R}(\mathcal{C})$ which is a quotient of the tensor algebra of $\mathcal{A}(\mathcal{C})$; first we need some notation. If M is an abelian group, we write $T(M)=\bigoplus_{n\geq 1}M^{\otimes n}$ for the (unaugmented) tensor algebra. Put

$$\mathcal{R}(\mathcal{C})=T(\mathcal{A}(\mathcal{C}))/<\{g\otimes f-g\circ f: f\in\hom_{\mathcal{C}}(a,b), g\in\hom_{\mathcal{C}}(b,c),\quad a,b,c\in\operatorname{ob}\mathcal{C}\}>$$

Note that any $\mathbb{Z}$-linear functor $\mathcal{C}\to\mathcal{D}\in\mathbb{Z}-\mathbf{Cat}$ defines a homomorphism $\mathcal{R}(\mathcal{C})\to\mathcal{R}(\mathcal{D})$. Thus we may regard $\mathcal{R}$ as a functor

$$\mathcal{R}:\mathbb{Z}-\mathbf{Cat}\to\mathbf{Ring},\quad \mathcal{C}\mapsto\mathcal{R}(\mathcal{C})$$

Observe that the canonical surjection $T(\mathcal{A}(\mathcal{C}))\to\mathcal{A}(\mathcal{C})$ factors through a map

$$\mathcal{R}(\mathcal{C})\twoheadrightarrow\mathcal{A}(\mathcal{C}) \tag{84}$$

whose kernel is the ideal generated by the elements $g\otimes f$ for non-composable g and f. In Lemma 3.2.10 we give conditions on a functor $\mathrm{E}:\mathbf{Ring}\to\mathbf{Spt}$ to send (84) to a weak equivalence; first we need some notation. A functor $\mathrm{E}:\mathbf{Ring}\to\mathbf{Spt}$ is called excisive if every ring is E-excisive. An excisive functor which satisfies Standing Assumption iii) is called matrix stable. Note that standing assumptions i)-ii) are automatically satisfied if $\mathrm{E}:\mathbf{Ring}\to\mathbf{Spt}$ is excisive and matrix stable. Any excisive functor E satisfies condition iv) for finite sums; if it satisfies it for arbitrary sums, we say that E is additive. For example if E is excisive and E_* commutes with filtering colimits, then E is additive.

For the proof of the next lemma we also need to recall the concept of multiplier ring which we borrow from [**16**]. The multiplier ring of a ring A is the ring $\mathcal{M}(A)$ whose elements are the pairs (f,g) of maps $A\to A$ such that f is a right A-module homomorphism, g is a left A-module homomorphism and the following compatibility condition is satisfied

$$g(a)b=af(b)$$

Multiplication in $\mathcal{M}(A)$ is defined by

$$(f_1,g_1)(f_2,g_2)=(f_1f_2,g_2g_1)$$

If $a\in A$ then the pair $m(a)=(L_a,R_a)$ given by left and right multiplication by a is an element of $\mathcal{M}(A)$, and $a\mapsto m(a)$ is a ring homomorphism $m:A\mapsto\mathcal{M}(A)$. The image of m is always an ideal of $\mathcal{M}(A)$; its kernel is the two-sided annihilator of A

$$\ker m=\{a\in A: aA=Aa=0\}$$

This kernel vanishes for example if A is s-unital.

Lemma 3.2.10. Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category and let $\mathrm{E}:\mathbf{Ring}\to\mathbf{Spt}$ be an excisive and matrix stable functor. Then E_* sends (84) to a naturally split surjection. Assume in addition that E is invariant under polynomial homotopy. Then E sends (84) to a weak equivalence.

Proof. Let $\operatorname{ob}_+\mathcal{C}=\operatorname{ob}\mathcal{C}\coprod\{+\}$ be the set of objects of $\mathcal{C}$ with a base point added. Consider the homomorphism

$$j:\mathcal{A}(\mathcal{C})\to M_{\operatorname{ob}_+\mathcal{C}}\mathcal{R}(\mathcal{C}),\quad j(f)=f\otimes e_{b,a}\qquad (f\in\hom_{\mathcal{C}}(a,b))$$

Write p for the map (84). Left and right multiplication by each of the following two matrices

$$W = \sum_{a\in \mathrm{ob}\, \mathcal{C}} 1_a \otimes e_{a,+}$$
$$V = \sum_{a\in \mathrm{ob}\, \mathcal{C}} 1_a \otimes e_{+,a}$$

leave $M_{\mathrm{ob}_+\,\mathcal{C}}\mathcal{A}(\mathcal{C})$ stable, and thus define elements $m(V), m(W) \in \mathcal{M}(M_{\mathrm{ob}_+\,\mathcal{C}}\mathcal{A}(\mathcal{C}))$. Moreover the map $m : M_{\mathrm{ob}_+\,\mathcal{C}}\mathcal{A}(\mathcal{C}) \to \mathcal{M}(M_{\mathrm{ob}_+\,\mathcal{C}}\mathcal{A}(\mathcal{C}))$ is injective, since $M_{\mathrm{ob}_+\,\mathcal{C}}\mathcal{A}(\mathcal{C})$ has local units. The composite $q = M_{\mathrm{ob}_+\,\mathcal{C}}(p) \circ j$ sends $f \in \mathcal{A}(\mathcal{C})$ to

$$q(f) = W(f \otimes e_{+,+})V$$

All this together with matrix invariance imply that $\mathrm{E}_*(q) = \mathrm{E}_*(?\otimes e_{+,+})$ is an isomorphism [**3**, 2.2.6]. This proves the first assertion of the Lemma. To prove the second, it suffices to show that $r = j \circ p$ is homotopic to the inclusion $\iota(a) = a \otimes e_{+,+}$. If $f \in \hom_{\mathcal{C}}(a, b)$, write $H(f) \in M_{\mathrm{ob}_+\,\mathcal{C}}(\mathcal{R}(\mathcal{C}))[t]$ for

$$H(f) = f \otimes (-t(t^3 - 2t)e_{+,+} + t(t^2 - 1)e_{+,a} + (1 - t^2)(t^3 - 2t)e_{b,+} + (1 - t^2)^2 e_{b,a})$$

Note that $\mathrm{ev}_0 H(f) = r(f)$, $\mathrm{ev}_1 H(f) = \iota(f)$. Further, one checks that if $g \in \hom_{\mathcal{C}}(b, c)$, then $H(gf) = H(g)H(f)$. Thus H induces a homomorphism $\mathcal{R}(\mathcal{C}) \to M_{\mathrm{ob}_+\,\mathcal{C}}(\mathcal{R}(\mathcal{C}))[t]$ which is a homotopy from r to ι. This concludes the proof. □

Example 3.2.11. Let R, S be unital rings, and let $\mathcal{C}$ be the $\mathbb{Z}$-linear category with two objects a and b such that $\hom_{\mathcal{C}}(a, b) = \hom_{\mathcal{C}}(b, a) = 0$, $\hom_{\mathcal{C}}(a, a) = R$ and $\hom_{\mathcal{C}}(b, b) = S$. Then $\mathcal{A}(\mathcal{C}) = R \oplus S$ and $\mathcal{R}(\mathcal{C}) = R \coprod S$ is the nonunital coproduct. By Lemma 3.2.10, any excisive, matrix stable, homotopy invariant functor $\mathrm{E} : \mathbf{Ring} \to \mathbf{Spt}$ sends $p : R \coprod S \to R \oplus S$ to a weak equivalence. We remark that the hypothesis on E are necessary; in particular there are functors $\mathrm{E} : \mathbb{Z} - \mathbf{Cat} \to \mathbf{Spt}$ which satisfy the standing assumptions, and which do not send p to a weak equivalence.

3. K-theory

3.1. The K-theory spectrum. Given a $\mathbb{Z}$-linear category $\mathcal{C}$, we denote by $\mathcal{C}_\oplus$ the $\mathbb{Z}$-linear category whose objects are finite sequences of objects of $\mathcal{C}$, and whose morphisms are matrices of morphisms in $\mathcal{C}$ with the obvious matrix product as composition. Concatenation of sequences yields a sum $\oplus$ and hence we obtain, functorially, an additive category; write $\mathrm{Idem}\,\mathcal{C}_\oplus$ for its idempotent completion. We shall also need Karoubi's cone $\Gamma(\mathcal{C})$ ([**18**, pp 270]). The objects of $\Gamma(\mathcal{C})$ are the sequences $x = (x_1, x_2, \dots)$ of objects of $\mathcal{C}$ such that the set

$$F(x) = \{c \in \mathcal{C} : (\exists n) \quad x_n = c\} \tag{85}$$

is finite. A map $x \to y$ in $\Gamma(\mathcal{C})$ is a matrix $f = (f_{i,j})$ of homomorphisms $f_{i,j} : x_j \to y_i$ such that

(1) There exists an N such that every row and every column of f has at most N nonzero entries.
(2) The set $\{f_{i,j} : i, j \in \mathbb{N}\}$ is finite.

Interspersing of sequences defines a symmetric monoidal operation $\boxplus : \Gamma(\mathcal{C})\times\Gamma(\mathcal{C}) \to \Gamma(\mathcal{C})$ and there is an endofunctor τ such that $1 \boxplus \tau \cong \tau$ (see [**17**, § III]). If $\mathcal{C}$ has finite direct sums, e.g. if $\mathcal{C} = \mathcal{D}_\oplus$ for some $\mathbb{Z}$-linear category $\mathcal{D}$, then the interspersing operation is naturally equivalent to the induced sum $(x \oplus y)_i = x_i \oplus y_i$ ([**17**, Lemme 3.3]). In particular, if $\mathcal{C}$ is additive, then $\Gamma\mathcal{C}$ is a flasque additive category; that is, there is an additive endofunctor $\tau : \mathcal{C} \to \mathcal{C}$ such that $\tau \oplus 1 \cong \tau$. A morphism f in $\Gamma(\mathcal{C})$ is finite if $f_{ij} = 0$ for all but finitely many (i, j). Finite morphisms form an ideal, and we write $\Sigma(\mathcal{C})$ for the category with the same objects as $\Gamma(\mathcal{C})$, and morphisms taken modulo the ideal of finite morphisms. The category $\Sigma(\mathcal{C})$ is Karoubi's suspension of $\mathcal{C}$. By [**27**, Thm. 5.3], if $\mathcal{C}$ is additive, we have a homotopy fibration sequence

$$K^Q(\operatorname{Idem}\mathcal{C}) \to K^Q(\Gamma(\operatorname{Idem}\mathcal{C})) \to K^Q(\Sigma(\operatorname{Idem}\mathcal{C})) \tag{86}$$

Here each of the categories is regarded as a semisimple exact category, and K^Q denotes the fibrant simplicial set for its algebraic K-theory. Because $\Gamma(\operatorname{Idem}\mathcal{C})$ is flasque, $K^Q(\Gamma(\operatorname{Idem}\mathcal{C}))$ is contractible, whence $K^Q(\operatorname{Idem}\mathcal{C}) \cong \Omega K^Q(\Sigma(\operatorname{Idem}\mathcal{C}))$. Now let $\mathcal{C}$ be any small $\mathbb{Z}$-linear category, possibly without direct sums. Consider the sequence of categories

$$\mathcal{C}^{(0)} = \operatorname{Idem}(\mathcal{C}_\oplus), \quad \mathcal{C}^{(n+1)} = \operatorname{Idem}(\Sigma\mathcal{C}^{(n)}) \tag{87}$$

Then we have a spectrum $K(\mathcal{C}) = \{{}_nK(\mathcal{C})\}$, with

$${}_nK(\mathcal{C}) \cong K^Q(\mathcal{C}^{(n)}) \tag{88}$$

Remark 3.3.1. If R is a unital ring, then by [**18**, Prop. 1.6], we have category equivalences

$$\operatorname{Idem}(\Gamma(\operatorname{proj}(R))) \cong \operatorname{proj}(\Gamma(R)) \text{ and } \operatorname{Idem}(\Sigma(\operatorname{proj}(R))) \cong \operatorname{proj}(\Sigma(R)) \tag{89}$$

Hence the spectrum $K(R)$ defined above is equivalent to the usual, Gersten-Karoubi-Wagoner spectrum of the ring R.

