INTERPOLATION IN JACOBI-WEIGHTED SPACES AND ITS APPLICATION TO A POSTERIORI ERROR ESTIMATIONS OF THE P-VERSION OF THE FINITE ELEMENT METHOD

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Abstract. The goal of this work is to introduce a local and a global interpolator in Jacobi-weighted spaces, with optimal order of approximation in the context of the p-version of finite element methods. Then, an a posteriori error indicator of the residual type is proposed for a model problem in two dimensions and, in the mathematical framework of the Jacobi-weighted spaces, the equivalence between the estimator and the error is obtained on appropriate weighted norm. **Keywords:** Jacobi-weighted Sobolev spaces, p finite element methods, a posteriori error estimates. MSC:65N15,65N30

1. Introduction

In this paper we show several results concerning the two dimensional Jacobi-weighted spaces and we introduce a local and a global interpolator with optimal order of approximation in the context of the p-version of finite element methods (FEM). Then, we consider a two dimensional model problem and we introduce an a posteriori error estimator of the residual type for the p-version of FEM, and we prove that the estimator is equivalent to the error on appropriate Jacobi-weighted norm up to higher order terms.

It is well known that the development of a posteriori error indicators and adaptive procedures play nowadays a relevant role in the numerical solution of partial differential equations. In contrast to the case of h refinement, it seems to be an open question whether uniform reliability and efficiency can be achieved for an hp a posteriori estimator of the residual type even in simple problems.

In the one dimensional case, analysis for a posteriori error indicators based on the residuals for the p and hp of FEM is well known [31, 21, 17]. More precisely, in [17] the authors obtain an error estimator of the residual type for the Poisson Problem and prove that the H^1 norm of the error is equivalent to the error estimator up to higher order terms. Moreover, they propose an adaptive algorithm and, since the error estimator is reliable and efficient, they prove that the algorithm leads to a uniform monotone decrease of the energy error in every step. It is important to point out that these kinds of results have not been established in high dimensions, and the techniques used for one-dimensional analysis can not be applied to higher dimensions.

In the two dimensional case, to the best of the authors' knowledge, the error estimators of the residual type present in the literature for the p and hp of FEM are equivalent to the error with constants depending on p (see [4, 5, 6, 30] and the references therein). Moreover, there is some numerical evidence that the p-gap can not be avoided [19]. In [30] the authors obtain an error estimator of the residual type for the Poisson Problem in two dimensions and, using optimal weighted inverse estimations, prove that the error is equivalent to the indicator and propose an hp strategy based on a predictor of the error in each element of the mesh. Using these error indicators, a particular algorithm is proposed in [16] and its convergence is reached assuming a data saturation which, due to the constant p dependence, becomes more restrictive for increasing polynomial degrees. In [2, 3] the authors proposed an hp refinement strategy in which in every step and for every element they decided to do h adaptivity or p adaptivity based on the local regularity of the solutions. On the other hand, in [15, 18] p-robust equilibrated residual error estimates are obtained for the Poisson problem and Elasticity problem respectively. We want to point out that although these estimators do not suffer from the p-gap frequently observed in the standard residual error estimators, some local problems have to be solved in order to get it.

In recent decades, the Jacobi-weighted Sobolev spaces have received increasing attention for the approximation theory of the p (and hp) version of the finite element methods. These spaces seem to be the appropriate functional spaces for a priori error analysis and play a crucial role in the analysis of the a posteriori error estimations (see [1, 7, 8, 9, 24], and the references therein). Indeed, the a priori error analysis and optimal convergence for the p-version of FEM in this context have been studied by several authors (see, for instance, [8, 9, 25]). More recently, increasing attention to this framework has developed because of the need for optimal a posteriori error estimates [10, 22]. Motivated by the results obtained by [17] in the one dimensional case and the a posteriori error analysis given by [22], in this paper we analyze the a priori and a posteriori approximation theory for the p-version in the mathematical framework of the Jacobi-weighted Sobolev spaces. In fact, we present several results concerning the interpolation theory for functions in Jacobi-weighted Sobolev spaces. In addition to that, for the two dimensional Poisson model problem, we develop an a posteriori error estimator of the residual type for the p-version of FEM. This estimator is similar to the standard estimators present in the bibliography (see, for example,[4, 5, 6, 16, 30] and the references therein).

We analyze the equivalence of this estimator with the error in a Jacobi-weighted norm and we prove quasi-optimal global reliability and local efficiency estimates, both up to higher order terms. According to our results, the typical p-gap between reliability and efficiency is essentially removed (up to p^δ, for arbitrary $\delta > 0$). As a consequence, our estimates are (see [22, 24]) the best result that one can have for this kind of error estimator based on residual in two dimensions, up to date.

The rest of the paper is organized as follows. In Section 2 we show several results concerning the Jacobi-weighted spaces. In Section 3 we present the p-approximation theory and the interpolations error. In Section 4 we consider a two dimensional model problem and we introduce the a posteriori error estimator and we prove its equivalence with the error in a Jacobi-weighted norm.

2. Jacobi-weighted Sobolev spaces

Let $Q = (-1,1)^2$ be the reference domain in $\mathbb{R}^2$. For $i = 1, 2$ let $\beta_i > -1$, $\alpha_i \geq 0$ be integer, $\beta = (\beta_1, \beta_2)$ and $\alpha = (\alpha_1, \alpha_2)$. We define the weighted function $W_{\beta,\alpha}$ in Q as follows

$$W_{\beta,\alpha}(x, y) = (1 - x^2)^{\beta_1+\alpha_1}(1 - y^2)^{\beta_2+\alpha_2}.$$

If $\alpha = \mathbf{0}$ we note $W_\beta = W_{\beta,\mathbf{0}}$.

For a function $u \in C^\infty(\bar{Q})$ and $k \geq 0$ integer, we define the following norm:

$$\|u\|^2_{H^{k,\beta}(Q)} = \sum_{|\alpha|\leq k} \int_Q |\partial^\alpha u|^2 W_{\beta,\alpha}.$$

The weighted Sobolev space $H^{k,\beta}(Q)$ is defined as the closure of the $C^\infty(\bar{Q})$ functions with this norm (see, for example, [8]), i.e.,

$$H^{k,\beta}(Q) = \overline{C^\infty(\bar{Q})}^{\|\cdot\|_{H^{k,\beta}(Q)}}.$$

With $|u|_{H^{k,\beta}(Q)}$ we denote the seminorms

$$|u|_{H^{k,\beta}(Q)} = \sum_{|\alpha|=k} \int_Q |\partial^\alpha u|^2 W_{\beta,\alpha}.$$

Let Ω be an open polygonal domain in $\mathbb{R}^2$, $\mathcal{T}$ an admissible partition of Ω in parallelograms. For any $K \in \mathcal{T}$, let $F : Q \to K$ be an affine transformation and $u \in C^\infty(\bar{K})$, then $\hat{u} = u \circ F \in C^\infty(\bar{Q})$. We define

$$\|u\|_{H^{k,\beta}(K)} = \|\hat{u}\|_{H^{k,\beta}(Q)}, \tag{1}$$

and

$$\|u\|^2_{H^{k,\beta}(\mathcal{T})} = \sum_{K\in\mathcal{T}} \|u\|^2_{H^{k,\beta}(K)}.$$

We observe that the norm depends on the partition that we are considering. Then, the Jacobi-weighted spaces for $\mathcal{T}$, a partition of Ω, are defined as

$$H^{k,\beta}(\mathcal{T}) = \overline{C^\infty(\bar{\Omega})}^{\|\cdot\|_{H^{k,\beta}(\mathcal{T})}}$$
$$H_0^{k,\beta}(\mathcal{T}) = \overline{C_0^\infty(\bar{\Omega})}^{\|\cdot\|_{H^{k,\beta}(\mathcal{T})}}.$$

From now on, we consider the case $\beta_i = \beta > -1$ for $i = 1, 2$.

In what follows we present the Jacobi projection and we enunciate its properties (see [25] and the references therein). In order to do that we need to introduce the Jacobi polynomials in one dimension (for details see [23]).

Let $I = (-1, 1)$, $\beta > -1$ and let p be a polynomial degree. For $x \in I$, let $J_p^\beta(x)$ be the Jacobi polynomial of degree p, i.e.,

$$J_p^\beta(x) = \frac{(1-x^2)^{-\beta}}{2^p p!} \frac{d^p(1-x^2)^{\beta+p}}{dx^p}.$$

It is well known that the Jacobi polynomials $J_p^\beta(x)$ are orthogonal with the Jacobi weight $W_\beta(x) = (1-x^2)^\beta$, i. e.,

$$\int_I J_p^\beta(x) J_m^\beta(x) W_\beta(x) = \begin{cases} \gamma_p^\beta, & p = m \\ 0, & p \neq m \end{cases}$$

with

$$\gamma_p^\beta = \frac{2^{2\beta+1}\Gamma^2(p+\beta+1)}{(2p+2\beta+1)\Gamma(p+1)\Gamma(p+2\beta+1)},$$

where Γ denotes the well known function Gamma given by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

Let $J^\beta_{p,k}(x) = \frac{d^k}{dx^k} J^\beta_p(x)$, then for $0 \le k \le p$ we have

$$J^\beta_{p,k}(x) = 2^{-k} \frac{\Gamma(p+2\beta+k+1)}{\Gamma(p+2\beta+1)} J^{\beta+k}_{p-k}(x)$$

which are orthogonal with the Jacobi weight $W_{\beta+k}(x)$;

$$\int_I J^\beta_{p,k}(x) J^\beta_{m,k}(x) W_{\beta+k}(x) = \begin{cases} \gamma^\beta_{p,k}, & p = m \ge k \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

with

$$\gamma^\beta_{p,k} = \frac{2^{2\beta+1}\Gamma(p+2\beta+k+1)\Gamma^2(p+\beta+1)}{(2p+2\beta+1)\Gamma(p+1-k)\Gamma^2(p+2\beta+1)}.$$

We note that if $k=0$ we obtain $\gamma^\beta_{p,0} = \gamma^\beta_p$.

Now, we will enunciate two important properties that provide us an estimation for the constants $\gamma^\beta_{p,k}$ and γ^β_p. For this purpose we need some previous lemmas.

The following lemma (which can be found, for example, in page 427 of [28]) gives a well known estimation for the function Γ.

Lemma 2.1. *For real x and $x \to +\infty$ the following applies*

$$\Gamma(x) \sim x^{x-1/2} e^{-x} \sqrt{2\pi}$$

where $\sim$ means the quotient of the left side by the right side tents to 1 as $x \to +\infty$.

Thus, we have the following useful results.

Corollary 2.1.

a) *For any $\alpha \in \mathbb{R}$, $\quad \Gamma(n+\alpha) \sim (n+\alpha)^{n+\alpha-1/2} e^{-(n+\alpha)} \sqrt{2\pi}, \quad$ for $n \to +\infty$*

b) *For any $\alpha \in \mathbb{R}$, $\quad \lim_{n\to+\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n) n^\alpha} = 1$,*

c) *Given $\alpha_0 > -1$, for any α such that $-1 < \alpha \le \alpha_0$, there exist positive constants A and B (depending on α_0 but independent of α) such that*

$$A \le \frac{\Gamma(n+\alpha)}{\Gamma(n) n^\alpha} \le B \quad \forall n \in \mathbb{N}$$

Therefore, the following estimate holds.

