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K-teoría Hermitiana Algebraica Bivariante

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires
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Hermitian Bivariant Algebraic K-theory
Summary

Consider a commutative ring ℓ with involution with an element $\lambda \in \ell$ such that $\lambda + \lambda^* = 1$; write Alg_ℓ^* for the category of ℓ-algebras with involution compatible with that of ℓ, which we call $*$-algebras. In this thesis we develop a triangulated category kk^h and a functor $j^h : Alg_\ell^* \to kk^h$ which we call *bivariant algebraic hermitian K-theory*; the functor j^h satisfies homotopy invariance, matrix and hermitian stability and is an excisive homology theory for extensions which are linearly split.

We also define a Weibel style homotopy invariant hermitian K-theory which we denote as KH_*^h. We show that the category kk^h recovers KH_0^h as a representable functor

$$\hom_{kk^h}(\ell, A) \cong KH_0^h(A).$$

We construct functors ${}_\varepsilon U$ and ${}_\varepsilon V$ which correspond to desuspensions of the functors U' and V' in Karoubi's Fundamental Theorem: for a unital $R \in Alg_\ell^*$ there is an element $\theta_0 \in K_2^h(U'^2 R)$ which the cup product induces an isomorphism

$${}_\varepsilon K_*^h(V'(R)) \cong {}_{-\varepsilon} K_{*+1}^h(U'(R)).$$

We prove an adjunction between kk^h and the bivariant algebraic K-theory kk as defined by Cortiñas and Thom and use it to prove a version of Karoubi's theorem in kk^h: the product with the image of θ_0 in $KH_0^h(U^2\ell)$ induces an isomorphism in kk^h

$$j^h({}_\varepsilon VA) \cong \Omega j^h({}_{-\varepsilon} UA)$$

for any $A \in Alg_\ell^*$. This allows us to obtain a bivariant homotopic version of the classical 12-term exact sequence of Karoubi for hermitian K-theory.

Keywords: hermitian algebraic K-theory, Karoubi's fundamental theorem, homotopy hermitian K-theory, bivariant algebraic K-theory, bivariant Witt groups

K-teoría Algebraica Hermitiana Bivariante
Resumen

Consideremos un anillo conmutativo ℓ con involución con un elemento $\lambda \in \ell$ tal que $\lambda + \lambda^* = 1$; sea Alg_ℓ^* la categoría de ℓ-algebras con involución compatible con la de ℓ que llamamos $*$-algebras. En esta tesis desarrollamos una categoría triangulada kk^h y un funtor $j^h : Alg_\ell^* \to kk^h$ que llamamos *K-teoría hermitiana algebraica bivariante*; el funtor j^h satisface invarianza homotópica, estabilidad matricial y hermitiana y es una teoría de homología escisiva para extensiones que se parten linealmente.

También definimos una versión invariante homotópica estilo Weibel de la K-teoría hermitiana que notamos como KH_*^h. Mostramos que la categoría kk^h recupera KH_0^h como funtor representable

$$\hom_{kk^h}(\ell, A) \cong KH_0^h(A).$$

Construimos funtores ${}_\varepsilon U$ y ${}_\varepsilon V$ que se corresponden con desuspensiones de los funtores U' y V' en el Teorema Fundamental de Karoubi: para $R \in Alg_\ell^*$ unital hay un elemento $\theta_0 \in K_2^h((U'^2)R)$ cuyo producto cup induce un isomorfismo

$${}_\varepsilon K_*^h(V'(R)) \cong {}_{-\varepsilon}K_{*+1}^h(U'(R)).$$

Probamos una adjunción entre kk^h y la K-teoría algebraica bivariante kk definida por Cortiñas y Thom y la usamos para probar una versión del teorema de Karoubi en kk^h: el producto con la imagen de θ_0 en $KH_0^h(U^2\ell)$ induce un isomorfismo en kk^h

$$j^h({}_\varepsilon VA) \cong \Omega j^h({}_{-\varepsilon}UA)$$

para todo $A \in Alg_\ell^*$. Esto nos permite obtener una versión bivariante homotópica de la clásica sucesión de 12 términos de Karoubi para la K-teoría hermitiana.

Palabras clave: K-teoría hermitiana algebraica, teorema fundamental de Karoubi, K-teoría hermitiana homotópica, K-teoría algebraica bivariante, grupos bivariantes de Witt

Acknowledgements

Introduction

Since the introduction of Kasparov's bivariant K-theory for C^*-algebras KK [Kas80], Higson's theorem on the universality of KK [Hig87] and Cuntz's foundational work [Cun87; Cun05], the development of bivariant versions of K-theory has been useful and important in many computations. This ranges from applications to the Baum-Connes conjecture, classification theory of C^*-algebras such as the Elliott program and the Kirchberg-Philips theorem but also to put some constructions in different versions of K-theory — between different topological versions such as C^*-algebras, Banach ($*$-)algebras and bornological algebras and also algebraic K-theory — on common ground. It also has been very fruitful in proving some cases of the Baum-Connes conjecture.

Cortiñas and Thom developed in [CT07] a bivariant version of algebraic K-theory with many similarities to KK, adapting them to an algebraic setting. Let ℓ be a commutative ring and write Alg_ℓ as the category of (associative) algebras over ℓ. Also fix an underlying category $\mathfrak{U}$ for Alg_ℓ such as that of sets or that of ℓ-modules and a forgetful functor $F : Alg_\ell \to \mathfrak{U}$. Cortiñas and Thom construct a triangulated category kk which has the same objects as Alg_ℓ together with a functor $j : Alg_\ell \to kk$ which is the identity on objects and satisfies:

- Matrix stability: the natural inclusion of $A \hookrightarrow M_\infty A$ on the upper left corner maps to an isomorphism through j.
- Polynomial homotopy invariance: the inclusion $A \to A[t]$ as constants maps to an isomorphism through j.
- The functor j is an excisive homology theory for extensions which are split in $\mathfrak{U}$, that is, for an extension

  $$0 \to A \to B \to C \to 0$$

  in Alg_ℓ which has a section $F(C) \to F(B)$, there is a natural (with respect to extensions) map $\partial : \Omega j(C) \to j(A)$ such that

  $$\Omega j(C) \to j(A) \to j(B) \to j(C)$$

  is a triangle in kk.

Moreover, for any triangulated category $\mathfrak{T}$ and functor $H : Alg_\ell \to \mathfrak{T}$ which satisfies the above mentioned properties, there is a unique triangulated functor $\overline{H} : kk \to \mathfrak{T}$

such that $H = \overline{H} \circ j$. A very important property of kk is that it recovers Weibel's homotopy K-theory as a representable functor

$$\hom_{kk}(\ell, A) = KH_0(A).$$

There have been alternative constructions of kk by Garkusha — who also constructed bivarant K-theory versions without matrix stability — [Gar13; Gar14; Gar16] and Rodríguez Cirone [Rod20]. Also, there have been generalizations of the original construction of kk to algebras with an action of a group and group graded algebras [Ell14] and to algebras with quantum group actions [Ell18].

In this thesis we construct a generalization of kk which incorporates algebras with involution: for a ring R, an involution is a ring morphism $(-)^* : R \to R^{op}$ with $(r^*)^* = r$. Suppose now that ℓ has an involution and an element λ which satisfies

$$\lambda + \lambda^* = 1. \tag{Intro.1}$$

Consider the category Alg_ℓ^* of ℓ-algebras with involution compatible with the involution of ℓ.

Let $R \in Alg_\ell^*$ unital and $\varepsilon \in R$ central unitary (i.e. $\varepsilon^{-1} = \varepsilon^*$). An element $\phi \in R$ is called *ε-hermitian* if $\phi^* = \varepsilon\phi$. For an invertible ε-hermitian element, we define R^ϕ as the $*$-ring which is the same as R as rings but with involution

$$r^\phi = \phi^{-1} r^* \phi.$$

When $A \trianglelefteq R$ is a $*$-ideal, this involution restricts to a new involution in A and we also write A^ϕ for A equipped with involution. We say a functor $H : Alg_\ell^* \to \mathfrak{C}$ is *hermitian stable* if for any $R \in Alg_\ell^*$ unital and $A \trianglelefteq R$ and invertible ε-hermitian elements $\phi, \psi \in R$ the inclusion on the upper left corner

$$i_\phi : A^\phi \to M_2(A)^{\phi \oplus \psi}$$

is mapped to an isomorphism through H.

In Chapter 3 we construct a triangulated category kk^h which has the same objects as Alg_ℓ^* together with a functor $j^h : Alg_\ell^* \to kk^h$ which is the identity on objects. One of the key pieces in this construction is the ability to fix some standard polynomial homotopies commonly occurring on K-theory (such as rotation homotopies) which are not involution preserving; this is mainly fixed with Lemma 1.2.3; the existence of the element (Intro.1) is essential. The main result in Chapter 3 is the following:

Theorem (Theorem 3.2.17 and Theorem 3.2.20) *There is a triangulated category kk^h and an excisive homology theory functor $j^h : Alg_\ell^* \to kk^h$ which is matricially and hermitian stable and polynomial homotopy invariant.*

Furthermore, the functor $j^h : Alg_\ell^* \to kk^h$ is universal between the matricially and hermitan stable, polynomial homotopy invariant excisive homology theories.

For a unital ring with involution R and a central unitary element $\varepsilon \in R$, recall the hermitan algebraic K-theory spectra ${}_\varepsilon K^h(R)$ as defined in [Lod76]. In Chapter

2 we define a Weibel style homotopy invariant version of ${}_\varepsilon K^h_*(R)$ which we denote ${}_\varepsilon KH^h_*(R)$.

In Chapter 4 we discuss some standard computations such as classification of the image through j^h of coproducts, the Toeplitz algebra, and the Cohn algebra of a finite graph and also prove the algebraic analogue of the Pimsner-Voiculescu sequence. We also show the following result:

Theorem(Theorem 4.2.1) There is a natural isomorphism

$$\hom_{kk^h}(\ell, A) \cong KH^h_0(A). \qquad \text{(Intro.2)}$$

For a unital $*$-ring R, there are natural maps between the K-theory spectra and the hermitian K-theory spectra induced by the hyperbolic and the forgetful maps

$$\text{hyp} : K(R) \to {}_\varepsilon K^h(R) \qquad \text{forg} : {}_\varepsilon K^h(R) \to K(R)$$

Write ${}_\varepsilon\mathcal{U}(R)$ and ${}_\varepsilon\mathcal{V}(R)$ for the homotopy fibers of these maps. Assume that R has an element as in (Intro.1). Karoubi's Fundamental Theorem for hermitian K-theory [Kar80] shows that there are natural homotopy equivalences

$${}_\varepsilon\mathcal{V}(R) \sim \Omega_{-\varepsilon}\mathcal{U}(R).$$

Moreover, Karoubi constructs functors U', V' for rings with involutions such that there are homotopy equivalences

$${}_\varepsilon K^h(U'R) \sim {}_\varepsilon\mathcal{U}(R) \text{ and } {}_\varepsilon K^h(V'R) \sim {}_\varepsilon\mathcal{V}(R).$$

Karoubi also shows that there is a natural equivalence

$${}_\varepsilon K^h(U'V'R) \sim {}_{-\varepsilon}K^h(R).$$

Thus, we can rephrase Karoubi's fundamental theorem as the equivalence

$${}_\varepsilon K^h(R) \sim \Omega^2{}_{-\varepsilon}K^h((U')^2R). \qquad \text{(Intro.3)}$$

The equivalence (Intro.3) is induced by the cup product with an element in $\theta_0 \in {}_{-1}K^h_2((U')^2(\mathbb{Z}))$.

In Chapter 5 we show that kk^h has a adjunction with kk which is analogue to the maps hyp and forg in the homotopy invariant setting. Then we construct functors ${}_\varepsilon U, {}_\varepsilon V : Alg^*_\ell \to Alg^*_\ell$ such that composing with the functor of homotopy hermitian algebraic K-theory we recover the homotopy versions of ${}_\varepsilon\mathcal{V}$ and ${}_\varepsilon\mathcal{U}$ up to a degree shift. Using the aforementioned adjunction we show that ${}_\varepsilon U$ and ${}_\varepsilon V$ have analogue properties in kk^h to those of U' and V' for hermitian K-theory. Write θ for the image of θ_0 in ${}_{-1}KH^h_0(U^2\ell)$. The main result of Chapter 5 is

Theorem (Theorem 5.3.1 and Corollary 5.3.2) The product with θ induces for every $A \in Alg^*_\ell$ an isomorphism in kk^h

$$j^h(A) \cong j^h({}_{-1}U^2(A)), \qquad \text{(Intro.4)}$$

which gives an isomorphism in kk^h

$$j^h({}_{\varepsilon}VA) \cong j^h({}_{-\varepsilon}UA).$$

Let R be a unital $*$-ring with an element λ which satisfies (Intro.1). The involution of R induces an involution $g \to (g^*)^{-1}$ in $\mathrm{GL}_\infty(R)$ which in turn induces a natural action of $\mathbb{Z}/2$ in $K_*(R)$; for $x \in K_n(R)$ write $\overline{x}$ for this action. Recall the Witt and coWitt groups ${}_{\varepsilon}W_n(R)$ and ${}_{\varepsilon}W'_n(R)$ and write $k_n(R)$ and $k'_n(R)$ for the $\mathbb{Z}/2$-Tate cohomology groups of $K_n(R)$ with the aforementioned action. Using the equivalence (Intro.3), Karoubi shows that there is a 12-term exact sequence:

$$\begin{array}{ccccccccccc}
k_{n+1}(R) & \longrightarrow & {}_{-\varepsilon}W_{n+2}(R) & \rightarrow & {}_{\varepsilon}W'_n(R) & \longrightarrow & k'_{n+1}(R) & \longrightarrow & {}_{-\varepsilon}W'_{n+1}(R) & \rightarrow & {}_{-\varepsilon}W_n(R) \\
\uparrow & & & & & & & & & & \downarrow \\
W_{n+1}(R) & \longleftarrow & W'_{n+1}(R) & \longleftarrow & k'_{n+1}(R) & \leftarrow & {}_{-\varepsilon}W'_n(R) & \longleftarrow & W_{n+2}(R) & \longleftarrow & k_{n+1}(R)
\end{array}$$

In the end of Chapter 5, we show that for bivariant adaptation of these groups (Definition 5.3.6) and we have a 12-term exact sequence (Theorem 5.3.7):

$$\begin{array}{ccccccccccc}
k_{n+1}(A,B) & \longrightarrow & {}_{-\varepsilon}W_{n+2}(A,B) & \rightarrow & {}_{\varepsilon}W'_n(A,B) & \longrightarrow & k'_{n+1}(A,B) & \longrightarrow & {}_{-\varepsilon}W'_{n+1}(A,B) & \rightarrow & {}_{-\varepsilon}W_{n+1}(A,B) \\
\uparrow & & & & & & & & & & \downarrow \\
{}_{\varepsilon}W_{n+1}(A,B) & \longleftarrow & {}_{\varepsilon}W'_{n+1}(A,B) & \longleftarrow & k'_{n+1}(A,B) & \leftarrow & {}_{-\varepsilon}W'_n(A,B) & \longleftarrow & {}_{\varepsilon}W_{n+2}(A,B) & \longleftarrow & k_n(A,B)
\end{array}$$

The rest of this thesis is outlined the following way. In Chapter 1 we discuss preliminary concepts and prove some useful lemmas that we will use throughout the thesis. In Chapter 2 we recall the construction of hermitian K-theory, we define KH^h and prove some of its basic properties; we also discuss the product structure of K^h and how it passes to KH^h. We end the chapter recalling Karoubi's Fundamental Theorem for hermitian algebraic K-theory. In Chapter 3 we construct the category kk^h and the functor $j^h : Alg^*_\ell \to kk^h$; first we prove the necessary technical lemmas to construct the morphism sets and then we show some of its properties as a triangulated category and how j^h is a universal excisive homology theory with matrix and hermitian stability and homotopy invariance. In Chapter 4 we proceed to develop some computations as a matter of examples and show (Intro.2). In Chapter 5 we show the adjunction between kk^h and kk and construct the functors U, V; we prove some of their properties in order to show (Intro.4) and obtain the 12-term exact sequence from it.

Chapter 1

Preliminaries

1.1 Rings and algebras with involution

Fix a commutative ring ℓ. An ℓ-algebra is a ring A together with a symmetric ℓ-module structure such that the product is ℓ-bilinear.

Suppose ℓ has an involution: a ring isomorphism $* : \ell \to \ell^{op} = \ell$, such that $(x^*)^* = x$, for all $x \in \ell$. A $*$-algebra over ℓ, is an ℓ-algebra A together with an involution $* : A \to A^{op}$ that is semilinear with respect to the module action:

$$(xa)^* = x^* a^* \text{ for } x \in \ell \text{ and } a \in A.$$

An ℓ-algebra morphism is a ring morphism that is also an ℓ-bimodule morphism. We write Alg_ℓ for the category of ℓ-algebras with ℓ-algebra morphisms and Alg_ℓ^* for the category of $*$-algebras over ℓ with $*$-morphisms, that is, ℓ-algebra morphisms that preserve the involution. A $*$-ideal in a $*$-algebra is a two-sided ideal that is closed under the action of ℓ and under the involution. For a $*$-ideal $I \trianglelefteq A$, the quotient A/I is also a $*$-algebra with the induced involution.

Example 1.1.1. For any commutative ring ℓ, the identity map $\mathrm{id} : \ell \to \ell$ is an involution; it is called the trivial involution. In the case of $\ell = \mathbb{Z}$ it is the only involution and $Alg_{\mathbb{Z}} = Rings$ is the category of rings; the category $Rings^* = Alg_\ell^*$ is called the category of $*$-rings.

Example 1.1.2. Let A and B be $*$-algebras over ℓ. The tensor product $A \otimes_\ell B$ is a $*$-algebra over ℓ with involution $(a \otimes b)^* = a^* \otimes b^*$. In some cases we write LA for $L \otimes_\ell A$ and write $L : Alg_\ell^* \to Alg_\ell^*$ for the functor given by tensoring with L. Except when explicitly noted, all tensor products will be over ℓ.

Example 1.1.3. Write M_n for the ring of $n \times n$ matrices over ℓ. The ℓ-algebra M_n has a natural involution $(a_{ij})^* = a_{ji}^*$.

More generally, let X be a set and define

$$\Gamma_X = \{a : X \times X \to \ell : \mathrm{im}(a) \text{ is finite and}$$
$$\exists N \text{ s.t. } \forall x \in X \ |\{y \in X : a(x,y) \neq 0\}|, |\{y \in X : a(y,x) \neq 0\}| \leq N\}.$$

with convolution product and conjugate transposition

$$(ab)(x,y) = \sum_{z\in X} a(x,z)b(z,y),$$
$$a^*(x,y) = a(y,x)^*$$

make Γ_X a $*$-algebra over ℓ. We write $M_X \trianglelefteq \Gamma_X$ for the $*$-ideal of finitely supported functions and Σ_X for the quotient Γ_X/M_X. We also write $\Gamma = \Gamma_{\mathbb{N}}$, $M_\infty = M_{\mathbb{N}}$ and $\Sigma = \Sigma_{\mathbb{N}}$. When X has cardinality n then $M_n \cong M_X = \Gamma_X$. For a $*$-algebra A we write $\Gamma_X A$, $M_X A$ and $\Sigma_X A$ for the tensor product of Γ_X, M_X and Σ_X with A respectively as in Example 1.1.2. We also write Σ_X^n for $\Sigma_X^{\otimes n}$

Example 1.1.4 (Unitalization)**.** Let A be a $*$-algebra and define $\widetilde{A} = A \oplus \ell$ as an ℓ-bimodule with the following multiplication and involution

$$(a,x)(b,y) = (ab + ay + xb, xy)$$
$$(a,x)^* = (a^*, x^*).$$

The $*$-algebra $\widetilde{A}$ is unital and has a natural morphism $A \to \widetilde{A}$, $a \mapsto (a,0)$ which maps A isomorphically to an ideal in $\widetilde{A}$. The quotient $\widetilde{A}/A$ is isomorphic to ℓ and the quotient map $\widetilde{A} \to \ell$ is split by $x \mapsto (0,x)$; whenever A is unital the unitalization $\widetilde{A}$ is isomorphic to $A \times \ell$ by means of this splitting.

Example 1.1.5 (Amalgamated coproducts and sums)**.** Let $A, B, C \in Alg_\ell^*$ and $i : C \to A$ and $j : C \to B$ two $*$-morphisms with retractions $\alpha : A \to C$ and $\beta : B \to C$ (i.e. $\alpha i = \mathrm{id}_C$ and $\beta j = \mathrm{id}_C$). The amalgamated coproduct of A and B over C is the ℓ-module

$$A \amalg_C B := C \oplus \ker\alpha \oplus \ker\beta \oplus (\ker\alpha \otimes_{\widetilde{C}} \ker\beta) \oplus (\ker\beta \otimes_{\widetilde{C}} \ker\alpha) \oplus \cdots$$

Where each summand beyond the first is given by the tensor product of $\ker\alpha$ and $\ker\beta$ in all possible orderings with an increasing number of tensor factors. This defines an ℓ-algebra with product given by concatenation of elementary tensors and extended by bilinearity. It also has an involution given by the involutions of A, B and C and twisting the elementary tensors appropriately. In the case $C = 0$, we write $A \amalg B$; this is simply the coproduct of A and B as ℓ-algebras.

The direct sum all tensors with two or more factors forms an ideal $K \trianglelefteq A \amalg_C B$ and we define the amalgamated direct sum as the quotient

$$A \oplus_C B := A \amalg_C B/K.$$

When $A = B$ and $C = 0$ we write $Q(A) := A \amalg A$ and $\iota_0, \iota_1 : A \to QA$ for the natural inclusions of A. The identity of $\mathrm{id}_A : A \to A$ induces a $*$-morphism $\mathrm{id}_A \amalg \mathrm{id}_A : Q(A) \to A$ and we write $q(A)$ for the kernel of this map. There are also two natural maps $\pi_0, \pi_1 : q(A) \to A$ which are the restrictions of $\mathrm{id}_A \amalg 0$ and $0 \amalg \mathrm{id}_A$ to $q(A)$.

Example 1.1.6 (Free involutions and induction)**.** Let A be a ring. Define $\mathrm{inv}(A) = A \oplus A^{\mathrm{op}}$ with involution $(a,b)^* = (b,a)$. This gives rise to an equivalence

$$\mathrm{inv} : Alg_\ell \to Alg^*_{\mathrm{inv}(\ell)}$$

with inverse $A \mapsto (1,0)A$. There is a natural $*$-morphism $\eta : \ell \to \mathrm{inv}(\ell)$ defined by $\eta(x) = (x, x^*)$. We can restrict the action of an $\mathrm{inv}(\ell)$-algebra to ℓ through η. Composing the functor inv with the restriction of scalars gives rise to a functor

$$\mathrm{ind} : Alg_\ell \to Alg^*_\ell.$$

This functor is right adjoint to the forgetful functor $\mathrm{res} : Alg^*_\ell \to Alg_\ell$ with unit and counit given by

$$\begin{aligned} \eta_A : A &\to \mathrm{ind}(\mathrm{res}(A)) = A \oplus A^{op} && (1.1.7)\\ a &\mapsto (a, a^*) \text{ and} \\ pr_1 : \mathrm{res}(\mathrm{ind}(B)) = B \oplus B^{op} &\to B && (1.1.8)\\ (x,y) &\mapsto x \end{aligned}$$

respectively.

Similarly, for an ℓ-algebra A define $\mathrm{ind}'(A) := A \amalg A$ with involution which permutes the copies of A. This gives a functor $\mathrm{ind}' : Alg_\ell \to Alg^*_\ell$ which is left adjoint to $\mathrm{res} : Alg^*_\ell \to Alg_\ell$ with unit and counit given by

$$\begin{aligned} \tilde{\eta}_A : A &\to \mathrm{res}(\mathrm{ind}'(A)) = A \amalg A && (1.1.9)\\ a &\mapsto \iota_0(a) + \iota_1(a) \text{ and} \\ \mathrm{id}_B \amalg 0 : \mathrm{ind}'(\mathrm{res}(B)) &= B \amalg B \to B && (1.1.10) \end{aligned}$$

respectively.

Definition 1.1.11 (Hermitian elements and involutions)**.** Let R be a unital ring with involution and $\varepsilon \in R$. We say that ε is *unitary* if it is invertible and $\varepsilon^* = \varepsilon^{-1}$ (e.g. $\varepsilon = \pm 1$).

For $\varepsilon \in R$ central unitary and $\phi \in R$, we say that ϕ is *ε-hermitian* if $\phi = \varepsilon\phi^*$. If $\phi \in R$ is invertible and ε-hermitian then we can define a new involution in R by

$$r \mapsto r^\phi := \phi^{-1} r^* \phi.$$

We write R^ϕ for the ring R with this new involution. If S is another unital $*$-algebra over ℓ and ψ is η-hermitian and invertible then $\phi \otimes \psi \in R \otimes_\ell S$ is $\varepsilon \otimes \eta$-hermitian and invertible and

$$(R \otimes_\ell S)^{\phi\otimes\psi} = R^\phi \otimes S^\psi. \tag{1.1.12}$$

Remark 1.1.13. Let R be a unital ring, $A \trianglelefteq R$ a $*$-ideal, $\varepsilon \in R$ central unitary and $\phi \in R$ an invertible ε-hermitian. The involution defined in Definition 1.1.11 restricts properly to an involution on A and we write A^ϕ for A equipped with this new involution.

Definition 1.1.14. Let A be a ring with involution and $u \in A$ unitary. The map

$$\begin{aligned} \operatorname{ad}(u) : A &\to A \\ x &\mapsto uxu^{-1} \end{aligned}$$

defines a $*$-isomorphism with inverse $\operatorname{ad}(u^*)$.