Lemma 3.3.2. Let $\mathcal{C}$ be an additive category, and let $\bullet$ be the only object of $\Gamma(\mathbb{Z})$. Consider the functor

$$\mu : \Gamma\mathbb{Z} \otimes \mathcal{C} \to \Gamma(\mathcal{C})$$
$$\mu(\bullet, c) = (c, c, \dots), \qquad \mu(f \otimes \alpha)_{ij} = f_{ij}\alpha$$

Then

i) The functor μ is fully faithful.

ii) Let $F(-)$ be as in (85). For every object $x \in \Gamma(\mathcal{C})$ there exist morphisms $\phi_c : \mu(\bullet, c) \to x$ and $\psi_c : x \to \mu(\bullet, c)$, $c \in F(x)$ such that $\sum_{c\in F(x)} \phi_c\psi_c = 1_x$.

iii) The functor μ induces a fully faithful functor $\bar{\mu} : \Sigma \otimes \mathcal{C} \to \Sigma(\mathcal{C})$.

Proof. Part i) is proved in [**5**, Lemma 4.7.1] for the case when $\mathcal{C}$ has only one object; the same argument applies in general. To prove ii), let $x \in \Gamma(\mathcal{C})$ be an object. If $c \in F(x)$, write $I(c) = \{n \in \mathbb{N} : x_n = c\}$, and let $\chi_{I(c)}$ be the characteristic function. Put

$$\phi_c : \mu(\bullet, c) \to x, \quad \psi_c : x \to \mu(\bullet, c), \quad (\phi_c)_{i,j} = (\psi_c)_{i,j} = \delta_{i,j}\chi_{I(c)}(j)1_c$$

One checks that

$$\sum_{c\in F(x)} \phi_c\psi_c = 1_x$$

This proves ii). Next, consider the exact sequence

$$0 \to M_\infty\mathbb{Z} \to \Gamma\mathbb{Z} \overset{\pi}{\to} \Sigma\mathbb{Z} \to 0$$

As is explained in [**5**, pp 92], it follows from results of Nöbeling [**26**] that the sequence above is split as a sequence of abelian groups. Hence if $c, d \in \mathcal{C}$, then

$$\ker(\pi\otimes 1 : \hom_{\Gamma\mathbb{Z}\otimes\mathcal{C}}((\bullet, c), (\bullet, d)) \to \hom_{\Sigma\mathbb{Z}\otimes\mathcal{C}}((\bullet, c), (\bullet, d))) = M_\infty\mathbb{Z}\otimes\hom_{\mathcal{C}}(c, d)$$

Next observe that if $\alpha \in \hom_{\mathcal{C}}(c, d)$ and $f \in M_\infty\mathbb{Z}$, then $\mu(f\otimes\alpha)$ is a finite morphism. Hence μ passes to the quotient, inducing a functor $\bar\mu : \Sigma\mathbb{Z}\otimes\mathcal{C} \to \Sigma(\mathcal{C})$. If $c, d \in \mathrm{B}\,\mathcal{C}$ and we put $x = \mu(\bullet, c)$, $y = \mu(\bullet, d)$ then we have a map of exact sequences

$$\begin{array}{ccccccc} 0 \to M_\infty\mathbb{Z}\otimes\hom_{\mathcal{C}}(c, d) & \longrightarrow & \Gamma\mathbb{Z}\otimes\hom_{\mathcal{C}}(c, d) & \longrightarrow & \Sigma\mathbb{Z}\otimes\hom_{\mathcal{C}}(c, d) \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to \hom_{\mathrm{Fin}(\mathcal{C})}(x, y) & \longrightarrow & \hom_{\Gamma(\mathcal{C})}(x, y) & \longrightarrow & \hom_{\Sigma(\mathcal{C})}(x, y) \to 0 \end{array}$$

Here $\mathrm{Fin}(\mathcal{C}) \subset \Gamma(\mathcal{C})$ is the subcategory of finite morphisms. The second vertical map is an isomorphism by part i). In particular the first map is injective; furthermore, one checks that it is onto. It follows that the third vertical map is an isomorphism; this proves iii). □

3.2. Comparing $K(\mathcal{C})$ with $K(\mathcal{A}(\mathcal{C}))$.

The operation $\Diamond$. Let X be a set and let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbb{Z}$-linear categories with $\mathrm{ob}\,\mathcal{C} = \mathrm{ob}\,\mathcal{D} = X$. Consider the category $\mathcal{C}\Diamond\mathcal{D}$ with set of objects $\mathrm{ob}(\mathcal{C}\Diamond\mathcal{D}) = X$, homomorphisms

$$\hom_{\mathcal{C}\Diamond\mathcal{D}}(x, y) = \hom_{\mathcal{C}}(x, y) \oplus \hom_{\mathcal{D}}(x, y)$$

and coordinate-wise composition. If $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are $\mathbb{Z}$-linear categories, we have

$$(\mathcal{C}\Diamond\mathcal{D})_\oplus = \mathcal{C}_\oplus\Diamond\mathcal{D}_\oplus$$

(90) $$\mathrm{Idem}((\mathcal{C}\Diamond\mathcal{D})_\oplus) = \mathrm{Idem}\,\mathcal{C}_\oplus \times \mathrm{Idem}\,\mathcal{D}_\oplus$$

(91) $$(\mathcal{C}\Diamond\mathcal{D})\otimes\mathcal{E} = (\mathcal{C}\otimes\mathcal{E})\Diamond(\mathcal{D}\otimes\mathcal{E})$$

Unitalization. We have already recalled the definition of the unitalization $\tilde{A}$ of a not necessarily unital ring A. Now we need a version of unitalization for $\mathbb{Z}$-linear categories; this can be more generally defined for nonunital $\mathbb{Z}$-categories, but we will have no occasion for that. Let $\mathcal{C} \in \mathbb{Z} - \mathbf{Cat}$; write $\tilde{\mathcal{C}}$ for the category with $\mathrm{ob}\,\tilde{\mathcal{C}} = \mathrm{ob}\,\mathcal{C}$ and with homomorphisms given by

$$\hom_{\tilde{\mathcal{C}}}(x, y) = \hom_{\mathcal{C}}(x, y) \oplus \delta_{x,y}\mathbb{Z} = \begin{cases} \hom_{\mathcal{C}}(x, y) & x \neq y \\ \hom_{\mathcal{C}}(x, x) \oplus \mathbb{Z} & x = y \end{cases}$$

Composition between $(f, \delta_{x,y}n) \in \hom_{\tilde{\mathcal{C}}}(x, y)$ and $(g, \delta_{y,z}m) \in \hom_{\tilde{\mathcal{C}}}(y, z)$ is defined by the formula

$$(g, \delta_{y,z}m) \circ (f, \delta_{x,y}n) = (gf + \delta_{y,z}mf + \delta_{x,y}gn, \delta_{x,y}\delta_{y,z}mn)$$

Observe that if R is a ring, considered as a $\mathbb{Z}$-linear category with one object, then

$$\tilde{R} \to R \times \mathbb{Z} = R\Diamond\mathbb{Z}, \qquad (r,n) \mapsto (r + n\cdot 1, n)$$

is an isomorphism. This isomorphism generalizes to $\mathbb{Z}$-categories as follows. Let $\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle \in \mathbb{Z} - \mathbf{Cat}$, be the $\mathbb{Z}$-linear category with the same objects as $\mathcal{C}$, homomorphisms given by

$$\hom_{\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle}(x,y) = \delta_{x,y}\mathbb{Z}$$

We have an isomorphism of linear categories

$$\mathcal{C}\Diamond\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle \to \tilde{\mathcal{C}} \tag{92}$$

which is the identity on objects, as well as on $\hom_{\mathcal{C}\Diamond\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle}(x,y)$ for $x \neq y$, and which sends

$$\hom_{\mathcal{C}\Diamond\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle}(x,x) \ni (f,n) \mapsto (f - n1_x, n) \in \hom_{\tilde{\mathcal{C}}}(x,x)$$

The map $K(\mathcal{C}) \to K(\mathcal{A}(\mathcal{C}))$. If $\mathcal{C}$ is a $\mathbb{Z}$-linear category, and $x, y \in \operatorname{ob}\mathcal{C}$, then by definition of $\mathcal{A}(\mathcal{C})$,

$$\hom_{\mathcal{C}}(x,y) \subset \mathcal{A}(\mathcal{C}) \tag{93}$$

and the inclusion is compatible with composition. We also have an inclusion

$$\hom_{\tilde{\mathcal{C}}}(x,x) \ni (f,n) \mapsto (f,n) \in \widetilde{\mathcal{A}(\mathcal{C})} \tag{94}$$

The inclusions (93) and (94) together with the only map $\operatorname{ob}\tilde{\mathcal{C}} \to \operatorname{ob}\widetilde{\mathcal{A}(\mathcal{C})} = \{\bullet\}$ define a functor

$$\phi : \tilde{\mathcal{C}} \to \widetilde{\mathcal{A}(\mathcal{C})} \tag{95}$$

Observe that $\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle \subset \tilde{\mathcal{C}}$ and that $\phi(\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle) \subset \mathbb{Z} \subset \widetilde{\mathcal{A}(\mathcal{C})}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{\phi} & \widetilde{\mathcal{A}(\mathcal{C})} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle & \longrightarrow & \mathbb{Z} \end{array}$$

Here the vertical maps are the obvious projections. By (92) and (90) we have an equivalence

$$K(\tilde{\mathcal{C}}) \xrightarrow{\sim} K(\mathcal{C}) \times K(\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle)$$

Under this equivalence the map induced by π_1 becomes the canonical projection; hence its fiber is $K(\mathcal{C})$. On the other hand, by definition, $K(\mathcal{A}(\mathcal{C}))$ is the fiber of $K(\pi_2)$. Hence ϕ induces a map

$$\varphi : K(\mathcal{C}) \to K(\mathcal{A}(\mathcal{C})) \tag{96}$$

PROPOSITION 3.3.3. Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category. Then the map (96) is an equivalence.

Proof. Because both the source and the target of (96) commute with filtering colimits, we may assume that $\mathcal{C}$ has finitely many objects. Then $\mathcal{A}(\mathcal{C})$ is unital, and thus we have an isomorphism $\widetilde{\mathcal{A}(\mathcal{C})} \cong \mathcal{A}(\mathcal{C}) \times \mathbb{Z}$. Recall that the idempotent completion of an additive category $\mathfrak{A}$ is the category whose objects are the idempotent endomorphisms in $\mathfrak{A}$ and where a map $f : e_1 \to e_2$ is an element of $\hom_{\mathfrak{A}}(\mathrm{dom} e_1, \mathrm{dom} e_2)$ such that $f = e_2 f e_1$. One checks that the composite

$$\mathcal{C}_\oplus \to \mathrm{Idem}\,\mathcal{C}_\oplus \overset{1\times 0}{\to} \mathrm{Idem}\,\mathcal{C}_\oplus \times \mathrm{Idem}\,\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle \cong$$
$$\mathrm{Idem}(\tilde{\mathcal{C}}_\oplus) \overset{\phi}{\to} \mathrm{Idem}(\widetilde{\mathcal{A}(\mathcal{C})}_\oplus) \cong \mathrm{Idem}(\mathcal{A}(\mathcal{C})_\oplus) \times \mathrm{Idem}(\mathbb{Z}_\oplus) \to \mathrm{Idem}(\mathcal{A}(\mathcal{C})_\oplus)$$

is the functor ψ which sends an object $(c_1, \dots, c_n)$ to the idempotent $\mathrm{diag}(1_{c_1}, \dots, 1_{c_n})$ and a map $f = (f_{i,j}) : (c_1, \dots, c_n) \to (d_1, \dots, d_m)$ to the corresponding matrix $(f_{i,j}) \in \hom_{\mathcal{A}(\mathcal{C})_\oplus}(\bullet^n, \bullet^m)$. Because ψ is fully faithful and cofinal, it induces an equivalence $K(\mathcal{C}) \to K(\mathcal{A}(\mathcal{C}))$. It follows that (96) is an equivalence. □

4. K-theory and the standing assumptions

In this section we prove some tecnical result to see that K-theory satisfy the standing assumptions.

4.1. The groups $\mathrm{Tor}_*^{\tilde{A}}(-, A)$. Let $M = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}$. Theorems of Suslin [**28**] (for $M = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$) and Suslin-Wodzicki (for $M = \mathbb{Q}$) establish that a ring A is excisive for K-theory with coefficients in M if and only if

$$\mathrm{Tor}_*^{\tilde{A}}(M, A) = 0$$

Example 3.4.1. A ring A is said to have the triple factorization property if for every finite family $a_1, \dots, a_n \in A$ there exist $b_1, \dots, b_n, c, d \in A$ such that

$$a_i = cdb_i \text{ and } \{a_i : a_i d = 0\} = \{a : acd = 0\} \qquad i = 1 \dots n$$

It was proved in [**29**, Theorem C] that rings having the triple factorization property are K-excisive. In particular, s-unital rings are K-excisive.