Lemma 2.2. *Let $-1 < \beta \le \beta_0$, $k \ge 0$ be integer and p a polynomial degree. There exist positive constants $A = A(\beta_0)$ and $B = B(\beta_0)$, independent of p and β, such that*

$$A\gamma^\beta_p p^{2k} \le \gamma^\beta_{p,k} \le B p^{2k} \gamma^\beta_p \quad \forall p \ge k.$$

Proof. If $k = 0$ we take $A = B = 1$. Suppose $k \ge 1$.

$$\begin{aligned}
\frac{\gamma^\beta_{p,k}}{\gamma^\beta_p p^{2k}} &= \frac{2^{2\beta+1}\Gamma(p+2\beta+k+1)\Gamma^2(p+\beta+1)}{(2p+2\beta+1)\Gamma(p+1-k)\Gamma^2(p+2\beta+1)} \frac{(2p+2\beta+1)\Gamma(p+1)\Gamma(p+2\beta+1)}{2^{2\beta+1}\Gamma^2(p+\beta+1)} \frac{1}{p^{2k}} \\
&= \frac{\Gamma(p+2\beta+k+1)\Gamma(p+1)}{\Gamma(p+1-k)\Gamma(p+2\beta+1)p^{2k}} \\
&= \frac{\Gamma((p+1-k)+(2\beta+2k))}{\Gamma(p+1-k)(p+1-k)^{2\beta+2k}} \frac{\Gamma(p)p^{2\beta+1}}{\Gamma(p+2\beta+1)} \Big(\frac{p+1-k}{p}\Big)^{2\beta+2k} \\
&= (I)(II)(III).
\end{aligned}$$

In (I) and (II) we apply Corollary 2.1 c), for (III) we observe that

$$\lim_{p\to\infty}\frac{p+1-k}{p}=1,$$

and, since $\frac{p+1-k}{p}>0$ for all $p\geq k$, we get

$$c_1\leq\frac{p+1-k}{p}\leq c_2,\quad\forall p\geq k,$$

for positive constants c_1 and c_2 and the proof concludes. □

Lemma 2.3. *For $-1<\beta<\beta_0$ and $p\geq 1$ a polynomial degree there exist positive constants $A=A(\beta_0)$ and $B=B(\beta_0)$, independent of p and β, such that*

$$Ap^{-1}\leq\gamma_p^\beta\leq Bp^{-1}.$$

Proof.

$$\begin{aligned}\gamma_p^\beta&=\frac{2^{2\beta+1}\Gamma(p+\beta+1)^2}{(2p+2\beta+1)\Gamma(p+1)\Gamma(p+2\beta+1)}\\&=\frac{2^{2\beta+1}}{(2p+2\beta+1)}\Big(\frac{\Gamma(p+\beta+1)}{\Gamma(p)p^{\beta+1}}\Big)^2\frac{p^{2\beta+1}\Gamma(p)}{\Gamma(p+2\beta+1)}\end{aligned}$$

By Corollary 2.1 c) there exist positive constants c_1 and c_2, independent of p and β, such that

$$c_1\leq\Big(\frac{\Gamma(p+\beta+1)}{\Gamma(p)p^{\beta+1}}\Big)^2\frac{p^{2\beta+1}\Gamma(p)}{\Gamma(p+2\beta+1)}\leq c_2,\forall p\geq 1,\forall -1<\beta<\beta_0$$

and the result holds. □

For any $\beta>-1$, $k\geq 0$ and $u\in H^{k,\beta}(Q)$, we have the Jacobi-Fourier expansion (see, for instance, [25])

$$u(x,y)=\sum_{i,j=0}^{\infty}c_{i,j}J_i^\beta(x)J_j^\beta(y)\tag{3}$$

with

$$c_{i,j}=\frac{1}{\gamma_i^\beta\gamma_j^\beta}\int_Q u(x)J_i^\beta(x)J_j^\beta(y)W_\beta(x,y).$$

Using the orthogonality of the Jacobi polynomials (2) we have that

$$\|u\|^2_{H^{0,\beta}(Q)}=\sum_{i,j=0}^{\infty}|c_{i,j}|^2\gamma_i^\beta\gamma_j^\beta\qquad\text{and}\qquad\int_Q|\partial^\alpha u|^2W_{\beta,\alpha}=\sum_{i\geq\alpha_1,j\geq\alpha_2}|c_{i,j}|^2\gamma^\beta_{i,\alpha_1}\gamma^\beta_{j,\alpha_2}.\tag{4}$$

Then, for $-1<\beta\leq\beta_0$, by Lemma 2.2 we deduce that

$$|u|^2_{H^{k,\beta}(Q)}=\sum_{|\alpha|=k}\sum_{i\geq\alpha_1,j\geq\alpha_2}|c_{i,j}|^2\gamma^\beta_{i,\alpha_1}\gamma^\beta_{j,\alpha_2}\simeq\sum_{|\alpha|=k}\sum_{i\geq\alpha_1,j\geq\alpha_2}|c_{i,j}|^2\gamma_i^\beta\gamma_j^\beta i^{2\alpha_1}j^{2\alpha_2},$$

where $A\simeq B$ means $c_1B\leq A\leq c_2B$ with positive constants c_1 and c_2 independent of β.

For $p\geq 0$ we define $\mathcal{Q}_p(Q)$ the set of all polynomials of degree less than or equal to p in each variable in Q. The Jacobi projection of u in $\mathcal{Q}_p(Q)$ is

$$\Pi_p^\beta u(x,y)=\sum_{i,j=0}^{p}c_{i,j}J_i^\beta(x)J_j^\beta(y).\tag{5}$$

Lemma 2.4. *Let $-1 < \beta \leq \beta_0$, then*

$$|J_p^\beta(-1)| = |J_p^\beta(1)| \sim \frac{1}{\Gamma(\beta+1)}(p+1)^\beta$$

with constants depending on β_0 but independent of p and β.

Proof. By equation (2.2) of [26] we know that

$$|J_p^\beta(-1)| = |J_p^\beta(1)|, \quad \text{and} \quad J_p^\beta(1) = \frac{\Gamma(p+\beta+1)}{\Gamma(p+1)\Gamma(\beta+1)}. \tag{6}$$

Then, we can write

$$J_p^\beta(1) = \frac{\Gamma(p+\beta+1)}{\Gamma(p+1)(p+1)^\beta}\frac{(p+1)^\beta}{\Gamma(\beta+1)},$$

and the result follows from Corollary 2.1 c) □

The following theorem states some approximation properties of the Jacobi projection. We want to recall that similar results can be found in [27, 20] for the case $\beta = 0$. However, these techniques can not applied for the case of β close to $-1/2$, which will be our focus of interest.

Theorem 2.1. *Let $I = (-1,1)$, Q be the reference domain in $\mathbb{R}^2$, $-1 < \beta \leq \beta_0$, and $u \in H^{1,\beta}(Q)$. Let $p \geq 1$ be a polynomial degree, $\Pi_p^\beta u \in \mathcal{Q}_p(Q)$ as in (5). Then there exists a positive constant $C = C(\beta_0)$, independent of u, β and p, such that*

$$\|u - \Pi_p^\beta u\|_{H^{0,\beta}(Q)} \leq C(p+1)^{-1}|u|_{H^{1,\beta}(Q)}. \tag{7}$$

If, in addition, $\beta \leq -1/2$ then

$$\begin{aligned} \|(u - \Pi_p^\beta u)(\pm 1, y)\|_{H^{0,\beta}(I)} &\leq \frac{C}{\Gamma(\beta+1)}(p+1)^{-1/2}|u|_{H^{1,\beta}(Q)}, \\ \|(u - \Pi_p^\beta u)(x, \pm 1)\|_{H^{0,\beta}(I)} &\leq \frac{C}{\Gamma(\beta+1)}(p+1)^{-1/2}|u|_{H^{1,\beta}(Q)}, \end{aligned} \tag{8}$$

and if, in addition, $u \in C^0(\bar{Q})$ and $\beta < -1/2$ then

$$|(u - \Pi_p^\beta u)(V)| \leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)^2}(p+1)^{\beta+1/2}|u|_{H^{1,\beta}(Q)}, \ \forall\ V \text{ vertex of } Q. \tag{9}$$

Proof. The proof of (7) is given in [25] with a constant that could depend on β. However, following the steps of that proof, we observe that a positive constant C can be chosen independent of β.

Now, we prove (8),i.e., the bound in the edges of Q. We assume that $u \in C^0(\bar{Q})$ since the general case can be followed by density arguments. We carry out the case $\|(u - \Pi_p^\beta u)(x, -1)\|_{H^{0,\beta}(I)}$, the other cases can be obtained analogously.

$$\begin{aligned} (u - \Pi_p^\beta u)(x, -1) &= \Big(\sum_{i \geq p+1, j \geq p+1} + \sum_{i \geq p+1, j < p+1} + \sum_{i < p+1, j \geq p+1}\Big) c_{i,j} J_i^\beta(x) J_j^\beta(-1) \\ &= \sum_{i \geq p+1} b_i^{[1]} J_i^\beta(x) + \sum_{i \geq p+1} b_i^{[2]} J_i^\beta(x) + \sum_{i < p+1} b_i^{[3]} J_i^\beta(x) \end{aligned}$$

where

$$\begin{aligned} b_i^{[1]} &= \sum_{j\geq p+1} c_{i,j} J_j^{\beta}(-1), \quad i \geq p+1 \\ b_i^{[2]} &= \sum_{j< p+1} c_{i,j} J_j^{\beta}(-1), \quad i \geq p+1 \\ b_i^{[3]} &= \sum_{j\geq p+1} c_{i,j} J_j^{\beta}(-1), \quad i < p+1. \end{aligned}$$

It is well known that, if $f : [p, +\infty] \to \mathbb{R}$ is a non-negative, decreasing and integrable function, then

$$\sum_{i=p+1}^{\infty} f(n) \leq \int_p^{\infty} f(x)dx. \tag{10}$$

Therefore, by Lemma 2.4, Hölder inequality, Lemma 2.3 and (10) we get,

$$\begin{aligned} |b_i^{[1]}|^2 &\leq \big(\sum_{j\geq p+1} |c_{i,j}||J_j^{\beta}(-1)|\big)^2 \\ &\leq \frac{C}{\Gamma(\beta+1)^2} \big(\sum_{j\geq p+1} |c_{i,j}|(j+1)^{\beta}(\gamma_j^{\beta})^{1/2}(\gamma_j^{\beta})^{-1/2} j j^{-1}\big)^2 \\ &\leq \frac{C}{\Gamma(\beta+1)^2} \big(\sum_{j\geq p+1} |c_{i,j}|^2 \gamma_j^{\beta} j^2\big)\big(\sum_{j\geq p+1} (j+1)^{2\beta}(\gamma_j^{\beta})^{-1} j^{-2}\big) \\ &\leq \frac{C}{\Gamma(\beta+1)^2} \big(\sum_{j\geq p+1} |c_{i,j}|^2 \gamma_j^{\beta} j^2\big)\big(\sum_{j\geq p+1} j^{-1+2\beta}\big) \\ &\leq \frac{C}{\Gamma(\beta+1)^2} \big(\sum_{j\geq p+1} |c_{i,j}|^2 \gamma_j^{\beta} j^2\big)\big(\int_p^{\infty} x^{-1+2\beta} dx\big) \\ &\leq \frac{C}{\Gamma(\beta+1)^2} p^{2\beta} \sum_{j\geq p+1} |c_{i,j}|^2 \gamma_j^{\beta} j^2. \end{aligned}$$

Then,

$$\begin{aligned} \| \sum_{i\geq p+1} b_i^{[1]} J_i^{\beta}(x)\|^2_{H^{0,\beta}(I)} &= \sum_{i\geq p+1} |b_i^{[1]}|^2 \gamma_i^{\beta} \leq \frac{C}{\Gamma(\beta+1)^2} p^{2\beta} \sum_{i\geq p+1, j\geq p+1} |c_{i,j}|^2 \gamma_j^{\beta} j^2 \gamma_i^{\beta} \\ &\leq \frac{C}{\Gamma(\beta+1)^2} p^{2\beta} \sum_{i\geq 0, j\geq 1} |c_{i,j}|^2 \gamma_j^{\beta} j^2 \gamma_i^{\beta} \leq \frac{C}{\Gamma(\beta+1)^2} p^{2\beta} |u|^2_{H^{1,\beta}(Q)}. \end{aligned}$$

Analogously,

$$
\begin{aligned}
|b_i^{[2]}|^2 &\leq \Big(\sum_{j<p+1} |c_{i,j}||J_j^\beta(-1)| \Big)^2 \\
&\leq \frac{C}{\Gamma(\beta+1)^2} \Big(\sum_{j<p+1} |c_{i,j}|(j+1)^\beta (\gamma_j^\beta)^{1/2} (\gamma_j^\beta)^{-1/2} i i^{-1} \Big)^2 \\
&\leq \frac{C}{\Gamma(\beta+1)^2} (p+1)^{-2} \Big(\sum_{j<p+1} |c_{i,j}|(j+1)^\beta (\gamma_j^\beta)^{1/2} (\gamma_j^\beta)^{-1/2} i \Big)^2 \\
&\leq \frac{C}{\Gamma(\beta+1)^2} (p+1)^{-2} \Big(\sum_{j<p+1} |c_{i,j}|^2 \gamma_j^\beta i^2 \Big) \Big(\sum_{j<p+1} (j+1)^{2\beta} (\gamma_j^\beta)^{-1} \Big) \\
&\leq \frac{C}{\Gamma(\beta+1)^2} (p+1)^{-2} \Big(\sum_{j<p+1} |c_{i,j}|^2 \gamma_j^\beta i^2 \Big) \Big(\sum_{j<p+1} (j+1)^{2\beta+1} \Big) \\
&\leq \frac{C}{\Gamma(\beta+1)^2} (p+1)^{-2} \Big(\sum_{j<p+1} |c_{i,j}|^2 \gamma_j^\beta i^2 \Big) (p+1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\| \sum_{i \geq p+1} b_i^{[2]} J_i^\beta(x) \|_{H^{0,\beta}(I)}^2 &= \sum_{i \geq p+1} |b_i^{[2]}|^2 \gamma_i^\beta \\
&\leq \frac{C}{\Gamma(\beta+1)^2} (p+1)^{-1} \sum_{i \geq p+1, j<p+1} |c_{i,j}|^2 \gamma_j^\beta i^2 \gamma_i^\beta \\
&\leq \frac{C}{\Gamma(\beta+1)^2} (p+1)^{-1} \sum_{i \geq 1, j \geq 0} |c_{i,j}|^2 \gamma_j^\beta i^2 \gamma_i^\beta \\
&\leq \frac{C}{\Gamma(\beta+1)^2} (p+1)^{-1} |u|_{H^{1,\beta}(Q)}^2.
\end{aligned}
$$

Similarly,

$$
\| \sum_{i<p+1} b_i^{[3]} J_i^{\beta_1}(x) \|_{H^{0,\beta}(I)}^2 \leq \frac{C}{\Gamma(\beta+1)^2} p^{-1} |u|_{H^{1,\beta}(Q)}^2.
$$

As a consequence,

$$
\|(u - \Pi_p^\beta u)\|_{H^{0,\beta}(I)} \leq \frac{C}{\Gamma(\beta+1)} (p+1)^{-1/2} |u|_{H^{1,\beta}(Q)}
$$

and (8) holds.

