ERROR ESTIMATES IN SOBOLEV SPACES FOR MOVING LEAST SQUARE APPROXIMATIONS *

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Abstract. The aim of this paper is to obtain error estimates for moving least square approximations in $\mathbb{R}^N$. We prove that, under appropriate hypothesis on the weight function and the distribution of points, the method produces optimal order error estimates in L^∞ and L^2 for the approximations of the function and its first derivatives. These estimates are important in the analysis of Galerkin approximations based on the moving least square method. In particular, our results provides error estimates, optimal in order and regularity, for second order coercive problems.

Key words. error estimates, moving least square, meshless method

AMS subject classifications. 65N15, 65N30, 65D10.

1. Introduction. The moving least square method, MLS ([10],[8]), has been used for the numerical solution of differential equations in several papers ([11], [2], [3], [1]). For this kind of application it is very important to obtain error estimates for the function and its derivatives which are not known up to now for the N-dimensional case.

In [9] Levin analyzes the MLS method for a particular weight function obtaining error estimates in the uniform norm for the approximation of a regular function in N dimensions. However, he does not obtain error estimates for the derivatives. In [1], Armentano and Durán prove error estimates in L^∞ for the function and its derivatives in the one dimensional case. They consider a weight function with compact support, as is usually done in the application of MLS for differential equations.

In this paper we obtain error estimates in L^∞ in N-dimensions which generalizes the result given in [1]. The arguments used in one dimensional case can not be extended straight forward to higher dimensions. Indeed, the proof given in [1], is based on the fact that the best polynomial approximation of degree m to a function u, interpolates it at $m+1$ points. As fas as we know, an analogous result is not known in more dimensions.

On the other hand, for the analysis of Galerkin approximations it is enough to have error estimates in L^2 based Sobolev norms. It is natural to expect that these kinds of estimates hold under less regularity assumptions on the function than those required for the approximation in the uniform norm. In this work we obtain error estimates in L^2 norm for the approximation of the function and its first derivatives by the moving least square method in N dimensions under optimal regularity assumptions. These estimates are obtained under hypotheses rather general on the set of points and on the weight function used.

So, these results provide the first proof of convergence of Galerkin approximation based on the MLS method for second order coercive problems in the N-dimensional case.

The paper is organized as follows. First, in the rest of this Section, we present the moving least square method. In Section 2 we obtain error estimates for the function and its first derivatives in L^∞. Section 3 deals with the error estimates for the function

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and its first derivatives in L^2. Finally, in Section 4 we analyze some particular cases for the bidimensional case and prove that they verify the hypotheses required in Section 2 and 3.

The Moving Least Square Method . The Moving Least Square Method can be described as follows: Given $R > 0$ let $0 \leq \Phi_R \leq 1$ be a function such that $supp\ \Phi_R \subset \overline{B_R(0)} = \{z|\ \|z\| \leq R\}$ and $\xi_1, \xi_2, \ldots, \xi_n$ points in $\Omega \subset \mathbb{R}^N$ a convex set and let $u_1, u_2, \ldots u_n$ be the values of the function u in those points, i.e, $u_j = u(\xi_j)$, $1 \leq j \leq n$.

We denote by $\mathcal{P}_m$ the set of all polynomials of degree m or less and $s = dim\ \mathcal{P}_m$. Let $\{p_1, \cdots, p_s\}$ be a basis of $\mathcal{P}_m$.
For each $x \in \Omega$ we consider $P^*(x,y) = \sum_{k=1}^{s} p_k(y)\alpha_k(x)$ where α is chosen such that

$$J_x(\alpha) = \sum_{j=1}^{n} \Phi_R(x - \xi_j)(u_j - \sum_{k=1}^{s} p_k(\xi_j)\alpha_k(x))^2$$

is minimized. We define the approximation of u as

$$\hat{u}(x) = P^*(x,x) = \sum_{k=1}^{s} p_k(x)\alpha_k(x) \tag{1.1}$$

This approximation will be well defined if we guarantee that the minimization problem has a solution. So we consider the following property:

Property P: For each $x \in \Omega$, there is a subset of s points of $\{\xi_1, \cdots, \xi_n\}$ where the Lagrange interpolation is possible and such that $\Phi_R(x - \xi_j) > 0$ for those points ξ_j.

REMARK 1.1. *For $N = 1$ it is enough to have $m + 1$ points among $\{\xi_1, \cdots, \xi_n\}$ such that $\Phi_R(x - \xi_j) > 0$ for those points ξ_j. So, in this particular case our hypothesis is the same used in [1].*

Therefore we can define $< f, g >_x = \sum_{j=1}^{n} \Phi_R(x - \xi_j) f(\xi_j) g(\xi_j)$ and $\|f\|_x^2 = < f, f >_x$. So by a classical result (see for example [6]) we have

THEOREM 1.1. *Assume that the weight function satisfies property P. Then, for any $x \in \Omega$ there exists $P^*(x, \cdot) \in \mathcal{P}_m$ which satisfies $\|u - P^*(x,.)\|_x \leq \|u - P\|_x$ for all $P \in \mathcal{P}_m$.*

Since the polynomial $P^*(x,y)$ can be obtained solving the normal equations for the minimization problem, an easy calculation shows that $P^*(x,x)$ may be written as

$$P^*(x,x) = \sum_{j=1}^{n} \beta_j(x) u_j \tag{1.2}$$

where β_j are functions with compact support and the same regularity than Φ ([8], section 2). We also note that if $u \in \mathcal{P}_k$ with $k \leq m$, then $\hat{u} = u$.

2. Error Estimates in L^∞ and $W^{1,\infty}$. In this section we will obtain error estimates in L^∞, in terms of the parameter R, in the approximation of u and its first derivatives.

We introduce the following properties on the weight function. In order to simplify notation we will drop the subscript R from the weight function Φ_R. All the constants appearing below are independent of R.

1. For each $x \in \Omega$ there exists a subset of s points of $\{\xi_1, \cdots, \xi_n\}$ in $B_{\frac{R}{2}}(x)$ where the Lagrange interpolation is possible.
2. $\exists c_0 > 0$ such that $\Phi(z) \geq c_0 \quad \forall z \in B_{\frac{R}{2}}(0)$.
3. $\exists c_\#$ such that for all $x \in \Omega$, $card\{\xi_j \in B_R(x), 1 \leq j \leq n\} < c_\#$.
4. For any $x \in \Omega$ there exists a constant c_L such that the Lagrange basis functions associated with the set of points of 1) are bounded for c_L in $B_{2R}(x)$.
5. For any $x \in \Omega$ there exists a constant c'_L such that the derivatives of the Lagrange basis functions associated with the set of points of 1) are bounded for $\frac{c'_L}{R}$ in $B_{2R}(x)$.
6. $\Phi \in C^1(B_R(0)) \cap W^{1,\infty}(\mathbb{R}^N)$ and $\exists c_1 > 0$ such that $\|\nabla\Phi(z)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{c_1}{R}$.