We shall introduce, for any abelian group M, a functorial abelian group $\bar{Q}(A, M)$ which computes $\mathrm{Tor}_*^{\tilde{A}}(M, A)$. Consider the functor $\perp : \tilde{A} - mod \to \tilde{A} - mod$,

$$\perp M = \bigoplus_{m \in M} \tilde{A}.$$

The functor $\perp$ is the free $\tilde{A}$-module cotriple [**33**, 8.6.6]. Let $Q(A) \to A$ be the canonical simplicial resolution by free $\tilde{A}$-modules associated to $\perp$ [**33**, 8.7.2]; by definition, its n-th term is $Q_n(A) = \perp^n A$. Put

$$\bar{Q}(A, M) = M \otimes_{\tilde{A}} Q(A).$$

We have

$$\pi_*(\bar{Q}(A, M)) = \mathrm{Tor}_*^{\tilde{A}}(M, A)$$

We abbreviate $\bar{Q}(A) = \bar{Q}(\mathbb{Z}, A)$. Note that

$$\bar{Q}(A, M) = M \otimes \bar{Q}(A)$$

We have

$$\bar{Q}_0(A) = \mathbb{Z}[A], \quad \bar{Q}_{n+1} = \mathbb{Z}[\tilde{A} \otimes \bar{Q}_n(A)].$$

LEMMA 3.4.2. Let $F \overset{\sim}{\twoheadrightarrow} A$ be a simplicial resolution in **Rings** and M an abelian group. Let $\operatorname{diag} \bar{Q}(F)$ be the diagonal of the bisimplicial abelian group $\bar{Q}(F)$. Then

$$\operatorname{Tor}_*^{\tilde{A}}(M, A) = \pi_*(M \otimes \operatorname{diag} \bar{Q}(F))$$

PROOF. Because $F \to A$ is a simplicial resolution in **Rings**, $\bar{Q}_0(F) = \mathbb{Z}[F] \to \mathbb{Z}[A] = \bar{Q}_0(A)$ is a free simplicial resolution in **Ab** of the free abelian group $\mathbb{Z}[A]$. Observe that if $G \to N$ is a free resolution of a free abelian group N, then $\tilde{A} \otimes G \to \tilde{A} \otimes N$ is a free simplicial $\tilde{A}$-module resolution, and $\mathbb{Z}[\tilde{A} \otimes G] \to \mathbb{Z}[\tilde{A} \otimes N]$ is a free simplical $\mathbb{Z}$-module resolution. Thus for each n, $\bar{Q}_n(F) \to \bar{Q}_n(A)$ is an equivalence of free simplicial abelian groups, and thus it remains an equivalence after tensoring by M. It follows that $M \otimes \operatorname{diag} \bar{Q}(F)$ computes $\operatorname{Tor}_*^{\tilde{A}}(M, A)$. □

PROPOSITION 3.4.3. Let $F \overset{\sim}{\twoheadrightarrow} A$ be a simplicial resolution and M an abelian group. Then there is a first quadrant spectral sequence

$$E^2_{p,q} = \pi_q(\operatorname{Tor}_p^{\tilde{F}}(M, F)) \Rightarrow \operatorname{Tor}_{p+q}^{\tilde{A}}(M, A)$$

PROOF. This is just the spectral sequence of the bisimplicial abelian group $([p], [q]) \mapsto \bar{Q}_p(M, F_q)$. □

COROLLARY 3.4.4. Let $F \overset{\sim}{\twoheadrightarrow} A$ be free simplicial a resolution in **Rings**. Then

$$\pi_*(M \otimes (F/F^2)) = \operatorname{Tor}_*^{\tilde{A}}(M, A)$$

PROOF. In view of the previous proposition, and of the fact that $\operatorname{Tor}_0^{\tilde{B}}(M, B) = M \otimes B/B^2$ for every ring B, it suffices to show that if V is a free abelian group, and TV the tensor algebra, then $\operatorname{Tor}_n^{\tilde{TV}}(M, TV) = 0$ for $n \geq 1$. But this is clear, since TV is free as a $\tilde{TV}$-module; indeed, the multiplication map $\tilde{TV} \otimes V \to TV$ is an isomorphism. □

4.2. Bar complex. Let A be a ring. Consider the complex $P(A)$ given by $P_n(A) = \tilde{A} \otimes A^{\otimes n+1}$ $(n \geq 0)$, with boundary map

$$b"(a_{-1} \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=-1}^{n-1} (-1)^i a_{-1} \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

The multiplication map $\mu : \tilde{A} \otimes A \to A$ gives a surjective quasi-isomorphism $\mu : P(A) \twoheadrightarrow A$ [**33**, 8.6.12]. A canonical $\mathbb{Z}$-linear section of μ is $j = 1 \otimes - : A \to \tilde{A} \otimes A$. Let $\epsilon : \tilde{A} \to A$, $\epsilon(a, n) = a$. A $\mathbb{Z}$-linear homotopy $j\mu \to 1$ is defined by

$$s : A^{\otimes n+1} \to A^{\otimes n+2}, \quad s(a_{-1} \otimes \cdots \otimes a_n) = 1 \otimes \epsilon(a_{-1}) \otimes a_0 \otimes \cdots \otimes a_n$$

Thus $P(A)$ is a resolution of A by $\tilde{A}$-modules, and moreover these $\tilde{A}$-modules are scalar extensions of $\mathbb{Z}$-modules. Hence if A is flat as $\mathbb{Z}$-module, then $C^{bar}(A) = \mathbb{Z} \otimes_{\tilde{A}} P(A)$ computes $\operatorname{Tor}_*^{\tilde{A}}(\mathbb{Z}, M)$ and $M \otimes C^{bar}(A)$ computes $\operatorname{Tor}_*^{\tilde{A}}(\mathbb{Z}, M)$. In general, the homology of $C^{bar}(A)$ can be interpreted as the Tor groups relative to the extension $\mathbb{Z} \to \tilde{A}$. For an arbitrary ring A, one can use the natural homotopy s to give a natural map

$$L(A) \to P(A)$$

The induced map $M \otimes \bar{Q}(A) \to M \otimes C^{bar}(A)$ is a quasi-homomorphism if A is flat as a $\mathbb{Z}$-module. In particular, we have the following.

LEMMA 3.4.5. Let $F \overset{\sim}{\twoheadrightarrow} A$ be a simplicial resolution by flat rings, and M an abelian group. Then

$$\mathrm{Tor}_*^{\tilde{A}}(M, A) = H_*(\mathrm{Tot}(M \otimes C^{bar}(F)))$$

4.3. H-unital rings. Let

$$0 \to A \to B \to C \to 0 \tag{97}$$

be an exact sequence of rings. We say that (97) is pure if for every abelian group V, the sequence of abelian groups

$$0 \to A \otimes V \to B \otimes V \to C \otimes V \to 0$$

A ring A is called H-unital if for every abelian group V, the complex $C^{bar}(A) \otimes V$ is acyclic. If $X(-)$ is a functorial chain complex, then we say that A is pure X-excisive if for every pure exact sequence (97),

$$X(A) \to X(B) \to X(C)$$

is a distinguished triangle. The following theorem was proved by M. Wodzicki in 3.4.6.

THEOREM 3.4.6. (Wodzicki) The following conditions are equivalent for a ring A.
i) A is H-unital.
ii) A is pure C^{bar}-excisive.
iii) A is pure HH-excisive.
iv) A is pure HC-excisive.

EXAMPLE 3.4.7. Any linearly split sequence (97) is pure. In particular, any sequence (97) with A a $\mathbb{Q}$-algebra is pure, since any $\mathbb{Q}$-vectorspace is injective as an abelian group. Thus for a $\mathbb{Q}$-algebra A, Wodzicki's theorem remains valid if we omit the word "pure" everywhere. Furthermore, the Suslin-Wodzicki theorem cited above, for A a $\mathbb{Q}$-algebra then the conditions of Theorem 3.4.6 are also equivalent to A being $K^{\mathbb{Q}}$-excisive. In fact it is well-known that for a $\mathbb{Q}$-algebra A, being $K^{\mathbb{Q}}$-excisive is equivalent to being K-excisive; as explained in [**4**, Lemma 4.1] this well-known fact follows from the main result of [**31**]. See [**29**, Lemma 1.9] for a different proof.

4.4. Colimits. The bar complex manifestly commutes with filtering colimits, and thus H-unital rings are closed under them. The next proposition establishes the analogue of this property for K-excisive rings.

PROPOSITION 3.4.8. Let $\{A_i\}$ be a filtering system of rings, and let M be an abelian group. Write $A = \mathrm{colim}\, A_i$. Then

$$\mathrm{Tor}_*^{\tilde{A}}(M, A) = \operatorname*{colim}_i \mathrm{Tor}_*^{\tilde{A}_i}(M, A_i)$$

Proof. Write $\perp$: **Rings** $\to$ **Rings**, $\perp B = T(\mathbb{Z}[B])$ for the cotriple associated with the forgetful functor **Rings** $\to$ **Set** and its adjoint. Write $F(A) \overset{\sim}{\twoheadrightarrow} A$ for the cotriple resolution $F(A)_n = \perp^{n+1} A$ ([**33**, Section 8/6]). We have $F(A) = \operatorname{colim}_i F(A_i)$. Thus $\operatorname{Tot}(M \otimes C^{bar} F(A)) = \operatorname{colim}_i M \otimes C^{bar} F(A_i)$. Hence we are done by Lemma 3.4.5. □

Corollary 3.4.9. *K-excisive rings are closed under filtering colimits.*

Let M^0 and M^1 be chain complexes of abelian groups, and let $f \in [1]^n$. Put

$$T^f(M^0, M^1) = M^{f(1)} \otimes \cdots \otimes M^{f(n)}$$

Let

$$M^0 \star M^1 = \bigoplus_{n\geq 0} \bigoplus_{f\in \operatorname{map}([n],[1])} T^f(M^0, M^1)$$

Lemma 3.4.10. *Let A and B be rings. Then*

$$C^{bar}(A \oplus B) = (C^{bar}(A)[-1] \star C^{bar}(B)[-1])[+1]$$

Proof. If D is a ring then $C^{bar}(D) = T(D[-1])[+1]$ as graded abelian groups. Hence for $\coprod$ the coproduct of rings, we have

$$\begin{aligned} C^{bar}(A \oplus B) =& T(A[-1] \oplus B[-1])[+1] \\ =& (T(A[-1]) \coprod T(B[-1]))[+1] \\ =& (C^{bar}(A)[-1] \star C^{bar}(B)[-1])[+1] \end{aligned}$$

It is is straightforward to check that the identifications above are compatible with boundary maps. □

Proposition 3.4.11. *Let $\{A_i\}$ be a family of rings and $A = \bigoplus_i A_i$. Then A is K-excisive if and only if each A_i is, and in that case $\bigoplus_i K(A_i) \to K(A)$ is an equivalence.*

Proof. Let B and C be rings, and $F \to B$ and $G \to C$ be free simplicial resolutions in **Rings**. Then $F \oplus G \to B \oplus C$ is a flat simplicial resolution. Fix $q \geq 0$, and put $C^0 = C^{bar}(F_q)$, $C^1 = C^{bar}(G_q)$. Let $p \geq 1$, and $f \in [1]^p$. Then by the Künneth formula

$$H_n(T^f(C^0[-1], C^1[-1])[+1]) = \\ T^f(H_*(C^0), H_*(C^1))_{n+1} = \begin{cases} T^f(F/F^2, G/G^2) & p = n+1 \\ 0 & p \neq n+1 \end{cases}$$

Hence the second page of the spectral sequence for the double complex of Lemma 3.4.5 is

$$E^2_{p,q} = \bigoplus_{f\in[1]^{p+1}} \pi_q(T^f(F/F^2, G/G^2))$$

If B and C are K-excisive, we have $E^2 = 0$, by the Künneth formula, and thus $B \oplus C$ is again K-excisive. It follows from this and from Proposition 3.4.8 that if A_i is a family of K-excisive rings as in the proposition, then A is K-excisive. If B and C are arbitrary, then

$$E^2_{0,q} = \operatorname{Tor}_q^{\tilde{B}}(\mathbb{Z}, B) \oplus \operatorname{Tor}_q^{\tilde{C}}(\mathbb{Z}, C)$$
$$E^2_{p,0} = \bigoplus_{f\in[1]^{p+1}} T^f(A/A^2, B/B^2)$$