Finally, we prove (9) for $V = (-1,-1)$, the same arguments can be used for the other vertices of Q.

$$
\begin{aligned}
|(u-\Pi_p^\beta u)(-1,-1)| &= |\Big(\sum_{i\ge p+1, j\ge p+1} + \sum_{i\ge p+1, j<p+1} + \sum_{i<p+1, j\ge p+1}\Big) c_{i,j} J_i^\beta(-1) J_j^\beta(-1)| \\
&\le \Big(\sum_{i\ge p+1, j\ge p+1} + \sum_{i\ge p+1, j<p+1} + \sum_{i<p+1, j\ge p+1}\Big) |c_{i,j}| |J_i^\beta(-1)| |J_j^\beta(-1)| \\
&\le \frac{C}{\Gamma(\beta+1)^2}\Big(\sum_{i\ge p+1, j\ge p+1} + \sum_{i\ge p+1, j<p+1} + \sum_{i<p+1, j\ge p+1}\Big) |c_{i,j}| (i+1)^\beta (j+1)^\beta \\
&= \frac{C}{\Gamma(\beta+1)^2}(I + II + III).
\end{aligned}
$$

At first we compute II and III. Due to Hölder inequality and Lemma 2.3 we obtain

$$
\begin{aligned}
II &\le \Big(\sum_{i\ge p+1, j<p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta i^2\Big)^{1/2} \Big(\sum_{i\ge p+1, j<p+1} (i+1)^{2\beta}(j+1)^{2\beta}(\gamma_i^\beta)^{-1}(\gamma_j^\beta)^{-1} i^{-2}\Big)^{1/2} \\
&\le C\Big(\sum_{i\ge p+1, j<p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta i^2\Big)^{1/2} \Big(\sum_{i\ge p+1, j<p+1} (i+1)^{2\beta+1}(j+1)^{2\beta+1} i^{-2}\Big)^{1/2} \\
&\le C\Big(\sum_{i\ge p+1, j<p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta i^2\Big)^{1/2} \Big(\sum_{i\ge p+1, j<p+1} (i+1)^{2\beta+1} i^{-2}\Big)^{1/2} \\
&\le C\Big(\sum_{i\ge p+1, j<p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta i^2\Big)^{1/2} \Big((p+1)\sum_{i\ge p+1} (i)^{2\beta-1}\Big)^{1/2}
\end{aligned}
$$

Now, from (10) it follows that

$$
\begin{aligned}
II &\le C\Big(\sum_{i\ge p+1, j<p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta i^2\Big)^{1/2} \Big((p+1)\int_p^\infty x^{-1+2\beta} dx\Big)^{1/2} \\
&\le C\Big(\sum_{i\ge p+1, j<p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta i^2\Big)^{1/2} \Big((p+1) p^{2\beta}\Big)^{1/2} \\
&\le C\Big(\sum_{i\ge p+1, j<p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta i^2\Big)^{1/2} \Big(p^{2\beta+1}\Big)^{1/2} \le C p^{\beta+1/2} |u|_{H^{1,\beta}(Q)}.
\end{aligned}
$$

For III we proceed analogously but changing the roles of i and j, and we can conclude that

$$
III \le C p^{\beta+1/2} |u|_{H^{1,\beta}(Q)}.
$$

In what follows we compute I.

$$
\begin{aligned}
I &\le \Big(\sum_{i\ge p+1, j\ge p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta ij\Big)^{1/2} \Big(\sum_{i\ge p+1, j\ge p+1} (i+1)^{2\beta}(j+1)^{2\beta}(\gamma_i^\beta)^{-1}(\gamma_j^\beta)^{-1} i^{-1} j^{-1}\Big)^{1/2} \\
&\le C\Big(\sum_{i\ge p+1, j\ge p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta ij\Big)^{1/2} \Big(\sum_{i\ge p+1} i^{2\beta} \sum_{j\ge p+1} (j+1)^{2\beta}\Big)^{1/2} \\
&\le C\Big(\sum_{i\ge p+1, j\ge p+1} |c_{i,j}|^2 \gamma_i^\beta \gamma_j^\beta ij\Big)^{1/2} \Big(\int_p^\infty x^{2\beta} dx \int_p^\infty y^{2\beta} dy\Big)^{1/2}.
\end{aligned}
$$

Since $\beta < -1/2$ the integrals converge, and it follows that

$$
\begin{aligned}
I &\le \frac{C}{-1-2\beta}\Big(\sum_{i\ge p+1, j\ge p+1} |c_{i,j}|^2\gamma_i^\beta\gamma_j^\beta ij\Big)^{1/2}\big(p^{2\beta+1}p^{2\beta+1}\big)^{1/2} \\
&\le \frac{C}{-1-2\beta}p^{2\beta+1}\Big(\sum_{i\ge p+1, j\ge p+1} |c_{i,j}|^2\gamma_i^\beta\gamma_j^\beta i^2 + \sum_{i\ge p+1, j\ge p+1} |c_{i,j}|^2\gamma_i^\beta\gamma_j^\beta j^2\Big)^{1/2} \\
&\le \frac{C}{-1-2\beta}p^{2\beta+1}|u|_{H^{1,\beta}(Q)}.
\end{aligned}
$$

Therefore,

$$
|(u-\Pi_p^\beta u)(-1,-1)| \le \frac{C}{(-1-2\beta)\Gamma(\beta+1)^2}p^{\beta+1/2}|u|_{H^{1,\beta}(Q)},
$$

and the proof concludes. □

Let K be a parallelogram of $\mathbb{R}^2$ and let $F_K : Q \to K$ be an affine transformation. For $p \ge 0$ we define

$$
\mathcal{Q}_p(K) = \{u | u\circ F_K \in \mathcal{Q}_p(Q)\}.
$$

Now, from (1) and Theorem 2.1 we have the following result.

Corollary 2.2. *Let K be a parallelogram of $\mathbb{R}^2$, $-1<\beta\le\beta_0$, and $u\in H^{1,\beta}(K)$. Let p be a polynomial degree, there exist $\Pi_{p,K}^\beta u \in \mathcal{Q}_p(K)$ and a positive constant C independent of p, β, and u such that*

$$
\|u-\Pi_{p,K}^\beta u\|_{H^{0,\beta}(K)} \le C(p+1)^{-1}|u|_{H^{1,\beta}(K)}, \tag{11}
$$

if, in addition, $\beta \le -1/2$ then

$$
\|u-\Pi_{p,K}^\beta u\|_{H^{0,\beta}(\gamma)} \le \frac{C}{\Gamma(\beta+1)}(p+1)^{-1/2}|u|_{H^{1,\beta}(K)} \quad \forall\ \gamma \text{ edge of } K, \tag{12}
$$

if, in addition, $u\in C^0(\bar K)$ and $\beta < -1/2$ then

$$
|(u-\Pi_p^\beta u)(V)| \le \frac{C}{(-1-2\beta)\Gamma(\beta+1)^2}(p+1)^{1/2+\beta}|u|_{H^{1,\beta}(K)},\ \forall\ V \text{ vertex of } K. \tag{13}
$$

3. p-Interpolation of smooth functions

The goal of this section is to introduce interpolation operators which are suitable to obtain a residual error estimation in the mathematical framework of Jacobi-weighted Sobolev spaces of the p-version of Finite Element Methods.

Let Ω be an open polygonal domain in $\mathbb{R}^2$, $\mathcal{T}$ an admissible partition of Ω in parallelograms. Let p be a polynomial degree, we denote

$$
\begin{aligned}
S^p(\mathcal{T}) &= \{u\in C^0(\Omega) \mid u|_K \in \mathcal{Q}_p(K) \quad \forall K\in\mathcal{T}\}, \\
S_0^p(\mathcal{T}) &= \{u\in C_0^0(\Omega) \mid u|_K \in \mathcal{Q}_p(K) \quad \forall K\in\mathcal{T}\}.
\end{aligned}
$$

We choose a polynomial degree p_K for each $K\in\mathcal{T}$ and we denote by $\mathbf{p}=(p_K)$ the vector of polynomials degrees. We assume that the polynomial degrees of neighboring elements are comparable, i.e., there exists a positive constant C such that

$$
p_K \le Cp_{K'} \quad K, K' \in \mathcal{T} \text{ with } \bar K \cap \bar K' \ne \emptyset. \tag{14}
$$

We introduce the following notation

$$\begin{aligned} S^{\mathbf{p}}(\mathcal{T}) &= \{u \in C^0(\Omega) \mid u|_K \in \mathcal{Q}_{p_K}(K) \quad \forall K \in \mathcal{T}\}, \\ S_0^{\mathbf{p}}(\mathcal{T}) &= \{u \in C_0^0(\Omega) \mid u|_K \in \mathcal{Q}_{p_K}(K) \quad \forall K \in \mathcal{T}\}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mathcal{E} &= \{\text{all edges } e \text{ in } \mathcal{T}\}, \\ \mathcal{E}^\circ &= \mathcal{E} \cap \Omega, \\ \mathcal{N} &= \{\text{all vertices } V \text{ in } \mathcal{T}\}. \end{aligned}$$

For $V \in \mathcal{N}$ we denote

$$\begin{aligned} \omega_V &= \bigcup \{\bar{K} | K \in \mathcal{T} \text{ and } \bar{K} \cap V \neq \emptyset\}, \\ \mathcal{T}_V &= \mathcal{T}|_{\omega_V}, \\ p_V &= \min\{p_K | V \in K\}, \\ \mathcal{E}_V &= \{\text{all edges } e \text{ of } \mathcal{E} \text{ such that } V \text{ is an endpoint of } e\}. \end{aligned}$$

For any $K \in \mathcal{T}$ or $e \in \mathcal{E}$ we define

$$\begin{aligned} \omega_K &= \bigcup \{\bar{K}' | K' \in \mathcal{T} \text{ and } \bar{K}' \cap \bar{K} \neq \emptyset\}, \\ \omega_e &= \bigcup \{\bar{K} | K \in \mathcal{T} \text{ and } \bar{K} \cap e \neq \emptyset\}, \\ p_e &= \min\{p_K | e \text{ edge of } K\}. \end{aligned}$$

Let $K \in \mathcal{T}$ and let $F_K : Q \to K$ be an affine transformation, for a function u in K we denote $\hat{u} = u \circ F_K$. Let $\hat{e} = I \times \{-1\})$, for any $e \in \mathcal{E}$ let $F_e : \hat{e} \to e$ be an affine transformation, then for a function u in e we denote $\hat{u} = u \circ F_e$

In order to introduce the local and the global interpolation operator we need some previous lemmas.

Lemma 3.1. *Let $I = (-1, 1)$, $-1 < \beta \leq \beta_0$ and p be a polynomial degree, there exists $g \in \mathcal{P}_p(\bar{I})$ such that $g(1) = 0$, $g(-1) = 1$ and*

$$\|g\|_{H^{0,\beta}(I)} \leq C\Gamma(\beta + 1)(p + 1)^{-(1+\beta)},$$

where the constant $C = C(\beta_0)$ is independent of p and β.