Remark 1.1.15. Let R be a unital $*$-algebra over ℓ, $\varepsilon \in R$ central unitary and $\phi, \psi \in R$ invertible ε-hermitian. If there exists $u \in R$ invertible such that $\psi = u^*\phi u$ then $\operatorname{ad}(u) : R^\psi \to R^\phi$ is a $*$-isomorphism.

Example 1.1.16. Let R_0 be an ℓ-algebra and $R = \operatorname{inv}(R_0) \in Alg_\ell^*$. If $\varepsilon = (\varepsilon_0, \varepsilon_1) \in R$ is central unitary then ε_0 and ε_1 are central and

$$\begin{aligned} (1,1) &= (\varepsilon_0, \varepsilon_1)^*(\varepsilon_0, \varepsilon_1) \\ &= (\varepsilon_1, \varepsilon_0)(\varepsilon_0, \varepsilon_1) \\ &= (\varepsilon_1\varepsilon_0, \varepsilon_0\varepsilon_1); \end{aligned}$$

therefore, $\varepsilon_1 = \varepsilon_0^{-1}$. We can deduce from this that any invertible ε-hermitian element $\phi \in R$ is of the form

$$\phi = (\phi_0, \varepsilon_0^{-1}\phi_0) = (1, \phi_0)^*(1, \varepsilon_0^{-1})(1, \phi_0).$$

It follows from Remark 1.1.15 that $R^\phi \cong R^{(1,\varepsilon_0^{-1})} = R$ since ε_0 is central.

Example 1.1.17. Let P be a finitely generated projective ℓ-module. An ε-hermitian bilinear form is a map $\psi : P \times P \to \ell$ which is ℓ-linear in the first coordinate and satisfies

$$\psi(x, y) = \varepsilon\psi(y, x)^*.$$

We say that ψ is non-degenerate if $\psi(-, y) : P \to P^*$ is an isomorphism for all $y \in P$; in this case we say that the pair (P, ψ) is an ε-hermitian module.

For an ε-hermitian module (P, ψ), the non-degenracy of ψ induces an involution on the ℓ-algebra of ℓ-linear endomorphisms $\operatorname{End}(P)$. This involution is determined by the following property: for $T \in \operatorname{End}(P)$ and $x, y \in P$ we have

$$\psi(T(x), y) = \psi(x, T^*(y)).$$

If $P = \ell^n$ is free, then $\operatorname{End}(P) \cong M_n$ and the involution induced by the ε-hermitian form ψ, corresponds to an ε-hermitan invertible h_ψ and the involution $(-)^{h_\psi}$.

Example 1.1.18. Consider the invertible -1-hermitian element

$$h_\pm = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2.$$

We write $M_\pm = (M_2)^{h_\pm}$ as in Example 1.1.11 and $M_\pm A$ for $M_\pm \otimes A$. We write $i_+, i_- : \ell \to M_\pm$ for the $*$-morphisms defined by the upper left and lower right corner inclusions respectively.

The element $h_\pm$ corresponds to the *hyperbolic* hermitian module: for $H(\ell) = \ell^2$, the -1-hermitian form

$$h((x_1, y_1), (x_2, y_2)) = x_1 y_2 - x_2 y_1.$$

It is well known [see for example KV73, Theorem 1.4] that for any hermitian module (P, ψ) then $(P, -\psi)$ is also a hermitian module and

$$(P, \psi) \oplus (P, -\psi) \cong H(\ell) \otimes P.$$

in such a way that the bilinear forms are preserved through this isomorphism.

Similarly, let $\varepsilon \in \ell$ be central unitary and consider the invertible ε-hermitian element

$$h_\varepsilon = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix} \in M_2.$$

We write ${}_\varepsilon M_2 = (M_2)^{h_\varepsilon}$ and ${}_\varepsilon M_2 A$ for ${}_\varepsilon M_2 \otimes A$.

Due to (1.1.12) we have the identity

$$ {}_\varepsilon M_2\, {}_\eta M_2 \cong {}_{\varepsilon\eta} M_2\, M_2. \tag{1.1.19}$$

Let X be an infinite set and fix a bijection $\{1, 2\} \times X \cong X$. This bijection together with (1.1.19) induces $*$-isomorphisms

$$ {}_\eta M_2\, {}_\varepsilon M_2\, M_X \cong {}_{\eta\varepsilon} M_2\, M_{\{1,2\}\times X} \cong {}_{\eta\varepsilon} M_2\, M_X. \tag{1.1.20}$$

Example 1.1.21 (Polynomial $*$-algebras)**.** We consider the polynomial ring $\ell[t]$ with the involution which fixes t. For any 1-hermitian element $\alpha \in A$ the evaluation map $\mathrm{ev}_\alpha : \ell[t] \to \ell$ that maps $t \mapsto \alpha$ is a $*$-morphism.

We write

$$\begin{aligned} P &= \ker(\mathrm{ev}_0 : \ell[t] \to \ell) \text{ and} \\ \Omega &= \ker(\mathrm{ev}_1 : P \to \ell). \end{aligned}$$

for the *path* and *loop* algebras respectively. We also consider the Laurent polynomial algebra $\ell[t, t^{-1}]$ with involution that interchanges t and t^{-1}, $t^* = t^{-1}$. For any unitary element $u \in \ell$ we have an evaluation map $\mathrm{ev}_u : \ell[t, t^{-1}] \to \ell$ which maps $t \mapsto u$.

As with matrices we write $A[t]$, $A[t, t^{-1}]$, PA and ΩA for $\ell[t] \otimes_\ell A$, $\ell[t, t^{-1}] \otimes_\ell A$, $P \otimes_\ell A$ and $\Omega \otimes A$ respectively. We write Ω^n for $\Omega^{\otimes n}$.

Example 1.1.22 (Simplicial $*$-algebras)**.** Let $n \in \mathbb{N}_0$ and

$$\ell[t_1, \ldots, t_n] = \ell[t_1] \otimes \cdots \otimes \ell[t_n]$$

be the polynomial algebra in n variables. We define

$$\ell^{\Delta^n} := \ell[t_0, \ldots, t_n]/\langle t_0 + \cdots + t_n - 1\rangle.$$

This defines a simplicial $*$-algebra

$$\ell^{\Delta}:\Delta^{op}\to Alg_{\ell}^{*}$$
$$[n]\mapsto \ell^{\Delta^n},$$

and we write A^{Δ} for $\ell^{\Delta}\otimes A$. Write $\mathfrak{S}$ for the category of simplicial sets. Let $X\in\mathfrak{S}$ and $B_{\bullet}:\Delta^{op}\to Alg_{\ell}^{*}$ be a simplicial $*$-algebra. The set $\hom_{\mathfrak{S}}(X,B_{\bullet})$ is an $*$-algebra. For $X\in\mathfrak{S}$ and $A\in Alg_{\ell}^{*}$ we define the $*$-algebra of functions on the simplicial set X as

$$A^{X}:=\hom_{\mathfrak{S}}(X,A^{\Delta}).$$

A pointed simplicial set (X,x) is a simplicial set X together with a map $x:\mathrm{pt}=\Delta^0\to X$. Write $\mathrm{ev}_x:A^X\to A^{\mathrm{pt}}$ for the induced $*$-morphism and define

$$A^{(X,x)}:=\ker(\mathrm{ev}_x).$$

Remark 1.1.23. Some of the $*$-algebras mentioned in Example 1.1.21 are particular cases of Example 1.1.22:

$$A^{\Delta^1}\cong A[t],$$
$$A^{(\Delta^1,\mathrm{pt})}\cong PA$$

and writing $S^1=\Delta^1/\Delta^0$ for the simplicial circle,

$$A^{(S^1,\mathrm{pt})}\cong\Omega A.$$

Throughout this thesis, we will often assume the following:

λ-assumption 1.1.24. the ring contains an element λ such that $\lambda+\lambda^*=1$.

Example 1.1.25. The λ-assumption 1.1.24 is satisfied for example when 2 is invertible in putting $\lambda=1/2$. Another example is when $\ell=\mathrm{inv}(\ell_0)$ for some ring ℓ_0 and $\lambda=(1,0)$.

Remark 1.1.26. Suppose that ℓ satisfies the λ-assumption 1.1.24 and let $\varepsilon\in\ell$ be unitary, R be a unital $*$-algebra and $\phi\in R$ be an invertible ε-hermitian element. Recall the matrices $h_{\pm}$ and h_{ε} from Example 1.1.18. The matrix

$$u_{\lambda}=\begin{pmatrix}1 & 1\\ \lambda\phi^* & -\lambda^*\phi^*\end{pmatrix} \tag{1.1.27}$$

satisfies $u_{\lambda}^*(h_{\varepsilon}\otimes 1)u_{\lambda}=h_{\pm}\otimes\phi$, whence $\mathrm{ad}(u_{\lambda}):M_{\pm}R^{\phi}\to {}_{\varepsilon}M_2R$ is a $*$-isomorphism. Taking $R=\ell$ and $\varepsilon=\phi=1$ we get $M_{\pm}\cong {}_{1}M_2$.

1.2 Algebraic homotopies

Definition 1.2.1. Let $A, B \in Alg_\ell^*$ and $f, g : A \to B$ two $*$-morphisms. We say that f and g are *elementary (algebraically) $*$-homotopic* if there exists a $*$-morphism $H : A \to B[t]$, called a $*$-homotopy, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{H} & B[t] \\ & \searrow^{(f,g)} & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ & & B \times B \end{array}$$

commutes. We say that f, g are *(algebraically) $*$-homotopic* if there exists a finite sequence $f_0, \dots, f_n : A \to B$ of $*$-morphisms such that $f_0 = f$, $f_n = g$ and f_i is elementary $*$-homotopic to f_{i+1} for $i = 0, \dots, n-1$; whenever f and g are $*$-homotopic we write $f \sim^* g$.

It is immediate from this definition that homotopy is an equivalence relation that is compatible with composition of $*$-morphisms. We write $[A, B]$ for set of equivalence classes of $*$-morphisms $A \to B$ modulo homotopy. The sets $[-, -]$ have a composition law and therefore are the arrows of a category $[Alg_\ell^*]$ which has $*$-algebras as objects.

Definition 1.2.2. Let $F : Alg_\ell^* \to \mathfrak{C}$ be a functor. We say that F is *homotopy invariant* if $F(f) = F(g)$ whenever $f \sim^* g$.

Let $C \in Alg_\ell$ and $A, B \subseteq C$ subalgebras. Suppose $u, v \in C$ satisfy

$$uAv \subseteq B \text{ and}$$
$$avua' = aa' \text{ for all } a, a' \in A.$$

Then

$$\mathrm{ad}(u, v) : A \to B$$
$$a \mapsto uav$$

is an algebra morphism. We say that the pair (u, v) *multiplies A into B*. Let $u_0, u_1, v_0, v_1 \in C$ such that (u_0, v_0) and (u_1, v_1) multiplies A into B. A *homotopy between the pairs (u_0, v_0) and (u_1, v_1)* is a pair $(u(t), v(t)) \in C[t]^2$ that multiplies A (as constants in $C[t]$) into $B[t]$ and that $(u(i), v(i)) = (u_i, v_i)$ (for $i = 0, 1$). In this case $\mathrm{ad}(u(t), v(t)) : A \to B[t]$ is a homotopy between $\mathrm{ad}(u_0, v_0)$ and $\mathrm{ad}(u_1, v_1)$. Suppose now that C is a $*$-algebra and that A, B are $*$-subalgebras; when $v = u^*$ and the pair (u, u^*) multiplies A into B, we have that $\mathrm{ad}(u, u^*)$ is a $*$-morphism. In this case we say that u $*$-multiplies A into B. If $u, w \in C$ both $*$-multiply A into B, a $*$-homotopy between u and w is an element $z(t) \in C[t]$ $*$-multiplying A into $B[t]$ such that $z(0) = u$ and $z(1) = w$. We shall often encounter examples of elements $u_0, u_1 \in C$ which $*$-multiply A into B that are homotopic via a pair $(u(t), v(t))$ with $u(t)^* \neq v(t)$ so that the homotopy $\mathrm{ad}(u(t), v(t))$ is not a $*$-morphism. This can be fixed as follows.

Lemma 1.2.3. *Suppose ℓ satisfies the λ-assumption 1.1.24. Let $C \in Alg^*_\ell$, $A, B \subseteq C$ $*$-subalgebras, $u_0, u_1 \in C$ that $*$-multiply A into B and $(v, w) \in C[t]^2$ a homotopy between (u_0, u_0^*) and (u_1, u_1^*). Assume as well that*

$$w^*Aw, vAv^* \subseteq B[t].$$

Then

$$c(v, w) = \begin{pmatrix} \lambda^*v + \lambda w^* & \lambda^*(v - w^*) \\ \lambda(v - w^*) & \lambda v + \lambda^* w^* \end{pmatrix} \in M_\pm C[t]$$

$*$-multiplies $i_+(A)$ into $M_\pm B[t]$ and $\mathrm{ad}(c(u, v), c(u, v)^*) \circ i_+$ *is a $*$-homotopy between* $i_+ \,\mathrm{ad}(u_0, u_0^*)$ *and* $i_+ \,\mathrm{ad}(u_1, u_1^*)$.

Proof. A straightforward computation shows that

$$c(v, w)^*c(v, w) = c(wv, wv).$$

Hence, for $a, a' \in A$ we have

$$\begin{aligned} i_+(a)c(v, w)^*c(v, w)i_+(a') &= i_+(a)c(wv, wv)i_+(a') \\ &= i_+(a(\lambda^*wv + \lambda(wv)^*)a') \\ &= i_+(\lambda^*awva' + \lambda a(wv)^*a') \\ &= i_+(\lambda^*aa' + \lambda(a'^*wva^*)^*) \\ &= i_+(aa'(\lambda^* + \lambda)) \\ &= i_+(aa'). \end{aligned}$$

Similarly, $c(v, w)i_+(A)c(v, w)^* \subseteq M_\pm B[t]$. Thus, $H = \mathrm{ad}(c(u, v))i_+ : A \to M_\pm B[t]$ is a $*$-morphism and for $i = 0, 1$ we get

$$\mathrm{ev}_i(c(u, v)) = c(u_i, u_i) = \begin{pmatrix} u_i & 0 \\ 0 & u_i \end{pmatrix},$$

so that $\mathrm{ev}_i H = i_+ \,\mathrm{ad}(u_i, u_i^*)$. □

Definition 1.2.4. Let $p, q \geq 0$ and $n = p + q$. Define

$$i_+^{p,q} := (M_\pm)^{\otimes p} \otimes i_+ \otimes (M_\pm)^{\otimes q} : M_\pm^{\otimes n} \to M_\pm^{\otimes n+1}$$

Lemma 1.2.5. *Let p, q and n be as above, and let $p', q' \geq 0$ be such that $p' + q' = n + 1$. Then $i_+^{p',q'} i_+^{p,q}$ is $*$-homotopic to $i_+^{0,n+1} i_+^{0,n}$.*

Proof. First observe that we have $i_+^{0,0} = i_+$ and $i_+^{1,0} i_+ = i_+^{0,1} i_+$. Therefore, tensoring with identity maps we get

$$i_+^{r,s+1} i_+^{r,s} = i_+^{r+1,s} i_+^{r,s} \tag{1.2.6}$$

for any $r, s \geq 0$. Next, under the identification $M_2 \otimes M_2 = M_{\{1,2\}^2}$, we have $i_+^{1,0}(e_{i,j}) = e_{(i,1),(j,1)}$ and $i_+^{0,1}(e_{i,j}) = e_{(1,i),(1,j)}$. One checks that the matrix

$$u = e_{(1,1),(1,1)} - e_{(1,2),(2,1)} + e_{(2,1),(1,2)} + e_{(2,2),(2,2)}$$

is a unitary element of $M_{\pm}^{\otimes 2}$ and satisfies $\mathrm{ad}(u)i_+^{1,0} = i_+^{0,1}$. Moreover by [CT07, Section 6.4], there exists an invertible element $u(t) \in M_{\pm}^{\otimes 2}[t]$ such that $u(0) = 1$ and $u(1) = u$. Hence the composites of $i_+^{0,2}$ with $i_+^{1,0}$ and $i_+^{0,1}$ are $*$-homotopic by Lemma 1.2.3. Tensoring on both sides with identity maps, we get that

$$i_+^{p,q+1} i_+^{p+1,q-1} \sim^* i_+^{p,q+1} i_+^{p,q}.$$

Let p', q' as in the statement. Permuting factors in the tensor product $M_{\pm}^{\otimes n+1}$ we obtain a $*$-isomorphism $\sigma : M_{\pm}^{\otimes n+1} \to M_{\pm}^{\otimes n+1}$ such that $\sigma i_+^{p,q+1} = i_+^{p',q'}$. Hence we have

$$i_+^{p',q'} i_+^{p+1,q-1} \sim^* i_+^{p',q'} i_+^{p,q} \tag{1.2.7}$$

for all p, q, p', q' as above. The lemma follows from (1.2.7) using the identity (1.2.6).

□

1.3 Ind-$*$-algebras

Definition 1.3.1. Let $\mathfrak{C}$ be a category. An *ind-object* in $\mathfrak{C}$ is a pair (C, I) consisting of an upward filtered poset I and a functor $C : I \to \mathfrak{C}$. We shall often write C_i for $C(i)$ and $(C_i)_{i\in I}$ or simply $C_\bullet$ for and ind-object $C : I \to \mathfrak{C}$.

The ind-objects of a category $\mathfrak{C}$ form a category $\mathrm{ind} - \mathfrak{C}$ whose morphisms sets are

$$\hom_{\mathrm{ind}-\mathfrak{C}}((C_i), (D_j)) = \varprojlim_i \operatorname*{\underrightarrow{colim}}_j \hom_{\mathfrak{C}}(C_i, D_j).$$

Any functor $F : \mathfrak{C} \to \mathfrak{D}$ extends to $F : \mathrm{ind} - \mathfrak{C} \to \mathrm{ind} - \mathfrak{D}$ by applying F indexwise; $F(C)_i = F(C_i)$.

Example 1.3.2. Let $i^n : M_n \to M_{n+1}$ be upper left corner inclusion and write $M_\bullet$ for the ind-$*$-algebra

$$\mathbb{N}_0 \to Alg_\ell^*$$
$$(n \to n+1) \mapsto (M_n \xrightarrow{i_n} M_{n+1}).$$

Similarly, recall Definition 1.2.4 and write $M_{\pm}^{\bullet}$ for the ind-$*$-algebra.

$$\mathbb{N}_0 \to Alg_\ell^*$$
$$(n \to n+1) \mapsto (M_{\pm}^{n} \xrightarrow{i_+^{0,n}} M_{\pm}^{n+1}).$$

For an infinite set X we write

$$\mathcal{M}_X = M_{\pm}^{\bullet} M_X.$$

Any bijection $f : X \to Y$ induces an isomorphism $f_* : \mathcal{M}_X \to \mathcal{M}_Y$ given by the corresponding isomorphism $M_X \cong M_Y$ and tensoring with the corresponding identities.

Example 1.3.3. For a finite simplicial set K, we write $\operatorname{sd} K$ for the barycentric subdivision. This defines a functor $\operatorname{sd} : \mathfrak{S} \to \mathfrak{S}$. The barycentric subdivision is equipped with a natural transformation $h : \operatorname{sd} \to \operatorname{id}_{\mathfrak{S}}$ so called the last vertex map [GJ99, Chapter III, Section 4, p.193]. Iterating this map, one obtains a system of simplicial sets

$$\cdots \xrightarrow{h} \operatorname{sd}^n K \xrightarrow{h} \operatorname{sd}^{n-1} K \to \cdots \to K.$$

Write $\operatorname{sd}^\bullet K$ for the (contravariant) functor

$$\mathbb{N}_0 \to s\mathbb{S}et$$
$$(n \to n+1) \mapsto (\operatorname{sd}^{n+1} K \xrightarrow{h} \operatorname{sd}^n K).$$

For each $A \in Alg_\ell^*$ the composed functor $A^{\operatorname{sd}^\bullet K}$ gives an ind-$*$-algebra. This construction also applies to pointed simplicial sets in a similar way.

Some particular examples of subdivision ind-$*$-algebras that we will use are

$$A^{\mathbb{S}^1} = A^{\operatorname{sd}^\bullet(S^1,\mathrm{pt})},$$
$$A^{\mathbb{S}^n} = (A^{\mathbb{S}^{n-1}})^{\mathbb{S}^1} \text{and}$$
$$\mathcal{P}A = A^{\operatorname{sd}^\bullet(\Delta,\mathrm{pt})}.$$

Remark 1.3.4. The two endpoint inclusions $\Delta^0 \to \Delta^1$ induce inclusions $\Delta^0 \to \operatorname{sd}^\bullet \Delta^1$ and evaluation maps $\mathrm{ev}_i : A^{\operatorname{sd}^\bullet \Delta^1} \to A^{\Delta^0} = A$. Let $f, g : A \to B$ be two homotopic $*$-morphisms. As such, there exists a chain of $*$-morphisms $f = f_0, f_1, \ldots, f_n = g$ and homotopies $H_i : A \to B[t]$, $i = 0, \ldots, n-1$ as in Definition 1.2.1. These homotopies can then be "concatenated" to an ind-$*$-morphism $H : A \to B^{\operatorname{sd}^\bullet \Delta^1}$. Conversely, it is easily seen that if two $*$-morphisms $f, g : A \to B$ can be recovered from an ind-$*$-morphism $H : A \to B^{\operatorname{sd}^\bullet \Delta^1}$ by composition with the evaluation maps

$$\mathrm{ev}_0 H = f \quad \mathrm{ev}_1 H = g$$

then f and g are homotopic.

Definition 1.3.5. Let $A, B \in \operatorname{ind} - Alg_\ell^*$, we write

$$[A, B] = \hom_{\operatorname{ind}-[Alg_\ell^*]}(A, B).$$

Lemma 1.3.6. *Let X, Y be sets and $f, g : X \to Y$ bijections. Write, $f_*, g_* : \mathcal{M}_X \to \mathcal{M}_Y$ as in Example 1.3.2. Then $[f_*] = [g_*] \in [\mathcal{M}_X, \mathcal{M}_Y]$.*

Proof. Since homotopy is compatible with composition, we can reduce to the case when $X = Y$ and $g = \mathrm{id}_X$. The matrix

$$u = \sum_{x \in X} e_{f(x),x}$$

is a unitary element of Γ_X and f_* is the restriction of $\operatorname{ad}(u)$ (tensored with the identity). Then $i_+ \operatorname{ad}(u) = \operatorname{ad}(u \oplus 1) i_+$. Using [CT07, Section 3.4] there is a homotopy $(v_0, v_1) \in M_2\Gamma_X[t]^2$ of multipliers between $\operatorname{ad}(u \oplus 1)$ and $\operatorname{ad}(1 \oplus u)$; thus, using Lemma 1.2.3, we have that $i_+^{0,2} \operatorname{ad}(u \oplus 1)$ is $*$-homotopic to $i_+^{0,2} \operatorname{ad}(1 \oplus u)$. Hence

$$i_+^{0,2} i_+ = i_+^{0,2} \operatorname{ad}(1 \oplus u) i_+ \sim i_+^{0,2} \operatorname{ad}(u \oplus 1) i_+ = i_+^{0,2} i_+ \operatorname{ad}(u)$$

and $\operatorname{ad}(u)$ induces the identity in $[\mathcal{M}_X, \mathcal{M}_X]$. □

1.4 Extensions

A $*$-algebra can be regarded as a set or an ℓ-module in each case with or without involution. Each of these four choices gives rise to an underlying category $\mathfrak{U}$ and a forgetful functor $F : Alg_\ell^* \to \mathfrak{U}$ which admits a left adjoint $\widetilde{T} : \mathfrak{U} \to Alg_\ell^*$ that is the free $*$-algebra functor for such F. We write $T = \widetilde{T}F$. For the rest of this thesis we will fix one of the four choices as above for $\mathfrak{U}$, F and $\widetilde{T}$.

An *extension* of $*$-algebras is a sequence in Alg_ℓ^*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{1.4.1}$$

where α is an isomorphism onto $\ker \beta$ and $C = \operatorname{im} \beta$.

We say that a surjective $*$-morphism is *split*, if it has a right inverse; we say that a surjective $*$-morphism f is *semi-split* if $F(f)$ has right inverse in $\mathfrak{U}$. We say an extension (1.4.1) is semi-split if β is.

For ind-$*$-algebras, a similar definition applies: a sequence in $\operatorname{ind} - Alg_\ell^*$

$$0 \to (A_i) \xrightarrow{\alpha} (B_j) \xrightarrow{\beta} (C_k) \to 0 \tag{1.4.2}$$

is an extension of ind-$*$-algebras if α a kernel for β and β is a cokernel for α. It is split if β admits a splitting and it is semi-split if $F(\beta)$ admits a splitting in $\operatorname{ind} - \mathfrak{U}$.

Remark 1.4.3. If the underlying category $\mathfrak{U}$ is the category of sets then every extension is semi-split, since every surjective map admits a section.

Remark 1.4.4. If ℓ satisfies the λ-assumption 1.1.24, then for a $*$-morphism $f : A \to B$ for which $F(f)$ admits a splitting s, the splitting can be averaged as $s' = \lambda s + \lambda^* s^*$ in order to have an *involution preserving* splitting. Therefore, in this case, if f admits an ℓ-linear splitting, then it is semi-split for any choice of $\mathfrak{U}$ and F.

Example 1.4.5. Let $A \in Alg_\ell^*$, we call the sequence

$$0 \to PA \to A[t] \xrightarrow{\mathrm{ev}_0} A \to 0 \tag{1.4.6}$$

the path extension. It is split by the inclusion $A \subset A[t]$.

We call the sequence

$$0 \to \Omega A \to PA \xrightarrow{\mathrm{ev}_1} A \to 0 \tag{1.4.7}$$

the loop extension. It admits an involution preserving ℓ-linear splitting $s(a) = ta$.