First, we obtain the approximation order of $\hat{u}$ to u. We observe that in one dimensional case $P^*(x, \cdot)$ is an interpolation polynomial of the function u in some points in $B_R(x)$ [1], but, as far as we know, it is not known whether this is true in more dimensions. So, we introduce for any $x \in \Omega$ $P_I(x, \cdot)$ the polynomial in $\mathcal{P}_m$ that interpolates u at the points $\in B_{\frac{R}{2}}(x)$ that satisfy property 1) and use it to obtain the error estimates.

THEOREM 2.1. *Let $x \in \Omega$, if properties 1) to 4) hold then, there exists a constant C which depends on $c_0, c_\#$ and c_L such that $\forall y \in B_R(x) \cap \Omega$:*

$$|u(y) - P^*(x, y)| \leq C\|u - P_I(x, \cdot)\|_{L^\infty(B_R(x)\cap\Omega)} \tag{2.1}$$

where $P_I(x, \cdot)$ is the polynomial in $\mathcal{P}_m$ that interpolates u at the points $\in B_{\frac{R}{2}}(x)$ that satisfy property 1) . So, in particular taking $y = x$ we have that

$$|u(x) - \hat{u}(x)| \leq C\|u - P_I(x, \cdot)\|_{L^\infty(B_R(x)\cap\Omega)} \tag{2.2}$$

where $\hat{u}(x) = P^*(x, x)$

Proof. Given $x \in \Omega$, in view of property 1) there exist points $\xi_{j_1}, \xi_{j_2}, \cdots, \xi_{j_s} \in B_{\frac{R}{2}}(x)$ such that the Lagrange basis function $l_{j_i}, 1 \leq i \leq s$ are well defined and the interpolation polynomial $P_I(x, \cdot)$ exists. Using property 2) and the fact that P^* attains the minimum we have

$$\begin{aligned}
\sum_{k\in\{j_i\}} (u(\xi_k) - P^*(x, \xi_k))^2 &\leq \frac{1}{c_0} \sum_{k\in\{j_i\}} \Phi(x - \xi_k)(u(\xi_k) - P^*(x, \xi_k))^2 \\
&\leq \frac{1}{c_0} \sum_{k=1}^{n} \Phi(x - \xi_k)(u(\xi_k) - P^*(x, \xi_k))^2 \\
&\leq \frac{1}{c_0} \sum_{k=1}^{n} \Phi(x - \xi_k)(u(\xi_k) - P_I(x, \xi_k))^2 \\
&= \frac{1}{c_0} \sum_{\xi_k\in B_R(x)\cap\Omega} \Phi(x - \xi_k)(u(\xi_k) - P_I(x, \xi_k))^2 \\
&\leq C(c_0, c_\#)\|u - P_I(x, \cdot)\|^2_{L^\infty(B_R(x)\cap\Omega)}
\end{aligned} \tag{2.3}$$

We note that $\forall y \in B_R(x) \cap \Omega$

$$|P_I(x, y) - P^*(x, y)|^2 \leq s \sum_{k\in\{j_i\}} |l_k(y)|^2 |P_I(x, \xi_k) - P^*(x, \xi_k)|^2 = s \sum_{k\in\{j_i\}} |l_k(y)|^2 |u(\xi_k) - P^*(x, \xi_k)|^2$$

so using property 4) and (2.3) we have that

$$|P_I(x, y) - P^*(x, y)| \leq C(c_0, c_\#, c_L)\|u - P_I(x, \cdot)\|_{L^\infty(B_R(x)\cap\Omega)} \tag{2.4}$$

and therefore

$$|u(y) - P^*(x,y)| \le C\|u - P_I(x,\cdot)\|_{L^\infty(B_R(x)\cap\Omega)} \quad \forall y \in B_R(x)\cap\Omega$$

In particular taking $y = x$ we conclude the proof. ∎

Now, we want to estimate the error in the approximation of the derivatives of u. This estimate is a consequence of the following Lemma which is a generalization of Lemma 3.1 in [1]

LEMMA 2.2. *Let $x \in \Omega$ such that $\frac{\partial P^*(x,y)}{\partial x_j}$ exists for $j = 1, \cdots, N$. If properties 1) to 6) hold then, there exists a constant $C = C(c_0, c_\#, c_1, c_L)$ such that $\forall y \in B_R(x)\cap\Omega$, $j = 1, \cdots, N$:*

$$|\frac{\partial P^*(x,y)}{\partial x_j}| \le \frac{C}{R}\|u - P_I(x,\cdot)\|_{L^\infty(B_{2R}(x)\cap\Omega)} \tag{2.5}$$

Proof. To simplify notation we prove the result for $j = 1$ (Clearly the same argument applies to any j). Given $x \in \Omega$, in view of property 1) there exist points $\xi_{j_1}, \xi_{j_2}, \cdots, \xi_{j_s} \in B_{\frac{R}{2}}(x)$ such that the polynomial interpolation exists. We will denote the set $\{j_1, \cdots, j_s\}$ by $\{j_i\}$. For any $h > 0$ we define

$$S(x) = \sum_{k\in\{j_i\}} |P^*((x_1+h, x_2, \cdots, x_N), \xi_k) - P^*(x, \xi_k)|^2 \tag{2.6}$$

Then, by property 2)

$$\begin{aligned}
S(x) &\le \frac{1}{c_0}\sum_{k\in\{j_i\}} \Phi(x-\xi_k)(P^*((x_1+h, x_2, \cdots, x_N), \xi_k) - P^*(x,\xi_k))^2 \\
&\le \frac{1}{c_0}\sum_{k=1}^{n} \Phi(x-\xi_k)(P^*((x_1+h, x_2, \cdots, x_N), \xi_k) - P^*(x,\xi_k))^2 \\
&= \frac{1}{c_0}\sum_{k=1}^{n} \Phi(x-\xi_k)(P^*((x_1+h, x_2, \cdots, x_N), \xi_k) - P^*(x,\xi_k))(P^*((x_1+h, x_2, \cdots, x_N), \xi_k) - u(\xi_k)) \\
&+ \frac{1}{c_0}\sum_{k=1}^{n} \Phi(x-\xi_k)(P^*((x_1+h, x_2, \cdots, x_N), \xi_k) - P^*(x,\xi_k))(u(\xi_k) - P^*(x,\xi_k))
\end{aligned}$$