Proof. Let $\xi_0^\beta = -1$, $\xi_p^\beta = 1$ and let ξ_j^β, $j = 1, \ldots, p-1$, be the Gauss-Lobatto-Jacobi points, i.e., the zeros of the polynomial $(J_p^\beta)'$ (see Theorem 19.8 of [14]). Let g be the Lagrange interpolation polynomial of degree $\leq p$ such that $g(\xi_0^\beta) = 1$ and $g(\xi_j^\beta) = 0$ for $j = 1, \ldots, p$. By formulas (2.14) of [13] and (19.31) of [14] we get

$$\|g\|^2_{H^{0,\beta}(I)} \leq \rho_0^\beta$$

where

$$\rho_0^\beta = 2^{2\beta+1}\Gamma(\beta + 1)\Gamma(\beta + 2)\frac{(p-1)!}{\Gamma(p + 2\beta + 2)}.$$

Finally, since $\Gamma(p) = (p-1)!$, by Corollary 2.1 c) we can conclude that there exists a constant $C = C(\beta_0)$, independent of p and β, such that

$$\rho_0^\beta \leq C\Gamma(\beta + 1)^2 p^{-2(1+\beta)}$$

and the proof concludes. □

Corollary 3.1. *Let K be a parallelogram and V_1, V_2, V_3 and V_4 its vertices. Let $-1 < \beta \le \beta_0$ and p be a polynomial degree. There exist a function $\xi_{K,l}$ and a positive constant $C = C(\beta_0)$ independent of p and β such that*

i) $\xi_{K,l}(V_j) = \delta_{lj}$ *(takes the value 1 in V_l and 0 in the others vertices),*
ii) $\xi_{K,l} \in \mathcal{Q}_p(K)$ *,*
iii) $\|\xi_{K,l}\|_{H^{0,\beta}(K)} \le C\Gamma(\beta+1)^2(p+1)^{-2-2\beta}$ *and*
iv) $\|\xi_{K,l}\|_{H^{0,\beta}(e)} \le C\Gamma(\beta+1)(p+1)^{-1-\beta} \quad \forall\ e$ *edge of K.*

Proof. Let $Q = (-1,1)^2$ be the reference rectangle, p a polynomial degree and g as in the Lemma 3.1, then the function $G(x,y) = g(x)g(y)$ is in $\mathcal{Q}_p(Q)$ and satisfies

$$\begin{aligned} &G(-1,-1) = 1 \text{ and takes the value 0 in the others vertices of } Q,\\ &\|G\|_{H^{0,\beta}(Q)} \le C\Gamma(\beta+1)^2(p+1)^{-2(1+\beta)},\\ &\|G\|_{H^{0,\beta}(e)} \le C\Gamma(\beta+1)(p+1)^{-(1+\beta)} \quad \forall\ e \text{ edge of } Q. \end{aligned}$$

Let $F_{K,V_l} : Q \to K$ be an affine transformation such that $F_{K,V_l}(-1,-1) = V_l$, then $\xi_{K,l} = G \circ F^{-1}_{K,V_l}$ satisfies i)-iv) and the proof concludes. □

Corollary 3.2. *Let K be a parallelogram and e an edge of K. Let $-1 < \beta \le \beta_0$ and p be a polynomial degree and $w \in \mathcal{P}_p(e)$ such that $w|_{\partial e} = 0$ (i.e. $w(V) = 0$ for all V vertex of e). There exist ψ an extension of w to K and a positive constant $C = C(\beta_0)$, independent of p and β, such that*

i) $\psi \in \mathcal{Q}_p(K)$
ii) $\psi|_e = w$,
iii) $\psi|_{\partial K \setminus e} = 0$ *and*
iv) $\|\psi\|_{H^{0,\beta}(K)} \le C\Gamma(\beta+1)(p+1)^{-(1+\beta)}\|w\|_{H^{0,\beta}(e)}$.

Proof. Given $Q = (-1,1)^2$, the reference rectangle and $\hat{e} = (-1,1) \times \{-1\}$ an edge of Q, we consider $F_{K,e} : Q \to K$ an affine transformation such that $F_{K,e}(\hat{e}) = e$. Hence, we define $\hat{w} \in \mathcal{P}_p(\hat{e})$ as $\hat{w} = w \circ F_{K,e}$ and

$$\hat{\psi}(\hat{x},\hat{y}) = \hat{w}(\hat{x},-1)g(\hat{y})$$

where g is the polynomial of degree p introduced in Lemma 3.1. Then $\hat{\psi} \in \mathcal{Q}_p(Q)$ and satisfies

$$\begin{aligned} \hat{\psi}|_{\hat{e}} &= \hat{w},\\ \hat{\psi}|_{\partial Q \setminus \hat{e}} &= 0,\\ \|\hat{\psi}\|_{H^{0,\beta}(Q)} &\le C\Gamma(\beta+1)(p+1)^{-(1+\beta)}\|\hat{w}\|_{H^{0,\beta}(\hat{e})}. \end{aligned}$$

Hence, the proof concludes by defining

$$\psi = \hat{\psi} \circ F^{-1}_{K,e}$$

□

For each $V \in \mathcal{N}$, in the following theorem we introduce a local operator $I^{\beta}_V : H^{1,\beta}_0(\Omega) \cap C^0(\omega_V) \to S^p(\mathcal{T}_V)$ and we present some local error estimates.

Theorem 3.1. *Let $-1 < \beta < -1/2$ and p a polynomial degree. For each $V \in \mathcal{N}$ and $u \in H_0^{1,\beta}(\Omega) \cap C^0(\omega_V)$ there exist an operator $I_V^\beta : H_0^{1,\beta}(\Omega) \cap C^0(\omega_V) \to S^p(\mathcal{T}_V)$ and a positive constant C independent of p, β and u such that*

$$\|u - I_V^\beta u\|_{H^{0,\beta}(K)} \leq \frac{C}{-1-2\beta}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(\mathcal{T}_V)} \quad \forall K \in \mathcal{T}_V,$$

$$\|u - I_V^\beta u\|_{H^{0,\beta}(e)} \leq \frac{C}{-1-2\beta}(p+1)^{-1/2}|u|_{H^{1,\beta}(\mathcal{T}_V)} \qquad \forall e \in \mathcal{E}_V^\circ.$$

If $\gamma \in (\mathcal{E} \setminus \mathcal{E}^\circ)$ is such that $\gamma \subset \partial\omega_V$ then $I_V^\beta u|_\gamma = u|_\gamma = 0$.

Proof. For each $K \in \mathcal{T}_V$ we consider $\Pi_{p,K}^\beta u$ as in Corollary 2.2. Now, we will define ϕ_K such that $\phi_K(V) = u(V)$ for all V vertex of K. Indeed, given V_1, V_2, V_3 and V_4 the vertexes of K, we define the polynomial ϕ_K of degree p as following:

$$\phi_K = \Pi_{p,K}^\beta u + \sum_{l=1}^{4}(u - \Pi_{p,K}^\beta u)(V_l)\xi_{K,l}$$

with $\xi_{K,l}$ the function defined in Corollary 3.1.

Therefore, using (11), (13) and Corollary 3.1 iii) we find that

$$\begin{aligned}\|u - \phi_K\|_{H^{0,\beta}(K)} &\leq \|u - \Pi_{p,K}^\beta u\|_{H^{0,\beta}(K)} + \sum_{l=1}^{4}|(u - \Pi_{p,K}^\beta u)(V_l)|\|\xi_{K,l}\|_{H^{0,\beta}(K)} \\ &\leq C(p+1)^{-1}|u|_{H^{1,\beta}(K)} + C\frac{1}{(-1-2\beta)}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K)} \\ &\leq \frac{C}{(-1-2\beta)}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K)},\end{aligned}$$

similarly, for any e edge of K, from (12), (13) and Corollary 3.1 iv), we have that

$$\|u - \phi_K\|_{H^{0,\beta}(e)} \leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-1/2}|u|_{H^{1,\beta}(K)}.$$

Let $e \in \mathcal{E}^\circ$ be such that $e \subset \mathcal{T}_V \cap (\omega_V)^\circ$ and let K_1 and K_2 be the elements of $\mathcal{T}_V$ that share the edge e, we consider

$$w = (\phi_{K_1} - \phi_{K_2})|_e \quad \in \mathcal{P}_p(e).$$

Observe that

$$\begin{aligned}\|w\|_{H^{0,\beta}(e)} &\leq \|u - \phi_{K_1}\|_{H^{0,\beta}(e)} + \|u - \phi_{K_2}\|_{H^{0,\beta}(e)} \\ &\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-1/2}(|u|_{H^{1,\beta}(K_1)} + |u|_{H^{1,\beta}(K_2)}).\end{aligned}$$

Hence, let $\psi \in \mathcal{Q}_p(K_1)$ be an extension of w to K_1 as in Corollary 3.2, then

$$\begin{aligned}\psi|_e &= w, \\ \psi|_{\partial K_1 \setminus e} &= 0, \\ \|\psi\|_{H^{0,\beta}(K_1)} &\leq C\Gamma(\beta+1)(p+1)^{-(1+\beta)}\|w\|_{H^{0,\beta}(e)}.\end{aligned}$$

We define $\tilde{\phi}_{K_1} = \phi_{K_1} - \psi$ and we note that this function satisfies:

$$\tilde{\phi}_{K_1}|_e = \phi_{K_2}|_e,$$
$$\tilde{\phi}_{K_1}|_{\partial K_1 \setminus e} = \phi_{K_1}|_{\partial K_1 \setminus e},$$

and therefore we have the following error estimates

$$\begin{aligned}
\|u - \tilde{\phi}_{K_1}\|_{H^{0,\beta}(K_1)} &\leq \|u - \phi_{K_1}\|_{H^{0,\beta}(K_1)} + \|\psi\|_{H^{0,\beta}(K_1)} \\
&\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K_1)} + C\Gamma(\beta+1)(p+1)^{-(1+\beta)}\|w\|_{H^{0,\beta}(e)} \\
&\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K_1)} \\
&+ C\Gamma(\beta+1)(p+1)^{-(1+\beta)}\frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-1/2}(|u|_{H^{1,\beta}(K_1)} + |u|_{H^{1,\beta}(K_2)}) \\
&\leq \frac{C}{-1-2\beta}(p+1)^{-(3/2+\beta)}(|u|_{H^{1,\beta}(K_1)} + |u|_{H^{1,\beta}(K_2)}),
\end{aligned}$$

and

$$\begin{aligned}
\|u - \tilde{\phi}_{K_1}\|_{H^{0,\beta}(e)} &\leq \|u - \phi_{K_1}\|_{H^{0,\beta}(e)} + \|\psi\|_{H^{0,\beta}(e)} \\
&= \|u - \phi_{K_1}\|_{H^{0,\beta}(e)} + \|w\|_{H^{0,\beta}(e)} \\
&\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-1/2}|u|_{H^{1,\beta}(K_1)} \\
&+ \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-1/2}(|u|_{H^{1,\beta}(K_1)} + |u|_{H^{1,\beta}(K_2)}) \\
&\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-1/2}(|u|_{H^{1,\beta}(K_1)} + |u|_{H^{1,\beta}(K_2)}).
\end{aligned}$$

Then, we can modify ϕ_{K_1} to $\tilde{\phi}_{K_1}$ and repeat this process in each $e \in \mathcal{E}^\circ$ such that $e \subset \mathcal{T}_V \cap (\omega_V)^\circ$.

If $\gamma \in (\mathcal{E} \setminus \mathcal{E}^\circ)$ is such that $\gamma \subset \partial\omega_V$, we consider $K \in \mathcal{T}_V$ such that $\gamma \in \partial K$. Let $w = \phi_K|_\gamma$, we observe that $w = u = 0$ in $\partial\gamma$. Now, let ψ be as in Corollary 3.2 an extension of w to K and $\tilde{\phi}_K = \phi_K - \psi$, then

$$\tilde{\phi}_K|_\gamma = 0,$$
$$\tilde{\phi}_K|_{\partial K \setminus \gamma} = \phi_K|_{\partial K \setminus \gamma},$$

and

$$\begin{aligned}
\|u - \tilde{\phi}_K\|_{H^{0,\beta}(K)} &\leq \|u - \tilde{\phi}_K\|_{H^{0,\beta}(K)} + \|\psi\|_{H0,\beta(K)} \\
&\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K)} + C\Gamma(\beta+1)(p+1)^{-(1+\beta)}\|w\|_{H^{0,\beta}(\gamma)} \\
&\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K)} \\
&+ C\Gamma(\beta+1)(p+1)^{-(1+\beta)}\big(\|u - \phi_K\|_{H^{0,\beta}(\gamma)} + \|u\|_{H^{0,\beta}(\gamma)}\big),
\end{aligned}$$

since $\|u\|_{H^{0,\beta}(\gamma)} = 0$ because $u|_\gamma = 0$ we have that

$$\begin{aligned}
\|u - \tilde{\phi}_K\|_{H^{0,\beta}(K)} &\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K)} \\
&\quad + C\Gamma(\beta+1)(p+1)^{-(1+\beta)}\|u - \phi_K\|_{H^{0,\beta}(\gamma)} \\
&\leq \frac{C}{(-1-2\beta)\Gamma(\beta+1)}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K)} \\
&\quad + \frac{C}{(-1-2\beta)}(p+1)^{-(1+\beta)}(p+1)^{-1/2}|u|_{H^{1,\beta}(K)} \\
&\leq \frac{C}{-1-2\beta}(p+1)^{-(3/2+\beta)}|u|_{H^{1,\beta}(K)}.
\end{aligned}$$

We can modify ϕ_K to $\tilde{\phi}_K$ and repeat the process for all γ such that $\gamma \in (\mathcal{E} \setminus \mathcal{E}^\circ)$ and $\gamma \subset \partial\omega_V$. Thus, the operator $I_V^\beta u|_K = \phi_K$ satisfies all the requirements. $\square$

Now, we are in conditions to introduce a global operator $Iu \in S_0^{\mathbf{p}}(\mathcal{T})$ which satisfies the following error estimates,

Theorem 3.2. *Let $-1 < \beta < -1/2$ and $u \in H_0^{1,\beta}(\Omega) \cap C^0(\Omega)$, then there exist $Iu \in S_0^{\mathbf{p}}(\mathcal{T})$ and a positive constant C such that*

$$\|u - Iu\|_{H^{0,\beta}(K)} \leq \frac{C}{-1-2\beta} p_K^{-(3/2+\beta)} |u|_{H^{1,\beta}(\mathcal{T}|_{\omega_K})} \quad \forall\ K \in \mathcal{T}, \tag{16}$$

$$\|u - Iu\|_{H^{0,\beta}(e)} \leq \frac{C}{-1-2\beta} p_e^{-1/2} |u|_{H^{1,\beta}(\mathcal{T}|_{\omega_e})} \quad \forall\ e \in \mathcal{E}. \tag{17}$$

The constant C is independent of u, β and $\mathbf{p}$.