Hence if $B \oplus C$ is excisive, $E^2_{*,0} = 0$. It follows that $E^2_{0,1} = 0$, and therefore $E^2_{*,1} = \pi_1(T^f(F/F^2, G/G^2))$ involves only tensor products of the form $E^2_{p,0} \otimes E^2_{0,1}$ and its symmetric, and both of these are zero. A recursive argument shows that $E^2 = 0$, whence both B and C are K-excisive. If now A and $\{A_i\}$ are as in the proposition, $A =$ is excisive, and $j \in I$ then setting $B = A_j$ and $C = \bigoplus_{i\neq j} A_i$ above, we obtain that A_i is K-excisive. The last assertion of the proposition is well-known if each A_i is unital. More generally, assume all A_i are K- excisive, and consider the exact sequence

$$0 \to A \to \bigoplus_i \tilde{A}_i \to \bigoplus_i \mathbb{Z} \to 0 \tag{98}$$

We have a commutative diagram with homotopy fibration rows

$$\begin{array}{ccccc}
\bigoplus_i K(A_i) & \longrightarrow & \bigoplus_i K(\tilde{A}_i) & \longrightarrow & \bigoplus_i K(\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
K(A) & \longrightarrow & K(\bigoplus_i \tilde{A}_i & \longrightarrow & K(\bigoplus_i \mathbb{Z})
\end{array}$$

Because the middle and right vertical arrows are equivalences, it follows that the left one is an equivalence too. □

Proposition 3.4.12. Let $\{A_i\}$ be a family of rings and $A = \bigoplus_i A_i$. Then A is H-unital if and only if each A_i is, and in that case $\bigoplus_i HH(A_i) \to HH(A)$ and $\bigoplus_i HC(A_i) \to HC(A)$ are quasi-isomorphisms.

Proof. The last assertion is proved by the same argument as its K-theoretic counterpart. By 3.4.6, if B and C are rings and B is H-unital, then $C^{bar}(B\oplus C)\otimes V \to C^{bar}(C)\otimes V$ is a quasi-isomorphism for every abelian group V. Thus if also C is H-unital, then so is $B \oplus C$. Using this and the fact that H-unitality is preserved under filtering colimits, it follows that if $\{A_i\}$ is a family of H-unital rings, then $A = \bigoplus_i A_i$ is H-unital. Suppose conversely that A is H-unital, and consider the pure extension (98). A similar argument as that of the proof of Proposition 3.4.11 shows that $\bigoplus_i HH(A_i) \to HH(A)$ is a quasi-isomorphism. Next fix an index j and let

$$0 \to A_j \to B \to C \to 0$$

be a pure extension. Then

$$0 \to A \to \bigoplus_{i\neq j} A_i \oplus \tilde{B} \to \bigoplus_{i\neq j} A_i \oplus \tilde{C} \to 0$$

is a pure extension. Applying HH yields a distinguished triangle quasi-isomorphic to

$$\bigoplus_i HH(A_i) \to \bigoplus_{i\neq j} A_i \oplus HH(B) \oplus HH(\mathbb{Z}) \to \bigoplus_{i\neq j} A_i \oplus HH(C) \oplus HH(\mathbb{Z})$$

Removing summands, we obtain a triangle

$$HH(A_j) \to HH(B) \to HH(C)$$

We have shown that A_j satisfies excision for pure extensions in Hochschild homology; by 3.4.6, this implies that A_j is H-unital. □

4.5. Tensor products. It was proved by Suslin and Wodzicki [**29**, Theorem 7.10] that the tensor product of H-unital rings is H-unital. Here we establish a weak analogue of this property for K-excisive rings.

Let A be a ring. Put

$$L_{-1}A = A, \qquad L_{n+1}A = \ker(A \otimes L_n(A) \to L_n(A)) \qquad (n \geq -1)$$

LEMMA 3.4.13. Let A be a K-excisive ring, and V an abelian group. Assume both A and V are flat over $\mathbb{Z}$. Then

$$\operatorname{Tor}_n^{\widetilde{A\otimes TV}}(\mathbb{Z}, A \otimes TV) = L_{n-1}A \otimes V^{\otimes n+1} \qquad n \geq 0$$

PROOF. If M is a left A-module such that

$$A \cdot M = M, \tag{99}$$

and $L(M) = \ker(A \otimes M \to M)$ is the kernel of the multiplication map, then we have a short exact sequence

$$0 \to L(M) \otimes T^{\geq n+1}V \to \widetilde{A \otimes TV} \otimes M \otimes V^{\otimes n} \to M \otimes T^{\geq n}V \to 0$$

By definition, $L_nA = L^{n+1}A$. By [**29**, Theorem 7.8 and Lemma 7.6], $M = L_nA$ satisfies (99) for all n, and moreover, it is a flat abelian group, by induction. Thus for $n \geq 1$, the sequence

$$0 \to L_{n-1}(M) \otimes T^{\geq n+1}V \to \widetilde{A \otimes TV} \otimes L_{n-2}M \otimes V^{\otimes n} \to L_{n-2}M \otimes T^{\geq n}V \to 0$$

is exact. Hence

$$\begin{aligned}\operatorname{Tor}_i^{\widetilde{A\otimes TV}}(\mathbb{Z}, A \otimes TV) =&\operatorname{Tor}_i^{\widetilde{A\otimes TV}}(\mathbb{Z}, L_{-1}A \otimes T^{\geq 1}V)\\ =&\operatorname{Tor}_0^{\widetilde{A\otimes TV}}(\mathbb{Z}, L_{i-1}A \otimes T^{\geq i+1}V)\\ =&L_{i-1}A \otimes V^{\otimes i+1}\end{aligned}$$

□

PROPOSITION 3.4.14. Let A and B be K-excisive rings, at least one of them flat as a $\mathbb{Z}$-module. Then $A \otimes B$ is K-excisive.

PROOF. Assume A is flat. Let $F \overset{\sim}{\twoheadrightarrow} B$ be a simplicial resolution by free rings. Then $A \otimes F \overset{\sim}{\twoheadrightarrow} A \otimes B$ is a resolution by flat rings. By Lemma 3.4.13, the second page of the spectral sequence of Proposition 3.4.3 is

$$E^2_{p,q} = \pi_q(A \otimes (F/F^2)^{\otimes p+1}) = A \otimes (\pi_q((F/F^2)^{\otimes p+1}))$$

which equals zero by Corollary 3.4.4 and the Künneth formula, since B is K-excisive by assumption. □

4.6. Crossed products. Let G be a group and $\pi : \mathbb{Z}[G] \to \mathbb{Z}$ the augmentation $g \mapsto 1$. Put

$$JG = \ker \pi$$

LEMMA 3.4.15. Let V be a $\mathbb{Z}$[G]-module, free as an abelian group. Then

$$\mathrm{Tor}_n^{\widetilde{TV \rtimes G}}(\mathbb{Z}, TV \rtimes G) = V^{\otimes n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G] \qquad n \geq 0$$

PROOF. Note that the subset

$$V^{\otimes n} \oplus TV^{\geq n+1} \rtimes G \subset TV \rtimes G$$

is a left ideal, and that the map

$$\widetilde{TV \rtimes G} \otimes V^{\otimes n} \to V^{\otimes n} \oplus TV^{\geq n+1} \rtimes G \tag{100}$$

$$1 \otimes y \mapsto y$$

$$x \rtimes g \otimes y \mapsto xg(y) \rtimes g$$

(101)

is a $\widetilde{TV \rtimes G}$-module isomorphism. Consider the map

$$V^{\otimes n} \otimes M \oplus (TV^{\geq n+1} \rtimes G) \otimes M \to TV^{\geq n} V \otimes M, \quad (x, (y \rtimes g) \otimes m) \mapsto x + y \otimes gm$$

Composing with the isomorphism (100), we obtain a $\mathbb{Z}$-split surjective map

$$\widetilde{TV \rtimes G} \otimes V^{\otimes n} \twoheadrightarrow TV^{\geq n} \otimes M$$

This map fits in an exact sequence

$$0 \to T^{\geq n+1} V \otimes JG \otimes M \to \widetilde{TV \rtimes G} \otimes V^{\otimes n} \to T^{\geq n} V \otimes M \to 0$$

If M is flat as an abelian group, then the middle term in the exact sequence above is a flat $\widetilde{TV \rtimes G}$-module. Applying this successively, starting with $M = \mathbb{Z}[G]$, we obtain

$$\begin{aligned}\mathrm{Tor}_n^{\widetilde{TV \rtimes G}}(\mathbb{Z}, TV \rtimes G) =& \mathrm{Tor}_0^{\widetilde{TV \rtimes G}}(\mathbb{Z}, TV^{\geq n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G]) \\ =& V^{\otimes n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G]\end{aligned}$$

□

PROPOSITION 3.4.16. Let G be a group and $A \in G-$**Rings**. Assume A is K-excisive. Then $A \rtimes G$ is K-excisive.

PROOF. Note that the forgetful functor from $G-$ **Rings** to sets has a left adjoint; namely $X \mapsto T(\mathbb{Z}[G \times X])$. Hence A admits a free resolution $F \overset{\sim}{\twoheadrightarrow} A$ such that each F_n is a G-ring; for example we may take the cotriple resolution associated to the adjoint pair just described. Since F is a simplicial G-ring, we can take its crossed product with G, to obtain a $\mathbb{Z}$-flat resolution $F \rtimes G \overset{\sim}{\twoheadrightarrow} A \rtimes G$. Now proceed as in the proof of Proposition 3.4.14, using Lemma 3.4.15. □

PROPOSITION 3.4.17. Let G be a group and $A \in G-$**Rings**. Assume A is H-unital. Then $A \rtimes G$ is H-unital.

Proof. The bar resolution $B(G,M)$ ([**33**, **S**6.5]) is functorial on the G-module M. Applying it dimensionwise to $C^{bar}(A)$, we obtain a simplicial chain complex $B(G, C^{bar}(A))$. We may view the latter as a double chain complex with $A^{\otimes q+1}\otimes\mathbb{Z}[G^{p+1}]$ in the (p,q) spot. Removing the first row and the first column yields a double complex whose total chain complex we shall call $M[-1]$. Note M is a chain complex of $A\rtimes G$-modules and homomorphisms. We have $M_0\cong (A\rtimes G)^{\otimes 2}$, and the multiplication map $(A\rtimes G)^{\otimes 2}\to A\rtimes G$ induces a surjection onto the kernel L of the augmentation $A\rtimes G\to A$, $a\rtimes g\to a$. Note that the hypothesis that A is H-unital implies that the augmented complex

$$\ldots\to M_1\to M_0\to L$$

is acyclic. Now proceed as in the proof of [**29**, Theorem 7.10]. □

Proposition 3.4.18. The functor $K:\mathbb{Z}-\mathbf{Cat}\to\mathbf{Spt}$ satisfies the standing assumptions.

Proof. Assumption iv) was proved in Proposition 3.3.3. By Example 3.4.1, s-unital rings are K-excisive; hence K-theory satisfies i). Assumption ii) holds by Proposition 3.4.16. If A is K-excisive and X is a set, then M_XA is K-excisive, by Proposition 3.4.14. Assumpton iii) follows from this and the fact that K-theory is matrix stable on unital rings. Assumption v) is proved in Proposition 3.4.11. □

5. Homotopy K-theory

If $\mathcal{C}$ is a $\mathbb{Z}$-linear category, then we write $\mathcal{C}^{\Delta^\bullet}$ for the simplicial $\mathbb{Z}$-linear category

$$\mathcal{C}^{\Delta^\bullet}:[n]\mapsto\mathcal{C}^{\Delta^n}=\mathcal{C}\otimes\mathbb{Z}[t_0,\ldots,t_n]/<t_0+\cdots+t_n-1>$$

Applying the functor K dimensionwise we get a simplicial spectrum whose total spectrum is the homotopy K-theory spectrum $KH(\mathcal{C})$. In particular if R is a unital ring, then $KH(R)$ was defined by Weibel in [**32**]. The following theorem was proved in [**32**]; see also [**3**, Section 5].