Let Q be the polynomial of degree $\le m$ defined by $Q(y) = P^*((x_1+h, x_2, \cdots, x_N), y) - P^*(x,y)$, then since P^* is the orthogonal projection in $<,>_x$ we have

$$< u(y) - P^*(x,y), Q(y) >_x = \sum_{k=1}^{n} \Phi(x-\xi_k)Q(\xi_k)(u(\xi_k) - P^*(x,\xi_k)) = 0 \tag{2.7}$$

Then,

$$S(x) \le \frac{1}{c_0}\sum_{k=1}^{n} \Phi(x-\xi_k)Q(\xi_k)(P^*((x_1+h, x_2, \cdots, x_N), \xi_k) - u(\xi_k)) \tag{2.8}$$

Form property 6) $\exists\, \theta_k = \theta(\xi_k, x)$ such that if h is small enough $\Phi(x-\xi_k) = \Phi((x_1+h, x_2, \cdots, x_N) - \xi_k) - \frac{\partial\Phi}{\partial x_1}(\theta_k)h$. Then, replacing in (2.8) we obtain

$$S(x) \le \frac{1}{c_0}\sum_{k=1}^{n} \Phi((x_1+h, x_2, \cdots, x_N) - \xi_k)Q(\xi_k)(P^*((x_1+h, x_2, \cdots, x_N), \xi_k) - u(\xi_k))$$

$$-\frac{h}{c_0}\sum_{k=1}^{n}\frac{\partial\Phi}{\partial x_1}(\theta_k)Q(\xi_k)(P^*((x_1+h,x_2,\cdots,x_N),\xi_k)-u(\xi_k))$$

Then using that $< u(y)-P^*((x_1+h,x_2,\cdots,x_N),y),Q(y) >_{(x_1+h,x_2,\cdots,x_N)}= 0$ and $\frac{\partial\Phi}{\partial x_1}$ has compact support together with property 6) we have

$$\begin{aligned}
S(x) &\leq \frac{h}{c_0}\sum_{k=1}^{n}\frac{\partial\Phi}{\partial x_1}(\theta_k)Q(\xi_k)(u(\xi_k)-P^*((x_1+h,x_2,\cdots,x_N),\xi_k))\\
&\leq \frac{h}{c_0}\sum_{k=1}^{n}|\frac{\partial\Phi}{\partial x_1}(\theta_k)||Q(\xi_k)||(P^*((x_1+h,x_2,\cdots,x_N),\xi_k)-u(\xi_k))|\\
&\leq \frac{hc_1}{Rc_0}\sum_{\xi_k\in B_{2R}(x)}|Q(\xi_k)||(P^*((x_1+h,x_2,\cdots,x_N),\xi_k)-u(\xi_k))|
\end{aligned}$$

From Theorem 2.1 we know that

$$\begin{aligned}
|(P^*((x_1+h,x_2,\cdots,x_N),\xi_k)-u(\xi_k))| &\leq C\|u-P_I((x_1+h,x_2,\cdots,x_N),\cdot)\|_{L^\infty(B_R(x_1+h,x_2,\cdots,x_N)\cap\Omega)}\\
&\leq C\|u-P_I((x_1+h,x_2,\cdots,x_N),\cdot)\|_{L^\infty(B_{2R}(x)\cap\Omega)}
\end{aligned}$$

where $P_I((x_1+h,x_2,\cdots,x_N),\cdot)$ is a polynomial that interpolates u at the points given by property 1). We can suppose for h small enough that the points $\{\xi_{j_1},\cdots,\xi_{j_s}\}\in B_{\frac{R}{2}}(x)$ are in $B_{\frac{R}{2}}((x_1+h,x_2,\cdots,x_N)$, so $P_I((x_1+h,x_2,\cdots,x_N),\cdot)$ can be chosen as $P_I(x,\cdot)$ then we have

$$S(x)\leq C\frac{h}{R}\||u-P_I(x,\cdot)\|_{L^\infty(B_{2R}(x)\cap\Omega)}\sum_{\xi_k\in B_{2R}(x)}|Q(\xi_k)| \tag{2.9}$$

Since Q is a polynomial of degree $\leq m$ it can be written as

$$Q(y)=\sum_{r\in\{j_i\}}Q(\xi_r)l_r(y) \tag{2.10}$$

where $l_r(y)$ are the Lagrange polynomial basis functions and therefore,

$$\sum_{\xi_k\in B_{2R}(x)}|Q(\xi_k)|\leq\sum_{r\in\{j_i\}}|Q(\xi_r)|(\sum_{\xi_k\in B_{2R}(x)}|l_r(\xi_k)|)$$

From property 4) there exist a constant c_L such that l_r are bounded $\forall r\in\{j_i\}$ and therefore

$$\begin{aligned}
(\sum_{k\in\{j_i\}}|Q(\xi_k)|)^2 &\leq s\sum_{k\in\{j_i\}}|Q(\xi_k)|^2 = sS(x)\\
&\leq C\frac{h}{R}\|u-P_I(x,\cdot)\|_{L^\infty(B_{2R}(x)\cap\Omega)}\sum_{k\in\{j_i\}}|Q(\xi_k)|
\end{aligned}$$

and therefore,

$$\sum_{k\in\{j_i\}}|Q(\xi_k)|\leq C\frac{h}{R}\|u-P_I(x,\cdot)\|_{L^\infty(B_{2R}(x)\cap\Omega)} \tag{2.11}$$

So, using this in (2.10) and property 4) again we obtain that $\forall y \in B_R(x) \cap \Omega$

$$\frac{|Q(y)|}{h} = \frac{|P^*((x_1+h, x_2, \cdots, x_N), y) - P^*(x,y)|}{h} \leq \frac{C}{R}||u - P_I(x,\cdot)||_{L^\infty(B_{2R}(x)\cap\Omega)}$$

and the proof concludes by taking $h \to 0$. ∎

The following Theorem states the order of approximation for the derivatives

THEOREM 2.3. *If properties 1) to 6) hold then, there exists $C = C(c_0, c_1, c_\#, c_L, c'_L)$ such that for any $x \in \Omega$ and $1 \leq j \leq N$*

$$|\frac{\partial}{\partial x_j}(u-\hat{u})(x)| \leq C\{\frac{1}{R}||u - P_I(x,\cdot)||_{L^\infty(B_{2R}(x)\cap\Omega)} + ||\frac{\partial}{\partial y_j}(u - P_I(x,\cdot))||_{L^\infty(B_R(x)\cap\Omega)}\}$$