Proof. A fundamental property of the space $S_0^{\mathbf{p}}(\mathcal{T})$ (see, for example, [29]) is that we can identify "nodal shape functions" that form a partition of unity, i.e., for each vertex $V \in \mathcal{N}$, we can find a function $\varphi_V \in S^1(\mathcal{T})$ such that

$$\varphi_V|_{\Omega\setminus\omega_V} \equiv 0, \quad \sum_{V\in\mathcal{N}} \varphi_V \equiv 1 \quad \text{and} \quad \sup_{x\in\Omega} |\varphi_V(x)| \leq 1.$$

We consider $I_V^\beta u \in S^{p_V-1}(\mathcal{T}_V)$ as in Theorem 3.1 with $p = p_V - 1$ and we define

$$Iu = \sum_{V\in\mathcal{N}} \varphi_V I_V^\beta u.$$

It is clear that $Iu \in C_0^0(\Omega)$ and

$$Iu|_K = \sum_{V\in K} \varphi_V I_V^\beta u|_K,$$

since $p_V \le p_K$, for all K such that $V \in K$, then $Iu|_K \in \mathcal{Q}_{p_K}(K)$ and

$$\begin{aligned}\|u - Iu\|_{H^{0,\beta}(K)} &= \|\sum_{V\in\mathcal{N}} \varphi_V u - \sum_{V\in\mathcal{N}} \varphi_V I_V^\beta u\|_{H^{0,\beta}(K)} \\ &= \|\sum_{V\in\mathcal{N}} \varphi_V (u - I_V^\beta u)\|_{H^{0,\beta}(K)} \le \sum_{V\in K} \|u - I_V^\beta u\|_{H^{0,\beta}(K)} \\ &\le \frac{C}{-1-2\beta} \sum_{V\in K} p_V^{-(3/2+\beta)} |u|_{H^{1,\beta}(\mathcal{T}_V)},\end{aligned}$$

since polynomial degrees of neighboring elements are comparable then $p_V^{-(3/2+\beta)} \le C p_K^{-(3/2+\beta)}$, therefore

$$\|u - Iu\|_{H^{0,\beta}(K)} \le \frac{C}{-1-2\beta} p_K^{-(3/2+\beta)} |u|_{H^{1,\beta}(\mathcal{T}|_{\omega_K})},$$

and the first estimate holds. Now, let $e \in \mathcal{E}$ we have that

$$\begin{aligned}\|u - Iu\|_{H^{0,\beta}(e)} &= \|\sum_{V\in\mathcal{N}} \varphi_V u - \sum_{V\in\mathcal{N}} \varphi_V I_V^\beta u\|_{H^{0,\beta}(e)} \le \sum_{V\in e} \|u - I_V^\beta u\|_{H^{0,\beta}(e)} \\ &\le \frac{C}{-1-2\beta} \sum_{V\in e} p_V^{-1/2} |u|_{H^{1,\beta}(\mathcal{T}_V)},\end{aligned}$$

since polynomial degrees of neighboring elements are comparable then $p_V^{-1/2} \le C p_e^{-1/2}$, therefore

$$\|u - Iu\|_{H^{0,\beta}(e)} \le \frac{C}{-1-2\beta} p_e^{-1/2} |u|_{H^{1,\beta}(\mathcal{T}|_{\omega_e})},$$

and we conclude the proof. $\square$

In what follows we denote by C_β a generic constant which depends on β but is independent of p.

We finish this Section recalling the following trace results which can be seen as a consequence of Theorem 2.1.

Corollary 3.3. *Let K be a parallelogram in $\mathbb{R}^2$, γ an edge of K and $-1 < \beta < 0$. There exist a unique lineal and continuous function $T : H^{1,\beta}(K) \to H^{0,\beta}(\gamma)$ such that*

$$\|T(u)\|_{H^{0,\beta}(\gamma)} \le C_\beta \|u\|_{H^{1,\beta}(K)},$$

For $u \in H^{1,\beta}(K)$ and γ an edge of K we denote $\|u\|_{H^{0,\beta}(\gamma)} = \|T(u)\|_{H^{0,\beta}(\gamma)}$.

Lemma 3.2. *Let $I = (-1,1)$ and $\alpha, \beta > -1$. Then, there exist $C_{1,\beta}$ and $C_{2,\alpha}$ such that for all polynomials $P \in \mathcal{P}_p(I)$*

$$\int_I P(x)^2 (1-x^2)^\beta dx \le C_{1,\beta} p^2 \int_I P(x)^2 (1-x^2)^{\beta+1} dx,$$
$$\int_I P'(x)^2 (1-x^2)^{\alpha+1} dx \le C_{2,\alpha} p^2 \int_I P(x)^2 (1-x^2)^{\alpha} dx.$$

If, in addition, $-1 < \alpha \le \alpha_0$ and $-1/2 \le \beta \le -1/4$ we can choose C_1 and $C_2 = C_2(\alpha_0)$ independent of β and α.

Proof. The first inequality can be found in, e.g. [14, 12]. The proof of the second inequality is analogous to the proof of Theorem 3.95 of [31]. Following the steps in those demonstrations we can see the dependence of β and α in the constants. $\square$

Lemma 3.3. *Let $I=(-1,1)$, $-1<\beta<1$ and $P\in\mathcal{P}_p(I)$, there exists $\hat v(\hat x,\hat y)$ which is defined on Q with the following properties:*

i) $\hat v(\hat x,-1)=P(\hat x)(1-\hat x^2)^\beta$, $\hat v|_{\partial Q\setminus\ell}=0$ *where* $\ell=I\times\{-1\}$*;*

ii) $\|\hat v\|_{H^{0,-\beta}(Q)}\le C_\beta(p+1)^{\beta-1}\|P\|_{H^{0,\beta}(I)}$*;*

iii) $\|\hat v\|_{H^{1,-\beta}(Q)}\le C_\beta(p+1)^{\beta}\|P\|_{H^{0,\beta}(I)}$.

If, in addition, $1/2\le\beta\le3/4$ we can choose C independent of β.

Proof. By Lemma 3.1 we know that there exists $g\in\mathcal{P}_p(I)$ such that $g(1)=0$, $g(-1)=1$ and $\|g\|_{H^{0,-\beta}(I)}\le C_\beta(p+1)^{-(1-\beta)}$. We define $\hat v(\hat x,\hat y)=P(\hat x)(1-\hat x^2)^\beta g(\hat y)$, this extension obviously satisfies condition i). To prove condition ii) we observe that

$$\begin{aligned}\int_Q \hat v(\hat x,\hat y)^2(1-\hat x^2)^{-\beta}(1-\hat y^2)^{-\beta}&=\int_I P(\hat x)^2(1-\hat x^2)^\beta d\hat x\int_I g(\hat y)^2(1-\hat y^2)^{-\beta}d\hat y\\&=\|P\|^2_{H^{0,\beta}(I)}\|g\|^2_{H^{0,-\beta}(I)}\le C_\beta(p+1)^{-2(1-\beta)}\|P\|^2_{H^{0,\beta}(I)},\end{aligned}$$

Hence,

$$\|\hat v\|_{H^{0,-\beta}(Q)}\le C_\beta(p+1)^{-(1-\beta)}\|P\|_{H^{0,\beta}(I)}.$$

To prove condition iii) we calculate $\|\frac{\partial\hat v}{\partial\hat x}\|_{H^{0,-\beta}(Q)}$ and $\|\frac{\partial\hat v}{\partial\hat y}\|_{H^{0,-\beta}(Q)}$

$$\frac{\partial\hat v}{\partial\hat x}(\hat x,\hat y)=\Big(\frac{\partial P}{\partial\hat x}(\hat x)(1-\hat x^2)^\beta+P(\hat x)\beta(1-\hat x^2)^{\beta-1}(-2\hat x)\Big)g(\hat y),$$

then

$$\begin{aligned}\int_Q\big(\frac{\partial\hat v}{\partial\hat x}(\hat x,\hat y)\big)^2(1-\hat x^2)^{1-\beta}(1-\hat y^2)^{-\beta}&\le C\big\{\int_Q\big(\frac{\partial P}{\partial\hat x}(\hat x)\big)^2(1-\hat x^2)^{\beta+1}(1-\hat y^2)^{-\beta}g(\hat y)\\&\quad+\int_Q P(\hat x)^2(1-\hat x^2)^{\beta-1}(1-\hat y^2)^{-\beta}g(\hat y)\big\},\end{aligned}$$

by inverse estimates of Lemma 3.2 we have that

$$\begin{aligned}\int_I\big(\frac{\partial P}{\partial\hat x}(\hat x)\big)^2(1-\hat x^2)^{\beta+1}d\hat x&\le C_\beta(p+1)^2\int_I P(\hat x)^2(1-\hat x^2)^\beta d\hat x\\\int_I P(\hat x)^2(1-\hat x^2)^{\beta-1}d\hat x&\le C_\beta(p+1)^2\int_I P(\hat x)^2(1-\hat x^2)^\beta d\hat x.\end{aligned}$$

Therefore

$$\begin{aligned}\int_Q\big(\frac{\partial\hat v}{\partial\hat x}(\hat x,\hat y)\big)^2(1-\hat x^2)^{1-\beta}(1-\hat y^2)^{-\beta}&\le C_\beta(p+1)^2\|P\|^2_{H^{0,\beta}(I)}\|g\|^2_{H^{0,-\beta}(I)}\\&\le C_\beta(p+1)^2(p+1)^{-2(1-\beta)}\|P\|^2_{H^{0,\beta}(I)}\\&=C_\beta(p+1)^{2\beta}\|P\|^2_{H^{0,\beta}(I)}.\end{aligned}$$

On the other hand,

$$\frac{\partial\hat v}{\partial\hat y}(\hat x,\hat y)=P(\hat x)(1-\hat x^2)^\beta\frac{\partial g}{\partial\hat y}(\hat y),$$

then

$$\int_Q \big(\frac{\partial \hat{v}}{\partial \hat{y}}(\hat{x},\hat{y})\big)^2(1-\hat{y}^2)^{1-\beta}(1-\hat{x}^2)^{-\beta} = \int_I P(\hat{x})^2(1-\hat{x}^2)^{\beta}d\hat{x}\int_I \left(\frac{\partial g}{\partial \hat{y}}(\hat{y})\right)^2(1-\hat{y}^2)^{1-\beta}d\hat{y},$$

by inverse estimates of Lemma 3.2

$$\int_I \frac{\partial g}{\partial \hat{y}}(\hat{y})^2(1-\hat{y}^2)^{1-\beta}d\hat{y} \le C_\beta(p+1)^2\int_I g(\hat{y})^2(1-\hat{y}^2)^{-\beta}d\hat{y},$$

thus

$$\begin{aligned}\int_Q \big(\frac{\partial \hat{v}}{\partial \hat{y}}(\hat{x},\hat{y})\big)^2(1-\hat{y}^2)^{1-\beta}(1-\hat{x}^2)^{-\beta} &\le C_\beta(p+1)^2\|P\|^2_{H^{0,\beta}(I)}\|g\|_{H^{0,-\beta}(I)}\\ &\le C_\beta(p+1)^{2\beta}\|P\|^2_{H^{0,\beta}(I)}.\end{aligned}$$

from which we conclude the proof. □

4. A posteriori error estimation

In this section we introduce an a posteriori error indicator of the residual type for the classical Poisson model problem and, by using the p-interpolation error estimates obtained in the previous sections, we show the equivalence between the indicator and the error in an appropriate Jacobi-weighted norm up to higher order terms.