Let $f : A \to B$ be a $*$-morphism. The *mapping path extension* of f is the extension induced by the pullback of the path extension of B along f

$$\begin{array}{ccccc} \Omega B & \longrightarrow & PB \times_B A & \xrightarrow{\pi_1} & A \\ \| & & \downarrow{\scriptstyle \pi_0} & \ulcorner & \downarrow{\scriptstyle f} \\ \Omega B & \longrightarrow & PB & \xrightarrow{\mathrm{ev}_1} & B. \end{array} \tag{1.4.8}$$

We call $P_f := PB \times_B A$ the *path algebra* of f. The mapping path extension has a natural ℓ-linear involution preserving splitting $s(a) = (tf(a), a)$. There is also natural inclusion $i_f : \ker(f) \to P_f$ given by $i_f(x) = (0, x)$. The same applies to the subdivided version which we write as $\mathcal{P}_f := \mathcal{P}B \times_B A$.

Example 1.4.9. Let X be a set and $A \in Alg_\ell^*$. We call the sequence

$$0 \to M_X A \to \Gamma_X A \to \Sigma_X A \to 0$$

the cone extension. By [CT07, first paragraph of p.92] it admits an ℓ-linear splitting.

Let $f : A \to B$ be a $*$-morphism. The *cone map extension* of f is the extension induced by the pullback of the cone extension of B along $\Sigma_X f$.

$$\begin{array}{ccccc} M_X B & \longrightarrow & \Gamma_X B \times_B \Sigma A & \xrightarrow{\pi_1} & \Sigma_X A \\ \| & & \downarrow{\scriptstyle \pi_0} & \ulcorner & \downarrow{\scriptstyle \Sigma_X f} \\ M_X B & \longrightarrow & \Gamma_X B & \longrightarrow & \Sigma_X B. \end{array} \tag{1.4.10}$$

We call $\Gamma_{X,f} := \Gamma_X B \times_B \Sigma A$ the *cone algebra* of f. The cone map extension has an ℓ-linear splitting given by composing $\Sigma_X f \times \mathrm{id} : \Sigma_X A \to \Sigma_X B \times \Sigma_X A$ and the splitting $\Sigma_X B \to \Gamma_X B$. As before, when $X = \mathbb{N}$ we omit it from notation.

For every algebra morphism $f : A \to B$, the underlying map in $\mathfrak{U}$, $F(f) : F(A) \to F(B)$ induces a map $\widetilde{f} : TA \to B$. In particular, for $\mathrm{id} : A \to A$, we have a natural surjective transformation $\eta_A : T(A) \to A$. Set

$$J(A) := \ker(\eta_A),$$

this defines a functor $J : Alg_\ell^* \to Alg_\ell^*$. The *universal extension* of A is the extension

$$0 \to J(A) \to T(A) \xrightarrow{\eta_A} A \to 0$$

which is semi-split by the natural inclusion $s : A \to T(A)$.

For a semi-split extension

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

and a splitting s of $F(g)$, define $\widehat{\xi} := \eta_B T'(s) : T(C) \to B$. The restriction of $\hat{\xi}$ to $J(C)$ maps to A since

$$g\hat{\xi} = g\eta_B T'(s) = \eta_C T(g) T'(s) = \eta_C.$$

Write ξ for the restriction of $\widehat{\xi}$ to $J(C)$. We call this map the *classifying map of the extension*. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \xi} & & \uparrow{\scriptstyle \hat{\xi}} & & \| & & \\ 0 & \longrightarrow & J(C) & \longrightarrow & T(C) & \xrightarrow{\eta_C} & C & \longrightarrow & 0. \end{array}$$

The definition of the classifying map ξ is clearly dependent of the splitting map s; however, its homotopy class does not depend on s. Let s_1 and s_2 two different splittings of g and ξ_1 and ξ_2 be the corresponding classifying maps. Define $H : F(C) \to F(A[t])$ as

$$H(c) = (1-t)\widehat{\xi_1}(c) + t\widehat{\xi_2}(c).$$

Extend H to a $*$-homomorphism $H : T(C) \to A[t]$ by adjunction. This map is an elementary $*$-homotopy between $\widehat{\xi_1}$ and $\widehat{\xi_2}$ and therefore ξ_1 and ξ_2 are homotopic; thus, the classifying map is natural up to homotopy. This shows the reasoning in calling the universal extension and the classifying map as such.

For an extension of ind-$*$-algebras, the same reasoning applies and thus for any extension of ind-$*$-algebras, there is also a unique classifying map in $\text{ind} - [Alg^*_\ell]$.

Remark 1.4.11. Take the following commutative diagram in Alg^*_ℓ

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

where each row is a semi-split extension. Let ξ be the classifying map associated to the first row extension and ξ' the classifying map associated the second row extension. Due to the uniqueness of the classifying map, the square

$$\begin{array}{ccc} J(C) & \xrightarrow{\xi} & A \\ \downarrow{\scriptstyle J(\gamma)} & & \downarrow{\scriptstyle \alpha} \\ J(C') & \xrightarrow{\xi'} & A' \end{array}$$

is commutative up to homotopy.

Example 1.4.12. Let $A, B \in Alg^*_\ell$ such that B is flat as an ℓ-module. Then, the extension

$$0 \to J(A) \otimes B \to T(A) \otimes B \to A \otimes B \to 0$$

is semi-split and we write the classifying map as

$$\phi_{A,B} : J(A \otimes B) \to J(A) \otimes B.$$

In the case the underlying category $\mathfrak{U}$ is the category of ℓ-modules, this map is natural in A and B (up to homotopy).

Taking $B = \ell^X$ for some simplicial set X, we obtain a map $J(A^X) \to J(A)^X$. Similarly, for a pointed simplicial set (X, x) we obtain a map $J(A^{(X,x)}) \to J(A)^{(X,x)}$.

The loop extension (1.4.7) is a particular case of this setting, taking into account the idenfications at the end of Example 1.1.22. We write the classifying map of the loop extension (1.4.7) as

$$\rho_A : J(A) \to \Omega A. \tag{1.4.13}$$

This map also induces an ind-$*$-algebra map by composing ρ_A with the last vertex map $h_* : \Omega A \to A^{\mathbb{S}^1}$. As an abuse of notation we will write it as $\rho_A : J(A) \to A^{\mathbb{S}^1}$.

For a map $f : A \to B$, the classifying map of the mapping path extension (1.4.8) is $\rho_f := \rho_B \circ J(f)$; this can be seen using Remark 1.4.11. The same applies for the subdivided version.

Example 1.4.14. For each A the sequence

$$0 \to J(A)^{\mathbb{S}^1} \to T(A)^{\mathbb{S}^1} \to A^{\mathbb{S}^1} \to 0$$

is a semi-split extension as in Example 1.4.12. We write

$$\gamma_A : J(A^{\mathbb{S}^1}) \to J(A)^{\mathbb{S}^1} \tag{1.4.15}$$

for the classifying map of said extension. For $m, n \geq 0$, write

$$\gamma_A^{1,n} : J(A^{\mathbb{S}^n}) \to J(A)^{\mathbb{S}^n}$$

for the composition

$$J(A^{\mathbb{S}^n}) \xrightarrow{\gamma_{A^{\mathbb{S}^n}}} J(A^{\mathbb{S}^{n-1}})^{\mathbb{S}^1} \xrightarrow{\gamma_{A^{\mathbb{S}^{n-1}}} \otimes \mathbb{S}^1} J(A^{\mathbb{S}^{n-2}})^{\mathbb{S}^2} \to \cdots \to J(A^{\mathbb{S}^1})^{\mathbb{S}^{n-1}} \xrightarrow{\gamma_A \otimes \mathbb{S}^{n-1}} J(A)^{\mathbb{S}^n},$$

and $\gamma_A^{m,n} : J^m(A^{\mathbb{S}^n}) \to J^m(A)^{\mathbb{S}^n}$ for the composition

$$J^m(A^{\mathbb{S}^n}) \xrightarrow{J^{m-1}(\gamma_A^{1,n})} J^{m-1}(J(A)^{\mathbb{S}^n}) \xrightarrow{J^{m-2}(\gamma_{J(A)}^{1,n})} J^{m-2}(J^2(A)^{\mathbb{S}^n}) \to \cdots \to J(J^{m-1}(A)^{\mathbb{S}^n}) \xrightarrow{\gamma_{J^{m-1}(A)}^{1,n}} J^m(A)^{\mathbb{S}^n}$$

1.5 $*$-Quasi-homomorphisms

Definition 1.5.1. Let $A, B \in Alg_\ell^*$, $C \trianglelefteq B$ a $*$-ideal and $f_+, f_- : A \to B$ two $*$-morphisms. We say that the pair $(f_+, f_-) : A \rightrightarrows B \trianglerighteq C$ is a *$*$-quasi-homomorphism* if $f_+(a) - f_-(a) \in C$ for every $a \in A$. This is equivalent to the following statement: if $\pi : B \to B/C$ is the quotient map, then $\pi f_+ = \pi f_-$.

Example 1.5.2. Recall from Example 1.1.5 the algebras $Q(A)$ and $q(A)$. By definition, there is a $*$-quasi-homomorphism induced by the inclusions $\iota_0, \iota_1 : A \to Q(A)$:

$$(\iota_0, \iota_1) : A \rightrightarrows Q(A) \trianglerighteq q(A).$$

This $*$-quasi-homomorphism is universal in the following sense: let $(f_+, f_-) : A \rightrightarrows B \trianglerighteq C$ be a $*$-quasi-homomorphism. Then there is a natural map $f_+ \amalg f_- : Q(A) \to B$. Since $f_+ \amalg f_-$ maps $q(A)$ into C, we can compose to get

$$f_+ \amalg f_- \circ \iota_0 = f_+ \text{ and}$$
$$f_+ \amalg f_- \circ \iota_1 = f_-.$$

We call the restriction of $f_+ \amalg f_-$ to $f : q(A) \to C$ the *classyfing map* of the $*$-quasi-homomorphism (f_+, f_-).

Let $\mathfrak{C}$ be an abelian category. A functor $H : Alg^*_\ell \to \mathfrak{C}$ is *split-exact* if for every split-exact extension

$$0 \to A \to B \to C \to 0$$

the sequence

$$0 \to H(A) \to H(B) \to H(C) \to 0$$

is exact in $\mathfrak{C}$.

Proposition 1.5.3 ([CMR07, Section 3.1.1])**.** *Let $\mathfrak{C}$ be an abelian category and $E : Alg^*_\ell \to \mathfrak{C}$ a split-exact functor.*

- *For every $*$-quasi-homomorphism $(f_+, f_-) : A \rightrightarrows B \trianglerighteq C$ there exists a morphism*
  $$E(f_+, f_-) : E(A) \to E(C)$$
  induced by $E(f_+) - E(f_-) : E(A) \to E(B)$.
- *$E(f_+, 0) = E(f_+)$.*
- *If $f_+ = f_- + g$ where $g(a)f_-(a) = f_-(a)g(a) = 0$ for every $a \in A$ then $E(f_+, f_-) = E(g)$.*
- *If $f : q(A) \to C$ is the classifying map of (f_+, f_-) then*
  $$E(f_+, f_-) = E(f) \circ E(\iota_0, \iota_1).$$

1.6 Stability

Definition 1.6.1. Let $F_1, F_2 : Alg^*_\ell \to Alg^*_\ell$, $G : Alg^*_\ell \to \mathfrak{C}$ be functors, $i : F_1 \to F_2$ be a natural transformation and $A \in Alg^*_\ell$. We say that the functor G is *i-stable at A* if the map $G(i_A) : G(F_1(A)) \to G(F_2(A))$ is an isomorphism. We say that G is i-stable if it is i-stable at every $A \in Alg^*_\ell$.

Example 1.6.2. A functor F is homotopy invariant as in Definition 1.2.2 if and only if it is stable for the canonical inclusion $A \to A[t]$.

Example 1.6.3. Let X be a set, $x, y \in X$ and $e_{x,y} \in M_X$ the matrix unit

$$e_{x,y}(z,w) = \delta_{(x,y),(z,w)}.$$

There is a natural map $i_x : \mathrm{id}_{Alg^*_\ell} \to M_X$ defined as

$$\begin{aligned} i_{x,A} : A &\to M_X A \\ a &\mapsto e_{x,x} \otimes a. \end{aligned}$$

Lemma 1.6.4. *Let X be a set and i_x be as in Example 1.6.3. If a functor $G : Alg^*_\ell \to \mathfrak{C}$ is i_x-stable for some x then it is i_y stable for any $y \in X$. Moreover $G(i_x) = G(i_y)$ for any $x, y \in X$.*

Proof. We follow [Cor11, Lemma 2.2.4]. There are permutation matrices $\sigma_2, \sigma_3 \in M_X A \otimes M_X A$ of orders two and three such that both conjugate $(i_{x,M_X A} \otimes \mathrm{id}_{M_X A}) i_{x,A}$ into $(i_{x,M_X A} \otimes \mathrm{id}_{M_X A}) i_{y,A}$. Since permutation matrices are unitary, conjugation by σ_2 and σ_3 are $*$-isomorphisms. After applying G we get

$$\begin{aligned} G(\mathrm{ad}(\sigma_2)) G((i_{x,M_X A} \otimes \mathrm{id}_{M_X A}) i_{x,A}) &= G((i_{x,M_X A} \otimes \mathrm{id}_{M_X A}) i_{y,A}) \\ &= G(\mathrm{ad}(\sigma_3)) G((i_{x,M_X A} \otimes \mathrm{id}_{M_X A}) i_{x,A}) \end{aligned} \tag{1.6.5}$$

Since the orders of σ_2 and σ_3 are coprime and all the maps in (1.6.5) are isomorphisms, it follows that $G(\mathrm{ad}(\sigma_2))$ and $G(\mathrm{ad}(\sigma_3))$ are equal to the identity. Furthermore, since $G(i_{x,M_X A} \otimes \mathrm{id}_{M_X A})$ is an isomorphism, we get that $G(i_{x,A}) = G(i_{y,A})$. □

Definition 1.6.6. We say that G is M_X-stable if it is i_x-stable for some (therefore, for any) $x \in X$. In this case we write i_X for any i_x. If the set X is fixed, we simply write i. When X has cardinality n we write $i_n : \mathrm{id} \to M_n$ for i_X.

Lemma 1.6.7. *Let X be a set and $x, y \in X$. Then the maps $i_+ i_x, i_+ i_y : \ell \to M_\pm M_X$ are $*$-homotopic.*

Proof. Assume $x \neq y$ and let $X' = X \setminus \{x, y\}$. Let

$$u = e_{y,x} - e_{x,y} + \sum_{z \in X'} e_{z,z}.$$

It is easily seen that u is unitary in Γ_X and satisfies $\mathrm{ad}(u) i_x = i_y$. Moreover, there is a rotational homotopy $u(t) \in \Gamma_X[t]$ [CT07, Section 3.4] such that $u(0) = 1$ and $u(1) = u$. Then, using Lemma 1.2.3 we obtain the desired statement. □

Lemma 1.6.8. *Let X be a set with at least two elements, $H : Alg^*_\ell \to \mathfrak{C}$ be an M_X-stable functor, $A \subseteq B \in Alg^*_\ell$ and $u \in B$ such that*

$$\begin{gathered} uA, Au^* \subseteq A \text{ and} \\ au^*ua' = aa' \text{ for any } a, a' \in A. \end{gathered}$$

Then $\mathrm{ad}(u) : A \to A$ is a $*$-homomorphism and $H(ad(u)) = \mathrm{id}_{H(A)}$.

Proof. The argument is as in [Cor11, Proposition 2.2.6]. We can assume B is unital (changing B for $\widetilde{B}$). Consider $u \oplus 1 \in M_2B$ and observe that $\operatorname{ad}(u \oplus 1) : M_2A \to M_2A$ is a $*$-homomorphism. Also, if $i_0 : A \to M_2A$ and $i_1 : A \to M_2A$ are the inclusions in the upper left corner and lower right corner respectively then

$$\begin{aligned} &\operatorname{ad}(u \oplus 1)i_0 = i_0 \operatorname{ad}(u) \text{ and} \\ &\operatorname{ad}(u \oplus 1)i_1 = i_1. \end{aligned}$$

Due to Lemma 1.6.4, applying G we get that $G(i_0) = G(i_1)$ are isomorphisms. Therefore,

$$\begin{aligned} G(i_0)G(\operatorname{ad}(u)) &= G(\operatorname{ad}(u \oplus 1))G(i_0) \\ &= G(\operatorname{ad}(u \oplus 1))G(i_1) \\ &= G(i_1) \\ &= G(i_0); \end{aligned}$$

so $G(\operatorname{ad}(u))$ is the identity. □

Lemma 1.6.9. *Let X be a set with at least two elements. Let $\mathfrak{C}$ be category enriched over abelian groups and $H : Alg_\ell^* \to \mathfrak{C}$ be an M_X-stable functor. Then the map for two distinct $x, y \in X$, the map*

$$H(A \oplus A) \xrightarrow{H(i_x \oplus i_y)} H(M_XA) \xrightarrow{H(i_x)^{-1} = H(i_y)^{-1}} H(A)$$

induces the additive operation on $H(A)$,

Proof. Write $D = i_x \oplus i_y : A \oplus A \to M_XA$ and $\nabla : H(A) \oplus H(A) \to H(A)$ for the operation in $\mathfrak{C}$. Using Lemma 1.6.4, the diagram

$$\begin{array}{ccc} H(A) \oplus H(A) & \xrightarrow{inc_1 \oplus inc_2} & H(A \oplus A) \\ \downarrow \nabla & & \downarrow H(D) \\ H(A) & \xrightarrow{H(i_x) = H(i_y)} & H(M_XA) \end{array}$$

commutes □

Lemma 1.6.10. *Let X, Y be two sets such that X has at least two elements and Y has greater cardinality than X. Then, any M_Y-stable functor $G : Alg_\ell^* \to \mathfrak{C}$ is also M_X-stable.*

Proof. Since a bijection between sets induces a $*$-isomorphism between their matrix algebras, we might assume that $X \subseteq Y$. We will prove the lemma in the case the coefficients are $A = \ell$, the same proof applies for any coefficients. Write $\operatorname{inc} : X \hookrightarrow Y$ for the natural inclusion map. Let $x \in X$ and $i = i_x : \ell \to M_X$. Since G is M_Y-stable, $G(\operatorname{inc} \circ i)$ is an isomorphism and therefore, $G(i)$ is an split monomorphism and $G(\operatorname{inc})$ is an split epimorphism.

Let $\tau : M_X \otimes M_Y \to M_Y \to M_X$ defined by $\tau(a \otimes b) = b \otimes a$. We have

$$\tau(i \otimes \mathrm{id}_{M_Y})\mathrm{inc} = \mathrm{inc} \otimes i. \tag{1.6.11}$$

Let $\sigma : Y \times X \to Y \times X$ be any bijection that restricts to coordinate permutation on $X \times \{x\}$. Also write σ for the corresponding permutation matrix in $M_{Y\times X} = M_Y \otimes M_X$. Then we have

$$\mathrm{ad}(\sigma)(\mathrm{inc} \otimes i) = i \otimes \mathrm{id}_{M_X}.$$

Since $G(\mathrm{ad}(\sigma))$ is the identity due to Lemma 1.6.4 and $G(i \otimes \mathrm{id}_{M_X})$ is an isomorphism, it follows that $G(\mathrm{inc}\otimes i)$ is an isomorphism. Using (1.6.11) we get that $G(\mathrm{inc})$ is also an split monomorphism, and therefore an isomorphism. Since $G(\mathrm{inc} \circ i)$ is an isomorphism it follows that $G(i)$ is an isomorphism and that concludes the proof. □

Definition 1.6.12. Let $A \in Alg^*_\ell$ and $G : Alg^*_\ell \to \mathfrak{C}$ be a functor. We say that G is *hermitian stable on* A if for every embedding $A \trianglelefteq R$ as a $*$-ideal in a unital $*$-algebra R, every central unitary element $\varepsilon \in R$ and any two invertible ε-hermitian elements $\phi, \psi \in R$, the functor G maps the upper left corner inclusion

$$i_\phi : A^\phi \to (M_2A)^{(\phi\oplus\psi)}$$

to an isomorphism.

Remark 1.6.13. Taking $\varepsilon = 1$, $R = \widetilde{A}$ and $\phi = \psi = 1$ in the previous definition, we get that any hermitian stable functor is also $i_2 : \mathrm{id} \to M_2$ stable

Remark 1.6.14. Let (P, ψ) and (Q, χ) be hermitian modules as in Example 1.1.17. Using (1.1.18), it follows that a hermitian stable functor G sends the map induced by the inclusion

$$\mathrm{End}(P) \otimes A \to \mathrm{End}(P \oplus Q) \otimes A$$

to an isomorphism.

Proposition 1.6.15 ([cf. Ell14, Proposition 3.1.9])**.** *Suppose ℓ satisfies the λ-assumption 1.1.24 and let $G : Alg^*_\ell \to \mathfrak{C}$ be a M_2-stable functor. Then $G\circ M_\pm$ is hermitian stable.*

Proof. Since ℓ satisfies the λ-assumption, we can use Remark 1.1.26 to get isomorphisms

$$M_\pm A^\phi \cong {}_\varepsilon M_2A \text{ and}$$
$$M_\pm(M_2A)^{(\phi\oplus\psi)} \cong {}_\varepsilon M_2M_2A.$$

Using the commutative diagram

$$\begin{array}{ccccc}
M_\pm A^\phi & \xrightarrow{\sim} & {}_\varepsilon M_2A & & \\
\downarrow{\scriptstyle \mathrm{id}_{M_\pm}\otimes i_\phi} & & \downarrow{\scriptstyle \mathrm{id}_{\varepsilon M_2}\otimes i_2} & \searrow{\scriptstyle i_2} & \\
M_\pm(M_2A)^{(\phi\oplus\psi)} & \xrightarrow{\sim} & {}_\varepsilon M_2M_2A & \xrightarrow{\sim} & M_2({}_\varepsilon M_2A)
\end{array}$$

and the fact that i_2 is mapped to an isomorphism through G, we get that $G \circ M_\pm(i_\phi)$ is an isomorphism as desired. □

Corollary 1.6.16. *Assuming ℓ and $G : Alg^*_\ell \to \mathfrak{C}$ as in Proposition 1.6.15, if G is also i_+-stable, then G is hermitian stable.*

Proof. Since we have the commutative diagram

$$\begin{array}{ccc} A^\phi & \xrightarrow{\quad i_+ \quad} & M_\pm A^\phi \\ \Big\downarrow{\scriptstyle i_\phi} & & \Big\downarrow{\scriptstyle \mathrm{id}_{M_\pm} \otimes i_\phi} \\ (M_2A)^{\phi\oplus\psi} & \xrightarrow{i_+} & M_\pm(M_2A)^{\phi\oplus\psi}, \end{array}$$

using that $G(i_+)$ is an isomorphism and by Proposition 1.6.15 we also have $G(\mathrm{id}_{M_\pm} \otimes i_\phi)$ is an isomorphism we get that $G(i_\phi)$ is an isomorphism. □

Chapter 2

Hermitian Algebraic K-theory

In this chapter we recall the definition of the hermitian algebraic K-theory spectra K^h together with some properties. We also recall the definition of the Karoubi-Villamayor hermitian K-theory KV^h and we construct the analogue to Weibel's homotopy K-theory for the hermitian case KH^h. In Section 2.2 we also recall the product structure of K^h and how it passes to KH^h. Finally in Section 2.3 we recall Karoubi's Fundamental Theorem with some associated reformulations and how it passes to KH^h; we will use this later in Chapter 5.

2.1 Definitions

Let A a $*$-ring. We write

$$\mathcal{U}(A) = \{x \in A : x^*x = xx^*, x + x^* + xx^* = 0\}.$$

The set $\mathcal{U}(A)$ is a group under the operation

$$x \cdot y = x + y + xy.$$

When A is unital, the group $\mathcal{U}(A)$ is isomorphic to the group of unitary elements of A via the map $x \to 1 + x$.

Let R be a unital ring, $A \trianglelefteq R$ a $*$-ideal and $\varepsilon \in R$ central unitary. Put

$${}_\varepsilon\mathcal{O}(A) = \mathcal{U}({}_\varepsilon M_2 M_\infty A).$$

By (1.1.20) we have a group isomorphism

$${}_\varepsilon\mathcal{O}(A) \cong {}_1\mathcal{O}({}_\varepsilon M_2 A). \tag{2.1.1}$$

The *ε-hermitian K-theory groups* of a unital $*$-ring R are the stable homotopy groups of a spectrum ${}_\varepsilon K^h R = \{{}_\varepsilon K^h R_n\}$ whose n-th space is ${}_\varepsilon K^h_n R_n = \Omega B_\varepsilon \mathcal{O}(\Sigma^{n+1} R)^+$, the loopspace of the $+$-construction [see Lod76, Section 3.1.6]. As usual we also write

$${}_\varepsilon K^h_n(R) = \pi_n({}_\varepsilon K^h R) \quad (n \in \mathbb{Z})$$

for the n-th stable homotopy group. When $\varepsilon = 1$ we drop it from the notation. For a nonunital $*$-ring A, we put

$$_{\pm 1}K_n^h(A) = \ker({}_{\pm 1}K_n^h(\tilde{A}_{\mathbb{Z}}) \to {}_{\pm}K_h^h(\mathbb{Z})). \tag{2.1.2}$$

If A is unital, these groups agree with those defined above since in that case $\widetilde{A}_{\mathbb{Z}} \cong A \times \mathbb{Z}$ and using the fact that $+$-construction is additive, the kernel in (2.1.2) recovers $_{\pm 1}K_n^h(A)$.

A ring A is called K-*excisive* if for any embedding $A \trianglelefteq R$ as an ideal of a unital ring R and every unital homomorphism $R \to S$ mapping A isomorphically onto an ideal of S, the map of relative K-theory spectra $K(R : A) \to K(S : A)$ is an equivalence. The definition of a $K^h = {}_1K^h$-excisive $*$-ring is analogous.