Proof. We want to estimate $|\frac{\partial}{\partial x_j}\{u(x) - P^*(x,x)\}|$ for $1 \leq j \leq N$. We note that

$$\frac{\partial}{\partial x_j}P^*(x,x) = \{\frac{\partial P^*(x,y)}{\partial x_j} + \frac{\partial P^*(x,y)}{\partial y_j}\frac{\partial y_j}{\partial x_j}\}\Big|_{y=x} \tag{2.12}$$

So, we will estimate $|\frac{\partial u(y)}{\partial y_j} - \frac{\partial P^*(x,y)}{\partial x_j} - \frac{\partial P^*(x,y)}{\partial y_j}|$, $\forall y \in B_R(x) \cap \Omega$.

$$|\frac{\partial u(y)}{\partial y_j} - \frac{\partial P^*(x,y)}{\partial x_j} - \frac{\partial P^*(x,y)}{\partial y_j}| \leq |\frac{\partial u(y)}{\partial y_j} - \frac{\partial P_I(x,y)}{\partial y_j}| + |\frac{\partial P_I(x,y)}{\partial y_j} - \frac{\partial P^*(x,y)}{\partial y_j}| + |\frac{\partial P^*(x,y)}{\partial x_j}| \tag{2.13}$$

Using (2.4) and property 5) we have that

$$|\frac{\partial P_I(x,y)}{\partial y_j} - \frac{\partial P^*(x,y)}{\partial y_j}| \leq \sum_{k\in\{j_i\}} |\frac{\partial l_k(y)}{\partial y_j}||P_I(x,\xi_k) - P^*(x,\xi_k)| \leq \frac{C}{R}||u - P_I(x,\cdot)||_{L^\infty(B_R(x)\cap\Omega)} \tag{2.14}$$

So, using this in (2.13) together with Lemma 2.2 we conclude the proof. ∎

Now, we will obtain the error estimates for the approximation of the function u and its first derivatives. In view of the lemmas it is enough to estimate the error for the interpolation polynomial.

Since the Lagrange basis associated with the points given by property 1) satisfy properties 4) and 5), the error estimates follows by standard arguments for interpolation. So, we omit the proof of the following theorem and refer, for example, [7]

THEOREM 2.4. *Let $x \in \Omega$, $u \in C^{m+1}(\Omega)$ and properties 1), 4) and 5) hold then there exists a constant $C = C(c_L, c'_L)$ such that $\forall y \in B_R(x) \cap \Omega$,*

$$|u(y) - P_I(x,y)| \leq C \; max_{|\alpha|=m+1}||D^\alpha u||_{L^\infty(\Omega)}R^{m+1}$$

$$max_{|\alpha|=1}|D^\alpha(u(y) - P_I(x,y))| \leq C \; max_{|\alpha|=m+1}||D^\alpha u||_{L^\infty(\Omega)}R^m$$

where $P_I(x,\cdot)$ is the polynomial which interpolates u at the points given by property 1).

So, as a consequence of this Theorem and the error estimates given by Theorem 2.1 and Theorem 2.3 we have

COROLLARY 2.1. *If properties 1) to 6) hold and $u \in C^{m+1}(\Omega)$ then, there exists C which depends on $c_0, c_\#, c_1, c_L, c'_L$ such that for each $x \in \Omega$*

$$|u(x) - \hat{u}(x)| \leq C max_{|\beta|=m+1}||D^\beta u||_{L^\infty(\Omega)}R^{m+1} \tag{2.15}$$

$$|\nabla(u-\hat{u})(x)| \leq C \; max_{|\beta|=m+1}||D^\beta u||_{L^\infty(\Omega)}R^m \tag{2.16}$$

3. Error Estimates in L^2 and W_2^1. In the previous section we obtained optimal order in L^∞ and $W^{1,\infty}$ under the assumption that $u \in C^{m+1}$. Clearly, these results imply optimal order error estimates in L^2 and H^1. However, since in many cases the function to approximate is less regular, it is of interest to prove that the same order of convergence holds under weaker regularity assumptions, namely $u \in H^{m+1}$. This is the goal of this section.

Let $\{\eta_1, \cdots, \eta_l\}$ points in Ω such that $\Omega \subset \cup_{i=1}^{l} B_R(\eta_i)$ and the number of the balls that overlap is bounded independent of R.

So, we will estimate $\|u - \hat{u}\|_{L^2(B_R(\eta_j)\cap\Omega)}$ for any $j \in \{1, \cdots, l\}$

THEOREM 3.1. *Let $m+1 > \frac{N}{2}$, if properties 1) to 4) hold and $u \in H^{m+1}(\Omega)$ there exists a constant C which depends on $c_0, c_\#, c_L$ such that $\forall j \in \{1, \cdots, l\}$*

$$\|u - \hat{u}\|_{L^2(B_R(\eta_j)\cap\Omega)} \le CR^{m+1}|u|_{H^{m+1}(B_{2R}(\eta_j)\cap\Omega)}$$

Proof. For any $j \in \{1, \cdots, l\}$ we want to estimate

$$\int_{B_R(\eta_j)\cap\Omega} |u(y) - \hat{u}(y)|^2 dy = \int_{B_R(\eta_j)\cap\Omega} |u(y) - P^*(y,y)|^2 dy$$

From property 1), for any $y \in B_R(\eta_j) \cap \Omega$ there exist $\xi_{j_1}(y), \cdots, \xi_{j_s}(y)$ a subset of $\{\xi_1, \cdots, \xi_n\}$ points in $B_{\frac{R}{2}}(y) \cap \Omega$ such that the Lagrange interpolation is possible. So, we denote by $P_I u(y, \cdot) = P_I(y, \cdot)$ the polynomial that interpolates u at those points. From Theorem 2.1 we have

$$\int_{B_R(\eta_j)\cap\Omega} |u(y) - P^*(y,y)|^2 dy \le C \int_{B_R(\eta_j)\cap\Omega} \|u(y) - P_I(y,\cdot)\|^2_{L^\infty(B_R(y)\cap\Omega)} dy$$