4.1. **Problem Statement.** Let $\Omega \subset \mathbb{R}^2$ be an open polygonal domain, $\Gamma = \partial\Omega$ and $f \in L^2(\Omega)$. We consider the classical Poisson problem: Find a function u such that:

$$\begin{cases} -\triangle u = f & \text{in } \Omega\\ \quad u = 0 & \text{on } \Gamma\end{cases} \tag{18}$$

The Variational Problem associated to (18) is: Find $u \in H_0^1(\Omega)$ such that:

$$a(u,v) = L(v) \quad \forall\, v \in H_0^1(\Omega) \tag{19}$$

where $a(u,v) = \int_\Omega \nabla u\cdot\nabla v$ and $L(v) = \int_\Omega fv$.

Let $\mathcal{T}$ be an admissible partition of Ω in parallelogram elements, and $S_0^{\mathbf{p}}(\mathcal{T})$ as in (15), the Discrete Variational Problem is defined as follows: Find $u \in S_0^{\mathbf{p}}(\mathcal{T})$ such that:

$$a(u,v) = L(v) \quad \forall\, v \in S_0^{\mathbf{p}}(\mathcal{T}). \tag{20}$$

Following the ideas introduced in [22], let $Q = (-1,1)^2$ be the reference domain, $\beta > -1$, $\alpha = (\alpha_1,\alpha_2)$ with $\alpha_i \ge 0$ integer, we define the weight function $\tilde{W}_{\beta,\alpha}$ in Q as follows:

$$\tilde{W}_{\beta,\alpha}(x_1,x_2) = (1-x_1^2)^{\beta-\alpha_1}(1-x_2^2)^{\beta-\alpha_2}.$$

Note that $\tilde{W}_{\beta,\alpha} = W_{\beta,-\alpha}$.

Then, the weighted Sobolev space $\tilde{H}^{k,\beta}(Q)$ is defined as the closure of the $C^\infty(\bar{Q})$ function with the norm

$$\|u\|^2_{\tilde{H}^{k,\beta}(Q)} = \sum_{|\alpha|\le k}\int_Q |\partial^\alpha u|^2\tilde{W}_{\beta,\alpha},$$

by $|u|^2_{\tilde{H}^{k,\beta}(Q)}$ we denote the semi-norm

$$|u|^2_{\tilde{H}^{k,\beta}(Q)} = \sum_{|\alpha|=k}\int_Q |\partial^\alpha u|^2\tilde{W}_{\beta,\alpha}.$$

Let K be a parallelogram and let $F : Q \to K$ be an affine transformation. Given $u \in C^\infty(\bar{K})$ we can define $\tilde{u} = u \circ F \ \in C^\infty(\bar{Q})$ and (see, for example [22])

$$\|u\|_{\tilde{H}^{k,\beta}(K)} = \|\tilde{u}\|_{\tilde{H}^{k,\beta}(Q)}.$$

Then, for $\mathcal{T}$ an admissible partition of Ω in parallelogram elements and $u \in C^\infty(\bar{\Omega})$, we define

$$\|u\|^2_{\tilde{H}^{k,\beta}(\mathcal{T})} = \sum_{K\in\mathcal{T}} \|u\|^2_{\tilde{H}^{k,\beta}(K)}.$$

Hence, the Jacobi-weighted spaces $\tilde{H}^{k,\beta}(\mathcal{T})$ and $\tilde{H}^{k,\beta}_0(\mathcal{T})$ for $\mathcal{T}$ a partition of Ω are defined as follows:

$$\tilde{H}^{k,\beta}(\mathcal{T}) = \overline{C^\infty(\bar{\Omega})}^{\|\cdot\|_{\tilde{H}^{k,\beta}(\mathcal{T})}}$$
$$\tilde{H}^{k,\beta}_0(\mathcal{T}) = \overline{C^\infty_0(\bar{\Omega})}^{\|\cdot\|_{\tilde{H}^{k,\beta}(\mathcal{T})}}.$$

Lemma 4.1. *Let $0 < \beta < 1$ and $u \in H^{1+s}(\Omega)$, with $s \geq \frac{1-\beta}{2}$, then $u \in \tilde{H}^{1,\beta}(\mathcal{T})$.*

Proof. Since the functions $C^\infty(\bar{\Omega})$ are dense in $H^{1+s}(\Omega)$, the result can be obtained from the inequality (1.8) of [11] on Q, i.e.,

$$\int_Q \frac{f^2(\hat{x},\hat{y})}{\rho((\hat{x},\hat{y}),\partial Q)^{2\lambda}} \leq C\|f\|^2_{H^\lambda(Q)} \quad \forall f \in H^\lambda(Q) \quad 0<\lambda<1/2 \tag{21}$$

where $\rho((\hat{x},\hat{y}),\partial Q)$ denotes the distance from $(\hat{x},\hat{y})$ to the boundary of Q.

Indeed, there exist $\varphi_n \in C^\infty(\bar{\Omega})$ such that $\varphi_n \longrightarrow u$ as $n \to \infty$ in $H^{1+s}(\Omega)$. Let $K \in \mathcal{T}$ and $F_K : Q \to K$ be an affine transformation, we note $\widehat{\varphi_n - u} = (\varphi_n - u)\circ F_K$. Since $\beta > 0$

$$\int_Q (\widehat{\varphi_n - u})^2(1-\hat{x}^2)^\beta(1-\hat{y}^2)^\beta \leq \int_Q (\widehat{\varphi_n - u})^2 \longrightarrow 0 \text{ as } n \to \infty,$$

then

$$\|\varphi_n - u\|_{H^{0,\beta}(K)} \longrightarrow 0 \text{ as } n \to \infty.$$

Since $0 < \beta < 1$ and $(1-\hat{x}^2) \geq \rho((\hat{x},\hat{y}),\partial Q)$ we get

$$\int_Q \big(\partial_{\hat{x}}(\widehat{\varphi_n - u})\big)^2(1-\hat{x}^2)^{\beta-1}(1-\hat{y}^2)^\beta \leq \int_Q \frac{\big(\partial_{\hat{x}}(\widehat{\varphi_n - u})\big)^2}{\rho((\hat{x},\hat{y}),\partial Q)^{2\lambda}}$$

with $\lambda = \frac{1-\beta}{2}$. Now, since $0 < \lambda < 1/2$ by (21) it follows that

$$\int_Q \frac{\big(\partial_{\hat{x}}(\widehat{\varphi_n - u})\big)^2}{\rho((\hat{x},\hat{y}),\partial Q)^{2\lambda}} \leq \|\partial_x(\widehat{\varphi_n - u})\|^2_{H^\lambda(Q)} \leq |\widehat{\varphi_n - u}|^2_{H^{1+\lambda}(Q)}$$

then, if $\lambda \leq s$ we have

$$|\widehat{\varphi_n - u}|^2_{H^{1+\lambda}(Q)} \longrightarrow 0 \text{ as } n \to \infty,$$

and therefore

$$\int_Q \big(\partial_{\hat{x}}(\widehat{\varphi_n - u})\big)^2(1-\hat{x}^2)^{\beta-1}(1-\hat{y}^2)^\beta \longrightarrow 0 \text{ as } n \to \infty.$$

We can proceed analogously for the other derivative and conclude that $\varphi_n \longrightarrow u$ as $n \to \infty$ in $\tilde{H}^{1,\beta}(K)$. Thus, the result follows since we can do this for any $K \in \mathcal{T}$. □

Let u be the solution of (19) and let u_N be the solution of (20). As in [22], we introduce the norm for the error $e = u - u_N$ denoted by $|||e|||$ and $|||e|||_K$ by

$$|||e|||_K = \sup_{\|v\|_{H^{1,-\beta}(K)}=1} |a(e,v)| \le \|e\|_{\tilde{H}^{1,\beta}(K)}$$

and

$$|||e||| = \sup_{\|v\|_{H_0^{1,-\beta}(\mathcal{T})}=1} |a(e,v)| \le \|e\|_{\tilde{H}^{1,\beta}(\mathcal{T})}.$$

Remark 4.1. *Let $0<\beta<1$ and u be the solution of (19), if $u \in H^{1+s}(\Omega)$ with $s \ge \frac{1-\beta}{2}$ then $e \in \tilde{H}^{1,\beta}(\mathcal{T})$ and, $|||e|||_K$ and $|||e|||$ are well defined.*

By (19) and (20) we can infer that

$$a(e, v_N) = 0 \quad \forall v_N \in S_0^{\mathbf{p}}(\mathcal{T}). \tag{22}$$

For each $l \in \mathcal{E}^\circ$ we choose a normal vector n_l and denote by K_{in} and K_{out} the elements that share the edge l. We define

$$\left[\!\left[\frac{\partial u_N}{\partial n}\right]\!\right]_\ell = \nabla(u_N|_{K_{out}}) \cdot n_\ell - \nabla(u_N|_{K_{in}}) \cdot n_\ell,$$

which corresponds to the jump of the normal derivative of u_N across the edge ℓ.

In what follows we assume that $1/2 < \beta < 1$ and u, the solution of (19), is such that $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ with $s \ge \frac{1-\beta}{2}$. Then, for $v \in H_0^{1,-\beta}(\mathcal{T})$ such that $\|v\|_{H_0^{1,-\beta}(\mathcal{T})} = 1$ we have that $e \in \tilde{H}^{1,\beta}(\mathcal{T})$ and

$$a(e,v) = \sum_{K\in\mathcal{T}} \Big(\int_K (f + \Delta u_N) v + \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \int_\ell \left[\!\left[\frac{\partial u_N}{\partial n}\right]\!\right]_\ell v \Big). \tag{23}$$

For $\epsilon > 0$ and $v_\epsilon \in C_0^\infty(\Omega)$ such that $\|v - v_\epsilon\|_{H_0^{1,-\beta}(\mathcal{T})} \le \epsilon$, by Theorem 3.3 $\|v - v_\epsilon\|_{H^{0,-\beta}(\ell)} \le C_\beta \epsilon$ for all $\ell \in \mathcal{E}$. Since $-1 < -\beta < -1/2$, if we take Iv_ϵ be as in Theorem 3.2, by using (22) and (23) we get

$$\begin{aligned}
a(e,v) &= a(e, v - Iv_\epsilon) + a(e, Iv_\epsilon) \\
&= \sum_{K\in\mathcal{T}} \Big(\int_K (f+\Delta u_N)(v - Iv_\epsilon) + \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \int_\ell \left[\!\left[\frac{\partial u_N}{\partial n}\right]\!\right]_\ell (v - Iv_\epsilon) \Big) \\
&= \sum_{K\in\mathcal{T}} \Big(\int_K (f+\Delta u_N)\big((v - v_\epsilon) + (v_\epsilon - Iv_\epsilon)\big) \\
&+ \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \int_\ell \left[\!\left[\frac{\partial u_N}{\partial n}\right]\!\right]_\ell \big((v - v_\epsilon) + (v_\epsilon - Iv_\epsilon)\big) \Big) \\
&\le \sum_{K\in\mathcal{T}} \Big(\|f + \Delta u_N\|_{H^{0,\beta}(K)} \big(\|v - v_\epsilon\|_{H^{0,-\beta}(K)} + \|v_\epsilon - Iv_\epsilon\|_{H^{0,-\beta}(K)} \big) \\
&+ \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \Big\| \left[\!\left[\frac{\partial u_N}{\partial n}\right]\!\right]_\ell \Big\|_{H^{0,\beta}(\ell)} \big(\|v - v_\epsilon\|_{H^{0,-\beta}(\ell)} + \|v_\epsilon - Iv_\epsilon\|_{H^{0,-\beta}(\ell)} \big) \Big).
\end{aligned}$$