Remark 2.1.3. Let A be a K-excisive ring that is a $*$-algebra over ℓ, and suppose that ℓ satisfies the λ-assumption 1.1.24. Let $A \trianglelefteq R$ be a $*$-ideal embedding into a unital $*$-algebra and $f : R \to S$ be a unital $*$-algebra homomorphism mapping and A isomorphically onto a $*$-ideal of S and $\varepsilon \in \ell$ be a central unitary. By [Bat11, Corollary 3.5.1] the map $_{\varepsilon}K^h(R : A) \to {}_{\varepsilon}K^h(S : A)$ is an equivalence. In particular, if A is K-excisive then it is also K^h-excisive. Taking all this into account, and assuming that ℓ satisfies the λ-assumption 1.1.24, we set, for any K-excisive $A \in Alg_{\ell}^{*}$, unitary $\varepsilon \in \ell$ and $n \in \mathbb{Z}$,

$$_{\varepsilon}K_n^h(A) = \ker({}_{\varepsilon}K_n^h(\tilde{A}) \to {}_{\varepsilon}K_n^h(\ell)). \tag{2.1.4}$$

Remark 2.1.5. For $n \leq 0$ and not necessarily K-excisive A, we take (2.1.4) as a definition. The non-positive hermitian K-groups agree with Bass' quadratic K-groups [Bas73] for the maximum form parameter. In particular, by [Bas73, Chapter III, Theorem 1.1] hermitian K-theory as defined above satisfies excision in non-positive dimensions.

Remark 2.1.6. Let R be a unital $*$-ring. Suppose that R has an element λ that satisfies the λ-assumption 1.1.24. Let $S \in \Sigma$ be the class of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Using the fact that the cup product with $[S] \in K_1^h(\Sigma)$ induces an isomorphism $K_0^h(R) \cong K_1^h(\Sigma R)$ [Lod76, Théorème 3.1.7], the group $K_0^h(R)$ can be described as the set of formal differences $[p] - [q]$ where $p, q \in {}_1M_2M_{\infty}R$ are projections and $[p] = [p']$ if there is a unitary matrix $u \in {}_1M_2M_nR$ such that u conjugates p into p' [KV73, Section 2].

For a class $x = [p] - [q] \in K_0^h(R)$ there are $*$-morphisms $p, q : \mathbb{Z} \to {}_1M_2M_{\infty}R$ mapping 1 to p and q respectively. These $*$-morphisms induce maps $p_*, q_* : K_0^h(\mathbb{Z}) \to K_0^h(R)$ sending the class of [1] to $[p]$ and $[q]$ respectively. This implies that the $*$-quasi-homomorphism $(p, q) : \mathbb{Z} \rightrightarrows {}_1M_2M_{\infty}R \trianglerighteq 0$, has an associated map $(p_*, q_*) =$

$p_* - q_* : K_0^h(\mathbb{Z}) \to K_0^h(R)$ which maps the class of [1] to x. This then implies that the set of $*$-quasi-homomorphisms $\{\mathbb{Z} \rightrightarrows {}_1M_2M_\infty R\}$ maps surjectively onto $K_0^h(R)$ sending each pair of $*$-quasi-homomorphisms to their corresponding associated map evaluated at the class [1]. Since K_0^h satisfies excision, it follows that the same applies to any $*$-ring A: the set

$$qq(\mathbb{Z}, A) := \{\mathbb{Z} \rightrightarrows {}_1M_2M_\infty \widetilde{A}_\mathbb{Z} \trianglerighteq {}_1M_2M_\infty A\}$$

maps surjectively to $K_0^h(A)$. If $A \in Alg_\ell^*$ then the same holds with ℓ substituted for $\mathbb{Z}$ and ℓ-linear, $*$-quasi-homomorphisms.

For a $*$-ring A and $\varepsilon = \pm 1$, Karoubi and Villamayor also introduce hermitian K-groups for $n \geq 1$. They agree with the homotopy groups of the simplicial group ${}_\varepsilon\mathcal{O}(A^\Delta)$ up to a degree shift

$$_\varepsilon KV_n^h(A) = \pi_{n-1\varepsilon}\mathcal{O}(A^\Delta) \quad (n \geq 1).$$

The argument of [Cor11, Proposition 10.2.1] shows that the definition above is equivalent to that given in [KV73]; we have

$$_\varepsilon KV_{n+1}^h(A) = {}_\varepsilon KV_1^h(\Omega^n A) \qquad (n \geq 1).$$

Similarly, if A is unital, for all $n \geq 1$ we have

$$_\varepsilon KV_n^h(A) = \pi_n B_\varepsilon\mathcal{O}(A^\Delta) = \pi_n B_\varepsilon\mathcal{O}(A^\Delta)^+ = \pi_n \Omega B_\varepsilon \mathcal{O}(\Sigma A^\Delta)^+. \tag{2.1.7}$$

Applying ${}_\varepsilon K_n^h$ to the path extension (1.4.6) and using excision, we obtain a natural map

$$_\varepsilon K_n^h(A) \to {}_\varepsilon K_{n-1}^h(\Omega A) \quad (n \leq 0).$$

For $n \in \mathbb{Z}$, the n^{th} *homotopy ε-hermitian K-theory* group of A is

$$_\varepsilon KH_n^h(A) = \underset{m \geq n}{\underrightarrow{\operatorname{colim}}}\, {}_\varepsilon K_{-m}^h(\Omega^{m+n} A).$$

Remark 2.1.8. One can also describe ${}_\varepsilon KH_n^h$ in terms of ${}_\varepsilon KV^h$; by [KV73, Théorème 4.1], ${}_\varepsilon KV^h$ satisfies excision for the cone extension (1.4.10). Hence we have a map

$$_\varepsilon KV_n^h(A) \to {}_\varepsilon KV_{n+1}^h(\Sigma A).$$

The argument of [CT07, Proposition 8.1.1] shows that

$$_\varepsilon KH_n^h(A) = \underset{m}{\underrightarrow{\operatorname{colim}}}\, {}_\varepsilon KV_{n+m}^h(\Sigma^m A).$$

Now assume that A is unital; let ${}_\varepsilon KH(A)$ be the total spectrum of the simplicial spectrum ${}_\varepsilon K^h(A^\Delta)$. We have

$$\pi_n({}_\varepsilon KH^h(A)) = \underset{m}{\underrightarrow{\operatorname{colim}}}\, \pi_{n+m}\Omega B_\varepsilon O(\Sigma^m A^\Delta)^+ = \underset{n}{\underrightarrow{\operatorname{colim}}}\, {}_\varepsilon KV_{n+m}^h(\Sigma^m A) = {}_\varepsilon KH_n^h(A).$$

For any exact sequence

$$0 \to A \to B \to B/A \to 0$$

there is a natural *index map* $\partial : K_1^h(B/A) \to K_0^h(A)$ [see Bas73, Chapter III].

Remark 2.1.9. There is a natural comparison map $c_m : K_m^h(A) \to KH_m^h(A)$. For $m \leq 0$ this is just mapping to the colimit. For $m > 0$ and A unital, using the description of (2.1.7) and the natural inclusion $A \to A^\Delta$, we get a comparison map $c'_m : K_m^h(A) \to KV_m^h$, then, by Remark 2.1.8, the comparison map factors

$$K_m^h(A) \xrightarrow{c'_m} KV_m^h(A) \to KH_m^h(A)$$

Using repeatedly the index map of the loop extension (1.4.7) we get maps

$$K_m^h(A) \to KV_m^h(A) \cong KV_1(\Omega^{m-1}A) \to KV_0^h(\Omega^m A) = K_0(\Omega^m A).$$

Finally, composing with the comparison map c_0 we arrive at $KH_0^h(\Omega^m A) \cong KH_m^h(A)$.

Lemma 2.1.10. *Homotopy hermitian K-theory is homotopy invariant, matricially stable and satisfies excision.*

Proof. The proof is the same as in non-hermitian K-theory [see Cor11, Theorem 5.1.1]. □

Lemma 2.1.11. *Let $\varepsilon \in \ell$ be unitary. If either $n \leq 0$ or A is K^h-excisive, then there is a canonical isomorphism*

$$_\varepsilon K_n^h(A) \cong K_n^h({}_\varepsilon M_2 A).$$

Moreover for all $A \in Alg_\ell^*$ we have a canonical isomorphism

$$_\varepsilon KH_n^h(A) \cong KH_n^h({}_\varepsilon M_2 A) \qquad (n \in \mathbb{Z}).$$

Proof. The isomorphism (2.1.1) is canonical up to the choices of an element $\lambda \in \ell$ in the λ-assumption 1.1.24 and a bijection $\{1,2\} \times X \to X$. By [Lod76, Lemme 1.2.7], if A is unital, then varying those choices has no effect on the homotopy type of the induced isomorphism $B_\varepsilon \mathcal{O}(A)^+ \cong B_1 \mathcal{O}({}_\varepsilon M_2 A)^+$. Applying this to $\Sigma^r A$ we obtain the statement of the lemma for unital A. The nonunital case follows from the unital one using split-exactness. The statement for $_\varepsilon KH^h$ follows by applying the former case for $\Omega^r A$ and from the definition. □

2.2 Cup products in KH^h

Hermitian K-theory of unital $*$-rings is equipped with products [Lod76, Chapitre III]. Using that K^h satisfies excision in nonpositive dimensions we obtain, for $R, A \in Alg_\ell^*$ with R unital, $m \in \mathbb{Z}$ and $n \leq 0$, a natural product

$$K_m^h(R) \otimes_{\mathbb{Z}} K_n^h(A) \xrightarrow{\star} K_{m+n}^h(R \otimes A). \tag{2.2.1}$$

If moreover $m \leq 0$, we also obtain the product above for not necessarily unital R.

Remark 2.2.2. Using Lemma 2.1.11, the product (2.2.1) also gives a product

$$_{\varepsilon}K^h_m(R) \otimes_{\mathbb{Z}} {}_{\eta}K^h_n(A) \xrightarrow{\star} {}_{\varepsilon\eta}K^h_{m+n}(R \otimes A). \tag{2.2.3}$$

Remark 2.2.4. Let R, S be unital $*$-rings that satisfy λ-assumption 1.1.24, using the description of Remark 2.1.6, the cup product

$$K^h_0(R) \otimes_{\mathbb{Z}} K^h_0(S) \xrightarrow{\star} K^h_0(R \otimes S)$$

corresponds to the natural extension of scalars of projections [cf. Lod76, Section 3.1.4]: for $[p] \in \mathcal{V}^h_\infty R$ and $[q] \in \mathcal{V}^h_\infty S$

$$[p] \star [q] = [p \otimes q].$$

Lemma 2.2.5. *Let $R, S \in Alg^*_\ell$ be unital that satisfy the λ-assumption 1.1.24 and let $I \trianglelefteq S$ be a $*$-ideal. Assume that the sequence*

$$0 \to R \otimes I \to R \otimes S \to R \otimes (S/I) \to 0$$

is exact and let ∂ be the associated index map. Then the following diagram commutes

$$\begin{array}{ccc} K^h_0(R) \otimes_{\mathbb{Z}} K^h_1(S/I) & \xrightarrow{\star} & K^h_1(R \otimes (S/I)) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\partial} & & \downarrow{\scriptstyle \partial} \\ K^h_0(R) \otimes_{\mathbb{Z}} K^h_0(I) & \xrightarrow{\star} & K^h_0(R \otimes I). \end{array}$$

Proof. Because R is unital and satisfies the λ-assumption, we may regard $K^h_0(R)$ as the group completion of the monoid $\mathcal{V}^h_\infty(R)$ as in Remark 2.1.6. If $g \in {}_1M_2M_n(S/I)$ is unitary, $p \in {}_1M_2M_nR$ is a self-adjoint idempotent and $\mathbb{1}_n \in {}_1M_2M_n$ is the identity matrix, then (see [Wei13, Corollary 1.6.1] for the non-hermitian case)

$$[p] \star [g] = [p \otimes g + (\mathbb{1}_n - p) \otimes \mathbb{1}_n] \in K^h_1(R \otimes (S/I)) \tag{2.2.6}$$

On the other hand, for any lift $h \in \mathcal{U}({}_1M_2M_{2n}S)$ of $g \oplus g^{-1}$ we have

$$\partial[g] = [h\mathbb{1}_n h^{-1}] - [\mathbb{1}_n].$$

Choosing the lift for (2.2.6) as

$$p \otimes h + (\mathbb{1}_{2n} - (p \oplus p)) \otimes \mathbb{1}_{2n}$$

we obtain $\partial([p] \star [g]) = [p] \star \partial[g]$. ☐

Lemma 2.2.7. *Suppose ℓ satisfies the λ-assumption 1.1.24. Let $m \in \mathbb{Z}$, $n \leq 0$ and $R, A \in Alg^*_\ell$ with R unital. Let ∂ be the connecting map associated to the path extension* (1.4.6). *Assume that* $\max\{n, m+n\} \leq 0$. *Then the following diagram commutes.*

$$\begin{array}{ccc} K^h_m(R) \otimes_{\mathbb{Z}} K^h_n(A) & \xrightarrow{\star} & K^h_{m+n}(R \otimes A) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\partial} & & \downarrow{\scriptstyle \partial} \\ K^h_m(R) \otimes_{\mathbb{Z}} K^h_{n-1}(\Omega A) & \xrightarrow{\star} & K^h_{m+n-1}(R \otimes \Omega A) \end{array}$$

Proof. Let $\jmath_2 : \ell \to \ell \oplus \ell$ be the inclusion in the second summand. The path and loop extensions, (1.4.6) and (1.4.7) respectively, are connected by a map of extensions

$$\begin{array}{ccccc} \Omega & \longrightarrow & P & \xrightarrow{\mathrm{ev}_1} & \ell \\ \| & & \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \jmath_2} \\ \Omega & \longrightarrow & \ell[t] & \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} & \ell \oplus \ell \end{array}$$

Let $i \leq 0$. Applying Lemma 2.2.5 with $S = \Sigma[t]$, $I = \Sigma\Omega$ and $R = \Sigma^{-i}\tilde{A}$, and using naturality and excision, we obtain that the boundary map $\partial : K_i^h(A) \to K_{i-1}^h(\Omega A)$ is the cup product with $\partial([1]) \in K_{-1}^h(\Omega)$. The proof now follows from associativity of $\star$. □

Corollary 2.2.8. *Suppose ℓ satisfies the λ-assumption 1.1.24. Let $R, A \in Alg_\ell^*$ with R unital and let $m, n \in \mathbb{Z}$.*

i) There is an associative product

$$\star : K_m^h(R) \otimes_{\mathbb{Z}} KH_n^h(A) \to KH_{m+n}^h(R \otimes A).$$

ii) Let $c_* : K_*^h(R) \to KH_*^h(R)$ be the comparison map. Then for all $m \in \mathbb{Z}$ and $\xi \in K_m^h(R)$, $c_m(\xi) = \xi \star c_0([1])$.

Proof. Part i) is immediate from Lemma 2.2.7 upon taking colimits. For $m \leq 0$, part ii) is clear from the construction of $\star$ and the definition of KH^h. For $m > 0$, this follows from Remark 2.1.9 and the fact that since $KV_{-1}^h = K_{-1}^h$, the diagram

$$\begin{array}{ccc} K_*^h(R) \otimes_{\mathbb{Z}} K_{-1}^h(\Omega) & \xrightarrow{\star} & K_{*-1}^h(\Omega R) \\ \downarrow{\scriptstyle c'_* \otimes \mathrm{id}} & & \downarrow{\scriptstyle c'_*} \\ KV_*^h(R) \otimes_{\mathbb{Z}} KV_{-1}^h(\Omega) & \xrightarrow{\star} & KV_{*-1}^h(\Omega R) \end{array}$$

commutes. □

Lemma 2.2.9. *Suppose ℓ satisfies the λ-assumption 1.1.24. Let $A, B \in Alg_\ell^*$ and $m, n \in \mathbb{Z}$. Then* (2.2.1) *induces an associative product*

$$KH_m^h(A) \otimes_{\mathbb{Z}} KH_n^h(B) \xrightarrow{\star} KH_{m+n}^h(A \otimes B).$$

If $m \leq 0$ or A is unital, then the following diagram commutes

$$\begin{array}{ccc} K_m^h(A) \otimes_{\mathbb{Z}} KH_n^h(B) & \xrightarrow{\star} & KH_{m+n}^h(B) \\ \downarrow{\scriptstyle c_m \otimes 1} & \nearrow{\scriptstyle \star} & \\ KH_m^h(A) \otimes_{\mathbb{Z}} KH_n^h(B) & & \end{array}$$

Proof. Lemma 2.2.7 shows that the boundary map $\partial : K^h_* \to K^h_{*-1} \circ \Omega$ is the cup product with $\partial([1]) \in K^h_{-1}(\Omega)$. It follows that for all $r \leq 0$, the following diagram commutes:

$$\begin{array}{ccc} K^h_r(A) \otimes_{\mathbb{Z}} K^h_r(B) & \xrightarrow{\quad\star\quad} & K^h_{2r}(A \otimes B) \\ \big\downarrow{\scriptstyle \partial\otimes\partial} & & \big\downarrow{\scriptstyle \partial^2} \\ K^h_{r-1}(\Omega A) \otimes_{\mathbb{Z}} K^h_{r-1}(\Omega B) & \xrightarrow{\star} & K^h_{2r-2}(\Omega^2 A \otimes B). \end{array}$$

Taking colimit along the columns we get the desired product map for $r = s = 0$. The general case is obtained from the latter applying the suspension and loop functors as many times as appropriate. Commutativity of the diagram in the statement follows from Corollary 2.2.8. $\square$

Corollary 2.2.10. *Let $A \in Alg^*_\ell$ and $n \in \mathbb{Z}$, then ${}_\varepsilon KH^h_n(A)$ is a $KH^h_0(\ell)$-module with the action induced by the product in Lemma 2.2.9.*

2.3 Karoubi's Fundamental Theorem

Let $A \in Rings^*$ and consider $\widehat{A} = \mathrm{ind}(\mathrm{res}(A)) = A \oplus A^{op}$ as in Example 1.1.6. There are natural $*$-morphisms

$$\begin{aligned} \phi_A : \widehat{A} &\to M_2(A) \\ (a,b) &\mapsto \begin{pmatrix} a & 0 \\ 0 & b^* \end{pmatrix}, \end{aligned} \tag{2.3.1}$$

$$\begin{aligned} \eta_A : A &\to \widehat{A} \\ a &\mapsto (a, a^*). \end{aligned} \tag{2.3.2}$$

Write $U'A = \Gamma_{\phi_A}$ and $V'A = \Gamma_{\eta_A}$ as in Example 1.4.9. This defines functors $U', V' : Rings^* \to Rings^*$ and write $(U')^n, (V')^n$ $(n \geq 0)$ for their repeated composition. As in Example 1.4.9, there are natural maps $U'A \to \Sigma\widehat{A}$ and $V'A \to \Sigma A$. The projection on the first coordinate $\widehat{A} \to A$ is *not* a $*$-morphism but is a ring morphism and as such it induces a map $K(\widehat{A}) \to K(A)$. Since for a unital ring (not necessarily with involution) $\mathcal{U}(\mathrm{inv}(R)) = \mathrm{GL}(R)$, we have that ${}_\varepsilon K^h(\mathrm{inv}(R)) \sim K(R)$ and therefore ${}_\varepsilon K^h(\widehat{R}) \sim K(R)$. It follows using the cone extension from Example 1.4.9 that there are maps

$$\Omega_\varepsilon K^h(U'R) \to K(R) \xrightarrow{(\phi_R)_*} {}_\varepsilon K^h(R)$$

$$\Omega_\varepsilon K^h(V'R) \to {}_\varepsilon K^h(R) \xrightarrow{(\eta_R)_*} K(R)$$

and that $\Omega_\varepsilon K^h(U'R)$ and $\Omega_\varepsilon K^h(V'R)$ are the homotopic fibers of the maps $(\phi_R)_*$ and $(\eta_R)_*$ respectively.

Theorem 2.3.3 (Karoubi, [Kar80]). *There is an element $\theta_0 \in {}_{-1}K_2^h((U')^2\mathbb{Z})$ such that:*

i) The composite

$$ {}_{-1}K_2^h((U')^2\mathbb{Z}) \to {}_{-1}K_2^h(\Sigma\widehat{U'\mathbb{Z}}) \cong {}_{-1}K_1^h(\widehat{U'\mathbb{Z}}) \cong K_1(U'\mathbb{Z}) \to K_1(\Sigma\widehat{\mathbb{Z}}) \cong K_0(\widehat{\mathbb{Z}}) \xrightarrow{pr_1} K_0(\mathbb{Z}) = \mathbb{Z} $$

maps θ_0 to 1.

ii) Assume that ℓ-satisfies the λ-assumption 1.1.24. Then, for every unital $*$-ℓ-algebra R, the product with θ_0 induces an isomorphism

$$ \theta_0 \star - : {}_{\varepsilon}K_*^h(R) \cong {}_{-\varepsilon}K_{*+2}^h((U')^2R). $$

Proof. The element θ_0 of the present theorem appears under the name of σ in the first line of [Kar80, Section 3.1]. Using the identifications

$$ \Omega_{\varepsilon}K^h(U'V'R) \sim \Omega_{\varepsilon}K^h(V'U'R) \sim {}_{\varepsilon}K^h(R) \tag{2.3.4} $$

as mentioned by Karoubi in [Kar80, Section 1.4], the current theorem is just another way of phrasing Karoubi's fundamental theorem

$$ {}_{\varepsilon}K^h(V'R) \sim \Omega_{-\varepsilon}K^h(U'R). $$

Furthermore, the theorem as stated here is equivalent to that proved in [Kar80, Section 3.5], which says that product with θ_0 induces an isomorphism

$$ {}_{\varepsilon}K_*^h(V'R) \cong {}_{-\varepsilon}K_{*+1}^h(U'R). $$ □

Remark 2.3.5. Using Lemma 2.1.11, the Theorem 2.3.3 is equivalent to the statement that θ_0 induces an isomorphism

$$ \theta_0 \star - : K_*^h(R) \cong {}_{-1}K_{*+2}^h((U')^2R). $$

Corollary 2.3.6. *Let $A \in Alg_{\ell}^*$ and assume that ℓ satisfies the λ-assumption 1.1.24. The element $\theta = c_2(\theta_0) \in {}_{-1}KH_2^h((U')^2\mathbb{Z})$ induces an isomorphism*

$$ \theta \star - : {}_{\varepsilon}KH_*^h(A) \to {}_{-\varepsilon}KH_{*+2}^h((U')^2A) $$

Proof. Using that K_n^h satisfies excision for $n \leq 0$ and Theorem 2.3.3 we get that for any $A \in Alg_{\ell}^*$, θ_0 induces an isomorphism

$$ \theta_0 : {}_{\varepsilon}K_*^h(A) \cong {}_{-\varepsilon}K_{*+2}^h((U')^2A). \qquad (* \leq -2) $$

This then follows from Corollary 2.2.8 upon taking colimits. □

The 12-term exact sequence

Definition 2.3.7. Let R be a unital $*$-ring. The involution of R induces an involution $g \to (g^*)^{-1}$ in $\mathrm{GL}_\infty(R)$ which in turn induces a natural action of $\mathbb{Z}/2$ in $K_*(R)$; for $x \in K_n(R)$ write $\overline{x}$ for this action. Define

$$\begin{aligned} {}_\varepsilon W_n(R) &:= \operatorname{coker}(K_n(R) \xrightarrow{(\phi_R)_*} {}_\varepsilon K_n^h(R)), \\ {}_\varepsilon W'_n(R) &:= \ker({}_\varepsilon K_n^h(R) \xrightarrow{(\eta_R)_*} K_n(R)), \\ k_n(R) &:= \{x \in K_n(R) : \overline{x} = x\}/\{x = y + \overline{y} \text{ for some } y\}, \text{ and} \\ k'_n(R) &:= \{x \in K_n(R) : \overline{x} = -x\}/\{x = y - \overline{y} \text{ for some } y\}. \end{aligned}$$

The groups ${}_\varepsilon W_n(R)$ and ${}_\varepsilon W'_n(R)$ are called the Witt and coWitt groups of R. The groups $k_n(R)$ and $k'_n(R)$ are the corresponding $\mathbb{Z}/2$-Tate cohomology groups of $K_n(R)$.

Theorem 2.3.8 (Suite exacte des douze, Karoubi [Kar80, Theoreme 4.3])**.** *Assume ℓ satisfies the λ-assumption 1.1.24 and let $R \in Alg^*_\ell$ be a unital $*$-algebra. There is an exact sequence*

$$\begin{array}{ccccccccccc} k_{n+1}(R) & \longrightarrow & {}_{-\varepsilon}W_{n+2}(R) & \longrightarrow & {}_\varepsilon W'_n(R) & \longrightarrow & k'_{n+1}(R) & \longrightarrow & {}_{-\varepsilon}W'_{n+1}(R) & \rightarrow & {}_{-\varepsilon}W_n(R) \\ \uparrow & & & & & & & & & & \downarrow \\ W_{n+1}(R) & \longleftarrow & W'_{n+1}(R) & \longleftarrow & k'_{n+1}(R) & \leftarrow & {}_{-\varepsilon}W'_n(R) & \longleftarrow & W_{n+2}(R) & \longleftarrow & k_{n+1}(R) \end{array}$$

Chapter 3

Bivariant Hermitian Algebraic K-theory

In this chapter we construct the bivariant hermitian algebraic K-theory category and develop some of its basic properties. This construction is based on the original bivariant algebraic K-theory $j : Alg_\ell \to kk$ made by Cortiñas and Thom in [CT07]. There are generalizations of kk to incorporate the action of groups and group graded algebras [Ell14] and also for algebras with actions of quantum groups [Ell18]. In Section 3.1 we develop the necessary results to construct kk^h as a category and the functor $j^h : Alg_\ell^* \to kk^h$. Then in Section 3.2 we show it is triangulated and prove how $j^h : Alg_\ell^* \to kk^h$ is the universal excisive homology theory (defined in such section) with matrix and hermitian stability and homotopy invariance.