For any $y \in \Omega$ let $T^m u$ be a polynomial of degree m such that if $u \in H^{m+1}(\Omega)$ we have that

$$\begin{aligned} &\|u - T^m u\|_{L^\infty(B_R(y)\cap\Omega)} \le CR^{m+1-\frac{N}{2}}|u|_{H^{m+1}(B_R(y)\cap\Omega)} \quad \text{for} \quad m+1 > \frac{N}{2} \\ &\|u - T^m u\|_{L^2(B_R(y)\cap\Omega)} \le CR^{m+1}|u|_{H^{m+1}(B_R(y)\cap\Omega)} \end{aligned} \tag{3.1}$$

We can choose, for example, $T^m u$ the Taylor polynomial averaged over $B_R(y) \cap \Omega$ (see [4]). Then

$$\|u - P_I u\|_{L^\infty(B_R(y)\cap\Omega)} \le \|u - T^m u\|_{L^\infty(B_R(y)\cap\Omega)} + \|P_I(T^m u - u)\|_{L^\infty(B_R(y)\cap\Omega)}$$

We observe that $P_I v(x) = \sum_{i=1}^{s} v(\xi_{j_i}) l_{j_i}(x)$ where l_{j_i} are the Lagrange basis, so for property 4) we have that

$$\|P_I v\|_{L^\infty(B_R(y)\cap\Omega)} \le C\|v\|_{L^\infty(B_R(y)\cap\Omega)}$$

where C depends of the bound of the Lagrange basis c_L.

Then, we have that

$$\begin{aligned} \|u - P_I u\|_{L^\infty(B_R(y)\cap\Omega)} &\le C\|u - T^m u\|_{L^\infty(B_R(y)\cap\Omega)} \\ &\le CR^{m+1-\frac{N}{2}}|u|_{H^{m+1}(B_R(y)\cap\Omega)} \end{aligned} \tag{3.2}$$

and therefore

$$\{\int_{B_R(\eta_j)\cap\Omega} |u(y)-P^*(y,y)|^2 dy\}^{\frac{1}{2}} \le CR^{m+1-\frac{N}{2}}\{\int_{B_R(\eta_j)\cap\Omega} |u|^2_{H^{m+1}(B_R(y)\cap\Omega)} dy\}^{\frac{1}{2}}$$
$$\le CR^{m+1-\frac{N}{2}}\{\int_{B_R(\eta_j)\cap\Omega} |u|^2_{H^{m+1}(B_{2R}(\eta_j)\cap\Omega)} dy\}^{\frac{1}{2}}$$
$$\le CR^{m+1}|u|_{H^{m+1}(B_{2R}(\eta_j)\cap\Omega)} \tag{3.3}$$

□

Now, we will obtain the error estimates for the derivatives

THEOREM 3.2. *Let $m+1 > \frac{N}{2}$, if properties 1) to 6) hold and $u \in H^{m+1}(\Omega)$. Then, there exists a constant C which depends on $c_0, c_{\#}, c_1, c_L, c'_L$ such that $\forall j \in \{1,\cdots,l\}$*

$$\|\nabla(u-\hat{u})\|_{L^2(B_R(\eta_j)\cap\Omega)} \le CR^m|u|_{H^{m+1}(B_{3R}(\eta_j)\cap\Omega)}$$

Proof. We will consider $\int_{B_R(\eta_j)\cap\Omega} |\frac{\partial u}{\partial y_k}(y) - \frac{\partial}{\partial y_k}P^*(y,y)|^2 dy$ for $k=1,\cdots,N$. By property 1), for any η_j , $1 \le j \le l$ there exists a polynomial $P_I(\eta_j,\cdot)$ that interpolates u at a subset of $\{\xi_1,\cdots,\xi_n\}$ in $B_{\frac{R}{2}}(\eta_j)\cap\Omega$ then,

$$\int_{B_R(\eta_j)\cap\Omega} |\frac{\partial u}{\partial y_k}(y) - \frac{\partial}{\partial y_k}P^*(y,y)|^2 dy \le c\{\int_{B_R(\eta_j)\cap\Omega} |\frac{\partial u}{\partial y_k}(y) - \frac{\partial P_I}{\partial y_k}(\eta_j,y)|^2 dy$$
$$+ \int_{B_R(\eta_j)\cap\Omega} |\frac{\partial P_I}{\partial y_k}(\eta_j,y) - \frac{\partial}{\partial y_k}P^*(y,y)|^2 dy\} \tag{3.4}$$

First we will consider the second term on the right hand side of (3.4). For any $y \in B_R(\eta_j)\cap\Omega$ and $z \in B_R(y)\cap\Omega$ we note that

$$\frac{\partial}{\partial y_k}P^*(y,y) = \{\frac{\partial P^*}{\partial y_k}(y,z) + \frac{\partial P^*}{\partial z_k}(y,z)\}\Big|_{z=y}$$

Then, we will estimate $|\frac{\partial P_I}{\partial z_k}(\eta_j,z) - (\frac{\partial P^*}{\partial y_k}(y,z) + \frac{\partial P^*}{\partial z_k}(y,z))|$ for any $z \in B_R(y)\cap\Omega$.

From Lemma 2.2 and the same argument used to obtain (3.2) we know that for any $z \in B_R(y)\cap\Omega$

$$|\frac{\partial P^*}{\partial y_k}(y,z)| \le \frac{C}{R}\|u-P_I(y,\cdot)\|_{L^\infty(B_R(y)\cap\Omega)} \le \frac{C}{R}\|u-T^m u\|_{L^\infty(B_R(y)\cap\Omega)} \le CR^{m-\frac{N}{2}}|u|_{H^{m+1}(B_{2R}(\eta_j)\cap\Omega)}$$

So we need to estimate $|\frac{\partial P_I}{\partial z_k}(\eta_j,z) - \frac{\partial P^*}{\partial z_k}(y,z)|$. We have that

$$|\frac{\partial P_I}{\partial z_k}(\eta_j,z) - \frac{\partial P^*}{\partial z_k}(y,z)| \le |\frac{\partial P_I}{\partial z_k}(\eta_j,z) - \frac{\partial P^*}{\partial z_k}(\eta_j,z)| + |\frac{\partial P^*}{\partial z_k}(\eta_j,z) - \frac{\partial P^*}{\partial z_k}(y,z)| \tag{3.5}$$

From (2.14) and the same argument used above we have, for the first term on the right hand side of (3.5), that

$$|\frac{\partial P_I}{\partial z_k}(\eta_j,z) - \frac{\partial P^*}{\partial z_k}(\eta_j,z)| \le \frac{C}{R}\|u-P_I(\eta_j,\cdot)\|_{L^\infty(B_R(\eta_j))} \le CR^{m-\frac{N}{2}}|u|_{H^{m+1}(B_R(\eta_j))}$$