Hence, by (16), (17), we obtain

$$\begin{aligned}
a(e,v) \le \sum_{K\in\mathcal{T}} \Big(&\|f+\Delta u_N\|_{H^{0,\beta}(K)} \big(\epsilon + \frac{C}{-1+2\beta} p_K^{-(3/2-\beta)} \|v_\epsilon\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_K})}\big) \\
&+ \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \Big\| \Big[\!\Big[\frac{\partial u_N}{\partial n} \Big]\!\Big]_\ell \Big\|_{H^{0,\beta}(\ell)} \big(C_\beta \epsilon + \frac{C}{-1+2\beta} p_\ell^{-1/2} \|v_\epsilon\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_\ell})}\big)\Big) \\
\le \sum_{K\in\mathcal{T}} \Big(&\|f+\Delta u_N\|_{H^{0,\beta}(K)} \big(\epsilon + \frac{C}{-1+2\beta} p_K^{-(3/2-\beta)} (\|v-v_\epsilon\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_K})} + \|v\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_K})})\big) \\
&+ \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \Big\| \Big[\!\Big[\frac{\partial u_N}{\partial n} \Big]\!\Big]_\ell \Big\|_{H^{0,\beta}(\ell)} \big(C_\beta \epsilon + \frac{C}{-1+2\beta} p_\ell^{-1/2} (\|v-v_\epsilon\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_\ell})} + \|v\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_\ell})})\big)\Big) \\
\le \sum_{K\in\mathcal{T}} \Big(&\|f+\Delta u_N\|_{H^{0,\beta}(K)} \big(\epsilon + \frac{C}{-1+2\beta} p_K^{-(3/2-\beta)} \epsilon + \frac{C}{-1+2\beta} p_K^{-(3/2-\beta)} \|v\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_K})}\big) \\
&+ \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \Big\| \Big[\!\Big[\frac{\partial u_N}{\partial n} \Big]\!\Big]_\ell \Big\|_{H^{0,\beta}(\ell)} \big(C_\beta \epsilon + C_\beta p_\ell^{-1/2}\epsilon + \frac{C}{-1+2\beta} p_\ell^{-1/2} \|v\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_\ell})}\big)\Big).
\end{aligned}$$

Since this holds for all $\epsilon > 0$ we conclude that

$$a(e,v) \le \frac{C}{-1+2\beta} \sum_{K\in\mathcal{T}} \Big(\|f+\Delta u_N\|_{H^{0,\beta}(K)} p_K^{-(3/2-\beta)} \|v\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_K})} + \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \Big\| \Big[\!\Big[\frac{\partial u_N}{\partial n} \Big]\!\Big]_\ell \Big\|_{H^{0,\beta}(\ell)} p_\ell^{-1/2} \|v\|_{H^{1,-\beta}(}$$

Therefore,

$$|||e||| \le \frac{C}{-1+2\beta} \sum_{K\in\mathcal{T}} \Big(\|f+\Delta u_N\|_{H^{0,\beta}(K)} p_K^{-(3/2-\beta)} + \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \Big\| \Big[\!\Big[\frac{\partial u_N}{\partial n} \Big]\!\Big]_\ell \Big\|_{H^{0,\beta}(\ell)} p_\ell^{-1/2} \Big).$$

Let $f_{p_K} = \Pi^\beta_{p_K,K} f$ as in Corollary 2.2, then

$$\begin{aligned}
|||e||| \le \frac{C}{-1+2\beta} \sum_{K\in\mathcal{T}} \Big(&\|f_{p_K}+\Delta u_N\|_{H^{0,\beta}(K)} p_K^{-(3/2-\beta)} + \|f-f_{p_K}\|_{H^{0,\beta}(K)} p_K^{-(3/2-\beta)} \\
&+ \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \Big\| \Big[\!\Big[\frac{\partial u_N}{\partial n} \Big]\!\Big]_\ell \Big\|_{H^{0,\beta}(\ell)} p_l^{-1/2} \Big),
\end{aligned}$$

Let $0 < \delta < 1/2$, we take $\beta = 1/2 + \delta$. Then

$$\begin{aligned}
|||e||| \le \frac{C}{\delta} \sum_{K\in\mathcal{T}} \Big(&\|f_{p_K}+\Delta u_N\|_{H^{0,\beta}(K)} p_K^{-(1-\delta)} + \|f-f_{p_K}\|_{H^{0,\beta}(K)} p_K^{-(1-\delta)} \\
&+ \frac{1}{2} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \Big\| \Big[\!\Big[\frac{\partial u_N}{\partial n} \Big]\!\Big]_\ell \Big\|_{H^{0,\beta}(\ell)} p_\ell^{-1/2} \Big),
\end{aligned}$$

C is independent of δ.

For each element $K \in \mathcal{T}$ the local error indicator is defined as:

$$\eta_K^2 = \eta_{B_K}^2 + \eta_{E_K}^2, \tag{24}$$

where

$$\eta_{B_K}^2 = \|f_{p_K}+\Delta u_N\|^2_{H^{0,\beta}(K)} p_K^{-2} \quad \text{and} \quad \eta_{E_K}^2 = \frac{1}{4} \sum_{\ell\subset\partial K\cap\mathcal{E}^\circ} \eta_\ell^2, \tag{25}$$

with

$$\eta_\ell^2 = \|R_\ell\|^2_{H^{0,\beta}(l)} p_\ell^{-1}, \quad R_\ell = \left[\!\!\left[\frac{\partial u_N}{\partial n} \right]\!\!\right]_\ell . \tag{26}$$

Therefore, the global estimator is given by

$$\eta^2 = \sum_{K \in \mathcal{T}} \eta_K^2. \tag{27}$$

Thus, we have the following theorem which proves the reliability of the error estimator up to higher order terms.

Theorem 4.1. *Let $\beta = 1/2 + \delta$ with $0 < \delta < \frac{1}{2}$, u be the solution of (19), u_N the solution of (20) and $e = u - u_N$. Let η be as in (27). Assume that $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ with $s \geq \frac{1-\beta}{2}$. Then, there exists a positive constant C such that*

$$|||e||| \leq \frac{C}{\delta} p_{max}^\delta \Big\{ \eta + \Big(\sum_{K \in \mathcal{T}} p_K^{-2} \|f - f_{p_K}\|^2_{H^{0,\beta}(K)} \Big)^{1/2} \Big\},$$

where $p_{max} = \max\{p_K | K \in \mathcal{T}\}$. The constant C is independent of $\mathbf{p}$ and δ.

In order to guarantee that the error indicator is efficient to guide an adaptive refinement scheme, our next goal is to prove that η_K is bounded by the weighted norm of the error up to higher order terms.

The following lemma provides an upper estimate for the first term in the definition of (24).

Theorem 4.2. *Let $\beta = 1/2 + \delta$ with $0 < \delta < \frac{1}{4}$, u be the solution of (19), u_N the solution of (20) and $e = u - u_N$. Let η_{B_K} be as in (25). Assume that $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ with $s \geq \frac{1-\beta}{2}$. Then there exists a constant $C(K)$ such that*

$$\eta_{B_K} \leq C(K) \big(|||e|||_K + \frac{1}{p_K} \|f_{p_K} - f\|_{H^{0,\beta}(K)} \big).$$

The constant $C(K)$ is not dependent on p_K and δ.

Proof.

$$\begin{aligned} \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)} &= \|\widehat{f_{p_K} + \Delta} u_N\|^2_{H^{0,\beta}(Q)} \\ &= \int_Q (\widehat{f_{p_K} + \Delta} u_N)^2 (1 - \hat{x}^2)^\beta (1 - \hat{y}^2)^\beta \\ &= \int_Q (\widehat{f_{p_K} + \Delta} u_N) \hat{v_K} \end{aligned}$$

where $\hat{v_K} = (\widehat{f_{p_K} + \Delta} u_N)(1 - \hat{x}^2)^\beta (1 - \hat{y}^2)^\beta$, then we define

$$v_K = \hat{v_K} \circ F_K^{-1} \text{ in } K \quad \text{ and } \quad v_K = 0 \text{ in } \Omega \setminus K.$$

Therefore

$$\begin{aligned} \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)} &= C(K) \int_K (f_{p_K} + \Delta u_N) v_K \\ &= C(K) \Big(\int_K f v_K + \int_K \Delta u_N v_K + \int_K (f_{p_K} - f) v_K \Big). \end{aligned}$$

Now, since $v_K \in H_0^1(\Omega)$ and u solution of (19) then $\int_K f v_K = \int_K \nabla u \cdot \nabla v_K$ and

$$\begin{aligned}
\|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)} &= C(K)\Big(\int_K \nabla u \cdot \nabla v_K - \int_K \nabla u_N \cdot \nabla v_K + \int_K (f_{p_K} - f) v_K\Big) \\
&= C(K)\Big(\int_K \nabla e \cdot \nabla v_K + \int_Q (f_{p_K} - f) v_K\Big) \\
&= C(K)\Big(a(e, v_K) + \int_K (f_{p_K} - f) v_K\Big) \\
&= C(K)\Big(\|v_K\|_{H^{1,-\beta}(K)} a\big(e, \frac{v_K}{\|v_K\|_{H^{1,-\beta}(K)}}\big) + \int_K (f_{p_K} - f) v_K\Big) \\
&\leq C(K)\big(\|v_K\|_{H^{1,-\beta}(K)} |||e|||_K + \|f_{p_K} - f\|_{H^{0,\beta}(K)} \|v_K\|_{H^{0,-\beta}(K)}\big).
\end{aligned}$$

Observe that

$$\begin{aligned}
\|v_K\|^2_{H^{0,-\beta}(K)} = \|\hat{v_K}\|^2_{H^{0,-\beta}(Q)} &= \int_Q \hat{v_K}^2 (1-\hat{x}^2)^{-\beta}(1-\hat{y}^2)^{-\beta} \\
&= \int_Q (\widehat{f_{p_K} + \Delta} u_N)^2 (1-\hat{x}^2)^{\beta}(1-\hat{y}^2)^{\beta} = \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)}.
\end{aligned} \tag{28}$$

On the other hand,

$$|v_K|^2_{H^{1,-\beta}}(K) = |\hat{v_K}|^2_{H^{1,-\beta}}(Q) = \int_Q \big(\frac{\partial \hat{v_K}}{\partial \hat{x}}\big)^2 (1-\hat{x}^2)^{1-\beta}(1-\hat{y}^2)^{-\beta} + \int_Q \big(\frac{\partial \hat{v_K}}{\partial \hat{y}}\big)^2 (1-\hat{x}^2)^{-\beta}(1-\hat{y}^2)^{1-\beta}.$$

We note that

$$\frac{\partial \hat{v_K}}{\partial \hat{x}} = \Big(\frac{\partial (\widehat{f_{p_K} + \Delta} u_N)}{\partial \hat{x}} (1-\hat{x}^2)^{\beta} + (\widehat{f_{p_K} + \Delta} u_N)\beta(1-\hat{x}^2)^{\beta-1}(-2\hat{x})\Big)(1-\hat{y}^2)^{\beta}, \tag{29}$$

and therefore

$$\begin{aligned}
\int_Q \big(\frac{\partial \hat{v_K}}{\partial \hat{x}}\big)^2 (1-\hat{x}^2)^{1-\beta}(1-\hat{y}^2)^{-\beta} &\leq C\Big(\int_Q \big(\frac{\partial (\widehat{f_{p_K} + \Delta} u_N)}{\partial \hat{x}}\big)^2 (1-\hat{x}^2)^{\beta+1}(1-\hat{y}^2)^{\beta} \\
&\quad + \int_Q (\widehat{f_{p_K} + \Delta} u_N)^2 (1-\hat{x}^2)^{\beta-1}(1-\hat{y}^2)^{\beta}\Big) \\
&= C(I + II).
\end{aligned}$$

Now, by the estimates given in Lemma 3.2, it follows that

$$I \leq C p_K^2 \int_Q (\widehat{f_{p_K} + \Delta} u_N)^2 (1-\hat{x}^2)^{\beta}(1-\hat{y}^2)^{\beta} = C p_K^2 \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)}$$

and

$$II \leq C p_K^2 \int_Q (\widehat{f_{p_K} + \Delta} u_N)^2 (1-\hat{x}^2)^{\beta}(1-\hat{y}^2)^{\beta} = C p_K^2 \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)}.$$

Therefore,

$$\int_Q \big(\frac{\partial \hat{v_K}}{\partial \hat{x}}\big)^2 (1-\hat{x}^2)^{1-\beta}(1-\hat{y}^2)^{-\beta} \leq C p_K^2 \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)}.$$

Analogously

$$\int_Q \big(\frac{\partial \hat{v_K}}{\partial \hat{y}}\big)^2 (1-\hat{y}^2)^{1-\beta}(1-\hat{x}^2)^{-\beta} \leq C p_K^2 \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)}.$$

Then,

$$\|v_K\|_{H^{1,-\beta}(K)} \le C p_K \|f_{p_K} + \Delta u_N\|_{H^{0,\beta}(K)}. \tag{30}$$

Hence, we can conclude that

$$\|f_{p_K} + \Delta u_N\|_{H^{0,\beta}(K)} \le C(K)\big(p_K |||e|||_K + \|f_{p_K} - f\|_{H^{0,\beta}(K)}\big),$$

and the result follows at once. □

Next, we prove an upper estimate for η_ℓ.