Theorem 3.5.1. (Weibel) The functor $KH:\mathbf{Ring}\to\mathbf{Spt}$ is excisive, matrix invariant, and invariant under polynomial homotopy.

Proposition 3.5.2. There is a natural weak equivalence $KH(\mathcal{C})\xrightarrow{\sim}KH(\mathcal{R}(\mathcal{C}))$.

Proof. We begin by observing that the inclusions (93) and (94) lift to inclusions $\hom_{\mathcal{C}}(x,y)\subset\mathcal{R}(\mathcal{C})$ and $\hom_{\tilde{\mathcal{C}}}(x,x)\subset\widetilde{\mathcal{R}(\mathcal{C})}$. Thus we have a functor

$$\phi':\tilde{\mathcal{C}}\to\widetilde{\mathcal{R}(\mathcal{C})}$$

Composing it with

$$\tilde{p}:\widetilde{\mathcal{R}(\mathcal{C})}\to\widetilde{\mathcal{A}(\mathcal{C})}$$

we obtain the map

$$\phi:\tilde{\mathcal{C}}\to\widetilde{\mathcal{A}(\mathcal{C})}$$

of (95) above. Tensoring with $\mathbb{Z}^{\Delta^\bullet}$ and applying $K(-)$ we obtain a commutative diagram

$$\begin{array}{ccc} KH(\tilde{\mathcal{C}}) & \longrightarrow & KH(\widetilde{\mathcal{R}(\mathcal{C})}) \\ & \searrow^{\phi} & \downarrow p \\ & & KH(\widetilde{\mathcal{A}(\mathcal{C})}) \end{array}$$

The diagram above maps to the diagram

$$\begin{array}{ccc} KH(\mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle) & \xrightarrow{\phi} & KH(\mathbb{Z}) \\ & \searrow^{\phi} & \Vert \\ & & KH(\mathbb{Z}) \end{array}$$

Taking fibers and using (90), (91) and (92), we obtain a homotopy commutative diagram

$$\begin{array}{ccc} KH(\mathcal{C}) & \xrightarrow{\varphi''} & KH(\mathcal{R}(\mathcal{C})) \\ & \searrow^{\varphi'} & \downarrow p \\ & & KH(\mathcal{A}(\mathcal{C})) \end{array}$$

Here φ' comes from a map of simplicial spectra

$$\varphi^\bullet : K(\mathcal{C}\otimes\mathbb{Z}^{\Delta^\bullet}) \xrightarrow{\sim} K(\tilde{\mathcal{C}}\otimes\mathbb{Z}^{\Delta^\bullet} : \mathbb{Z}\langle \operatorname{ob}\mathcal{C}\rangle\otimes\mathbb{Z}^{\Delta^\bullet}) \to K(\widetilde{\mathcal{A}(\mathcal{C})}\otimes\mathbb{Z}^{\Delta^\bullet} : \mathbb{Z}\otimes\mathbb{Z}^{\Delta^\bullet}),$$

and $\varphi^0 = \varphi$ is the map (96), which is an equivalence by Proposition 3.3.3. The same argument of the proof of Proposition 3.3.3 shows that φ^n is an equivalence for every n. On the other hand, by Theorem 3.5.1 and Lemma 3.2.10, the map $p : KH(\mathcal{R}(\mathcal{C})) \to KH(\mathcal{A}(\mathcal{C}))$ is an equivalence. It follows that φ'' is an equivalence too. □

PROPOSITION 3.5.3. The functor $KH : \mathbb{Z}-\mathbf{Cat} \to \mathbf{Spt}$ satisfies the standing assumptions.

PROOF. This follows from Theorem 3.5.1, Proposition 3.5.2, and Lemma 3.2.10. □

6. Proper G-rings

6.1. Extending polynomial functions and excision properties.

LEMMA 3.6.1. If X is a locally finite simplicial set, then $\mathbb{Z}^{(X)}$ is a free abelian group.

PROOF. By [**5**, 3.1.3] the lemma is true when X is finite. Hence if X is any simplicial set, and $\sigma \in X$ is a simplex, then $\mathbb{Z}^{<\sigma>}$ is free. If X locally finite, then by Example 2.2.4, $\mathbb{Z}^{(X)}$ is a subgroup of a free group, and therefore it is free. □

THEOREM 3.6.2. Let X be a simplicial set, $Y \subset X$ a sub-simplicial set and A a ring. Let $\phi \in A^Y$ and $K = \operatorname{supp}\phi$. There there exists $\psi \in A^X$ with $\operatorname{supp}\psi \subset \overline{\operatorname{St}}_X K$ such that $\psi_{|\operatorname{Link}_X(K)} = 0$ and $\psi_{|Y} = \phi$.

PROOF. We have $K \subset \mathrm{St}_Y K \subset \overline{\mathrm{St}}_Y K$, whence $\phi_{|\mathrm{Link}_Y(K)} = 0$. Note $\mathrm{St}_X K \cap Y = \mathrm{St}_Y K$; thus ϕ vanishes on $\mathrm{Link}_X(K) \cap Y$. Hence we may extend ϕ to a $\phi' : Y' = Y \cup \mathrm{Link}_X(K) \to A^{\Delta^\bullet}$ by $\phi'_{|\mathrm{Link}_X(K)} = 0$. Put $Y" = Y \cup \overline{\mathrm{St}}_X K$. Because $Y' \subset Y"$ is a cofibration and $A^{\Delta^\bullet} \twoheadrightarrow 0$ is a trivial fibration, we may further extend ϕ' to a $\phi" : Y" \to A^{\Delta^\bullet}$. By construction, $\{\sigma \in X : \phi"(\sigma) \neq 0\} \subset \mathrm{St}_X K$, and $\phi"$ vanishes on $\mathrm{Link}_X K$. Hence we may further extend $\phi"$ to a $\psi : X \to A^{\Delta^\bullet}$, by letting $\psi(\sigma) = 0$ if $\sigma \notin \overline{\mathrm{St}}_X K$. This concludes the proof. □

COROLLARY 3.6.3. If X is locally finite and $Y \subset X$ is a subsimplicial set, then the restriction map $A^{(X)} \to A^{(Y)}$ is surjective.

PROOF. It follows from Theorem 3.6.2, using 2.2.1. □

PROPOSITION 3.6.4. (Compare [**7**, Lemma 2.5]) Let A be a nonzero ring. The following are equivalent for a simplicial set X.
i) For every simplex $\sigma \in X$ there exists $\phi \in A^{(X)}$ such that $\phi(\sigma) \neq 0$.
ii) X is locally finite.

PROOF. Observe that if $\sigma, \tau \in X$ are simplices, with $< \tau > \supset < \sigma >$ and $\phi \in A^X$ satisfies $\phi(\sigma) \neq 0$, then $\phi(\tau) \neq 0$. If X is not locally finite, then by Lemma 2.2.1, there exists a simplex $\sigma \in X$ which is contained in infinitely many nondegenerate simplices. By the previous observation, $\phi(\sigma) = 0$ for every $\phi \in A^{(X)}$. We have proved that i)$\Rightarrow$ii). Assume conversely that X is locally finite, and let σ be a simplex of X. We want to show that there exists $\phi \in A^{(X)}$ such that $\phi(\sigma) \neq 0$. We may assume that σ is nondegenerate. Let $Y =< \sigma > \subset X$ be the sub-simplicial set generated by σ; by Corollary 3.6.3, it suffices to show that $A^Y \neq 0$. Now Y is an n-dimensional quotient of Δ^n, whence $S^n = \Delta^n / \partial \Delta^n$ is a quotient of Y. So we may further reduce to showing A^{S^n} is nonzero. Now

$$A^{S^n} = Z_n A^{\Delta^\bullet} = \bigcap_{i=0}^{n} \ker(d_i : A^{\Delta^n} \to A^{\Delta^{n-1}})$$

But if $0 \neq a \in A$, then $a t_0 \dots t_n$ is a nonzero element of $Z_n A^{\Delta^\bullet}$. □

PROPOSITION 3.6.5. If X is a locally finite simplicial set, then $\mathbb{Z}^{(X)}$ is s-unital.

PROOF. Let $\phi_1, \dots, \phi_n \in \mathbb{Z}^{(X)}$, and let $K = \bigcup_i \mathrm{supp}(\phi)$. By Theorem 3.6.2 there is $\mu \mathbb{Z}^{(X)}$ such that $\mu_{|K} = 1$ is the constant map. Thus

$$\phi_i = \phi_i \mu \qquad (\forall i). \tag{6.1}$$

□

PROPOSITION 3.6.6. If A is K-excisive and X is locally finite, then $\mathbb{Z}^{(X)} \otimes A$ is K-excisive.

PROOF. Follows from Lemma 3.6.1 and Propositions 3.6.5 and 3.4.14. □

REMARK 3.6.7. If A is a ring and X a locally finite simplicial set, then there is a natural map

$$\mathbb{Z}^{(X)} \otimes A \to A^{(X)}$$

It was proved in [**5**, 3.1.3] that this map is an isomorphism if X is finite.

6.2. Proper rings over a G-simplicial set. Fix a group G and consider rings equipped with an action of G by ring automorphisms. We write G $-$**Rings** for the category of such rings and equivariant ring homomorphisms. If $C \in$ G $-$**Rings** is commutative but not necessarily unital and $A \in$ G $-$**Rings**, then by a compatible (G, C)-algebra structure on A we understand a C-bimodule structure on A such that the following identities hold for $a, b \in A$,$c \in C$, and $g \in \mathrm{G}$:

$$\begin{aligned} c \cdot a &= a \cdot c \\ c \cdot (ab) &= (c \cdot a)b = a(c \cdot b) \\ g(c \cdot a) &= g(c) \cdot g(a) \end{aligned} \tag{6.2}$$

(6.3)

If X is a G-simplicial set and $A \in$ G $-$**Rings**, then we say that A is proper over X if it carries a compatible $(\mathrm{G}, \mathbb{Z}^{(X)})$ algebra structure such that

$$\mathbb{Z}^{(X)} \cdot A = A \tag{6.4}$$

If $\mathcal{F}$ is a family of subgroups of G, we say that A is $(\mathrm{G}, \mathcal{F})$-proper if it is proper over some $(\mathrm{G}, \mathcal{F})$ complex X.

Example 3.6.8. Fix a group G, a family of subgroups $\mathcal{F}$ and a $(\mathrm{G}, \mathcal{F})$-complex X. By Proposition 3.6.5, we have $\mathbb{Z}^{(X)} \cdot \mathbb{Z}^{(X)} = \mathbb{Z}^{(X)}$; thus $\mathbb{Z}^{(X)}$ is proper. If A is proper over X, and B is any ring, then $A \otimes B$ is proper over X. In particular, $\mathbb{Z}^{(X)} \otimes B$ is proper.

Let A be a G-ring, proper over a locally finite G-simplicial set X. We write A_X for the unitalization of A as an algebra over $\mathbb{Z}^{(X)}$; this is the abelian group

$$A_X = A \oplus \mathbb{Z}^{(X)} \tag{6.5}$$

equipped with the multiplication law given by the formula 79.

Lemma 3.6.9. Let A be a G-ring, proper over a locally finite G-simplicial set X. Then the ring A_X of (6.5) is s-unital.

Proof. Immediate from Proposition 3.6.5 and condition (6.4). □

Let X be a locally finite simplicial set, and $K \subset X$ a subobject. Put

$$I(K) = \{\phi : \mathrm{supp}\phi \subset K\} \lhd \mathbb{Z}^{(X)}$$

If $A \in$ **Ring** has a compatible $\mathbb{Z}^{(X)}$ structure, we put

$$A(K) = I(K) \cdot A \lhd A$$

Lemma 3.6.10. Let A be a G-ring. Assume that A is $(\mathrm{G}, \mathcal{F})$-proper. Then $\{A_i\}$ of ideals of A such that $A = \cup_i A_i$ and such that each A_i is proper over a finite $(\mathrm{G}, \mathcal{F})$-complex.