For the second term on the right hand side of (3.5) we note that for any $y \in B_R(\eta_j) \cap \Omega$, $\frac{\partial P^*}{\partial z_k}(\eta_j, z)$ and $\frac{\partial P^*}{\partial z_k}(y, z)$ are polynomials in z of degree $m-1$. Let $\{\xi_{j_1}, \cdots, \xi_{j_s}\} \in B_{\frac{R}{2}(y)}$ be the points given by property 1). using Lemma 2.2 (it is easy to check that the result (2.5) holds $\forall y \in B_{2R}(x)$) together with property 4) we have

$$\begin{aligned}|\frac{\partial P^*}{\partial z_k}(y,z) - \frac{\partial P^*}{\partial z_k}(\eta_j, z)| &= |\sum_{i=1}^{s} \frac{\partial l_{j_i}}{\partial z_k}(z)(P^*(\eta_j, \xi_{j_i}) - P^*(y, \xi_{j_i}))| \\ &\leq \frac{C}{R}\sum_{i=1}^{s} |\nabla_y P^*(\theta_j, \xi_{j_i}) \cdot (\eta_j - y)| \leq \frac{C}{R}\|u - P_I(\theta_j, \cdot)\|_{L_\infty(B_{2R}(\theta_j)\cap\Omega)} \\ &\leq CR^{m-\frac{N}{2}}|u|_{H^{m+1}(B_{3R}(\eta_j)\cap\Omega)}\end{aligned}$$

So, for any $z \in B_R(y) \cap \Omega$ we have that

$$|\frac{\partial P_I}{\partial z_k}(\eta_j, z) - (\frac{\partial P^*}{\partial y_k}(y,z) + \frac{\partial P^*}{\partial z_k}(y,z))| \leq CR^{m-\frac{N}{2}}|u|_{H^{m+1}(B_{3R}(\eta_j))}$$

in particular taking $z = y$ we have

$$\{\int_{B_R(\eta_j)\cap\Omega} |\frac{\partial P_I}{\partial y_k}(\eta_j, y) - \frac{\partial}{\partial y_k}(P^*(y,y))|^2 dy\}^{\frac{1}{2}} \leq CR^m |u|_{H^{m+1}(B_{3R}(\eta_j)\cap\Omega)}$$

Now, we will consider the first term on the right hand side of (3.4).

$$\|\frac{\partial u}{\partial y_k} - \frac{\partial P_I}{\partial y_k}(\eta_j, \cdot)\|_{L^2(B_R(\eta_j)\cap\Omega)} \leq \|\frac{\partial u}{\partial y_k} - \frac{\partial T^m u}{\partial y_k}\|_{L^2(B_R(\eta_j)\cap\Omega)} + \|\frac{\partial P_I}{\partial y_k}(u - T^m u)\|_{L^2(B_R(\eta_j)\cap\Omega)} \tag{3.6}$$

But, since the Lagrange basis satisfies property 5) we have

$$\|\frac{\partial P_I}{\partial y_k}(v)\|_{L^2(B_R(\eta_j))} \leq \|v\|_{L^\infty(B_R(\eta_j)\cap\Omega)} \sum_{i=1}^{s} \|\frac{\partial l_{j_i}}{\partial y_k}(y)\|_{L^2(B_R(\eta_j)\cap\Omega)} \leq C\|v\|_{L^\infty(B_R(\eta_j)\cap\Omega)} R^{\frac{N}{2}-1}$$

Then, if $v \in H^2(B_R(\eta_j) \cap \Omega)$ using the Sobolev imbedding theorem we have that

$$\|v\|_{L^\infty(B_R(\eta_j)\cap\Omega)} \leq c\{R^{-\frac{N}{2}}\|v\|_{L^2(B_R(\eta_j)\cap\Omega)} + R^{-\frac{N}{2}+1}|D^\alpha v|_{L^2(B_R(\eta_j)\cap\Omega)} + R^{-\frac{N}{2}+2}|D^\beta v|_{L^2(B_R(\eta_j)\cap\Omega)}\}_{|\alpha|=1, |\beta|=2}$$

Then, using (3.1) the second term on right hand side of (3.6) satisfies

$$\|\frac{\partial P_I}{\partial y_k}(u - T^m u)\|_{L^2(B_R(\eta_j)\cap\Omega)} \leq \frac{C}{R}\|u - T^m u\|_{L^2(B_R(\eta_j)\cap\Omega)} \leq CR^m|u|_{H^{m+1}(B_R(\eta_j)\cap\Omega)}$$

For the first term on right hand side of (3.6) we have that [4]

$$\|\frac{\partial u}{\partial y_k} - \frac{\partial T^m u}{\partial y_k}\|_{L^2(B_R(\eta_j)\cap\Omega)} \leq CR^m|u|_{H^{m+1}(B_R(\eta_j)\cap\Omega)}$$

So, the theorem is proved. $\square$

4. Bounds for the Lagrange basis. In this section we will consider the case $N = 2$ and we will show different cases in which the Lagrange basis for $m = 1, 2$ satisfy properties 4) and 5).

For our subsequent analysis we introduce the following definition.

DEFINITION 4.1. *For any $x \in \Omega$, let T be a triangle of vertices $a^1, a^2, a^3 \in B_{\frac{R}{2}}(x)$ and ρ the diameter of the largest ball contained in T. We say that T satisfies the regularity condition with constant $\sigma > 0$, independent of R and ρ, if $\frac{R}{\rho} \leq \sigma$*

First, we consider the case $m = 1$. An easy calculation shows that the following results holds ([4],[5])

LEMMA 4.2. *Let $m = 1$. Given $x \in \Omega$, if there are a^1, a^2, a^3 in $B_{\frac{R}{2}}(x)$ such that the triangle T of vertices a^1, a^2, a^3 verifies the regularity condition then, there exists a constant $C = C(\sigma)$ such that the Lagrange basis l_i associated with these points satisfy, $\forall y \in B_{2R}(x) \cap \Omega$ and $i = 1, 2, 3$*

$$|l_i(y)| \leq C \tag{4.1}$$

$$|\frac{\partial l_i(y)}{\partial y_j}| \leq \frac{C}{R} \quad 1 \leq j \leq 2 \tag{4.2}$$

Now, we consider the case $m = 2$. We will obtain bounds for the Lagrange basis and its first derivatives for some distribution of points rather general as in Figure 1.