From this chapter on we will assume that ℓ satisfies the λ-assumption 1.1.24 without further mention.

3.1 The kk^h category

Fix an infinte set X. A bijection $X \amalg X \cong X$ induces a $*$-homomorphism $\mathcal{M}_X \oplus \mathcal{M}_X \to \mathcal{M}_X$; write $\boxplus$ for its ind-$*$-homotopy class. By Lemma 1.3.6, $\boxplus$ is independent of the choice of bijection above.

Lemma 3.1.1 ([cf. CT07, Section 4.1])**.** *The map* $\boxplus$ *together with the zero map, makes* $\mathcal{M}_X$ *an abelian monoid object in* $\operatorname{ind} - [Alg_\ell^*]$

Proof. Since any chosen bijection $X \amalg X \cong X$ also induces a bijection $X \amalg X \amalg X \cong X$ in any possible association and these choices induce the same class in $\operatorname{ind} - [Alg_\ell^*]$, it is clear that $\boxplus$ is associative. Similarly, the permutation of copies of X in $X \amalg X$ induce the same isomorphism as $\boxplus$ and therefore it is commutative.

Let $X_0, X_1 \subseteq X$ be the corresponding subsets to $X \amalg \emptyset$ and $\emptyset \amalg X$ through the bijection $X \amalg X \cong X$. Write $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ the corresponding bijections. Then, we have

$$[(f_0 \amalg \emptyset)^*](\mathrm{id} \boxplus 0) = \mathrm{id}_{\mathcal{M}_X}$$
$$[(\emptyset \amalg f_1)^*](0 \boxplus \mathrm{id}) = \mathrm{id}_{\mathcal{M}_X}.$$

Therefore the zero map is a neutral element for $\boxplus$. □

Similarly, any choice of bijection $X \times X \cong X$ gives rise to the same ind-$*$-homotopy class of a $*$-homomorphism $\mathcal{M}_X \otimes \mathcal{M}_X \to \mathcal{M}_X$; we write μ for this ind-$*$-homotopy class.

Lemma 3.1.2. *The map μ is an associative and commutative product in* $\mathrm{ind}-[Alg^*_\ell]$ *and the inclusion $i : \ell \to \mathcal{M}_X$ is an identity map for μ. Furthermore μ distributes over $\boxplus$ and therefore $(\mathcal{M}_X, \boxplus, \mu, 0, [i])$ is a semi-ring object in* $\mathrm{Ind} - [Alg^*_\ell]$.

Proof. Associativity and commutativity are proven in the same way as in the previous lemma and it is clear that i is an identity for μ. Finally, the fact that μ distributes over $\boxplus$ can be derived from the fact that there is a natural bijection

$$X \times (X \amalg X) \cong (X \times X) \amalg (X \times X)$$

and Lemma 1.3.6. □

Let $A, B \in \mathrm{ind} - Alg^*_\ell$. Put

$$\{A, B\} := [A, \mathcal{M}_X B]; \tag{3.1.3}$$

the monoid operation $\boxplus$ on $\mathcal{M}_X$ induces one on $\{A, B\}$.

Lemma 3.1.4. *The product μ induces a bilinear, associative composition law:*

$$\begin{aligned} \star : \{B, C\} \times \{A, B\} &\to \{A, C\} \\ ([f], [g]) &\mapsto [\mu \otimes \mathrm{id}_C] \circ [(\mathrm{id}_{\mathcal{M}_X} \otimes f)] \circ [g]. \end{aligned}$$

Proof. Since changing the representative of the class $[f]$ does not change the class of $[\mathrm{id}_{\mathcal{M}_X} \otimes f]$, it is clear that $\star$ is well defined. The fact that $\star$ is bilinear follows from the fact that μ distributes over $\boxplus$. Finally associativity follows from observing that for any map $h : C \to \mathcal{M}_X D$, the diagram

$$\begin{array}{ccccc} \mathcal{M}_X\mathcal{M}_X C & \xrightarrow{\mathrm{id}_{\mathcal{M}_X}\otimes \mathrm{id}_{\mathcal{M}_X}\otimes h} & \mathcal{M}_X\mathcal{M}_X\mathcal{M}_X D & \xrightarrow{\mathrm{id}_{\mathcal{M}_X}\otimes \mu \otimes \mathrm{id}_D} & \mathcal{M}_X\mathcal{M}_X D \\ {\scriptstyle \mu\otimes \mathrm{id}_D}\downarrow & & & & \downarrow{\scriptstyle \mu\otimes \mathrm{id}_D} \\ \mathcal{M}_X C & \xrightarrow{\mathrm{id}_{\mathcal{M}_X}\otimes h} & \mathcal{M}_X\mathcal{M}_X D & \xrightarrow{\mu\otimes \mathrm{id}_D} & \mathcal{M}_X D \end{array}$$

commutes due to the associativity of μ. □

Definition 3.1.5. Let $\{\mathrm{ind} - Alg^*_\ell\}_X$ be the category with the same objects as $\mathrm{ind} - Alg^*_\ell$, where morphisms sets are given by (3.1.3) and which is enriched over the category of abelian monoids. Lemma 3.1.2 also shows that for $A \in \mathrm{ind} - Alg^*_\ell$ the inclusion $i : A \to \mathcal{M}_X A$ is the identity. Write $\{Alg^*_\ell\}_X$ for full subcategory of $\{\mathrm{ind} - Alg^*_\ell\}_X$ where the objects are in Alg^*_ℓ instead of $\mathrm{ind} - Alg^*_\ell$.

Remark 3.1.6. Let $A, B \in Alg^*_\ell$. The algebra B^Δ has natural binary operation called concatenation $\bullet : B^\Delta \times B^\Delta \to B^\Delta$; this induces a binary operation in $[A, B^{\mathbb{S}^1}]$: for maps $f, g : A \to B^{\mathbb{S}^1}$ we write $f \bullet g$ for their concatenation. The zero map is a neutral element for this operation and the reversing map

$$\begin{aligned} \ell[t] &\to \ell[t] \\ t &\mapsto 1-t \end{aligned} \tag{3.1.7}$$

induces a $*$-morphism $a : B^{\mathbb{S}^1} \to B^{\mathbb{S}^1}$ such that $[f \bullet af] = [0]$. Concatenation and $\boxplus$ distribute over each other in $\{A, B^{\mathbb{S}^1}\}$ [see CT07, Section 3.3].

Lemma 3.1.8. *Let $A, B \in Alg^*_\ell$. For $n \geq 1$, the concatenation and $\boxplus$ operations coincide in $\{A, B^{\mathbb{S}^n}\}$ and it is an abelian group with such operation.*

Proof. As said in Remark 3.1.6, $\bullet$ and $\boxplus$ distribute over each other, due to the Eckmann-Hilton argument, both operations coincide. Since concatenation has an inverse as discussed in the same remark, the abelian monoid $\{A, B^{\mathbb{S}^n}\}$ is a group. □

There is a canonical functor $[Alg^*_\ell] \to \{Alg^*_\ell\}$, which is the identity on objects and sends the class of a map f to that of if.

Lemma 3.1.9. *The composite functor* can : $Alg^*_\ell \to [Alg^*_\ell] \to \{Alg^*_\ell\}$ *is homotopy invariant, M_X-stable and i_+-stable. Moreover any functor $H : Alg^*_\ell \to \mathfrak{C}$ which is homotopy invariant M_X-stable and i_+-stable, factors uniquely through* can.

Proof. Since can factors through $[Alg^*_\ell]$, it is homotopy invariant by definition. Moreover for any functor H as in the statement, H factors through $Alg^*_\ell \to [Alg^*_\ell]$.

To see M_X stability, for any $x \in X$, the inclusions $i_{x,A} : A \to M_X A$ maps to $i_x, A : A \to M_X A$ in $\{Alg^*_\ell\}$. The identity map $M_X A \to M_X A$ induces a map $M_X A \to A$ in $\{Alg^*_\ell\}$ using the isomorphism $M_X M_X \cong M_X$. It is immediate that these maps are inverses to each another. induces the identity in $\{Alg^*_\ell\}$ so can is M_X-stable. Similarly, since the ind-system $\mathcal{M}_X$ is built with repeated composition of i_+, using Lemma 1.2.5 we get that it is i_+ stable.

Finally, for a functor H as in the statement of the lemma, as said before H factors through $[H] : [Alg^*_\ell] \to \mathfrak{C}$. Since H is M_X-stable and i_+-stable, for any $B \in Alg^*_\ell$, the map $[H](i_B : B \to \mathcal{M}_X B)$ is an isomorphism in $\mathfrak{C}$, so we can define

$$\{H\}([f : A \to \mathcal{M}_X B]) = [H](i_B)^{-1} \circ [H]([f]).$$

It is easy to see that $\{H\}$ defines a functor $\{H\} : \{Alg^*_\ell\} \to \mathfrak{C}$ that factors H through can. □

Lemma 3.1.10. *The canonical functor* can : $Alg^*_\ell \to \{Alg_\ell\}$ *is hermitian stable.*

Proof. Since ℓ satisfies the λ-assumption 1.1.24, the proof follows from Lemma 3.1.9 and Corollary 1.6.16. □

Lemma 3.1.11. *Let R be a unital $*$-algebra, $A \trianglelefteq R$ a $*$-ideal and $\lambda_1, \lambda_2 \in R$ be central elements satisfying the requirements of the element λ in the λ-assumption 1.1.24. Let*

$$p_i = p_{\lambda_i} = \begin{pmatrix} \lambda_i^* & 1 \\ \lambda_i\lambda_i^* & \lambda_i \end{pmatrix}$$

and let $\iota_i : A \to {}_1M_2A$, $\iota_i(a) = p_i a$. Then $\operatorname{can}(\iota_1) = \operatorname{can}(\iota_2)$ *is an isomorphism in* $\{Alg_\ell^*\}$.

Proof. Let $u_i = u_{\lambda_i}$ be as in (1.1.27) of Remark 1.1.26. Under the isomorphism ${}_1M_2 \cong M_\pm$, ι_i corresponds to ι_+. Thus $\operatorname{can}(\iota_i)$ is an isomorphism. Moreover, since $u = u_2u_1^{-1} \in {}_1M_2R$ is unitary, $\operatorname{can}(\operatorname{ad}(u)) = \operatorname{id}_{{}_1M_2A}$ by Lemma 1.6.8, we get

$$\operatorname{can}(\iota_2) = \operatorname{can}(\operatorname{ad}(u_2u_1^{-1}))\operatorname{can}(\iota_1) = \operatorname{can}(\iota_1).$$ □

Lemma 3.1.12. *The functor $J : Alg_\ell^* \to Alg_\ell^*$ passes down to a functor $J : \{Alg_\ell^*\} \to \{Alg_\ell^*\}$.*

Proof. For a map $[f] \in [A, B]$, it easy to check using the universal extension that the class $[J(f)] \in [J(A), J(B)]$ does not depend on the representative of the class f.

Recall the map $\phi_{M_X,B} : J(M_XB) \to M_XJ(B)$ from Example 1.4.12. This induces a map $[\phi] \in \{J(\mathcal{M}_XB), B\}$. For a map $\xi = [f] \in \{A, B\}$, define $J(\xi) \in \{A, B\}$ as the class of the composition

$$J(A) \xrightarrow{J(f)} J(\mathcal{M}_XB) \xrightarrow{\phi} \mathcal{M}_XJ(B).$$

Using Remark 1.4.11, it is clear that this defines a functor. □

From here on, we shall abuse notation and use the same letter for the homotopy class of a map $f : A \to B \in \operatorname{ind} - Alg_\ell^*$ and for its image in $\{A, B\}$, and in case the latter is an abelian group (e.g. if $B = C^{\mathbb{S}^n}$) we put $-f$ for the inverse of $\operatorname{can}(f)$ in that group.

Lemma 3.1.13. *Let $A, B \in \operatorname{ind} - Alg_\ell^*$ and $f \in [A, B]$. The the square*

$$\begin{array}{ccc} J(A) & \xrightarrow{J(f)} & J(B) \\ \downarrow{\scriptstyle \rho_A} & & \downarrow{\scriptstyle \rho_B} \\ A^{\mathbb{S}^1} & \xrightarrow{\operatorname{id}_{\mathbb{S}^1}\otimes f} & B^{\mathbb{S}^1} \end{array}$$

is homotopy commutative.

Proof. This is direct consequence of Remark 1.4.11. □

Lemma 3.1.14 ([cf. CMR07, Lemma 6.30])**.** *Let $A \in Alg_\ell^*$. Recall the maps $\rho_A : J(A) \to \Omega A$ and $\gamma_A : J(A^{\mathbb{S}^1}) \to J(A)^{\mathbb{S}^1}$ from* (1.4.13) *and* (1.4.15) *respectively. Then the following diagram commutes in* $\{\operatorname{ind} - Alg_\ell^*\}$.

$$\begin{array}{ccc} J^2(A) & \xrightarrow{-\rho_{JA}} & J(A)^{\mathbb{S}^1} \\ {\scriptstyle J(\rho_A)}\downarrow & \nearrow{\scriptstyle \gamma_A} & \\ J(A^{\mathbb{S}^1}). & & \end{array}$$

Proof. Recall the reversing map $a : \ell[t] \to \ell[t]$ from (3.1.7). For an element $p \in \ell[t]$, observe that $\mathrm{ev}_0(p) = \mathrm{ev}_1(a(p))$ and $\mathrm{ev}_1(p) = \mathrm{ev}_0(a(p))$. Writing $P'\ell = \ker \mathrm{ev}_1$, we get that $a(P\ell) = P'\ell$ and $a(P'\ell) = P\ell$. Passing to the subdivision versions, for any $A \in Alg_\ell^*$, a induces an isomorphism $\mathcal{P}A \cong \mathcal{P}'A$. Observe as well that since $P\ell \cap P'\ell = \Omega$ then $\ker(\mathrm{ev}_0 : \mathcal{P}'A \to A) \cong A^{\mathbb{S}^1}$.

Define

$$I = \ker(\mathcal{P}T(A) \xrightarrow{\mathrm{ev}_1} T(A) \xrightarrow{\eta_A} A) = \{p \in \mathcal{P}T(A) : \mathrm{ev}_1(p) \in J(A)\}$$

and E as the pullback of the diagram

$$\begin{array}{ccc} E & \xrightarrow{pr_2} & I \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ \mathcal{P}'J(A) & \xrightarrow{\mathrm{ev}_0} & J(A). \end{array}$$

The surjection $pr_2 : E \to I$ is semi-split by $p \mapsto (p, t\,\mathrm{ev}_0(p))$ and its kernel is

$$\{(q,0) \in E : \mathrm{ev}_0(q) = 0)\} \cong \ker \mathrm{ev}_0(\mathcal{P}'J(A) \to J(A)) \cong J(A)^{\mathbb{S}^1}.$$

Therefore, there is a semi-split extension

$$0 \to J(A)^{\mathbb{S}^1} \xrightarrow{i_1} E \xrightarrow{pr_2} I \to 0. \tag{3.1.15}$$

Also, by definition of I, there is a semi-split extension

$$0 \to I \to \mathcal{P}T(A) \xrightarrow{\eta_A \mathrm{ev}_1} A \to 0. \tag{3.1.16}$$

Therefore there are maps of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J(A)^{\mathbb{S}^1} & \xrightarrow{i_1} & E & \xrightarrow{pr_2} & I & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \\ 0 & \longrightarrow & J(A)^{\mathbb{S}^1} & \longrightarrow & \mathcal{P}'J(A) & \xrightarrow{\mathrm{ev}_0} & J(A) & \longrightarrow & 0. \end{array} \tag{3.1.17}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{P}T(A) & \xrightarrow{\eta_A \mathrm{ev}_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \| & & \\ 0 & \longrightarrow & J(A) & \longrightarrow & T(A) & \xrightarrow{\eta_A} & A & \longrightarrow & 0. \end{array} \tag{3.1.18}$$

Let $\xi : J(I) \to J(A)^{\mathbb{S}^1}$ be the classifying map of the extension (3.1.15). Using Remark 1.4.11 and Remark 3.1.6, it follows that the classifying map of the bottom row of (3.1.17) is $-\rho_{J(A)} : J^2(A) \to J(A)^{\mathbb{S}^1}$. So from Remark 1.4.11 it follows that the map of extensions (3.1.17) gives the equality

$$\xi = \mathrm{id}_{J(A)^{\mathbb{S}^1}} \circ \xi = -\rho_{J(A)} \circ J(\mathrm{ev}_1). \tag{3.1.19}$$

Similarly let $\zeta : J(A) \to I$ the classifying map of extension (3.1.16); the map of extensions (3.1.18) gives

$$\mathrm{ev}_1 \circ \zeta = \mathrm{id}_{J(A)} \tag{3.1.20}$$

On the other hand, write $\overline{\eta} : I \to A^{\mathbb{S}^1}$ for the restriction of the map $(\mathrm{id}_{\mathcal{P}} \otimes \eta_A) \bullet 0 : \mathcal{P}T(A) \to \mathcal{P}A$ (where $\bullet$ is concatenation). The map $\overline{\eta}$ lifts to a map $q : E \to T(A)^{\mathbb{S}^1}$ by concatenation of paths in I and paths in $\mathcal{P}'J(A)$ which by definition of E they coincide in the endpoints. This gives maps of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J(A)^{\mathbb{S}^1} & \xrightarrow{i_1} & E & \xrightarrow{pr_2} & I & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \overline{\eta}} & & \\ 0 & \longrightarrow & J(A)^{\mathbb{S}^1} & \longrightarrow & T(A)^{\mathbb{S}^1} & \xrightarrow{\mathrm{id}_{\mathbb{S}^1} \otimes \eta_A} & A^{\mathbb{S}^1} & \longrightarrow & 0. \end{array} \tag{3.1.21}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathcal{P}T(A) & \xrightarrow{\eta_A \mathrm{ev}_1} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\eta}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{P}} \otimes \eta} & & \| & & \\ 0 & \longrightarrow & A^{\mathbb{S}^1} & \longrightarrow & \mathcal{P}A & \xrightarrow{\mathrm{ev}_1} & A & \longrightarrow & 0. \end{array} \tag{3.1.22}$$

Since the classifying map of the bottom row of (3.1.21) is $\gamma_A : J(A^{\mathbb{S}^1}) \to J(A)^{\mathbb{S}^1}$, we get

$$\xi = \mathrm{id}_{J(A)^{\mathbb{S}^1}} \circ \xi = \gamma_A \circ J(\overline{\eta}). \tag{3.1.23}$$

Also, since the classifying map of the bottom row of (3.1.22) is ρ_A, we have

$$\rho_A = \rho_A \circ \mathrm{id}_A = \overline{\eta} \circ \zeta \tag{3.1.24}$$

Using (3.1.19),(3.1.20),(3.1.23) and (3.1.24):

$$\begin{aligned} \gamma_A \circ J(\rho_A) &= \gamma_A \circ J(\overline{\eta}) \circ J(\zeta) \\ &= \xi \circ J(\zeta) \\ &= -\rho_{J(A)} \circ J(\mathrm{ev}_1) \circ J(\zeta) \\ &= -\rho_{J(A)} \end{aligned}$$

□

Remark 3.1.25. The analogue of Lemma 3.1.14 for algebras without involution also holds as stated (this will be later deduced from the fact that kk is equivalent kk^h for a particular choice of ℓ). This corrects a mistake in [CT07, Lemma 6.2.2], where the sign is missing. A sign is also missing in the definition of composition in the category kk [CT07, Theorem 6.2.3], which is fixed below.

Let $A, B \in Alg^*_\ell$. As in [CT07, Section 6.1], using the functor $J : \{Alg^*_\ell\} \to \{Alg^*_\ell\}$ of Lemma 3.1.12, there is a map

$$\begin{aligned} \{A, B\} &\to \{JA, B^{\mathbb{S}^1}\} \\ \xi &\mapsto \rho_B \star J(\xi). \end{aligned}$$

Thus one can form the colimit

$$kk^h(A, B) = kk^h(A, B) = \underrightarrow{\mathrm{colim}}_n \{J^n A, B^{\mathbb{S}^n}\}$$

Lemma 3.1.26. *Let* $\xi = [f] \in \{J^m B, C^{\mathbb{S}^m}\}$ *and* $\eta = [g] \in \{J^n A, B^{\mathbb{S}^n}\}$*; put*

$$\xi \circ \eta = [(\mathrm{id}_{\mathbb{S}^n} \otimes f)] \star (-1)^{mn}[\gamma_B^{m,n}] \star [J^m(g)] \in \{J^{m+n}(A), C^{\mathbb{S}^{n+m}}\}.$$

This defines a bilinear composition law

$$kk^h(B,C) \otimes_{\mathbb{Z}} kk^h(A,B) \to kk^h(A,C)$$
$$\xi \otimes \eta \mapsto \xi \circ \eta$$

Proof. This follows from Lemma 3.1.13 and Lemma 3.1.14. □

Therefore, the sets $kk^h(-,-)$ are the morphism sets of a category kk^h with the same objects as Alg_ℓ^*, where the identity map of $A \in Alg_\ell^*$ is represented by the class of $i : A \to \mathcal{M}_X A$. Define a functor $\{Alg_\ell^*\} \to kk^h$ as the identity on objects and as the canonical map to the colimit $\{A, B\} \to kk^h(A, B)$ on arrows. Composing the latter with the functor $Alg_\ell^* \to \{Alg_\ell^*\}$ we obtain a functor

$$j^h : Alg_\ell^* \to kk^h. \tag{3.1.27}$$

The category kk^h together with the functor j^h is called *bivariant algebraic hermitian K-theory.* We will often use the term *kk^h-equivalence* between two $*$-algebras to mean that their corresponding images in kk^h are isomorphic.

3.2 j^h as an excisive homology theory

A triangulated category is a triple $(\mathfrak{T}, \Omega_{\mathfrak{T}}, \mathcal{T})$ where $\mathfrak{T}$ is an additive category, $\Omega_{\mathfrak{T}} : \mathfrak{T} \to \mathfrak{T}$ is a self-equivalence functor called the loop functor and $\mathcal{T}$ is a class of sequences of morphisms in $\mathfrak{T}$

$$\Omega_{\mathfrak{T}} C \to A \to B \to C$$

called *(distinguished) triangles* such that they satisfy the following axioms:

TR0 The class $\mathcal{T}$ is closed under isomorphisms and the sequence

$$\Omega_{\mathfrak{T}} A \to 0 \to A \xrightarrow{\mathrm{id}_A} A$$

is a distinguished triangle.

TR1 For any map $\alpha : A \to B$ in $\mathfrak{T}$, there is a distinguished triangle

$$\Omega_{\mathfrak{T}} B \to C \to A \xrightarrow{\alpha} B.$$

TR2 For the sequences

$$\Omega_{\mathfrak{T}} C \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C, \tag{3.2.1}$$

$$\Omega_{\mathfrak{T}} B \xrightarrow{-\Omega_{\mathfrak{T}} h} \Omega_{\mathfrak{T}} C \xrightarrow{f} A \xrightarrow{g} B \tag{3.2.2}$$

one is a distinguished triangle if and only if the other is. In this case we say that (3.2.2) is a rotation of (3.2.1).

TR3 For any commutative diagram between distinguished triangles

$$\begin{array}{ccccccc}
\Omega_{\mathfrak{T}}C & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\
\downarrow{\scriptstyle \Omega_{\mathfrak{T}}\gamma} & & & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
\Omega_{\mathfrak{T}}C' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C'
\end{array}$$

there exists a map $\alpha : A \to A'$ which makes the whole diagram commute.

TR4 Let $\alpha : A \to B$ and $\beta : B \to C$ be maps in $\mathfrak{T}$. There is a commutative diagram

$$\begin{array}{ccccccc}
\Omega^2_{\mathfrak{T}}C & \longrightarrow & \Omega_{\mathfrak{T}}D & \longrightarrow & \Omega_{\mathfrak{T}}B & \xrightarrow{\Omega_{\mathfrak{T}}\beta} & \Omega_{\mathfrak{T}}C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & D''' & = & D''' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow \\
\Omega_{\mathfrak{T}}C & \xrightarrow{j} & D'' & \longrightarrow & A & \xrightarrow{\beta\alpha} & C \\
\| & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \| \\
\Omega_{\mathfrak{T}}C & \longrightarrow & D' & \longrightarrow & B & \xrightarrow{\beta} & C
\end{array}$$

in which each row and column is a distinguished triangle. Furthermore, the square

$$\begin{array}{ccc}
\Omega_{\mathfrak{T}}B & \xrightarrow{\Omega_{\mathfrak{T}}\beta} & \Omega_{\mathfrak{T}}C \\
\downarrow & & \downarrow{\scriptstyle j} \\
D''' & \xrightarrow{h} & D''
\end{array}$$

commutes.

Remark 3.2.3. Usually the axioms for triangulated categories are defined using the inverse to the loop functor, called the *suspension* functor. In this thesis we present the axiom in this way since it will be more natural to work with the loop functor.