Proof. By hypothesis, there exists a $(\mathrm{G}, \mathcal{F})$-complex X such that A is proper over X. For each G-finite $(\mathrm{G}, \mathcal{F})$-subcomplex $K \subset X$, consider $I(K)$ and $A(K)$. It is clear that $\{I(K)\}$ and $\{A(K)\}$ are filtering systems of ideals and that $\cup_K I(K) = \mathbb{Z}^{(X)}$. We claim furthermore that $A = \cup_K A(K)$. By definition of $\mathbb{Z}^{(X)}$-algebra, $A = \mathbb{Z}^{(X)} \cdot A$. Hence if $a \in A$, then there exist $\phi_1, \dots, \phi_n \in \mathbb{Z}^{(X)}$ and $a_1, \dots, a_n \in A$ such that $a = \sum_i \phi_i a_i$. Hence $a \in A(K)$ for $K = \cup_i \mathrm{G} \cdot \mathrm{supp}(\phi_i)$. □

Lemma 3.6.11. (cf. [**11**, pp. 51]) Let $A \in \mathrm{G}-\mathbf{Rings}$ be proper over a locally finite G-simplicial set X, and let $f : X \to Y$ be an equivariant map with Y locally finite. Then the map $f^* : \mathbb{Z}^Y \to \mathbb{Z}^X$ induces a compatible $(G, \mathbb{Z}^{(Y)})$-algebra structure on A which makes it proper over Y.

Proof. We begin by showing that the compatible $(\mathrm{G}, \mathbb{Z}^{(X)})$-algebra structure on A extends to a compatible $(\mathrm{G}, \mathbb{Z}^X)$-module structure. By the lemma above, if $a \in A$ then there exists a finite subsimplicial set $K \subset X$ such that $a \in A(K) = I(K) \cdot A$. By Theorem 3.6.2 there exists $\mu_K \in Z^X$, with $\mathrm{supp}\mu_K \subset \overline{\mathrm{St}}(K))$ such that

$$\mu_K a = a \qquad \forall a \in A(K). \tag{6.6}$$

Because X is locally finite, $\overline{\mathrm{St}}(K)$ is finite and $\mu_K \in \mathbb{Z}^{(X)}$. Thus we have a map $A(K) \to I(\overline{\mathrm{St}}(K))) \otimes A(K)$, $a \mapsto \mu_K \otimes a$. Now $I(\overline{\mathrm{St}}(K))$ is an ideal in $\mathbb{Z}^X$ by (38); using the multiplication of $\mathbb{Z}^X$ we obtain a map

$$\mathbb{Z}^X \otimes A(K) \to A(\overline{\mathrm{St}}(K)), \quad \phi \otimes a \mapsto (\phi \cdot \mu_K)a. \tag{6.7}$$

If $L \supset K$, and we choose an element μ_L as above, then for $a \in A(K)$ and $\phi \in \mathbb{Z}^X$ we have:

$$(\phi \cdot \mu_L) \cdot a = (\phi \cdot \mu_L) \cdot (\mu_K \cdot a) = (\phi \cdot \mu_K)a$$

This shows that (6.7) is independent of the choice of the element μ_K of (6.6), and that we have a well-defined action $\mathbb{Z}^X \otimes A \to A$. Compatibility with the G-action follows from the fact that $g \cdot \mu_K$ is the identity on $g \cdot K$. The remaining compatibility conditions are immediate. Now A becomes an $\mathbb{Z}^{(Y)}$-module through f^*. If $K \subset X$ is a finite subsimplicial set, then $L = f(K) \subset Y$ is finite, and since Y is locally finite, there is a $\mu_L \in \mathbb{Z}^{(Y)}$ which is the identity on L, and thus $f^*(\mu_L)$ is the identity on K. It follows that the action of $\mathbb{Z}^{(Y)}$ on A satisfies (6.4). The remaining $(\mathrm{G}, \mathbb{Z}^{(Y)})$-compatibility conditions of (6.2) are straightforward. □

If $C, A \in \mathrm{H}-\mathbf{Rings}$ with C commutative and we have a compatible (H, C)-algebra structure on A, then $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ carries a compatible $(\mathrm{G}, \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(C))$-algebra structure, given by

$$\xi_{\mathrm{H}}(s, c) \cdot \xi_{\mathrm{H}}(t, a) = \begin{cases} \xi_{\mathrm{H}}(s, c \cdot a) & s = t \\ 0 & s\mathrm{H} \neq t\mathrm{H} \end{cases}$$

If moreover $C \cdot A = A$, then $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(C) \cdot \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A) = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$. We record a particular case of this in the following

Lemma 3.6.12. If $A \in \mathrm{H}-\mathbf{Rings}$ is proper over an H-simplicial set X, then the G-ring $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(A)$ is proper over $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(X)$.

6.3. Compression. Let $A \in \mathrm{G}-\mathbf{Rings}$, and $\mathrm{H} \subset \mathrm{G}$ a subgroup. Assume that A is proper over G/H. Let $\chi_{\mathrm{H}} \in \mathbb{Z}^{(\mathrm{G}/\mathrm{H})}$ be the characteristic function of H. The compression of A over H is the subring

$$\mathrm{Comp}_{\mathrm{H}}^{\mathrm{G}}(A) = \chi_{\mathrm{H}} \cdot A$$

Note the action of G on A restricts to an action of H on $\mathrm{Comp}_{\mathrm{H}}^{\mathrm{G}}(A)$, which makes it into an object of $\mathrm{H}-\mathbf{Rings}$.

Proposition 3.6.13. (Compare [**11**] Lemma 12.3, and paragraph after 12.4)

i) If $B \in \mathrm{H}-\mathbf{Rings}$, then $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B)$ is proper over G/H, and

$$B \to \mathrm{Comp}_{\mathrm{H}}^{\mathrm{G}} \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} B, \quad b \mapsto \xi_{\mathrm{H}}(1, b)$$

is an H-equivariant isomorphism.
ii) If $A \in \mathrm{G}-\mathbf{Rings}$ is proper over G/H, then

$$\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}} \mathrm{Comp}_{\mathrm{H}}^{\mathrm{G}}(A) \to A, \quad \xi_{\mathrm{H}}(s, \chi_{\mathrm{H}} a) \mapsto \chi_{s\mathrm{H}} s(a)$$

is a G-equivariant isomorphism.

Proof. Any $B \in \mathrm{H}-\mathbf{Rings}$ is proper over the 1-point space $*$. Hence $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(B)$ is proper over $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(*) = \mathrm{G}/\mathrm{H}$, by Lemma 3.6.12. The proof that the maps of i) and ii) are isomorphisms is straightforward; to show equivariance, one uses (44) and (46). □

7. Induction and equivariant homology

Lemma 3.7.1. Let G be a group, $\mathrm{K} \subset \mathrm{G}$ a subgroup, A a K-ring, and $\mathrm{E} : \mathbb{Z} - \mathbf{Cat} \to \mathbf{Spt}$ a functor satisfying the standing assumptions. Then A is E-excisive if and only if $\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A)$ is E-excisive.

Proof. The map (47) gives a nonequivariant isomorphism

$$\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \cong \mathbb{Z}^{(\mathrm{G}/\mathrm{K})} \otimes A = \bigoplus_{x \in \mathrm{G}/\mathrm{K}} A$$

The equivalence of the lemma follows from assumption v). □

Let G, K and A be as in the Lemma 3.7.1, and let X be a G-simplicial set. If A is unital, then for each subgroup $S \subset \mathrm{K}$ we have a functor

$$A \rtimes \mathcal{G}^{\mathrm{K}}(\mathrm{K}/S) \to \mathrm{Ind}_{\mathrm{K}}^{G}(A) \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/S)$$
$$kS \mapsto kS,$$
$$a \rtimes k \mapsto \xi_{\mathrm{K}}(1, a) \rtimes k$$

If A is any E-excisive ring, the map above is defined for the unitalization $\tilde{A}$; applying E, taking fibers relative to the augmentation $\tilde{A} \to \mathbb{Z}$, and using the standing assumptions, we get a map

$$\mathrm{E}(A \rtimes \mathcal{G}^{\mathrm{K}}(\mathrm{K}/S)) \to \mathrm{E}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/S)).$$

The maps

$$X_+^S \wedge \mathrm{E}(A \rtimes \mathcal{G}^{\mathrm{K}}(\mathrm{K}/S)) \to X_+^S \wedge \mathrm{E}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/S)) \to H^{\mathrm{G}}(X, \mathrm{E}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}} A))$$

assemble to

$$\mathrm{Ind} : H^{\mathrm{K}}(X, \mathrm{E}(A)) \to H^{\mathrm{G}}(X, \mathrm{E}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A))) \tag{7.1}$$

Proposition 3.7.2. (Compare [**11**, Proposition 12.9]) Let A be an E-excisive G-ring. Then the map (7.1) is an equivalence.

Proof. Because equivariant homology satisfies excision on X [**8**, Section 8], and because X is obtained by gluing together cells of the form $\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\Delta^n)$, $\mathrm{H} \in \mathcal{A}ll$, it suffices to prove the proposition for $X = \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(T)$ where H acts trivially on T. Let $\mathcal{R}$ be a full set of representatives of $\mathrm{K}\backslash\mathrm{G}/\mathrm{H}$. We have

$$\begin{aligned}\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(T) &= T \times \mathrm{G}/\mathrm{H} \\ &= \coprod_{\theta\in\mathcal{R}} T \times \mathrm{K}\,\theta\mathrm{H} \\ &\cong \coprod_{\theta\in\mathcal{R}} T \times \mathrm{K}\,/\,\mathrm{K}_\theta\end{aligned}$$

Here as in (49), $\mathrm{K}_\theta = c_\theta(\mathrm{H}) \cap \mathrm{K}$. Thus

$$H^{\mathrm{K}}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(T), \mathrm{E}(A)) = T_+ \wedge \bigvee_{\theta\in\mathcal{R}} \mathrm{E}(A \rtimes \mathcal{G}^{\mathrm{K}}(\mathrm{K}\,/\,\mathrm{K}_\theta))$$

On the other hand,

$$H^{\mathrm{G}}(\mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(T), \mathrm{E}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A)) = T_+ \wedge \mathrm{E}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$$

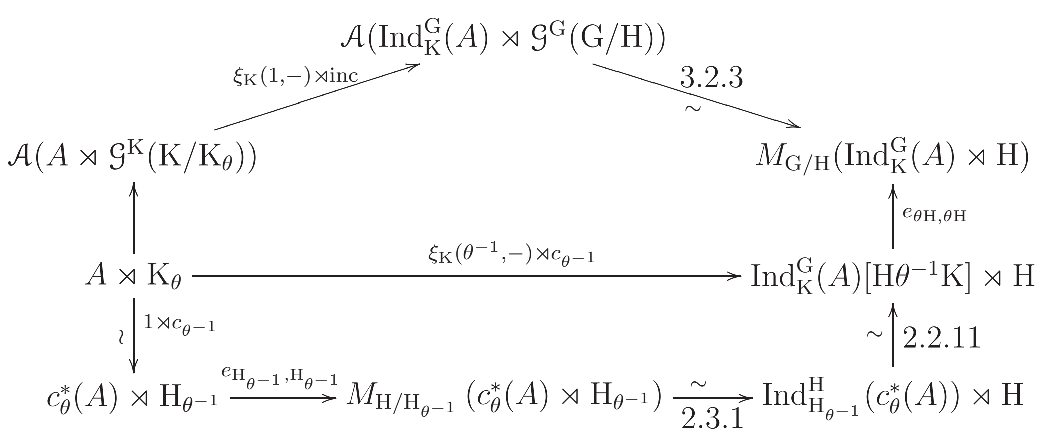
We have to show that

$$\bigvee_{\theta\in\mathcal{R}} E(A \rtimes \mathcal{G}^{\mathrm{K}}(\mathrm{K}\,/\,\mathrm{K}_\theta)) \to \mathrm{E}(\mathrm{Ind}_{K}^{\mathrm{G}}(A) \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$$

is an equivalence. By standing assumptions iv) and v) we may replace the map above by that induced by the corresponding ring homomorphism

$$\bigoplus_{\theta\in\mathcal{R}} \mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{K}}(\mathrm{K}/\mathrm{K}_\theta)) \to \mathcal{A}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H})) \tag{7.2}$$

Here $\mathcal{A}(A \rtimes \mathcal{G}^{\mathrm{K}}(\mathrm{K}/\mathrm{K}_\theta)) \to \mathcal{A}(\mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathcal{G}^{\mathrm{G}}(\mathrm{G}/\mathrm{H}))$ is induced by $\xi_{\mathrm{K}}(1,-) : A \to \mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A)$ and by the inclusions $\mathrm{K} \subset \mathrm{G}$ and $\mathrm{K}/\mathrm{K}_\theta \to \mathrm{G}/\mathrm{H}$, $k\mathrm{K}_\theta \mapsto k\theta\mathrm{H}$. One checks that the following diagram commutes