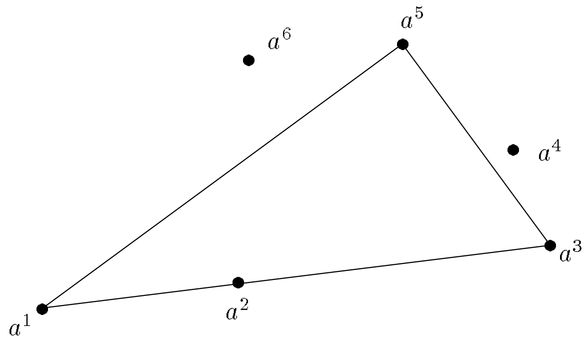

Figure 1

In order to present the ideas we will consider first two particular cases of the situation of Figure 1.

LEMMA 4.3. *Let $m = 2$. Given $x \in \Omega$ we suppose that there are points $a^j, 1 \leq j \leq 6$ in $B_{\frac{R}{2}}(x)$ such that we have some of the following situations*

i) $a^j, 1 \leq j \leq 6$ are distributed as in Figure 2 and the triangles of vertices $\{a^1, a^2, a^6\}$, $\{a^2, a^3, a^4\}$ and $\{a^4, a^5, a^6\}$ satisfy the regularity condition *ii)* $a^j, 1 \leq j \leq 6$ *are distributed as in Figure 3 and the triangles of vertices* $\{a^1, a^2, a^6\}$, $\{a^2, a^3, a^4\}$, $\{a^4, a^5, a^6\}$ *and* $\{a^1, a^2, a^5\}$ *satisfy the regularity condition*

then, there exists a constant $C = C(\sigma)$ *such that* $\forall y \in B_{2R}(x) \cap \Omega$ *the Lagrange basis* l_i, $1 \leq i \leq 6$ *satisfy*

$$\begin{aligned} |l_i(y)| &\leq C \\ |\frac{\partial l_i}{\partial y_j}(y)| &\leq \frac{C}{R} \qquad 1 \leq j \leq 2 \end{aligned}$$

Proof. Let $x \in \Omega$, we consider a triangle T of vertices a^1, a^3, a^5 in $B_{\frac{R}{2}}(x)$. Let L_{ij} be non-trivial linear functions that define the lines such that $a^i, a^j \in L_{ij}$.
First we will consider the case i). Let a^2, a^4, a^6 points on the lines L_{13}, L_{35}, L_{15} respectively as in Figure 2.

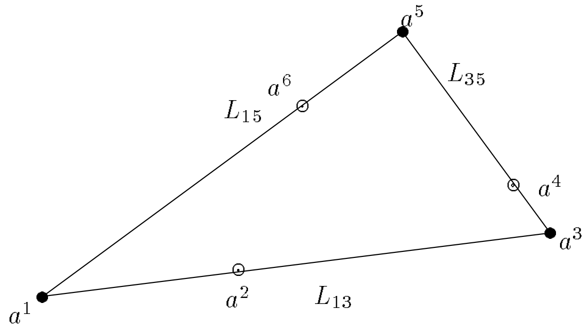

Figure 2

We will show that the Lagrange basis $l_1 \in \mathcal{P}_2$ is bounded (the same argument can be used to obtain the bounds for the other Lagrange basis).

Since $l_1 \in \mathcal{P}_2$ vanishes in $a^3, a^4, a^5 \in L_{35}$ it follows that, $l_1 = 0$ in L_{35} so we can write [4] $l_1 = L_{35} q_1$ where $q_1 \in \mathcal{P}_1$. But l_1 vanishes in a^2, a^6 then $q_1 = 0$ in L_{26} and consequently $l_1 = c L_{35} L_{26}$ and using that $l_1(a^1) = 1$ we determine the constant c. Therefore, an easy calculation show that $\forall y \in B_{2R}(x) \cap \Omega$ we have

$$\begin{aligned} |l_1(y)| &\leq c \frac{R^4}{(\text{area of } T_{135})(\text{area of } T_{126})} \\ |\frac{\partial l_1}{\partial y_j}(y)| &\leq c \frac{R^3}{(\text{area of } T_{135})\ (\text{area of } T_{126})} \quad 1 \leq j \leq 2 \end{aligned}$$

where T_{ijk} is the triangle of vertices $\{a^i, a^j, a^k\}$. Let ρ_{ijk} be the diameter of the largest ball contained in T_{ijk}. We note that $\rho_{135} \geq \rho_{234}$. So, since T_{234} satisfies the regularity condition we obtain the bounds.

Now we consider the case ii). Note that we can build $l_j \in \mathcal{P}_2$ such that $l_j(a^i) = \delta_{ij}$ for $i, j = 1, 2, 4, 5, 6$ by the same argument used above for the case i). So, we have that $l_1 = c_1 L_{35} L_{26}$, $l_2 = c_2 L_{35} L_{16}$, $l_4 = c_4 L_{13} L_{56}$, $l_5 = c_5 L_{13} L_{46}$ and $l_6 = c_6 L_{13} L_{35}$ where the constants c_j are determined using that $l_j(a^j) = 1$. So, if the triangles of vertices $\{a^1, a^2, a^5\}$, $\{a^1, a^2, a^6\}$, $\{a^4, a^5, a^6\}$ and $\{a^2, a^3, a^4\}$, satisfy the regularity condition then, an easy calculation shows that $|l_j| \leq C$ and $|\frac{\partial l_j}{\partial y_k}| \leq \frac{C}{R}$, $k = 1, 2$ and $j = 1, 2, 4, 5, 6$.

This argument fails when we need to build the basis l_3 because there is not a line which include 3 points different to a^3. However, we can obtain $b_3 \in \mathcal{P}_1$ independent of $\{l_1, l_2, l_4, l_5, l_6\}$ such that $b_3(a^3) = 1$ and $b_3\big|_{L_{15}} = 0$ and therefore and easy calculation shows that $\{l_1, l_2, b_3, l_4, l_5, l_6\}$ satisfy the bounds.

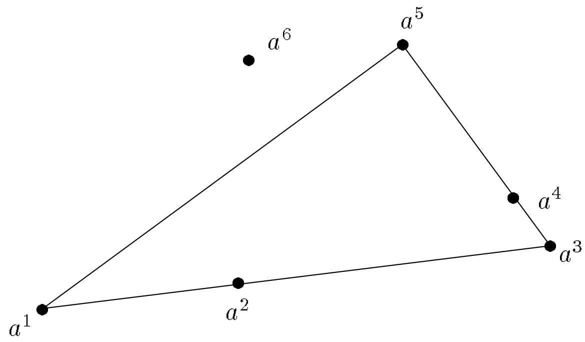

Figure 3

Then the Lagrange basis l_3 is $l_3(y) = b_3(y) - \sum_{j=1,\cdots,6, j\neq 3} b_3(a^j) l_j(y)$ and consequently l_3 satisfies the bounds $\square$.