Let $\mathfrak{T}$ be a triangulated category; write $[n]$ for the n-fold loop functor in $\mathfrak{T}$. Let $\mathcal{E}$ be the class of all semi-split extensions

$$0 \to A \to B \to C \to 0. \qquad (E)$$

An *excisive homology theory* on Alg^*_ℓ (with coefficients in $\mathfrak{T}$) is a functor $H : Alg^*_\ell \to \mathfrak{T}$ together with a family of maps

$$\{\partial_E : H(C)[1] \to H(A) : E \in \mathcal{E}\}$$

such that for every $E \in \mathcal{E}$, the sequence

$$H(C)[1] \xrightarrow{\partial_E} H(A) \to H(B) \to H(C)$$

is a triangle in $\mathfrak{T}$ and the maps $\{\partial_E\}$ are compatible with maps of extensions in the sense that for a commutative diagram between semi-split extensions

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 & & (E) \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 & & (E') \end{array}$$

the following diagram

$$\begin{array}{ccc} H(C)[1] & \xrightarrow{\partial_E} & H(A) \\ \downarrow{\scriptstyle H(f_3)} & & \downarrow{\scriptstyle H(f_1)} \\ H(C')[1] & \xrightarrow{\partial_{E'}} & H(A') \end{array}$$

commutes.

Remark 3.2.4. In a triangulated category, a sequence

$$\Omega C \to A \to B \xrightarrow{f} C$$

with a splitting $g : C \to B$ (i.e. $\mathrm{id}_C = fg$), is always isomorphic to the split distinguished triangle

$$\Omega C \xrightarrow{0} A \xrightarrow{i_1} A \oplus C \xrightarrow{pr_2} C.$$

In particular, the first sequence is a distinguished triangle [Nee01, Remark 1.2.7].

In what follows we will see that there is a natural triangulation of kk^h which makes the functor j^h a homology theory.

Lemma 3.2.5. *Let $L \in Alg_\ell^*$ be flat as an ℓ-module. The functor $L = L \otimes - : Alg_\ell^* \to Alg_\ell^*$ induces a functor $L : kk^h \to kk^h$.*

Proof. Using the universal property described in Lemma 3.1.9, the functor descents to $\{L\} : \{Alg_\ell^*\} \to \{Alg_\ell^*\}$.

Next, recall the map

$$\phi_{A,L} : J(L \otimes A) \to L \otimes J(A)$$

from Example 1.4.12. Write ϕ_L^n for the composition

$$J^n(L \otimes A) \xrightarrow{J^{n-1}(\phi_{A,L})} J^{n-1}(L \otimes J(A)) \to \cdots \xrightarrow{\phi_{J^{n-1}(A),L}} L \otimes J^n(A).$$

For a map $\alpha \in kk^h(A, B)$ represented by $[f : J^n(A) \to \mathcal{M}_X B^{\mathbb{S}^n}]$ define $L \otimes \alpha \in kk^h(L \otimes A, L \otimes B)$ as the class of the composition

$$J^n(L \otimes A) \xrightarrow{\phi_L^n} L \otimes J^n(A) \xrightarrow{L \otimes f} L \otimes \mathcal{M}_X B^{\mathbb{S}^n} \cong \mathcal{M}_X (L \otimes B)^{\mathbb{S}^n}.$$

Using Remark 1.4.11, it is clear that this definition gives a functor $L : kk^h \to kk^h$. □

Corollary 3.2.6 ([cf. CT07, Section 6.6])**.** *The functors $\Omega, \Sigma_X : Alg^*_\ell \to Alg^*_\ell$ induce functors $\Omega, \Sigma_X : kk^h \to kk^h$*

Proof. This follows from the previous lemma since Ω and Σ_X are flat. □

Lemma 3.2.7. *Let $f : A \to B$ be semi-split $*$-morphism. Then, for any subdivision $\mathcal{P}_{n,f} = B^{\mathrm{sd}^n \Delta^1} \times_B A$ of the path algebra, the inclusion $i_f : \ker(f) \to \mathcal{P}_{n,f}$ induced by the inclusion $i_f : \ker(f) \to P_f$ and the last vertex map is invertible in kk^h.*

Proof. The same proof as in [CT07, Lemma 6.3.2] in the non-hermitian case works verbatim. □

Corollary 3.2.8. *The last vertex map $h : \Omega A \to A^{\mathrm{sd}^n S^1}$ is invertible in kk^h; it follows that in the ind-object $A^{\mathbb{S}^1}$ all the transition maps are kk^h-equivalences.*

Proof. This follows from Lemma 3.2.7 by considering the loop extension (1.4.7) and that if $\mathcal{P}^n A$ is the n-th subdivision of PA then the kernel of the induced map $\mathrm{ev}_1^n : \mathcal{P}^n A \to A$ is isomorphic to $A^{\mathrm{sd}^n S^1}$ and that $P_{\mathrm{ev}_1^n}$ is $\mathcal{P}_{n,\mathrm{ev}_1}$. □

Definition 3.2.9. For a semi-split extension

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{E}$$

the Lemma 3.2.7 gives an kk^h-equivalence $A \to P_g$. Define the *connecting map* as the following morphism in $kk^h(\Omega C, A)$:

$$\partial_E : \Omega C \to P_g \xleftarrow{\sim} A, \tag{3.2.10}$$

the composition of the natural map $\Omega C \to P_g$ of the mapping path extension of Example 1.4.5 and the inverse of $A \xrightarrow{\sim} P_g$.

Lemma 3.2.11. *For a semi-split extension (E), the sequences*

$$kk^h(D, \Omega B) \xrightarrow{\Omega j^h(g)_*} kk^h(D, \Omega C) \xrightarrow{(\partial_E)_*} kk^h(D, A) \xrightarrow{j^h(f)_*} kk^h(D, B) \xrightarrow{j^h(g)_*} kk^h(D, C)$$

$$kk^h(C, D) \xrightarrow{j^h(g)^*} kk^h(B, D) \xrightarrow{j^h(f)^*} kk^h(A, D) \xrightarrow{(\partial_E)^*} kk^h(\Omega C, D) \xrightarrow{\Omega j^h(g)^*} kk^h(\Omega B, D)$$

are exact.

Proof. This is proved in [CT07, Theorem 6.3.6 and Theorem 6.3.7] in the non-hermitian case. The same proof works verbatim. □

Corollary 3.2.12. *For any $D \in Alg^*_\ell$, the functors*

$$kk^h(D, -), kk^h(-, D) : Alg^*_\ell \to \mathfrak{Ab}$$

are split exact.

Proof. This is [CT07, Corollary 6.3.4] in the non-hermitian case; again, the same proof works. □

For $R \in Alg^*_\ell$ unital, the $*$-algebra $\Gamma_X R$ is what is known as a $*$-*infinite-sum* algebra: define

$$\alpha = \sum_{n\in\mathbb{N}} e_{n,2n} \text{ and } \beta = \sum_{n\in\mathbb{N}} e_{n,2n+1};$$

these elements satisfy the identities

$$\begin{aligned}\alpha^*\alpha = 1 = \beta^*\beta\\ \alpha\alpha^* + \beta\beta^* = 1.\end{aligned}$$

For $a, b \in \Gamma_X R$, define

$$\begin{aligned}a \oplus b &= \alpha^* a\alpha + \beta^* b\beta\\ a^\infty &= \sum_{n\in\mathbb{N}} (\beta^*)^n \alpha^* a\alpha\beta^n,\end{aligned}$$

and for $f, g : B \to \Gamma_X R$, write $f \oplus g : B \to \Gamma_X R$ and $f^\infty : B \to \Gamma_X R$ for

$$\begin{aligned}f \oplus g(b) &= f(b) \oplus g(b)\\ f^\infty(b) &= f(b)^\infty.\end{aligned}$$

Then, it is straightforward to compute that

$$\mathrm{id}_{\Gamma_X R} \oplus \mathrm{id}^\infty_{\Gamma_X R} = \mathrm{id}_{\Gamma_X R}.$$

Lemma 3.2.13. *There exists a unitary matrix $Q \in M_3\Gamma_X R$ such that for any $a, b \in \Gamma_X A$*

$$Q^* \begin{pmatrix} a\oplus b & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} Q = \begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & 0\end{pmatrix}$$

Proof. The matrix Q in [Wag72, p.355] can easily seen to be unitary in our case. □

Corollary 3.2.14. *For any $A \in Alg^*_\ell$, the $*$-algebra $\Gamma_X A$ is isomorphic to 0 in kk^h.*

Proof. Assume A unital, the general case follows from split-exactness. From Lemma 3.2.13, Lemma 1.6.8 and Lemma 1.6.9 we get that

$$j^h(\mathrm{id}_{\Gamma_X A} \oplus \mathrm{id}^\infty_{\Gamma_X A}) = j^h(\mathrm{id}_{\Gamma_X A}) + j^h(\mathrm{id}^\infty_{\Gamma_X A});$$

since $\mathrm{id}_{\Gamma_X A} \oplus \mathrm{id}^\infty_{\Gamma_X A} = \mathrm{id}_{\Gamma_X A}$, it follows that $j^h(\mathrm{id}_{\Gamma_X A}) = 0$ and therefore $\Gamma_X A$ is kk^h-equivalent to 0. □

Corollary 3.2.15. *There is a natural kk^h-equivalence $\Omega\Sigma_X A \cong A$. Since the functors Σ_X and Ω commute, it follows that they are inverse equivalences on kk^h.*

Proof. Write $q : \Gamma_X A \to \Sigma_X A$ for the quotient map. Using Lemma 3.2.7, there is a natural kk^h-equivalence $M_X A \cong P_q$. On other hand, considering the mapping path extension of q, there is a natural map $\Omega\Sigma_X A \to P_q$. Since $\Gamma_X A$ is kk^h-equivalent to 0 for any A, it follows from Lemma 3.2.11, that for any $D \in Alg^*_\ell$, the inclusion $\Omega\Sigma_X A \to P_q$ induces isomorphisms

$$kk^h(\Omega\Sigma_X A, D) \cong kk^h(P_q, D).$$

Therefore there are kk^h-equivalences $\Omega\Sigma A \cong P_q \cong M_X A \cong A$. □

Lemma 3.2.16. *The classifying map $\rho_A : J(A) \to \Omega A$ is an kk^h-equivalence.*

Proof. The algebras $T(A)$ and PA are *contractible*: there are $*$-homotopies $H_0 : T(A) \to T(A)[s]$ and $H_1 : PA \to PA[s]$ such that

$$\begin{aligned} \mathrm{ev}_1 H_0 &= \mathrm{id}_{T(A)} \quad \mathrm{ev}_0 H_0 = 0 \\ \mathrm{ev}_1 H_1 &= \mathrm{id}_{PA} \quad \mathrm{ev}_0 H_1 = 0. \end{aligned}$$

These are defined as follows: H_0 is the adjoint to the ℓ-linear map

$$\begin{aligned} A &\to T(A)[s] \\ a &\mapsto sa; \end{aligned}$$

similarly, H_1 is defined by

$$\begin{aligned} PA &\to PA[s] \\ p(t) &\mapsto p(st). \end{aligned}$$

Therefore, using the loop (1.4.7) and the universal extensions in Lemma 3.2.11, there are natural equivalences $p_{loop} : \Omega A \to \Omega A$ and $p_{univ} : \Omega A \to J(A)$. Using naturality of these maps and the map of extensions from the universal to the loop extension that defines ρ_A, the statement of the theorem follows. □

Let $\mathcal{T}$ be the class of sequences in kk^h

$$\Omega C \to A \to B \to C$$

which are isomorphic (as sequences) to the image of some mapping path extension

$$\Omega B' \to P_f \to A' \xrightarrow{f} B'.$$

Theorem 3.2.17. *The triple $(kk^h, \Omega, \mathcal{T})$ is a triangulated category.*

Proof. This is proved in [CT07, Theorem 6.5.2] for the non-hermitian case. The same proof works verbatim. □

Theorem 3.2.18. *The functor $j^h : Alg^*_\ell \to kk^h$ together with the connecting maps $\{\partial_E\}$ form an excisive homology theory which is homotopy invariant and M_X and hermitian stable.*

Proof. The fact that j^h is homotopy invariant and M_X and hermitian stable follows from Lemma 3.1.9 and Lemma 3.2.11. By definition of the connecting map, for a semi-split extension (E) the sequence

$$\Omega C \xrightarrow{\partial_E} A \to B \to C$$

is isomorphic (as a sequence) to the mapping path triangle of the extension. Moreover, the maps ∂_E are clearly natural on the extension (E). □

Remark 3.2.19. Theorem 3.2.18 corrects an error in [CT07, Example 6.6.1] in which the connecting map is wrongly defined.

Theorem 3.2.20. *The functor $j^h : Alg_\ell^* \to kk^h$ is universal in the following sense: for any excisive homology theory $H : Alg_\ell^* \to \mathfrak{T}$ that is homotopy invariant, M_X and hermitian stable, there is a unique triangulated functor $\overline{H} : kk^h \to \mathfrak{T}$ such that following diagram commutes*

$$\begin{array}{ccc} Alg_\ell^* & \xrightarrow{H} & \mathfrak{T} \\ {\scriptstyle j^h}\downarrow & \nearrow_{\overline{H}} & \\ kk^h & & \end{array}$$

Proof. This is [CT07, Theorem 6.6.2] in the non-hermitian case. The same proof works. □

Remark 3.2.21. As explained in Remark 1.4.4, the classes of extensions which are semi-split with respect to the underlying categories of sets and ℓ-modules agree with those semi-split with respect to sets with involution and ℓ-modules with involution. Hence by Theorem 3.2.20, the corresponding kk-theories are the same whether involutions are included in the underlying category or not.

Let $\mathfrak{C}$ be an abelian category. A functor $H : Alg_\ell^* \to \mathfrak{C}$ is *half-exact* if for an extension in Alg_ℓ^*

$$0 \to A \to B \to C \to 0$$

the sequence

$$H(A) \to H(B) \to H(C)$$

is exact.

Proposition 3.2.22. *Let $\mathfrak{C}$ be an abelian category and $H : Alg_\ell^* \to \mathfrak{C}$ a functor. Assume that H is half-exact, homotopy invariant and M_X and hermitian stable. Then there is a unique homological functor $\overline{H} : kk^h \to \mathfrak{C}$ such that $\overline{H} \circ j^h = H$.*

Proof. Again, the proof is the same as in [CT07, Theorem 6.6.6]. □

Remark 3.2.23. For a map $\alpha \in kk^h(A, B)$ we will show how to describe $\overline{H}(\alpha)$ for a functor H as in Proposition 3.2.22: first extend H to $\{H\}$ as in Lemma 3.1.9; next realize α as a class of a map $f : J^n(A) \to M_X M_\pm^{\otimes k} B^{\mathrm{sd}^r S^n}$. Composing with the inverse of $J^n(A) \to \Omega^n A$ and using M_X-stability, hermitian stability and Corollary 3.2.8 we get a map $\overline{f} : \Omega^n A \to \Omega^n B$ in kk^h. It is immediate to see that $\overline{f}$ induces the class of $\Omega^n(\alpha)$, and therefore $\overline{H}(\Omega^n(\alpha))$ is determined by $\{H\}(\overline{f})$ and in turn $\overline{H}(\alpha) = \{H\}(\Sigma_X^n \overline{f})$.

From here on, we will fix $X = \mathbb{N}$.

Remark 3.2.24. Let $f : A \to B$ be a semi-split $*$-morphism. One can also fit f into other equivalent triangles instead of the one induced by P_f. For example, take the pullback of the natural map $T(B) \to B$ along f

$$\begin{array}{ccccc} J(B) & \longrightarrow & T(B) \times_B A & \longrightarrow & A \\ \downarrow & & \downarrow & \ulcorner & \downarrow f \\ J(B) & \longrightarrow & T(B) & \longrightarrow & B. \end{array}$$

Write $T_f := T(B) \times_B A$. Then, we have a commutative diagram

$$\begin{array}{ccccccc} J(B) & \longrightarrow & T_f & \longrightarrow & A & \xrightarrow{f} & B \\ \downarrow \rho_B & & \downarrow & & \| & & \| \\ \Omega B & \longrightarrow & P_f & \longrightarrow & A & \xrightarrow{f} & B \end{array} \tag{3.2.25}$$

By Lemma 3.2.16, the vertical map $JB \to \Omega B$ is a kk^h-equivalence. Since the first three terms of the top row in (3.2.25) form an extension, using the five lemma it follows that the vertical map $T_f \to P_f$ is a kk^h-equivalence. Thus, the top row is kk^h-isomorphic to the bottom row, and is thus a triangle in kk^h.

In a similar case, let Γ_f be as in Example 1.4.9. By Corollary 3.2.14, the classifying map $J\Sigma B \to M_\infty B$ of the cone extension is a kk^h-equivalence, and therefore $T_{\Sigma f} \to \Gamma_f$ is a kk^h-equivalence by the same reasoning as before. Thus the vertical maps in the commutative diagram below form an isomorphism of triangles in kk^h:

$$\begin{array}{ccccccc} J\Sigma B & \longrightarrow & T_{\Sigma f} & \longrightarrow & \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \\ \downarrow & & \downarrow & & \| & & \| \\ M_\infty B & \longrightarrow & \Gamma_f & \longrightarrow & \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \end{array} \tag{3.2.26}$$

Therefore, the bottom row of (3.2.26) is a distinguished triangle in kk^h. The map (3.2.26) together with that of (3.2.25) with $\Sigma(f)$ substituted for f is a zig-zag of kk^h-equivalences. In particular Γ_f is kk^h-equivalent to $P_{\Sigma f}$). Since ΣP_f is isomorphic to $P_{\Sigma f}$, the bottom row of (3.2.26) is isomorphic in kk^h to the suspension of the mapping path extension 1.4.9 associated to Σf. Thus, we have an isomorphism of triangles:

$$\begin{array}{ccccccc} M_\infty B & \longrightarrow & \Gamma_f & \longrightarrow & \Sigma A & \xrightarrow{j^h(\Sigma f)} & \Sigma B \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ B & \longrightarrow & \Sigma P_f & \longrightarrow & \Sigma A & \xrightarrow{\Sigma j^h(f)} & \Sigma B \end{array}$$

Remark 3.2.27. Let $(Alg^*_\ell)_f \subset Alg^*_\ell$ and $kk^h_f \subset kk^h$ be the full subcategories whose objects are the $*$-algebras that are flat as ℓ-modules and let $j^h_f : (Alg^*_\ell)_f \to kk^h_f$ be the restriction of j^h. Observe that $(Alg_\ell^*)_f$ is closed under J and under mapping path extensions; hence kk_f^h is triangulated and j_f^h is excisive, homotopy invariant, ι_+-stable and M_X-stable. Moreover, in the same way as in Theorem 3.2.20, the functor j_f^h is universal among such functors.

Example 3.2.28. Let ℓ_0 be any commutative ring and let $\ell = \mathrm{inv}(\ell_0)$ and $\mathrm{inv} : Alg_{\ell_0} \to Alg_\ell^*$ be as in Example 1.1.6. Recall the universal excisive matrix stable and homotopy invariant homology theory $j : Alg_{\ell_0} \to kk$. Then, the composition $j^h \circ \mathrm{inv} : Alg_\ell^* \to kk^h$ is excisive, homotopy invariant and M_X-stable; by universality of j it induces a triangulated functor $\mathrm{inv} : kk_{\ell_0} \to kk_\ell^h$. Similarly, for the inverse functor to inv,

$$\begin{aligned} \mathrm{res} : Alg_\ell^* &\to Alg_{\ell_0} \\ B &\mapsto (1,0)B \end{aligned}$$

the composition $j \circ \mathrm{res}$ is excisive, homotopy invariant, M_X-stable and by Example 1.1.16 it is also hermitian stable. Hence it induces a functor $\mathrm{res} : kk_\ell^h \to kk_{\ell_0}$ which is inverse to inv. This shows that kk is a particular case of kk^h.

Similarly, for an arbitrary ℓ, recall the adjunctions from Example 1.1.6:

$$\mathrm{res} : Alg_\ell^* \leftrightarrow Alg_\ell : \mathrm{ind},$$
$$\mathrm{ind}' : Alg_\ell \leftrightarrow Alg_\ell^* : \mathrm{res}.$$

The same reasoning as before gives adjunctions

$$\mathrm{res} : kk^h \leftrightarrow kk : \mathrm{ind},$$
$$\mathrm{ind}' : kk \leftrightarrow kk^h : \mathrm{res}.$$

Example 3.2.29. Let $L \in Alg_\ell^*$; then $L \otimes -$ preserves semi-split extensions with linear splittings if either L is flat as ℓ-module or every semi-split extension is ℓ-linearly split. In either case, $j^h(L \otimes -) : Alg_\ell^* \to kk^h$ is homotopy invariant, matricially stable, hermitian stable and excisive, and therefore induces a triangulated functor $L \otimes - : kk^h \to kk^h$. By a similar argument, for kk_f^h as in Remark 3.2.27, any $L \in Alg_\ell^*$ induces a triangulated functor $L \otimes - : kk_f^h \to kk^h$.

Proposition 3.2.30. *Let $A_1, A_2 \in Alg_\ell^*$ such that $A_i \otimes -$ $(i = 0, 1)$ preserve linearly split extensions. Then we have a natural bilinear, associative product*

$$kk^h(A_1, A_2) \times kk^h(B_1, B_2) \to kk^h(A_1 \otimes B_1, A_2 \otimes B_2),\ (\xi, \eta) \mapsto \xi \otimes \eta$$

that is compatible with composition in all variables.

Proof. Suppose first the case that A_1, A_2 are flat as ℓ-modules. By Example 3.2.29, $A_i \otimes -$ and $- \otimes B_i$ extend to functors $A_i \otimes - : kk^h \to kk^h$ and $- \otimes B_i : kk_f^h \to kk^h$. For $\xi \in kk^h(A_1, A_2)$ and $\eta \in kk^h(B_1, B_2)$, set

$$\xi \otimes \eta = (\xi \otimes \mathrm{id}_{B_2}) \circ (\mathrm{id}_{A_1} \otimes \eta).$$

It is straightforward to check that the product above has all the desired properties.

In the case semi-split extensions are always linearly split, then $- \otimes B_i$ extend to the functors $- \otimes B_i : kk^h \to kk^h$ and use the same definition as before. $\square$

Definition 3.2.31. Let $\varepsilon \in \ell$ be a unitary, $A, B \in Alg_\ell^*$ and $n \in \mathbb{Z}$. Put

$$kk_n^h(A,B) := \begin{cases} kk^h(A, \Sigma^n B) & \text{if } n \geq 0 \\ kk^h(A, \Omega^{-n} B) & \text{if } n < 0 \end{cases}$$
$$_\varepsilon kk_n^h(A,B) := kk_n^h(A, {}_\varepsilon M_2 B)$$

Remark 3.2.32. Since Ω and Σ are inverse functors in kk^h there are natural isomorphisms

$$kk_n^h(A,B) \cong \begin{cases} kk^h(\Omega^n A, B) & \text{if } n \geq 0 \\ kk^h(\Sigma^{-n} A, B) & \text{if } n < 0 \end{cases}$$

Remark 3.2.33. Due to Remark 1.1.26, there is a $*$-isomorphism ${}_1M_2 \cong M_\pm$. It follows from this and from Theorem 3.2.20 that for all $A, B \in Alg_\ell^*$, $i_+ : \ell \to M_\pm$ induces a canonical isomorphism

$$_1kk_*^h(A,B) \cong kk_*^h(A,B).$$

Example 3.2.34. The functor $KH_0^h : Alg_\ell^* \to KH_0^h(\ell) - \mathrm{Mod}$ satisfies the hypothesis of Proposition 3.2.22. Hence the functor $\overline{KH}_0^h$ of the proposition induces a natural homomorphism

$$kk^h(A,B) \to \hom_{KH_0^h(\ell)}(KH_0^h(A), KH_0^h(B))$$

Setting $A = \ell$ we obtain a natural map

$$kk^h(\ell, B) \to KH_0^h(B).$$

Proposition 3.2.35. *The product from Proposition 3.2.30 maps to the cup product from Lemma 2.2.9 under the map from Example 3.2.34. In other words, there is a commutative diagram*

$$\begin{array}{ccc} kk^h(\ell,A) \otimes_{\mathbb{Z}} kk^h(\ell,B) & \longrightarrow & KH_0^h(A) \otimes_{\mathbb{Z}} KH_0^h(B) \\ \downarrow{\scriptstyle\otimes} & & \downarrow{\scriptstyle\star} \\ kk^h(\ell, A \otimes B) & \longrightarrow & KH_0^h(A \otimes B). \end{array}$$

Proof. Assume A, B unital and let $\alpha \in kk^h(\ell, A)$ and $\beta \in kk^h(\ell, B)$. Using Remark 3.2.23, the corresponding elements in $KH_0^h(A)$ and $KH_0^h(B)$ are determined by maps

$$\Omega^n(\alpha)_* : KH_0^h(\Omega^n) \to KH_0^h(\Omega^n A),$$
$$\Omega^m(\beta)_* : KH_0^h(\Omega^m) \to KH_0^h(\Omega^m B).$$

and evaluation at $[1] \in KH_0^h(\ell)$. Since the product

$$kk^h(\ell, A) \otimes_{\mathbb{Z}} kk^h(\ell, B) \to kk^h(\ell, A \otimes B)$$

extends the tensor product of algebras and due to Remark 2.2.4 the cup product corresponds to the extension of scalars, it follows that

$$\Omega^{n+m}(\alpha \otimes \beta)_*[1] = \Omega^n(\alpha)_*[1] \star \Omega^m(\beta)_*[1].$$

From this, the statement follows in the unital case. The non-unital case follows from the unital one and excision. □

Chapter 4

Computations and the comparison with KH^h

In this chapter we show some computations in kk^h as a matter of examples: in Section 4.1 we characterize the image of the coproducts, of the Toeplitz algebra and of the Cohn algebra of a graph and also give an algebraic analogue of the Pimsner-Voiculescu sequecen. In Section 4.2 we show that the natural map $kk^h(\ell, A) \to KH_0^h(A)$ as described in Example 3.2.28 is an isomorphism.