Because the lower rectangle commutes, $\mathrm{E}(A \rtimes \mathrm{K}_\theta \to \mathrm{Ind}_{\mathrm{K}}^{\mathrm{G}}(A)[\mathrm{H}\theta\mathrm{K}] \rtimes \mathrm{H})$ is an equivalence, by matrix stability. Again by matrix stability and by Lemma 3.2.3, applying E to the top left vertical arrow is an equivalence. Hence to prove that E applied to (7.2) is an equivalence, it suffices to show that E applied to

$$\operatorname{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathrm{H} = \bigoplus_{\theta\in\mathcal{R}} \operatorname{Ind}_{\mathrm{K}}^{\mathrm{G}}(A)[\mathrm{H}\theta\mathrm{K}] \rtimes \mathrm{H} \xrightarrow{\sum_\theta e_{\theta\mathrm{H},\theta\mathrm{H}}} M_{\mathrm{G}/\mathrm{H}}(\operatorname{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathrm{H}) \tag{7.3}$$

is one. But another application of matrix stability (using Remark 1.7.15) shows that (7.3) induces the same map in Ho**Spt** as the inclusion

$$e_{\mathrm{H},\mathrm{H}} : \operatorname{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathrm{H} \to M_{\mathrm{G}/\mathrm{H}}(\operatorname{Ind}_{\mathrm{K}}^{\mathrm{G}}(A) \rtimes \mathrm{H}).$$

This concludes the proof. □

Theorem 3.7.3. Let $\mathrm{E} : \mathbb{Z}-\mathbf{Cat} \to \mathbf{Spt}$ be a functor satisfying the standing assumptions 3.2.5. Also let G be a group, $\mathcal{F}$ a family of subgroups of G and B an E-excisive ring, proper over a 0-dimensional $(\mathrm{G},\mathcal{F})$-complex X. Then $H^{\mathrm{G}}(-,\mathrm{E}(B))$ maps $(\mathrm{G},\mathcal{F})$-equivalences to equivalences. In particular, the assembly map

$$H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(B)) \to \mathrm{E}(B \rtimes \mathrm{G})$$

is an equivalence.

Proof. We have $X = \coprod_i \mathrm{G}/\mathrm{K}_i$ for some $\mathrm{K}_i \in \mathcal{F}$, and $\mathbb{Z}^{(X)} = \bigoplus_i \mathbb{Z}^{(\mathrm{G}/\mathrm{K}_i)}$. The ring $B_i = \mathbb{Z}^{(\mathrm{G}/\mathrm{K}_i)} \cdot B$ is proper over G/K_i, and is excisive by standing assumption v). Again by standing assumption v), it suffices to prove the assertion of the theorem individually for each B_i; in other words, we may assume $X = \mathrm{G}/\mathrm{K}$ for some $\mathrm{K} \in \mathcal{F}$. Hence for $A = \operatorname{Comp}_{\mathrm{G}}^{\mathrm{K}} B$ we have $B = \operatorname{Ind}_{\mathrm{K}}^{\mathrm{G}} A$, by Proposition 3.6.13. Moreover, by Lemma 3.7.1, A is E-excisive. Let $Y \to Z$ be a $(\mathrm{G},\mathcal{F})$-equivalence. We have a commutative diagram

$$\begin{array}{ccc} H^{\mathrm{G}}(Y,\mathrm{E}(B)) & \longrightarrow & H^{\mathrm{G}}(Z,\mathrm{E}(B)) \\ {\scriptstyle \mathrm{Ind}}\uparrow & & \uparrow{\scriptstyle \mathrm{Ind}} \\ H^{\mathrm{K}}(Y,\mathrm{E}(A)) & \longrightarrow & H^{\mathrm{K}}(Z,\mathrm{E}(A)) \end{array}$$

The bottom horizontal arrow is an equivalence because $\mathrm{K} \in \mathcal{F}$. The two vertical arrows are equivalences by Proposition 3.7.2. It follows that the top horizontal arrow is an equivalence too. □

8. Assembly as a connecting map

Throughout this section, we consider a fixed functor $\mathrm{E} : \mathbb{Z} - \mathbf{Cat} \to \mathbf{Spt}$, and –except when otherwise stated– we assume that, in addition to the standing assumptions, it satisfies the following:

Sectional Assumptions 3.8.1.
vi) E_* commutes with filtering colimits.
vii) If A is E-excisive and L has local units and is flat as a $\mathbb{Z}$-module, then $L \otimes A$ is E-excisive.

8.1. Preliminaries.

Mapping cones. Let $f : A \to B$ be a ring homomorphism, the mapping cone of f is defined as the pullback

$$\begin{array}{ccc} \Gamma_f & \longrightarrow & \Gamma B \\ \downarrow & & \downarrow \\ \Sigma A & \xrightarrow[\Sigma f]{} & \Sigma B \end{array}$$

LEMMA 3.8.2. Let $\mathrm{E} : \mathbb{Z}-\mathbf{Cat} \to \mathbf{Spt}$ be a functor satisfying both the standing and the sectional assumptions, and $f : A \to B$ a homomorphism of strongly E-excisive rings. Then

i) $\mathrm{E}(\Gamma B)$ is weakly contractible.

ii) $\mathrm{E}(\Sigma B) \xrightarrow{\sim} \Sigma\,\mathrm{E}(B)$.

ii) Assume that Γ_f is E-excisive. Then the following is a distinguished triangle in Ho**Spt**

$$\mathrm{E}(B) \to \mathrm{E}(\Gamma_f) \to \Sigma\,\mathrm{E}(A) \to \Sigma\,\mathrm{E}(B)$$

PROOF. By Lemma 3.3.2, $\Gamma B = \Gamma\mathbb{Z} \otimes B$, whence it is E-excisive. Part i) follows from matrix stability and the fact that $\Gamma\mathbb{Z}$ is a ring with infinite sums (see e.g. [**3**, Prop. 2.3.1]). Parts ii) and iii) follow from i) and excision. □

Matrix rings and group actions.

LEMMA 3.8.3. Let G be a group, A a G-ring and X a G-set. Write $M_{\underline{X}}$ for the ring M_X equipped with the G-action

$$g(e_{x,y}) = e_{gx,gy}$$

The map

$$(M_{\underline{X}}A) \rtimes \mathrm{G} \to M_X(A \rtimes \mathrm{G}), (e_{x,y} \otimes a) \rtimes g \mapsto e_{x,g^{-1}y} \otimes (a \rtimes g)$$

is a G-equivariant isomorphism of rings.

8.2. Dirac extensions. Let G be a group, $\mathcal{F}$ a family of subgroups, $\mathrm{E} : \mathbb{Z}-\mathbf{Cat} \to \mathbf{Spt}$ a functor satisfying the standing assumptions, and A an E-excisive ring. A Dirac extension for $(\mathrm{G}, \mathcal{F}, A, \mathrm{E})$ consists of an extension of E-excisive G-rings

$$0 \to B \to Q \to P \to 0 \tag{8.1}$$

together with a zig-zag

$$A = Z_0 \xrightarrow{f_0} Z_1 \xleftarrow{f_2} Z_2 \xrightarrow{f_3} \cdots \qquad Z_n = B$$

such that

a) $\mathrm{E}(f_i \rtimes \mathrm{H})$ is an equivalence for every subgroup $\mathrm{H} \subset \mathrm{G}$.
b) $\mathrm{E}_*(Q \rtimes \mathrm{H}) = 0$ for every $\mathrm{H} \in \mathcal{F}$.
c) The assembly map $H^{\mathrm{G}}(\mathcal{E}(\mathrm{G}, \mathcal{F}), \mathrm{E}(P)) \to \mathrm{E}(P \rtimes \mathrm{G})$ is an equivalence.

PROPOSITION 3.8.4. Let $\mathrm{E} : \mathbb{Z}-\mathbf{Cat} \to \mathbf{Spt}$ be a functor satisfying the standing assumptions, G a group, $\mathcal{F}$ a family of subgroups of G, and A a G-ring. Let (8.1) be a Dirac extension for $(\mathrm{G}, \mathcal{F}, A, \mathrm{E})$. Then there are an exact sequence

$$\mathrm{E}_{*+1}(A \rtimes \mathrm{G}) \to \mathrm{E}_{*+1}(Q \rtimes \mathrm{G}) \to \mathrm{E}_{*+1}(P \rtimes \mathrm{G}) \xrightarrow{\partial} \mathrm{E}_*(A \rtimes G)$$

an isomorphism $H_*^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(A)) \cong \mathrm{E}_{*+1}(P \rtimes \mathrm{G})$, and a commutative diagram

$$\begin{array}{ccc} H_*^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(A)) & \xrightarrow{\text{Assembly}} & \mathrm{E}_*(A \rtimes \mathrm{G}) \\ & {\scriptstyle\cong}\searrow \quad \nearrow{\scriptstyle\partial} & \\ & \mathrm{E}_{*+1}(P \rtimes \mathrm{G}) & \end{array}$$

PROOF. Condition a) together with standing assumptions iii) and iv) and Lemma 3.2.3 imply that the zig-zag $f = \{f_i\}$ induces an equivalence

$$H^{\mathrm{G}}(X,\mathrm{E}(A)) \xrightarrow{\sim} H^{\mathrm{G}}(X,\mathrm{E}(B))$$

for every G-space X. Hence by Proposition 3.2.6 we have a distinguished triangle

$$H^{\mathrm{G}}(X,\mathrm{E}(A)) \longrightarrow H^{\mathrm{G}}(X,\mathrm{E}(Q)) \longrightarrow H^{\mathrm{G}}(X,\mathrm{E}(P)) \xrightarrow{\partial^X} \Sigma H^{\mathrm{G}}(X,\mathrm{E}(A))$$

The proposition follows by comparison of the long exact sequence of homotopy associated to the triangles for $X = \mathcal{E}(\mathrm{G},\mathcal{F})$, and $X = *$, and noting that condition b) implies that $H_*^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(Q)) = 0$. $\square$

8.3. A canonical Dirac extension. Let G be a group and $\mathcal{F}$ a family of subgroups. Consider the discrete G-simplicial sets

$$X = X_{\mathcal{F}} = \coprod_{H\in\mathcal{F}} \mathrm{G}/\mathrm{H}, \quad Y = \mathrm{G}/\mathrm{G} \coprod X$$

The group G acts on Y and thus on the ring M_Y of $Y \times Y$-matrices with finitely many nonzero integral coefficients. The point y_0 corresponding to the unique orbit of G/G is fixed by G, whence the map $\iota : \mathbb{Z} \to M_{\underline{Y}}$, $\lambda \to \lambda e_{y_0,y_0}$ is G-equivariant. In particular we have a directed system of G-rings $\{\mathrm{id} \otimes \iota : (M_\infty M_{\underline{Y}})^{\otimes n} \to (M_\infty M_{\underline{Y}})^{\otimes n+1}\}_n$. Put

$$\mathfrak{F}^0 = \operatorname*{colim}_n (M_\infty M_{\underline{Y}})^{\otimes n}$$

Since X is discrete, the ring of finitely supported functions breaks up into a sum

$$\mathbb{Z}^{(X)} = \bigoplus_{x\in X} \mathbb{Z}\chi_x$$

Multiplication by an element of $M_{\underline{Y}}$ gives an $\mathbb{Z}$-linear endomorphism of $\mathbb{Z}^{(Y)}$. This defines a monomorphism

$$M_Y \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^{(Y)})$$

whose image consists of those linear transformations T such that the matrix of T with respect to the basis $\{\chi_y : y \in Y\}$ has finitely many nonzero entries. Note that multiplication by χ_x in $\mathbb{Z}^{(X)} \subset \mathbb{Z}^{(Y)}$ is in this image. Thus we have an injective ring homomorphism

$$\rho : \mathbb{Z}^{(X)} \to M_{\underline{Y}}$$

For each $n \geq 1$, consider the G-ring

$$\mathfrak{F}^n = \left(\bigotimes_{i=1}^{n} \Gamma_\rho\right) \otimes \mathfrak{F}^0$$