Finally, we will obtain the bounds for the Lagrange basis for a most general case given in Figure 1.

Lemma 4.4. *Let $x \in \Omega$, $m = 2$ we suppose that there are points $a^j, 1 \leq j \leq 6$ in $B_{\frac{R}{2}}(x)$ as in Figure 1 such that the triangles of vertices $\{a^1, a^2, a^5\}$, $\{a^4, a^5, a^6\}$, $\{a^1, a^2, a^6\}$ $\{a^2, a^3, a^4\}$ and $\{a^2, a^3, a^5\}$ satisfy the regularity condition then, there exists a constant $C = C(\sigma)$ such that $\forall y \in B_{2R}(x) \cap \Omega$ the Lagrange basis l_i, $1 \leq i \leq 6$ satisfy*

$$\begin{aligned} &|l_i(y)| \leq C \\ &|\frac{\partial l_i}{\partial y_j}(y)| \leq \frac{C}{R} \quad 1 \leq j \leq 2 \end{aligned} \tag{4.3}$$

Proof. By the same argument used in the Lemma given above we have that the Lagrange basis $l_j \in \mathcal{P}_2$, $4 \le j \le 6$ can be constructed as $l_4 = c_4 L_{13} L_{56}$, $l_5 = c_5 L_{13} L_{46}$, and $l_6 = c_6 L_{13} L_{45}$, where c_j can be obtained using that $l_j(a^j) = 1$, $4 \le j \le 6$. Also, we can obtain $b_1, b_2 \in \mathcal{P}_1$ such that $b_1 = c_1 L_{35}$ and $b_2 = c_2 L_{15}$ where c_j are obtained using that $b_j(a^j) = 1$, $1 \le j \le 2$ Then $\{b_1, b_2, l_4, l_5, l_6\}$ are independent and satisfy (4.3) if the triangle of vertices $\{a^1, a^2, a^5\}$, $\{a^4, a^5, a^6\}$, $\{a^1, a^2, a^6\}$ and $\{a^2, a^3, a^4\}$ satisfy the regularity condition.

Therefore we need to construct b_3 independent of the others, i.e, we will show that there exists $b_3 \in \mathcal{P}_2$ with $b_3(a^3) = 1$ such that the following matrix be nonsingular.

$$\begin{pmatrix} 1 & 0 & b_3(a^1) & 0 & 0 & 0 \\ b_1(a^2) & 1 & b_3(a^2) & 0 & 0 & 0 \\ 0 & b_2(a^3) & 1 & 0 & 0 & 0 \\ b_1(a^4) & b_2(a^4) & b_3(a^4) & 1 & 0 & 0 \\ 0 & 0 & b_3(a^5) & 0 & 1 & 0 \\ b_1(a^6) & b_2(a^6) & b_3(a^6) & 0 & 0 & 1 \end{pmatrix}$$

It is enough to obtain $b_3 \in \mathcal{P}_2$ such that the submatrix

$$\begin{pmatrix} 1 & 0 & b_3(a^1) \\ b_1(a^2) & 1 & b_3(a^2) \\ 0 & b_2(a^3) & 1 \end{pmatrix}$$

be nonsingular. To obtain b_3 we will work in the following reference triangle $\hat{T}$ (Figure 4)

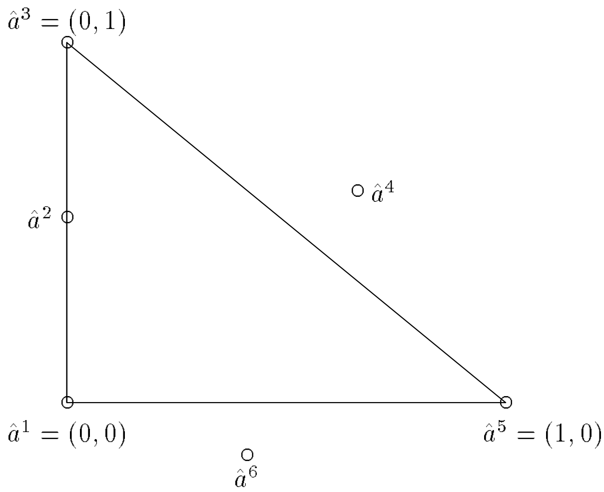

Figure 4

In this case the basis $\hat{b}_1$ and $\hat{b}_2$ defined above are $\hat{b}_1(\hat{x}, \hat{y}) = 1 - \hat{x} - \hat{y}$ and $\hat{b}_2(\hat{x}, \hat{y}) = \frac{\hat{y}}{\hat{a}_2^2}$, where $\hat{a}^2 = (0, \hat{a}_2^2)$ (we observe that since the triangle of vertices

$\{\hat{a}^1, \hat{a}^2, \hat{a}^5\}$ and $\{\hat{a}^2, \hat{a}^3, \hat{a}^5\}$ satisfy the regularity condition then $\hat{a}^2_2$ is different from 0 and 1). Then we take $\hat{b}_3 \in \mathcal{P}_2$ such that $\hat{b}_3|_{L_{15}} = 0$, $\hat{b}_3(\hat{a}^3) = 1$ independent of the others.

An easy calculation shows that we can choose $\hat{b}_3(\hat{x}, \hat{y}) = \hat{y}^2$. Let T be the triangle of vertices $\{a^1, a^3, a^5\}$ and let F be the invertible affine mapping such that $F(\hat{T}) = T$ then, $(x, y) = F(\hat{x}, \hat{y})$ and $b_3 = \hat{b}_3 \circ F^{-1}$, i.e, $b_3(x, y) = \hat{b}_3(\hat{x}, \hat{y})$.

Since the triangle of vertices $\{a^1, a^2, a^5\}$ satisfies the regularity condition then, the triangle of vertices $\{a^1, a^3, a^5\}$ also satisfies it. Then, the basis b_3 is bounded independently of R.

Finally, we can obtain the Lagrange basis l_1, l_2, l_3 as a linear combination of b_1, b_2, b_3, l_4, l_6 with coefficients bounded by a constant which depends on σ. So, we conclude the proof $\square$.

REMARK 4.1. *The hypothesis of having three aligned points could be seen as a strong restriction. However, since the Lagrange basis depend continuously on the interpolation points it follows that the bounds obtained hold when the points have a distribution close enough to that in Figure 1.*

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