4.1 Computations

Coproducts

Proposition 4.1.1. *Let $A, B \in Alg_\ell^*$. Then the natural map $A \amalg B \to A \oplus B$ is a kk^h-equivalence.*

Proof. Define $f : A \oplus B \to M_2(A \amalg B)$ as

$$f(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

We will show that

$$j^h(\mathrm{id}_{M_2} \otimes \pi \circ f) = j^h(i_2 : A \oplus B \to M_2(A \oplus B)) \tag{4.1.2}$$

and that

$$j^h(f \circ \pi) = j^h(i_2 : \amalg B \to \mathbb{M}_2(A \amalg B)); \tag{4.1.3}$$

it follows that $j^h(\pi)$ is left and right inversible and therefore an isomorphism in kk^h.

Identify $A \amalg B$ with its image through i_2 in $M_2(A \amalg B)$. Let $u(t) \in M_2(\widetilde{A \amalg B}[t])$ defined by

$$u_t = \mathrm{id}_{M_2 A} \amalg \begin{pmatrix} 1-t^2 & t \\ (t^3 - 2t) & 1-t^2 \end{pmatrix}.$$

It is easily shown that u_t is invertible, $u_0 = \mathrm{id}$ and

$$\begin{aligned} u_1 i_2(a) u_1^* &= f \circ \pi(a) \qquad (a \in A) \\ u_1 i_2(b) u_1^* &= f \circ \pi(b) \qquad (b \in B); \end{aligned}$$

therefore, it follows from Lemma 1.2.3 that the equality in (4.1.3) stands. Similarly, using the matrix $\pi(u_t) \in M_2(\widetilde{A \oplus B})$ we can conclude the equality in (4.1.2). □

Corollary 4.1.4. *The natural map $Q(A) \to A \oplus A$ is a kk^h-equivalence and it induces a kk^h-equivalence $\pi_0 : q(A) \to A$.*

Proof. This follows from Proposition 4.1.1 and the commutative diagram between split triangles in kk^h (which are distinguished by Remark 3.2.4):

$$\begin{array}{ccccccc} \Omega A & \xrightarrow{0} & q(A) & \longrightarrow & Q(A) & \longrightarrow & A \\ \| & & \downarrow{\scriptstyle \pi_0} & & \wr\downarrow & & \| \\ \Omega A & \xrightarrow{0} & A & \xrightarrow{\jmath_1} & A \oplus A & \xrightarrow{pr_2} & A \end{array}$$

□

The fundamental theorem

Recall from Example 1.1.21 the Laurent polynomial algebra $A[t, t^{-1}]$. Write

$$\sigma A = \ker(\mathrm{ev}_1 : A[t, t^{-1}] \to A).$$

The *Toeplitz* algebra τ (over ℓ) is the $*$-algebra generated by an element S such that $S^*S = 1$. We write τ_0 for the kernel of the map $\tau \to \ell$ that sends S to 1.

Proposition 4.1.5. *Let A be an algebra in Alg^*_ℓ. Then $A[t, t^{-1}]$ and $A \oplus \Sigma A$ are kk^h-equivalent.*

Proof. Consider the split extension

$$0 \to \sigma A \to A[t, t^{-1}] \to A \to 0;$$

therefore, from Remark 3.2.4, it follows that $A[t, t^{-1}]$ is kk^h equivalent to $A \oplus \sigma A$. We will show that σA is kk^h equivalent to ΣA. Since the coefficient ring A does not matter in the following proof, we omit it from notation. The proof follows like [CT07, Theorem 7.3.1 and Lemma 7.3.2].

Let $f : \tau \to A[t, t^{-1}]$ be the $*$-morphism defined by $S \mapsto t$. This morphism restricts to $f| : \tau_0 \to \sigma$. On the other hand there is also a natural $*$-morphism $g : \tau \to \Gamma$ sending S to the matrix

$$S \mapsto \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

It is easy to see that g is injective; thus, τ identifies with a $*$-subalgebra of Γ. In this identification, the kernel of $f|$ is mapped to M_∞. This gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_\infty & \longrightarrow & \tau_0 & \longrightarrow & \sigma & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_\infty & \longrightarrow & \Gamma & \longrightarrow & \Sigma & \longrightarrow & 0. \end{array}$$

If we show that τ_0 is kk^h-equivalent to 0, since we know that Γ is kk^h-equivalent to 0 by Corollary 3.2.14, we can use the five lemma and conclude that σ is kk^h-equivalent to Σ. For this we will construct a $*$-homotopy from τ_0 to $M_\infty\tau[t]$ that when evaluated at $t = 0$ is the natural inclusion and is null when evaluated at $t = 1$.

First we define several $*$-morphisms $\psi, \varphi_1, \varphi_2, \varphi_3 : \tau \to \tau \otimes \tau$ which are given by defining them on the generator S as

$$\begin{aligned} \psi(S) &= S^2S^* \otimes 1 \\ \varphi_1(S) &= S^2S^* \otimes 1 + (1 - SS^*) \otimes S \\ \varphi_2(S) &= S \otimes 1 \\ \varphi_3(S) &= S^2S^* \otimes 1 + (1 - SS^*) \otimes 1 \end{aligned} \tag{4.1.6}$$

All of these morphisms agree modulo the ideal $M_\infty\tau$. Identify τ with its image in Γ and define elements $u_t, v_t \in (\tau \otimes \tau)[t] \subseteq \Gamma\tau[t]$ by

$$u_t = \begin{pmatrix} 1 - SS^*t^2 & (t^3 - 2t)S & 0 & 0 & \cdots \\ tS^* & 1 - t^2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$v_t = \begin{pmatrix} 1 - t^2 & (t^3 - 2t) & 0 & 0 & \cdots \\ t & 1 - t^2 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

It is readily checked that u_0, u_1, v_0 and v_1 are unitary matrices and that $1 - u_t$ and $1 - v_t$ belong in the ideal $M_\infty\tau[t]$. Write $U_t = c(u_t, u_t^{-1})$ and $V_t = c(v_t, v_t^{-1})$ as in Lemma 1.2.3. Define $\Phi_1, \Phi_2 : \tau \to M_\pm(\tau \otimes \tau)[t]$ as

$$\begin{aligned} \Phi_1(S) &= U_t i_+(S \otimes 1) \\ \Phi_2(S) &= V_t i_+(S \otimes 1). \end{aligned}$$

The following identities are then satisfied:

$$\begin{aligned} &\mathrm{ev}_0 \circ \Phi_1 = \mathrm{ev}_1 \circ \Phi_2 = i_+\varphi_2, \\ &\mathrm{ev}_1 \circ \Phi_1 = i_+\varphi_1 \text{ and} \\ &\mathrm{ev}_1 \circ \Phi_2 = i_+\varphi_3 \end{aligned}$$

Thus, restricting to τ_0 there are $*$-quasi-homomorphisms

$$(\Phi_1, i_+\psi), (\Phi_2, i_+\psi) : \tau_0 \rightrightarrows M_\pm \tau \otimes \tau[t] \trianglerighteq M_\pm M_\infty \tau[t].$$

Using Proposition 1.5.3 the $*$-quasi-homomorphisms $(i_+\varphi_1, i_+\psi)$ and $(i_+\varphi_3, i_+\psi)$ induce the same morphisms in kk^h. Therefore, using hermitian stability the $*$-quasi-homomorphisms (φ_1, ψ) and (φ_3, ψ) induce the same morphisms in kk^h.

Finally, since φ_1 is the orthogonal sum of ψ and the inclusion $\tau_0 \to M_\infty \tau$ and φ_3 agrees with ψ when restricted to τ_0, using Proposition 1.5.3 this means that (φ_1, ψ) induces the same morphism as the inclusion $\tau_0 \to M_\infty \tau$ in kk^h and (φ_3, ψ) induces the null morphism. Thus the inclusion $\tau_0 \to M_\infty \tau$ is null on kk^h. By M_∞-stability this then implies that the inclusion $\tau_0 \to \tau$ is null, which implies that τ_0 is kk^h equivalent to 0 since the following extension is split:

$$0 \to \tau_0 \to \tau \to A \to 0$$ □

Pimsner-Voiculescu sequence

In topological K-theory, the Pimsner-Voiculescu sequence relates the K-theory groups of a crossed product $A \rtimes \mathbb{Z}$ with those of A. Here we present the algebraic analogue of this sequence in our setting.

Given a $*$-automorphism $\sigma : A \to A$ we define the crossed product $A \rtimes_\sigma \mathbb{Z}$ as the ℓ-module $A[t, t^{-1}]$ but with multiplication given by the relation

$$tat^{-1} = \sigma(a)$$

and involution $(at)^* = t^{-1}\sigma(a)^*$.

Consider the $*$-subalgebra τ_σ of $\tau \otimes_\ell (A \rtimes_\sigma \mathbb{Z})$ generated by $1 \otimes A$ and $S \otimes t$. This gives a semi-split extension

$$0 \to M_\infty A \to \tau_\sigma \to A \rtimes_\sigma \mathbb{Z} \to 0. \tag{4.1.7}$$

Proposition 4.1.8. *Let A be an algebra in Alg^*_ℓ. Then the sequence* (4.1.7) *induces the distinguished triangle in kk^h*

$$\Omega A \to A \xrightarrow{\mathrm{id} - j^h(\sigma^{-1})} A \to A \rtimes_\sigma \mathbb{Z}.$$

Proof. Write $\kappa : A \to \tau_\sigma$ for the canonical inclusion. The same argument (with the obvious modifications) as in [Cun05, Propositions 14.1 and 14.2] shows that there is a commutative diagram in kk^h

$$\begin{array}{ccccccc}
\Omega A \rtimes_\sigma \mathbb{Z} & \longrightarrow & A & \xrightarrow{\mathrm{id} - j^h(\sigma^{-1})} & A & \longrightarrow & A \rtimes_\sigma \mathbb{Z} \\
\| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \kappa} & & \| \\
\Omega A \rtimes_\sigma \mathbb{Z} & \longrightarrow & M_\infty A & \longrightarrow & \tau_\sigma & \longrightarrow & A \rtimes_\sigma \mathbb{Z},
\end{array}$$

so the statement of the proposition follows. □

Cohn algebra of a graph

Let E be a directed graph, that is, a cuadruple (E^0, E^1, r, s) where E^0 is the set of vertices of the graph and E^1 is the set of edges, $r, s : E^1 \to E^0$ are the source and range of the edges. A path in E is a sequence of edges $e_1e_2\cdots e_n$ where $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$; in this case we call n the length of the path. We define vertices to be paths of length 0. We define $\mathcal{P}(E)$ as the set of finite paths in E; the range and source functions extend to $r, s : \mathcal{P}(E) \to E^0$ in the obvious way.

The *Cohn path algebra* $C(E)$ of a graph E is the $*$-algebra generated by E^0 and E^1 subject to the relations

$$\begin{aligned} v \cdot w &= \delta_{v,w} v \\ v^* &= v \\ s(e) \cdot e = e &= r(e) \cdot e \\ e^* f &= \delta_{e,f} r(e) \end{aligned}$$

for v, w in E^0 and $e, f \in E^1$.

There is a natural morphism $\varphi : \ell^{(E^0)} \to C(E)$ sending ev_v to v. For a vertex $v \in E^0$ such that $s^{-1}(v)$ is a finite set, we define

$$C(E) \ni m_v = \begin{cases} \sum_{e \in s^{-1}(e)} ee^* & \text{if } s^{-1}(v) \neq \emptyset \\ 0 & \text{if } s^{-1}(v) = \emptyset. \end{cases}$$

The elements m_v satisfy the identities

$$\begin{aligned} m_v = m_v^*,\ m_v^2 = m_v,\ m_v w = \delta_{v,w} m_v, \\ m_v e = \delta_{v,s(e)} e \quad (w \in E^0, e \in E^1). \end{aligned} \tag{4.1.9}$$

Write $C^m(E)$ for the algebra obteined from $C(E)$ by formally adjoining an element m_v for each vertex in E such that $s^{-1}(v)$ is infinite, subject to the identities (4.1). Let $q_v = v - m_v \in C^m(E)$ and write

$$\mathcal{K}(E) = \langle q_v | v \in E^0\ s^{-1}(v) \text{ is finite non empty} \rangle \subseteq \widehat{\mathcal{K}}(E) = \langle q_v | v \in E^0 \rangle$$

for the corresponding ideals in $C^m(E)$.

Write $\widehat{i} : \ell^{(E^0)} \to \widehat{\mathcal{K}}(E)$ for the $*$-morphism that maps ev_v to q_v and let $\xi : C(E) \to C^m(E)$ be determined by

$$\xi(v) = m_v; \quad \xi(e) = e m_{r(e)}.$$

The same aregument as in [CM18, Remark 4.9] shows that $\widehat{i}$ is a kk^h-equivalence. On other hand, the canonical inclusion $i : C(E) \to C^m(E)$ and ξ determine a $*$-quasi-homomorphism $(i, \xi) : C(E) \rightrightarrows C^m(E) \trianglerighteq \widehat{\mathcal{K}}(E)$. It is straightforward to see that $i\varphi = \xi\varphi + \widehat{i}$, therefore, using Proposition 1.5.3, we get

$$j^h(i, \xi) j^h(\varphi) = j^h(i\varphi, \xi\varphi) = j^h(\xi\varphi, \xi\varphi) + j^h(\widehat{i}, 0) = j^h(\widehat{i})$$

Hence, $j^h(\varphi)$ has a left inverse $j^h(\widehat{i})^{-1}j^h(i,\xi)$. We will show that $j^h(\varphi)$ is right inversible and therefore an isomorphism.

Consider $M_{\mathcal{P}(E)}$, the matrix ring indexed on the set $\mathcal{P}(E)$ and write $\epsilon_{\alpha,\beta}$ for its units. Define $\widehat{i}_\tau : C(E) \to M_{\mathcal{P}(E)}C(E)$ given on generators by

$$\widehat{i}_\tau(v) = \epsilon_{v,v} \otimes v, \quad \widehat{i}_\tau(e) = \epsilon_{s(e),r(e)} \otimes e$$

for $v \in E^0$ and $e \in E^1$. Also define $\widehat{\varphi} : \widehat{\mathcal{K}}(E) \to M_{\mathcal{P}(E)}C(E)$ by

$$\widehat{\varphi}(\alpha q_v \beta^*) = \epsilon_{\alpha,\beta} \otimes v.$$

There is a commutative diagram

$$\begin{array}{ccc} \ell^{(E^0)} & \xrightarrow{\widehat{i}} & \widehat{\mathcal{K}}(E) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\widehat{\varphi}} \\ C(E) & \xrightarrow{\widehat{i}_\tau} & M_{\mathcal{P}(E)}C(E). \end{array}$$

Lemma 4.1.10. *Let $\alpha \in \mathcal{P}(E)$ be a path and $i_\alpha : C(E) \to M_{\mathcal{P}(E)}C(E)$ be the inclusion in the α-diagonal coordinate $(e_{\alpha,\alpha})$. Then $\widehat{i}_\tau$ and i_α induce the same isomorphism in kk^h.*

Proof. Using Lemma 1.6.8, the class of i_α does not depend on α since i_α and i_β are conjugates. So we assume $\alpha = w \in E^0$. For each $v \in E^0$, $v \neq w$ write

$$\begin{aligned} a_v &= (1-t^2)\epsilon_{w,w} + (t^3-2t)\epsilon_{w,v} + t\epsilon_{v,w} + (1-t^2)\epsilon_{v,v} \\ b_v &= (1-t^2)\epsilon_{w,w} + (2t-t^3)\epsilon_{w,v} - t\epsilon_{v,w} + (1-t^2)\epsilon_{v,v} \\ a_w &= b_w = \epsilon_{w,w}. \end{aligned}$$

Define $C_v = c(a_v, b_v)$ as in Lemma 1.2.3. Then we have a $*$-homotopy $H : C(E) \to M_\pm M_{\mathcal{P}(E)}C(E)[t]$ given by

$$\begin{aligned} H(v) &= C_v i_+(\epsilon_{v,v} \otimes v) C_v \\ H(e) &= C_{s(e)} i_+(\epsilon_{s(e),r(e)} \otimes e) C^*_{r(e)} \end{aligned}$$

which satisfies $\mathrm{ev}_0 H = i_+\widehat{i}_\tau$ and $\mathrm{ev}_1 H = i_+ i_w$. Using hermitian stability we conclude that $\widehat{i}_\tau$ and i_w are the same in kk^h. □

Write $\mathfrak{A} \subseteq M_{\mathcal{P}(E)}C(E)$ for the ℓ-submodule generated by

$$\mathfrak{A} = \mathrm{span}\{e_{\gamma,\delta} \otimes \alpha\beta^* \in M_{\mathcal{P}(E)}C(E) : s(\alpha) = r(\gamma),\ s(\beta) = r(\delta),\ r(\alpha) = r(\beta)\}.$$

It is readily checked that $\mathfrak{A}$ is a $*$-subalgebra of $M_{\mathcal{P}(E)}C(E)$, and $\mathrm{Im}\,\widehat{\mathrm{i}}_\tau, \mathrm{Im}\,\widehat{\varphi} \subseteq \mathfrak{A}$. In particular, $\widehat{i}_\tau$ restricted to $\mathfrak{A}$ induces a monomorphism in kk^h.

Let $\Gamma_{\mathcal{P}(E)}$ be the cone algebra indexed by $\mathcal{P}(E)$. There is a $*$-morphism $\rho : C^m(E) \to \Gamma_{\mathcal{P}(E)}$ given by

$$\rho(v) = \sum_{s(\alpha)=v} \epsilon_{\alpha,\alpha}$$
$$\rho(e) = \sum_{r(\alpha)=r(e)} \epsilon_{e\alpha,\alpha}$$
$$\rho(m_w) = \sum_{\substack{r(\alpha)=w \\ \text{length } \alpha\geq 1}} \epsilon_{\alpha,\alpha}.$$

Consider the $*$-morphism $\rho' = \rho \otimes 1 : C^m(E) \to \Gamma_{\mathcal{P}(E)}\widetilde{C^m(E)}$. Then $\mathfrak{A}$ is closed by multiplication by elements on the image of ρ' on both sides, so we can form the semi-direct product $C^m(E) \ltimes \mathfrak{A}$. Define the algebra D as the quotient of $C^m(E) \ltimes \mathfrak{A}$ by the $*$-ideal

$$\langle \alpha q_v \beta^*, -\epsilon_{\alpha,\beta} \otimes v : v = r(\alpha) = r(\beta)\rangle.$$

It is shown on [CM18, Lemma 4.19] that $\mathfrak{A}$ maps injectively to D, meaning it is isomorphic to an ideal inside D. We then have a commutative diagram

$$\begin{array}{ccccc} \ell^{(E^0)} & \xrightarrow{\widehat{i}} & \widehat{\mathcal{K}}(E) & \longrightarrow & C^m(E) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\widehat{\varphi}} & & \downarrow{\scriptstyle\Xi} \\ C(E) & \xrightarrow{\widehat{i}_\tau} & \mathfrak{A} & \longrightarrow & D, \end{array} \tag{4.1.11}$$

where Ξ is given by the composition of the inclusion $C^m(E) \to C^m(E) \ltimes \mathfrak{A}$ and the projection $C^m(E) \rtimes \mathfrak{A} \to D$. Define $\psi_0 = \Xi$, $\psi_1 = \Xi\xi$. It is easy to check that ψ_1 is orthogonal to $\widehat{i_\tau}$ so we can define $\psi_{1/2} = \psi_1 + \widehat{i_\tau}$. These $*$-morphisms define $*$-quasi-homomorphisms

$$(\psi_0, \psi_1), (\psi_0, \psi_{1/2}), (\psi_{1/2}, \psi_1) : C(E) \rightrightarrows D \rhd \mathfrak{A}$$

Lemma 4.1.12. *The $*$-quasi-homomorphism $(\psi_0, \psi_{1/2})$ induces the zero map in kk^h.*

Proof. For each $e \in E^1$ consider the matrices in $\Gamma_{\mathcal{P}(E)}C(E)[t]$

$$u_t^e = \epsilon_{s(e),s(e)}(1-t^2) \otimes ee^* + \epsilon_{e,s(e)} \otimes te^*$$
$$v_t^e = \epsilon_{s(e),s(e)}(1-t^2) \otimes ee^* + \epsilon_{s(e),e} \otimes (2t-t^3)e.$$

Observe that multiplying by u_t^e and v_t^e preserves $\mathfrak{A}$. Put $U_t^e = c((0,u_t^e),(0,v_t^e)) \in M_\pm D[t]$ and define a $*$-homotopy $H : C(E) \to M_\pm D[t]$ determined by

$$H(v) = i_+(v,0) \qquad (v \in E^0)$$
$$H(e) = i_+(em_{r(e)},0) + U_t^e i_+(0, \epsilon_{s(e),r(e)} \otimes e) \qquad (e \in E^1)$$

Then H is a $*$-homotopy between ψ_0 and $\psi_{1/2}$ and the $*$-quasi-homomorphism $(H, i_+\psi_{1/2})$ is a $*$-homotopy between $(\psi_0,\psi_{1/2})$ and $(\psi_{1/2},\psi_{1/2})$. Therefore, by Proposition 1.5.3 $j^h(\psi_0,\psi_{1/2})$ is the zero morphism □

Theorem 4.1.13. *The morphism $\varphi : \ell^{(E^0)} \to C(E)$ is a kk^h-equivalence.*

Proof. We have already checked that

$$j^h(\widehat{i})^{-1} j^h(i,\xi) j^h(\varphi) = j^h(\mathrm{id}_{\ell^{(E^0)}}).$$

The commutative diagram (4.1.11) and the previous lemma show that

$$j^h(\widehat{\varphi}) j^h(i,\xi) = j^h(\psi_0,\psi_1) = j^h(\psi_0,\psi_{1/2}) + j^h(\psi_{1/2},\psi_1) = j^h(\psi_{1/2},\psi_1) = j^h(\widehat{i_\tau})$$

And on other hand

$$j^h(\widehat{\varphi}) j^h(i,\xi) = j^h(\widehat{i_\tau}) j^h(\varphi) j^h(\widehat{i})^{-1} j^h(i,\xi)$$

hence

$$j^h(\widehat{i_\tau}) = j^h(\widehat{i_\tau}) j^h(\varphi) j^h(\widehat{i})^{-1} j^h(i,\xi),$$

and since $j^h(\widehat{i_\tau})$ is a monomorphism, this shows that

$$j^h(\mathrm{id}_{C(E)}) = j^h(\varphi) j^h(\widehat{i})^{-1} j^h(i,\xi)$$

as we wanted. □

4.2 Comparison of with KH^h

Theorem 4.2.1 ([cf. CT07, Theorem 8.2.1])**.** *The map from Example 3.2.34 gives an isomorphism*

$$KH_0^h(A) \cong kk^h(\ell, A).$$

Proof. Suppose first A unital, the general case follows from excision. Recall from Remark 2.1.6 the set of $*$-quasi-homomorphisms $qq(\ell, A)$ and the surjective map

$$qq(\ell, A) \to K_0^h(A).$$

Using Example 1.5.2, for $(e_0, e_1) \in qq(\ell, A)$, this $*$-quasi-homomorphism also induces a map $(e_i) : q\ell \to {}_1M_2^h M_\infty A$ which induces a map in kk^h,

$$\begin{aligned} qq(\ell, A) &\to kk^h(q\ell, {}_1M_2 M_\infty A) \cong kk^h(q\ell, A) \\ (e_0, e_1) &\mapsto [e_i] \end{aligned} \tag{4.2.2}$$

Using Lemma 1.6.8, the map (4.2.2) sends equivalent classes in K_0^h to the same morphism in kk^h, so the map then factors as

$$\begin{array}{ccc} qq(\ell, A) & \longrightarrow & kk^h(q\ell, A) \\ \big\downarrow & \nearrow & \\ K_0^h(A) & & \end{array}$$

Using Corollary 4.1.4 we have that $\pi_0 : q\ell \to \ell$ is induces a kk^h-equivalence, therefore we can compose to get

$$K_0^h(A) \to kk^h(\ell, A).$$

Using excision for K_n^h for $n \leq 0$, this gives a map

$$\alpha : KH_0^h(A) = \varinjlim_n K_0^h(\Sigma^n \Omega^n A) \to \varinjlim_n kk^h(\ell, \Sigma^n \Omega^n A) = kk(\ell, A).$$

Write

$$\beta : kk^h(\ell, A) \to KH_0^h(A)$$

for the map in Example 3.2.34. We will show that α and β are inverses of each other.

Using the description of β given in Remark 3.2.23, for a self-adjoint idempotent $e \in {}_1M_2^h M_\infty A$, where A is unital, is immediate to see that $\beta\alpha(c_0[e]) = c_0[e]$. This implies that $\beta\alpha$ is the identity in $KH_0^h(A)$.