Note that $\mathfrak{F}^n = (\bigotimes_{i=1}^n \Gamma_\rho) \otimes \mathfrak{F}^0 = (\bigotimes_{i=1}^n \Gamma_\rho) \otimes M_\infty M_{\underline{Y}} \otimes \mathfrak{F}^0$. The inclusion $M_\infty M_{\underline{Y}} \to \Gamma_\rho$ induces an inclusion $\mathfrak{F}^n \subset \mathfrak{F}^{n+1}$ for each $n \geq 0$. Put

$$\mathfrak{F}^\infty = \bigcup_{n\geq 0} \mathfrak{F}^n$$

If $A \in \mathbf{Rings}$, we also write $\mathfrak{F}^n A = \mathfrak{F}^n \otimes A$ $(n \geq 0)$. We have

LEMMA 3.8.5.
i) $\mathfrak{F}^n \subset \mathfrak{F}^\infty$ is an ideal $(n < \infty)$.
ii) For each $n \geq 0$, $\mathfrak{F}^n$ and $\mathfrak{F}^{n+1}/\mathfrak{F}^n \cong \Sigma\mathbb{Z}^{(X)} \otimes \mathfrak{F}^n$ have local units, and are $(\mathrm{G}, \mathcal{F})$-proper rings and are flat as abelian groups.
iii) If $\mathrm{H} \in \mathcal{F}$, $x \in \mathrm{G}/\mathrm{H}$, and A is a G-ring, we have a commutative diagram

$$\begin{array}{ccc}
(\mathbb{Z}^{(\mathrm{G}/\mathrm{H})} \otimes \mathfrak{F}^n A) \rtimes \mathrm{H} \subset (\mathbb{Z}^{(X)} \otimes \mathfrak{F}^n A) \rtimes H & \xrightarrow{(\rho\otimes 1)\rtimes \mathrm{id}} & (M_{\underline{Y}}\mathfrak{F}^n A) \rtimes \mathrm{H} \\
\chi_x \otimes 1 \uparrow & & \wr \downarrow (3.8.3) \\
\mathfrak{F}^n A \rtimes \mathrm{H} & \xrightarrow[e_{x,x}\otimes -]{} & M_Y(\mathfrak{F}^n A \rtimes \mathrm{H})
\end{array}$$

PROOF. Part i) is clear. Because $M_{\underline{Y}}$ is proper over Y, $\mathfrak{F}^n$ is proper over Y for all n, by 3.6.8. Similarly,

$$\mathfrak{F}^{n+1}/\mathfrak{F}^n = \Sigma\mathbb{Z}^{(X)} \otimes \mathfrak{F}^n \tag{8.2}$$

is proper. That $\mathfrak{F}^n$ is flat is clear for $n = 0$; the general case follows by induction, using (8.2). The ring $\mathfrak{F}^0$ has local units because $M_{\underline{Y}}$ and M_∞ do. To prove that $\mathfrak{F}^n$ has local units for $n \geq 1$, it suffices to show that Γ_ρ does. We may and do identify Γ_ρ with the inverse image of $\Sigma(\rho(\mathbb{Z}^{(X)}))$ under the projection $\pi : \Gamma M_{\underline{Y}} \to \Sigma M_{\underline{Y}}$; thus

$$\Gamma_\rho = \Gamma\rho(\mathbb{Z}^{(X)}) + M_\infty M_{\underline{Y}} \subset \Gamma M_{\underline{Y}}$$

One checks that if $\phi_1, \ldots, \phi_r \in \Gamma_\rho$, then there are finite subsets $F_1 \subset X$ and $F_2 \subset \mathbb{N}$ such that for $y_0 = G/G \in Y$, the element

$$e = 1 \otimes \sum_{x\in F_1} e_{x,x} + \sum_{p\in F_2} e_{p,p} \otimes e_{y_0,y_0} \in \Gamma_\rho$$

satisfies $e^2 = e$ and $e\phi_i = \phi_i e = \phi_i$ for all $i = 1, \ldots, r$. This proves part ii); part iii) is straightforward. $\square$

THEOREM 3.8.6. (Compare [**23**, Theorem 5.18]) Let $\mathrm{E} : \mathbb{Z} - \mathbf{Cat} \to \mathbf{Spt}$ be a functor satisfying both the standing and the sectional assumptions. Let G a group, $\mathcal{F}$ a family of subgroups, and A a G-ring. Then

$$\mathfrak{F}^0 A \to \mathfrak{F}^\infty A \to \mathfrak{F}^\infty A/\mathfrak{F}^0 A$$

is a Dirac extension for $(\mathrm{G}, \mathcal{F}, A, \mathrm{E})$.

PROOF. The three rings in the extension of the theorem are E-excisive, by Lemma 3.8.5 ii) and the hypothesis on A. The map $\mathrm{E}(A) \to \mathrm{E}(\mathfrak{F}^0 A)$ is an equivalence by standing assumption iii) and the assumption that E_* commutes with filtering colimits. Next we prove that the assembly map $H^{\mathrm{G}}(\mathcal{E}(\mathrm{G}, \mathcal{F}), \mathrm{E}(\mathfrak{F}^\infty A/\mathfrak{F}^0 A)) \to \mathrm{E}(\mathfrak{F}^\infty A/\mathfrak{F}^0 A \rtimes \mathrm{G})$ is an equivalence. By excision and the hypothesis that E_* commutes with filtering colimits, it suffices to show that

$$H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(\mathfrak{F}^nA/\mathfrak{F}^0A))\to\mathrm{E}(\mathfrak{F}^nA/\mathfrak{F}^0A\rtimes\mathrm{G}) \tag{8.3}$$

is an equivalence. Consider the extension

$$0\to\mathfrak{F}^nA/\mathfrak{F}^0A\to\mathfrak{F}^{n+1}A/\mathfrak{F}^0A\to\mathfrak{F}^{n+1}A/\mathfrak{F}^nA\to 0$$

By Proposition 3.2.6, assembly gives a map of homotopy fibration sequences

$$\begin{array}{ccc}
H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(\mathfrak{F}^nA/\mathfrak{F}^0A)) & \longrightarrow & \mathrm{E}(\mathfrak{F}^nA/\mathfrak{F}^0A\rtimes\mathrm{G}) \\
\downarrow & & \downarrow \\
H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(\mathfrak{F}^{n+1}A/\mathfrak{F}^0A)) & \longrightarrow & \mathrm{E}(\mathfrak{F}^{n+1}A/\mathfrak{F}^0A\rtimes\mathrm{G}) \\
\downarrow & & \downarrow \\
H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(\mathfrak{F}^{n+1}A/\mathfrak{F}^nA)) & \longrightarrow & \mathrm{E}(\mathfrak{F}^{n+1}A/\mathfrak{F}^nA\rtimes\mathrm{G})
\end{array}$$

By Lemma 3.8.5 and Theorem 3.7.3, the bottom horizontal map is an equivalence. Hence (8.3) is an equivalence for each n, by induction. It remains to show that $\mathrm{E}_*(\mathfrak{F}^\infty A\rtimes\mathrm{H})=0$ for each $\mathrm{H}\in\mathcal{F}$. Because E_* preserves filtering colimits by assumption, we may further restrict ourselves to proving that the map $j_n:\mathrm{E}_*(\mathfrak{F}^nA\rtimes\mathrm{H})\to\mathrm{E}_*(\mathfrak{F}^{n+1}A\rtimes\mathrm{H})$ induced by inclusion is zero for all n. By Lemma 3.8.2 we have a long exact sequence ($q\in\mathbb{Z}$)

$$\begin{array}{ccccc}
\mathrm{E}_q(\mathfrak{F}^nA\rtimes\mathrm{H}) & \xrightarrow{j_n} & \mathrm{E}_q(\mathfrak{F}^{n+1}A\rtimes\mathrm{H}) & \longrightarrow & \mathrm{E}_{q-1}(\mathbb{Z}^{(X)}\otimes\mathfrak{F}^nA\rtimes\mathrm{H}) \\
 & & & & \downarrow{\scriptstyle\partial} \\
 & & & & \mathrm{E}_{q-1}(\mathfrak{F}^nA\rtimes\mathrm{H})
\end{array}$$

where $\partial=\mathrm{E}_{q-1}(\rho\otimes 1\rtimes 1)$. By Lemma 3.8.5, part iii), ∂ is a split surjection. It follows that $j_n=0$; this concludes the proof. □

Example 3.8.7. The hypothesis of Theorem 3.8.6 are satisfied, for example, by the functorial spectra K and KH.

9. Isomorphism conjectures with proper coefficients

9.1. The excisive case.

Theorem 3.9.1. Let $\mathrm{E}:\mathbb{Z}-\mathbf{Cat}\to\mathbf{Spt}$ be a functor. Assume that E is excisive, additive and matrix-stable. Let $A\in\mathrm{G}-\mathbf{Rings}$ be proper over a locally finite, finite dimensional $(\mathrm{G},\mathcal{F})$-complex X. Then the assembly map

$$H^{\mathrm{G}}(\mathcal{E}(\mathrm{G},\mathcal{F}),\mathrm{E}(A))\to\mathrm{E}(A\rtimes\mathrm{G})$$

is a weak equivalence.

PROOF. If $\dim X = 0$, this follows from Theorem 3.7.3. Let $n > 0$ and assume the theorem true in dimensions $< n$. If $\dim X = n$, and $Y \subset X$ is the $n-1$-skeleton, we have a pushout diagram

$$\begin{array}{ccc} \coprod_i \operatorname{Ind}_{\mathrm{H}_i}^{\mathrm{G}}(\Delta^n) & \longrightarrow & X \\ \uparrow & & \uparrow \\ \coprod_i \operatorname{Ind}_{\mathrm{H}_i}^{\mathrm{G}}(\partial\Delta^n) & \longrightarrow & Y \end{array}$$

Here $\mathrm{H}_i \in \mathcal{F}$ and the horizontal arrows are proper, since X is assumed locally finite. Hence we obtain a pullback diagram

$$\begin{array}{ccc} \bigoplus_i \mathbb{Z}^{(\Delta^n)} \otimes \mathbb{Z}^{(\mathrm{G}/\mathrm{H}_i)} & \longleftarrow & \mathbb{Z}^{(X)} \\ \downarrow & & \downarrow \\ \bigoplus_i \mathbb{Z}^{(\partial\Delta^n)} \otimes \mathbb{Z}^{(\mathrm{G}/\mathrm{H}_i)} & \longleftarrow & \mathbb{Z}^{(Y)} \end{array}$$

Let $I = \ker(\mathbb{Z}^{(X)} \to \mathbb{Z}^{(Y)})$ be the kernel of the restriction map; because the diagram above is cartesian, $I \cong \bigoplus_i \ker(\mathbb{Z}^{(\Delta^n)} \otimes \mathbb{Z}^{(\mathrm{G}/\mathrm{H}_i)} \to \bigoplus_i \mathbb{Z}^{(\partial\Delta^n)} \otimes \mathbb{Z}^{(\mathrm{G}/\mathrm{H}_i)})$. The quotient $A/I \cdot A$ is proper over Y, and $I \cdot A$ is proper over $\coprod_i \operatorname{Ind}_{\mathrm{H}_i}^{\mathrm{G}}(\Delta^n)$, whence also over the zero-dimensional $\coprod_i \mathrm{G}/\mathrm{H}_i$, by Lemma 3.6.11. Thus the theorem is true for both $A/I \cdot A$ and $I \cdot A$; because E is excisive by hypothesis, this implies that the theorem is also true for A. $\square$

THEOREM 3.9.2. Let $\mathrm{E} : \mathbb{Z} - \mathbf{Cat} \to \mathbf{Spt}$ be a functor. Assume that E is excisive and matrix stable, and that E_* commutes with filtering colimits. Let $\mathcal{F}$ be a family of subgroups of a group G. Let $A \in \mathrm{G}-\mathbf{Rings}$ be proper over a locally finite $(\mathrm{G}, \mathcal{F})$-complex X. Then the assembly map

$$H_*^{\mathrm{G}}(\mathcal{E}(\mathrm{G}, \mathcal{F}), \mathrm{E}(A)) \to \mathrm{E}_*(A \rtimes \mathrm{G})$$

is an isomorphism.

PROOF. By Lemma 3.6.10, we may write A as a filtering colimit $A = \operatorname{colim}_i A_i$ such that each A_i is proper over a finite $(\mathrm{G}, \mathcal{F})$ complex. Because E_* commutes with filtering colimits by hypothesis, we may therefore restrict to the case when X is a finite $(\mathrm{G}, \mathcal{F})$-complex. Now apply Theorem 3.9.1. $\square$

EXAMPLE 3.9.3. Homotopy K-theory satisfies the hypothesis of Theorem 3.9.2.

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