To complete the proof we will show that α is surjective. Let $\varphi : J^n(\ell) \to M_\infty M_\pm^{\otimes k} A^{\mathrm{sd}^r S^n}$ represent a class in $kk^h(\ell, A)$. Using the map $\ell \to \Sigma^n J^n(\ell)$ in kk^h consider the induced map on KH_0^h

$$KH_0^h(\ell) \to KH_0^h(\Sigma^n J^n(\ell))$$

and write $e \in KH_0^h(\Sigma^n J^n(\ell))$ for the image of the element $[1] \in KH_0^h(\ell)$. It follows from the definition of α, that $\alpha([1])$ equals id_ℓ in $kk^h(\ell, \ell)$. Let then $\kappa : q\ell \to {}_1M_2 M_\infty \Sigma^n J^n(\ell)$ be the associated map to a $*$-quasi-homomorphism that induces e. Thus we have the follows equality in $kk^h(\ell, \Sigma^n J^n \ell)$

$$j^h(\kappa) j^h(\pi_0)^{-1} = \alpha(e). \tag{4.2.3}$$

In turn, this shows that $j^h(\kappa) j^h(\pi_0)^{-1}$ is the morphism that induces the kk^h equivalence $\ell \sim \Sigma^n J^n(\ell)$. On other hand, consider the commutative diagram in kk^h

$$\begin{array}{ccc} \Sigma^n J^n(\ell) & \xrightarrow{\Sigma^n \varphi} & \Sigma^n A^{\mathrm{sd}^r S^n} \\ \wr\uparrow & & \wr\uparrow \\ \ell & \xrightarrow{\quad\varphi\quad} & A \end{array}$$

where the right arrow is an isomorphism because of Corollary 3.2.8. It follows that $(\Sigma^n \varphi)_*(e) \in KH_0^h(\Sigma^n A^{\mathrm{sd}^r S^n}) \cong KH_0^h(A)$ is the same class as $j^h(\varphi)_*([1]) \in KH_0^h(A)$. Therefore, using (4.2.3) we have following equalities in $kk^h(\ell, A)$:

$$\alpha(\varphi_*([1])) = \alpha((\Sigma^n \varphi)_*(e)) = (\Sigma^n \varphi)\alpha(e) = (\Sigma^n \varphi) j^h(\kappa) j^h(\pi_0)^{-1} = \varphi.$$

This concludes the proof. □

Chapter 5

Karoubi's Fundamental Theorem in kk^h

In this chapter we prove an analogous result to Theorem 2.3.3 in the category kk^h. For this, we develop some preliminary results about the induction and restriction functors in Section 5.1; we then define functors $U, V : Alg_\ell^* \to Alg_\ell^*$ in Section 5.2, which are similar to the functors U', V' described in Section 2.3 and which in kk^h give equivalent functors up to suspension/looping; we also show that functors U, V satisfy analogous properties to the ones discussed in Section 2.3. Finally in Section 5.3 we use the functors U, V and the properties that were discussed in Section 5.2 to conclude Theorem 5.3.1 and Theorem 5.3.7.

5.1 The functors ind, res and Λ

Recall the functors $\mathrm{res} : kk^h \to kk$, and $\mathrm{ind}, \mathrm{ind}' : kk \to kk^h$ from Example 3.2.28.

Proposition 5.1.1. *The functors* $\mathrm{res} : kk^h \leftrightarrow kk : \mathrm{ind}$ *are both right and left adjoint to one another; in other words, for every* $A \in Alg_\ell^*$ *and* $B \in Alg_\ell$ *there are natural isomorphisms*

$$kk(\mathrm{res}(A), B) \cong kk^h(A, \mathrm{ind}(B)) \text{ and } kk^h(\mathrm{ind}(B), A) \cong kk(B, \mathrm{res}(A)).$$

Proof. Using Proposition 4.1.1, the functors $\mathrm{ind}, \mathrm{ind}' : kk \to kk^h$ are naturally equivalent, since one is right adjoint to res and the other is left adjoint, the result follows. □

Remark 5.1.2. The unit and counit maps of the second adjunction in Proposition 5.1.1 are obtained from those of the adjunction between ind' and res using the projection $\pi : \mathrm{ind}' \to \mathrm{ind}$ and the diagonal map $\mathrm{ind} \to M_2\mathrm{ind}'$ as in the proof of Proposition 4.1.1.

Let $\Lambda = \ell \oplus \ell$ equipped with involution

$$(\lambda, \mu)^* = (\mu^*, \lambda^*).$$

For $A \in Alg^*_\ell$ write ΛA for $\Lambda \otimes A$ and $\Lambda : Alg^*_\ell \to Alg^*_\ell$ for the associated functor.

Recall from Section 2.3 that for $A \in Alg^*_\ell$ we write $\widehat{A} = \operatorname{ind}(\operatorname{res}(A))$. Then $\widehat{A} \cong \Lambda A$ via the isomorphism

$$\begin{aligned} \Lambda A &\to \widehat{A} \\ (x, y) &\mapsto (x, y^*). \end{aligned}$$

Under this identification, the maps η_A of (2.3.2) and φ_A of (2.3.1) become the scalar extensions of the embeddings

$$\begin{aligned} \eta : \ell &\to \Lambda & (5.1.3) \\ x &\mapsto (x, x), \\ \phi : \Lambda &\to {}_1M_2 & (5.1.4) \\ (x, y) &\mapsto \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}. \end{aligned}$$

Remark 5.1.5. The functor induced by tensoring with $\Lambda : kk^h \to kk^h$ is left and right adjoint to itself since $\Lambda \cong \operatorname{ind}\operatorname{res}(\ell)$. Also, Proposition 5.1.1 shows that

$$kk^h(\cdot, \Lambda(\cdot)) \cong kk(\operatorname{res}(\cdot), \operatorname{res}(\cdot)).$$

In other words, Λ represents kk. In particular, we have

$${}_\varepsilon kk^h(\cdot, \Lambda(\cdot)) \cong kk^h(\cdot, \Lambda(\cdot))$$

for any unitary $\varepsilon \in \ell$. Moreover by Remark 1.1.26, if $R \in Alg^*_\ell$ is unital and $\varepsilon \in R$ is central unitary and $\Psi \in R$ is an invertible ε-hermitian element, then we have an $*$-isomorphism

$$\operatorname{ad}(1, \Psi^{-1}) : \Lambda R \to \Lambda R^\Psi.$$

In particular, we have $*$-isomorphisms

$$\Lambda M_\pm \cong \Lambda({}_\varepsilon M_2) \cong \Lambda M_2.$$

Remark 5.1.6. Let $t : \Lambda \to \Lambda$ defined as $t(x, y) = (y, x)$.Then t is a $*$-automorphism, with $t^2 = \mathrm{id}_\Lambda$; moreover using Remark 5.1.5 one checks that the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{(\mathrm{id}_{{}_1M_2} \otimes \eta)\phi} & {}_1M_2\Lambda \\ & \searrow^{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & t \end{pmatrix}} & \downarrow \wr \\ & & M_2\Lambda. \end{array}$$

Thus, using Lemma 1.6.9 we get

$$j^h(i_2)^{-1} j^h(\eta) j^h(\phi) = j^h(\mathrm{id}_\Lambda) + j^h(t).$$

5.2 The functors U and V

Consider the path algebras (Example 1.4.5) of the maps (5.1.4) and (5.1.3),

$$U = P_\phi \text{ and } \quad V = P_\eta.$$

For $A \in Alg_\ell^*$, write $UA = U \otimes A$ and $VA = V \otimes A$; these are, respectively, the path algebras of $\phi \otimes \mathrm{id}_A : \Lambda A \to {}_1M_2A$ and $\eta \otimes \mathrm{id}_A : A \to \Lambda A$. Because U and V are flat ℓ-modules, they define functors $U, V : kk^h \to kk^h$ by Example 3.2.29.

Remark 5.2.1. Recall the functors U', V' from Section 2.3. Using Remark 3.2.24 and the isomorphism $\widehat{A} \cong \Lambda A$, it follows that there are kk^h-equivalences

$$U \sim \Omega U'\ell$$
$$V \sim \Omega V'\ell.$$

In Lemmas 5.2.2 and 5.2.6 we recast the equivalences of (2.3.4) into the framework of kk^h.

Lemma 5.2.2. *There are kk^h-equivalences*

$$U\Lambda \sim \Lambda \text{ and}$$
$$V\Lambda \sim \Omega\Lambda.$$

Proof. Let us prove the first equivalence. To ease the notation we omit the functor j^h. Let

$$\Omega_1 M_2\Lambda \to U\Lambda \to \Lambda^2 \xrightarrow{\phi\otimes\mathrm{id}_\Lambda} {}_1M_2\Lambda$$

be the triangle in kk^h induced by the extension which defines $U\Lambda$. We have an isomorphism

$$\tau : \Lambda^2 \cong \Lambda \oplus \Lambda \tag{5.2.3}$$
$$(x_1, x_2) \otimes (x_3, x_4) \mapsto (x_1x_3, x_2x_4, x_1x_4, x_2x_3).$$

Put $\lambda_1 = (0, 1)$, $\lambda_2 = (1, 0)$ and $\iota_i : \Lambda \to {}_1M_2\Lambda$ as in Lemma 3.1.11. Let $\jmath_i : \Lambda \to \Lambda \oplus \Lambda$ $(i = 1, 2)$ be the inclusions in each coordinate. Observe that

$$((\phi \otimes \mathrm{id}_\Lambda) \circ \tau^{-1} \circ \jmath_1)(x, y) = \begin{pmatrix} (x, 0) & (0, 0) \\ (0, 0) & (0, y) \end{pmatrix}.$$

The matrix

$$u = \begin{pmatrix} (1, -1) & (1, 1) \\ (0, 0) & (-1, 1) \end{pmatrix} \in {}_1M_2(\mathrm{ind}(\tilde{B})) \tag{5.2.4}$$

is unitary and satisfies

$$\mathrm{ad}(u) \circ \iota_1 = (\phi \otimes \mathrm{id}_\Lambda) \circ \tau^{-1} \circ \jmath_1 : \Lambda \to {}_1M_2\Lambda.$$

So by Lemma 1.6.8, the following diagram commutes in kk^h

$$\begin{array}{ccc} \Lambda & \xrightarrow{\jmath_1} & \Lambda\oplus\Lambda \\ & \underset{\iota_1}{\overset{\sim}{\searrow}} & \Big\downarrow{\scriptstyle (\phi\otimes \mathrm{id}_\Lambda)\circ\tau^{-1}} \\ & & {}_1M_2\Lambda. \end{array}$$

Similarly, the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\jmath_2} & \Lambda\oplus\Lambda \\ & \underset{\iota_2}{\overset{\sim}{\searrow}} & \Big\downarrow{\scriptstyle (\phi\otimes \mathrm{id}_\Lambda)\circ\tau^{-1}} \\ & & {}_1M_2\Lambda \end{array}$$

commutes in kk^h. Let $pr_i : \Lambda\oplus\Lambda \to \Lambda$ $(i = 1,2)$ be the projections on each coordinate. By Lemma 3.1.11 we have $j^h(\iota_1) = j^h(\iota_2)$; thus, using the previous diagrams, the following solid arrow diagram commutes in kk^h:

$$\begin{array}{ccccc} U\Lambda & \longrightarrow & \Lambda^2 & \xrightarrow{\varphi\otimes\mathrm{id}_\Lambda} & {}_1M_2\Lambda \\ \Big\uparrow\vdots & & {\scriptstyle\tau^{-1}}\Big\uparrow\wr & & {\scriptstyle \iota_1=\iota_2}\Big\uparrow \\ \Lambda & \xrightarrow{\jmath_1-\jmath_2} & \Lambda\oplus\Lambda & \xrightarrow{\pi_1+\pi_2} & \Lambda. \end{array}$$

Since the lower row is split, it completes to a triangle by Remark 3.2.4. Then, because the middle and right vertical arrows are isomorphisms in kk^h, we get that the dashed map is an isomorphism in kk^h.

Next we prove the second isomorphism of the statement. Let

$$\Omega\Lambda^2 \to V\Lambda \to \Lambda \xrightarrow{\eta\otimes\mathrm{id}_\Lambda} \Lambda^2$$

be the triangle in kk^h induced by the extension defining $V\Lambda$. Let t be as in (5.1.6); one checks that the following square commutes

$$\begin{array}{ccc} \Lambda & \xrightarrow{\eta\otimes\mathrm{id}_\Lambda} & \Lambda^2 \\ \Big\| & & \Big\downarrow{\scriptstyle\tau} \\ \Lambda & \xrightarrow{\jmath_1+\jmath_2 t} & \Lambda\oplus\Lambda. \end{array} \tag{5.2.5}$$

The map $\jmath_1 + \jmath_2 t$ completes to a split distinguished triangle in kk^h

$$\Omega\Lambda \to \Lambda \xrightarrow{\jmath_1+\jmath_2 t} \Lambda\oplus\Lambda \xrightarrow{\pi_1 - t\pi_2} \Lambda.$$

Rotating the split triangle above we get the triangle

$$\Omega(\Lambda\oplus\Lambda) \to \Omega\Lambda \xrightarrow{0} \Lambda \xrightarrow{\jmath_1+\jmath_2 t} \Lambda\oplus\Lambda.$$

Finally, (5.2.5) extends to a commutative diagram in kk^h:

$$\begin{array}{ccccc} V\Lambda & \longrightarrow & \Lambda & \xrightarrow{\eta\otimes\mathrm{id}_\Lambda} & \Lambda^2 \\ \vdots\Big\downarrow & & \Big\| & & \Big\downarrow{\scriptstyle\tau} \\ \Omega\Lambda & \xrightarrow{0} & \Lambda & \xrightarrow{\jmath_1+\jmath_2 t} & \Lambda. \end{array}$$

It follows that the dashed map is an isomorphism. $\square$

Lemma 5.2.6. *There is a kk^h-equivalence*

$$\Sigma VU \sim \ell.$$

In particular, $VU \sim \Omega$.

Proof. As before, we omit j^h from the notation. In view of Lemma 3.1.11, it suffices to show that ΣVU is kk^h-equivalent to ${}_1M_2$. Let

$$\Omega\Lambda U \to VU \to U \xrightarrow{\eta\otimes \mathrm{id}_U} \Lambda U$$

be the triangle in kk^h induced by the extension that defines VU. The kk^h isomorphism between $\Lambda U = U\Lambda$ and Λ established in Lemma 5.2.2 is induced by mapping Λ^2 to $\Lambda \oplus \Lambda$ and then retracting onto the first coordinate. Using this fact we get that there is a map of triangles in kk^h

$$\begin{array}{ccccccc} U & \xrightarrow{\eta\otimes \mathrm{id}_U} & \Lambda U & \longrightarrow & \Sigma VU & \longrightarrow & \Sigma U \\ \| & & \wr\downarrow & & \downarrow & & \| \\ U & \longrightarrow & \Lambda & \xrightarrow{\phi} & {}_1M_2 & \longrightarrow & \Sigma U. \end{array}$$

It follows that the dashed kk^h-map is an isomorphism. □

Remark 5.2.7. By Example 3.2.29, the isomorphisms of Lemmas 5.2.2 and 5.2.6 induce kk^h-equivalences $U\Lambda A \sim \Lambda A$, $V\Lambda A \sim \Omega\Lambda A$ and $VUA \sim \Omega A$ for every $A \in Alg^*_\ell$.

5.3 Bivariant version of Karoubi's Fundamental Theorem

Recall from Corollary 2.3.6, the element $\theta \in KH^h_2({}_{-1}M_2(U')^2\ell)$. Using Remark 5.2.1 and Theorem 4.2.1, we get an element $\theta \in kk^h(\ell, {}_{-1}M_2U^2)$. Also, recall the product induced by the tensor product from Proposition 3.2.30.

Theorem 5.3.1. *For all $A \in Alg^*_\ell$, the product with θ induces a natural isomorphism*

$$\theta_A := \theta \otimes j^h(\mathrm{id}_A) : j^h(A) \cong j^h({}_{-1}M_2U^2A).$$

Proof. By Example 3.2.29, it suffices to show that $\theta = \theta_\ell$ is an isomorphism. Equivalently, we need to see that

$$kk^h(\ell,\theta)_* : kk^h(\ell,\ell) \to kk^h(\ell, {}_1M_2U^2) \text{ and}$$
$$kk^h({}_{-1}M_2U^2,\theta)_* : kk^h({}_{-1}M_2U^2,\ell) \to kk^h({}_{-1}M_2U^2, {}_{-1}M_2U^2)$$

are isomorphisms.

Taking into account hermitian stability and using Lemma 5.2.6, we see that $kk^h({}_{-1}M_2U^2, \theta)_*$ is an isomorphism if and only if

$$kk^h(\ell, \theta_{{}_{-1}M_2(\Sigma V)^2})_* : kk^h(\ell, {}_{-1}M_2(\Sigma V)^2) \to kk(\ell, {}_{-1}M_2(\Sigma VU)^2)$$

is an isomorphism. Hence the theorem will follow if we prove that $(\theta_A)_* := kk^h(\ell, \theta_A)$ is an isomorphism for all A.

By Proposition 3.2.35 and the isomorphism of Theorem 4.2.1, the map $(\theta_A)_*$ corresponds to the cup-product with θ, which by Corollary 2.3.6 is an isomorphism. □

Corollary 5.3.2. *Let $\varepsilon \in \ell$ be unitary. For every $A \in Alg^*_\ell$, there is a kk^h-equivalence*

$${}_{\varepsilon}M_2VA \sim {}_{-\varepsilon}M_2U\Omega A.$$

Proof. It is immediate from Theorem 5.3.1, Lemma 5.2.6 and Remark 5.2.7 that $VA \sim {}_{-1}M_2U\Omega A$. The corollary follows from this applied to ${}_{\varepsilon}M_2A$ using the isomorphism

$${}_{-1}M_2({}_{\varepsilon}M_2) \cong M_{\pm}({}_{-\varepsilon}M_2)$$

and hermitian stability. □

Lemma 5.3.3. *Consider the kk^h-equivalences $U\Lambda \sim \Lambda$ of Lemma 5.2.2 and $M_2\Lambda \cong {}_{-1}M_2\Lambda$ of Remark 5.1.5. Then the following diagram commutes in kk^h:*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\theta_\Lambda} & {}_{-1}M_2U^2\Lambda \\ \downarrow{\scriptstyle i_2} & & \wr\downarrow \\ M_2\Lambda & \xrightarrow{\sim} & {}_{-1}M_2\Lambda \end{array}$$

Proof. By part i) of Theorem 2.3.3, we have a commutative diagram in kk^h, where as usual we have omitted j^h,

$$\begin{array}{ccccc} \ell & \xrightarrow{\theta} & {}_{-1}M_2U^2 & \longrightarrow & {}_{-1}M_2\Lambda U \\ \downarrow{\scriptstyle i_2\eta} & & & & \downarrow \\ M_2\Lambda & & \xrightarrow{\sim} & & {}_{-1}M_2\Lambda. \end{array} \tag{5.3.4}$$

Let $p = pr_1 \circ \tau : \Lambda^2 \to \Lambda$; we have

$$p((x_1, x_2) \otimes (x_3, x_4)) = (x_1x_3, x_2x_4).$$

Tensoring (5.3.4) with Λ and composing the resulting vertical maps with those induced by p, we get another commutative diagram

$$\begin{array}{ccccc} \Lambda & \xrightarrow{\theta_\Lambda} & {}_{-1}M_2U^2\Lambda & \longrightarrow & {}_{-1}M_2\Lambda U\Lambda \\ \downarrow{\scriptstyle i_2} & & & & \downarrow \\ M_2\Lambda & & \xrightarrow{\sim} & & {}_{-1}M_2\Lambda. \end{array} \tag{5.3.5}$$

Using the fact that the kk^h-equivalence $U\Lambda \sim \Lambda$ is induced by first mapping to Λ^2 and then applying p, we obtain a commutative diagram in kk^h

$$\begin{array}{ccc} U^2\Lambda & \longrightarrow & \Lambda U\Lambda \\ \downarrow & \swarrow & \\ \Lambda & & \end{array}$$

Tensoring with ${}_{-1}M_2$ we obtain that the composite ${}_{-1}M_2U^2\Lambda \to {}_{-1}M_2\Lambda$ in diagram (5.3.5) is the map in the diagram of the proposition, finishing the proof. □

The bivariant 12-term exact sequence

Definition 5.3.6 (cf. Definition 2.3.7)**.** Let $A, B \in Alg^*_\ell$, $\varepsilon \in \ell$ unitary, ${}_\varepsilon kk^h(A,B)$ as in Definition 3.2.31 and t as in (5.1.6). Let $\eta : \ell \to \Lambda$ and $\varphi : \Lambda \to {}_1M_2$ be as in (5.1.3) and (5.1.4). Put $\overline{\varphi} = j^h(\iota_1)^{-1} \circ j^h(\varphi)$. Set

$$\begin{aligned} {}_\varepsilon W(A,B) &:= \operatorname{coker}({}_\varepsilon kk^h(A,\Lambda B) \xrightarrow{\overline{\varphi}_*} {}_\varepsilon kk^h(A,B)) \\ {}_\varepsilon W'(A,B) &:= \ker({}_\varepsilon kk^h(A,B) \xrightarrow{\eta_*} {}_\varepsilon kk^h(A,\Lambda B)) \\ k(A,B) &:= \{x \in kk^h(A,\Lambda B) : x = t_*x\}/\{x = y + t_*y \text{ for some } y\} \\ k'(A,B) &:= \{x \in kk^h(A,\Lambda B) : x = -t_*x\}/\{x = y - t_*y \text{ for some } y\} \end{aligned}$$

If $\varepsilon = 1$ we omit it from the notation. Note that k and k' do not need the ε prescript due to the isomorphism in Remark 5.1.5.

Theorem 5.3.7 ([cf. Kar80, Théorème 4.3])**.** *There is an exact sequence*

$$\begin{array}{ccccccccccc} k(A,\Omega B) & \to & {}_{-1}W(A,\Omega^2 B) & \to & W'(A,B) & \to & k'(A,\Omega B) & \to & {}_{-1}W'(A,\Omega B) & \to & {}_{-1}W(A,\Omega B) \\ \uparrow & & & & & & & & & & \downarrow \\ W(A,\Omega B) & \leftarrow & W'(A,\Omega B) & \leftarrow & k'(A,\Omega B) & \leftarrow & {}_{-1}W'(A,B) & \leftarrow & W(A,\Omega^2 B) & \leftarrow & k(A,\Omega B) \end{array}$$

Proof. As above, we omit j^h in our notation. Write ν for the map obtained upon tensoring the canonical map $U \to \Lambda$ with $\Omega_{-1}M_2$. Consider the following distinguished triangles in kk^h

$$\begin{array}{ccccccc} \Omega\Lambda & \xrightarrow{\partial} & V & \longrightarrow & \ell & \xrightarrow{\eta} & \Lambda \\ & & \wr\downarrow\theta & & & & \\ \Omega^2{}_{-1}M_2 & \xrightarrow{\delta} & \Omega_{-1}M_2U & \xrightarrow{\nu} & \Omega_{-1}M_2\Lambda & \xrightarrow{\Omega_{-1}M_2\phi} & \Omega_{-1}M_2. \end{array}$$

Recall $\tau : \Lambda^2 \cong \Lambda \oplus \Lambda$ from (5.2.3) and let $\widetilde{\tau} : \Omega_{-1}M_2\Lambda^2 \to \Omega(\Lambda \oplus \Lambda)$ be the composite in kk^h of the isomorphism Remark 5.1.5, the inverse of the corner inclusion, and

$\Omega\tau$. Using Lemma 5.3.3 we get the following commutative diagram in kk^h:

$$\begin{array}{ccccccc}
\Omega\Lambda & \xrightarrow{\partial} & V & \xrightarrow[\sim]{\theta} & \Omega_{-1}M_2U & \xrightarrow{\nu} & \Omega_{-1}M_2\Lambda \\
\downarrow{\scriptstyle \Omega\Lambda\eta} & & \downarrow{\scriptstyle V\eta} & & \downarrow{\scriptstyle \Omega_{-1}M_2U\eta} & & \downarrow{\scriptstyle \Omega(_{-1}M_2\Lambda)\eta} \\
\Omega\Lambda^2 & \xrightarrow{\partial} & V\Lambda & \xrightarrow[\sim]{\theta} & \Omega_{-1}M_2U\Lambda & \xrightarrow{\nu} & \Omega_{-1}M_2\Lambda^2 \\
\downarrow{\scriptstyle \Omega\tau} & & \wr\downarrow & & \wr\downarrow & & \downarrow{\scriptstyle \widetilde{\tau}} \\
\Omega(\Lambda\oplus\Lambda) & \xrightarrow{\Omega(\pi_1 - t\pi_2)} & \Omega\Lambda & = & \Omega\Lambda & \xrightarrow[\Omega(\jmath_1-\jmath_2)]{} & \Omega(\Lambda\oplus\Lambda).
\end{array}$$

A direct computation shows that $\tau\circ(\Lambda\eta):\Lambda\to\Lambda\oplus\Lambda$ is the diagonal map. Hence from the diagram we get following equality in $kk^h(\Omega\Lambda,\Omega(\Lambda\oplus\Lambda))$

$$\widetilde{\tau}(\Omega_{-1}M_2\eta)\nu\theta\partial = \Omega((\jmath_1-\jmath_2))(\pi_1-t\pi_2)(\jmath_1+\jmath_2)). \tag{5.3.8}$$

Similarly, for h_{-1} as in Example 1.1.18 and i_2 the upper left-hand corner inclusion, we have in $kk^h(\Omega_{-1}M_2\Lambda,\Omega(\Lambda\oplus\Lambda)$

$$\widetilde{\tau}(\Omega_{-1}M_2\eta) = \Omega(\jmath_1+\jmath_2)(i_2)^{-1}\operatorname{ad}(1,h_{-1}^{-1}).$$

Therefore, composing both sides of the equality (5.3.8) on the left with the projection onto the first coordinate, we get

$$(\iota_1)^{-1}\operatorname{ad}(1,h_{-1}^{-1})\nu\theta\partial = \Omega(\pi_1-t\pi_2)(\jmath_1+\jmath_2)) = \mathrm{id}-t.$$

Thus, after using Remark 5.1.5 and hermitian stability, with the identification

$$kk^h(\Omega\Lambda,\Omega_{-1}M_2\Lambda)\cong kk^h(\Omega\Lambda,\Omega\Lambda),$$

the composition $\nu\theta\partial$ corresponds to $\mathrm{id}-t$.

Because the $*$-algebras involved in the argument above are flat, for any $B\in Alg_\ell^*$ we map apply the functor $-\otimes B$ of Example 3.2.29 to obtain the same identity in $kk^h(\Omega\Lambda B,\Omega\Lambda B)$.

Finally, apply the functor $kk^h(A,-)$ and the rest of the proof proceeds exactly as in [Kar80, Théorème 4.3]. □

Corollary 5.3.9. *Let $\mathfrak{C}$ and $H: Alg_\ell^*\to\mathfrak{C}$ be as in Proposition 3.2.22. The same argument as in Theorem 5.3.7 proves an analogous exact sequence for the groups obtained substituting $H(-)$ for $kk^h(A,-)$ in Definition 5.3.6.*

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