Theorem 4.3. *Let $\beta = 1/2 + \delta$ with $0 < \delta < \frac{1}{4}$, u be the solution of (19), u_N the solution of (20) and $e = u - u_N$. Let $\ell \in \mathcal{E}^\circ$, K_1 and $K_2 \in T$ such that $\ell = K_1 \cap K_2$ and η_ℓ as in (26). Assume that $u \in H^{1+s}(\Omega) \cap H^1_0(\Omega)$ with $s \ge \frac{1-\beta}{2}$. Then, there exists a constant $C(\ell)$ such that*

$$\begin{aligned}\eta_\ell \le C(\ell)\Big(&|||e|||_{K_1} p_\ell^\delta + \|f_{p_{K_1}} - f\|_{H^{0,\beta}(K_1)} p_{K_1}^{-1+\delta} + |||e|||_{K_2} p_\ell^\delta \\ &+ \|f_{p_{K_2}} - f\|_{H^{0,\beta}(K_2)} p_{K_2}^{-1+\delta} + |||e|||_{\mathcal{T}|_{\omega_\ell}} p_\ell^\delta\Big).\end{aligned}$$

The constant $C(\ell)$ is independent of $\mathbf{p}$ and δ.

Proof. Let $\omega_\ell = K_1 \cup K_2$ and $\hat{v}$ be as in Lemma 3.3 with $P(\hat{x}) = \hat{R}_\ell(\hat{x}, -1)$, $p = p_\ell$. Let $F_{K_1} : Q \to K_1$ and $F_{K_2} : Q \to K_2$ be affine transformations such that $F_{K_i}(I \times \{-1\}) = \ell$. We define v_ℓ such that $v_\ell|_{K_i} = \hat{v} \circ F_{K_i}^{-1}$, then

i) $\widehat{(v_\ell|_\ell)}(\hat{x}, -1) = \hat{R}_\ell(\hat{x})(1 - \hat{x}^2)^\beta, \quad v_\ell|_{\partial\omega_\ell \setminus \ell} = 0$;
ii) $\|v_\ell\|_{H^{0,-\beta}(\mathcal{T}|_{\omega_\ell})} \le C(p_\ell + 1)^{-(1-\beta)} \|R_\ell\|_{H^{0,\beta}(\ell)}$;
iii) $\|v_\ell\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_\ell})} \le C(p_\ell + 1)^{\beta} \|R_\ell\|_{H^{0,\beta}(\ell)}$.

Therefore

$$\begin{aligned}\|R_\ell\|^2_{H^{0,\beta}(\ell)} &= \int_I \hat{R}_\ell(\hat{x})^2 (1 - \hat{x}^2)^\beta d\hat{x} = \int_I \hat{R}_\ell \hat{v}_\ell = C(\ell) \int_\ell R_\ell v_\ell \\ &= C(\ell)\Big(- \int_{\omega_\ell} \Delta u_N v_\ell - \int_{\omega_\ell} \nabla u_N \cdot \nabla v_\ell\Big) \\ &= C(\ell)\Big(- \int_{K_1} \Delta u_N v_\ell - \int_{K_1} \nabla u_N \cdot \nabla v_\ell - \int_{K_2} \Delta u_N v_\ell - \int_{K_2} \nabla u_N \cdot \nabla v_\ell\Big) \\ &= C(\ell)\Big(- \int_{K_1} (f_{p_{K_1}} + \Delta u_N) v_\ell - \int_{K_1} \nabla u_N \cdot \nabla v_\ell + \int_{K_1} (f_{p_{K_1}} - f) v_\ell \\ &\quad - \int_{K_2} (f_{p_{K_2}} + \Delta u_N) v_\ell - \int_{K_2} \nabla u_N \cdot \nabla v_\ell + \int_{K_2} (f_{p_{K_2}} - f) v_\ell + \int_{\omega_\ell} f v_\ell\Big).\end{aligned}$$

It is easy to see that $v_\ell \in H^1_0(\omega_\ell)$, then since u is a solution of (19) we have that $\int_{\omega_\ell} f v_\ell = \int_{\omega_\ell} \nabla u \cdot \nabla v_\ell$ and

$$\begin{aligned}
\|R_\ell\|^2_{H^{0,\beta}(\ell)} &= C(\ell)\Big(-\int_{K_1} (f_{p_{K_1}} + \Delta u_N) v_\ell - \int_{K_1} \nabla u_N \cdot \nabla v_\ell + \int_{K_1} (f_{p_{K_1}} - f) v \\
&\quad - \int_{K_2} (f_{p_{K_2}} + \Delta u_N) v_\ell - \int_{K_2} \nabla u_N \cdot \nabla v_\ell + \int_{K_2} (f_{p_{K_2}} - f) v_\ell + \int_{\omega_\ell} \nabla u \cdot \nabla v_\ell \Big) \\
&= C(\ell)\Big(-\int_{K_1} (f_{p_{K_1}} + \Delta u_N) v_\ell + \int_{K_1} (f_{p_{K_1}} - f) v_\ell - \int_{K_2} (f_{p_{K_2}} + \Delta u_N) v_\ell \\
&\quad + \int_{K_2} (f_{p_{K_2}} - f) v_\ell + \int_{\omega_\ell} \nabla e \cdot \nabla v_\ell \Big),
\end{aligned} \tag{31}$$

consequently

$$\begin{aligned}
\|R_\ell\|^2_{H^{0,\beta}(\ell)} \le C(\ell)\Big(&\|f_{p_{K_1}} + \Delta u_N\|_{H^{0,\beta}(K_1)} \|v_\ell\|_{H^{0,-\beta}(K_1)} + \|f_{p_{K_1}} - f\|_{H^{0,\beta}(K_1)} \|v_\ell\|_{H^{0,-\beta}(K_1)} \\
&+ \|f_{p_{K_2}} + \Delta u_N\|_{H^{0,\beta}(K_2)} \|v_\ell\|_{H^{0,-\beta}(K_2)} + \|f_{p_{K_2}} - f\|_{H^{0,\beta}(K_2)} \|v_\ell\|_{H^{0,-\beta}(K_2)} \\
&+ \|v_\ell\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_\ell})} a(e, \frac{v}{\|v_\ell\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_\ell})}}) \Big).
\end{aligned}$$

From ii) and iii) we get

$$\begin{aligned}
\|R_\ell\|_{H^{0,\beta}(\ell} \le C(\ell)\Big(&\|f_{p_{K_1}} + \Delta u_N\|_{H^{0,\beta}(K_1)} (p_\ell + 1)^{-1/2+\delta} + \|f_{p_{K_1}} - f\|_{H^{0,\beta}(K_1)} (p_\ell + 1)^{-1/2+\delta} \\
&+ \|f_{p_{K_2}} + \Delta u_N\|_{H^{0,\beta}(K_2)} (p_\ell + 1)^{-1/2+\delta} + \|f_{p_{K_2}} - f\|_{H^{0,\beta}(K_2)} (p_\ell + 1)^{-1/2+\delta} \\
&+ (p_\ell + 1)^{1/2+\delta} |||e|||_{\mathcal{T}|_{\omega_\ell}} \Big),
\end{aligned}$$

then

$$\begin{aligned}
p_\ell^{-1/2} \|R_\ell\|_{H^{0,\beta}(\ell)} \le C(\ell)\Big(&\|f_{p_{K_1}} + \Delta u_N\|_{H^{0,\beta}(K_1)} p_\ell^{-1+\delta} + \|f_{p_{K_1}} - f\|_{H^{0,\beta}(K_1)} p_\ell^{-1+\delta} \\
&+ \|f_{p_{K_2}} + \Delta u_N\|_{H^{0,\beta}(K_2)} p_\ell^{-1+\delta} + \|f_{p_{K_2}} - f\|_{H^{0,\beta}(K_2)} p_\ell^{-1+\delta} + |||e|||_{\mathcal{T}|_{\omega_\ell}} p_\ell^{\delta} \Big).
\end{aligned}$$

Since the polynomial degrees of neighboring elements are comparable, from Theorem 4.2 we can deduce that

$$\begin{aligned}
p_\ell^{-1/2} \|R_\ell\|_{H^{0,\beta}(\ell)} \le C(\ell)\Big(&|||e|||_{K_1} p_\ell^{\delta} + \|f_{p_{K_1}} - f\|_{H^{0,\beta}(K_1)} p_{K_1}^{-1+\delta} + |||e|||_{K_2} p_\ell^{\delta} \\
&+ \|f_{p_{K_2}} - f\|_{H^{0,\beta}(K_2)} p_{K_2}^{-1+\delta} + |||e|||_{\mathcal{T}|_{\omega_\ell}} p_\ell^{\delta} \Big),
\end{aligned}$$

and we conclude the proof. ∎

Corollary 4.1. *Under the same assumptions of Theorem 4.2 and 4.3, we have concluded that*

$$\eta^2 \le p_{max}^{\delta} \sum_{K \in \mathcal{T}} C(K)\big(|||e|||_K^2 + \frac{1}{p_K^2} \|f_{p_K} - f\|^2_{H^{0,\beta}(K)} \big).$$

We observe that this estimation involves the local norms of the error while the reliability, obtained in Theorem 4.1, is in terms of the global norm. However, the following theorem shows that, the efficiency of the error indicator could be also obtained in terms of the global norm which allows us to conclude simultaneously the reliability and efficiency.

Theorem 4.4. *Let $\beta = 1/2 + \delta$ with $0 < \delta < 1/4$, u the solution of (19), u_N the solution of (20), $e = u - u_N$ and η as in (27). Assume that $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ with $s \geq \frac{1-\beta}{2}$. Then there exists a constant C independent of $\mathbf{p}$ and δ such that*

$$\eta \leq C p_{max}^{\delta} \Big\{ |||e||| + \big(\sum_{K \in \mathcal{T}} p_K^{-2} \|f_{p_K} - f\|^2_{H^{0,\beta}(K)} \big)^{1/2} \Big\}.$$

Proof. Let $K \in \mathcal{T}$, and v_K be as in the proof of Theorem 4.2, in this proof we show that

$$\|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)} = C(K) a(e, v_K) + C(K) \int_K (f_{p_K} - f) v_K,$$

then,

$$\eta^2_{B_K} = a(e, (p_K+1)^{-2} C(K) v_K) + (p_K+1)^{-2} C(K) \int_K (f_{p_K} - f) v_K,$$

let $v_{\mathcal{T}} = \sum_{K \in \mathcal{T}} (p_K+1)^{-2} C(K) v_K$, from (28) and (30) we find that

$$\begin{aligned}
\sum_{K \in \mathcal{T}} \eta^2_{B_K} &= a(e, v_{\mathcal{T}}) + \sum_{K \in \mathcal{T}} (p_K+1)^{-2} C(K) \int_K (f_{p_K} - f) v_K \\
&= \|v_{\mathcal{T}}\|_{H^{1,-\beta}(\mathcal{T})} a(e, \frac{v_{\mathcal{T}}}{\|v_{\mathcal{T}}\|_{H^{1,-\beta}(\mathcal{T})}}) + \sum_{K \in \mathcal{T}} (p_K+1)^{-2} C(K) \int_K (f_{p_K} - f) v_K \\
&\leq |||e||| \sum_{K \in \mathcal{T}} (p_K+1)^{-2} C(K) \|v_K\|_{H^{1,-\beta}(K)} \\
&+ \sum_{K \in \mathcal{T}} (p_K+1)^{-2} C(K) \|f_{p_K} - f\|_{H^{0,\beta}(K)} \|v_K\|_{H^{0,-\beta}(K)} \\
&\leq C\big(|||e||| \sum_{K \in \mathcal{T}} (p_K+1)^{-1} \|f_{p_K} + \Delta u_N\|_{H^{0,\beta}(K)} \\
&+ \sum_{K \in \mathcal{T}} (p_K+1)^{-2} \|f_{p_K} - f\|_{H^{0,\beta}(K)} \|f_{p_K} + \Delta u_N\|_{H^{0,\beta}(K)} \big) \\
&\leq C\Big(|||e||| + \big(\sum_{K \in \mathcal{T}} (p_K+1)^{-2} \|f_{p_K} - f\|^2_{H^{0,\beta}(K)} \big)^{1/2} \Big) \big(\sum_{K \in \mathcal{T}} (p_K+1)^{-2} \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)} \big)^{1/2} \Big),
\end{aligned}$$

hence,

$$\big(\sum_{K \in \mathcal{T}} \eta^2_{B_K} \big)^{1/2} \leq C\Big(|||e||| + \big(\sum_{K \in \mathcal{T}} (p_K+1)^{-2} \|f_{p_K} + \Delta u_N\|^2_{H^{0,\beta}(K)} \big)^{1/2} \Big). \tag{32}$$

Let $\ell \in \mathcal{E}^\circ$, and v_ℓ be as in the proof of Theorem 4.3, from (31) we have that

$$\begin{aligned}
\|R_\ell\|^2_{H^{0,\beta}(\ell)} = a(e, C(\ell) v_\ell) + C(\ell) \Big(&- \int_{K_1} (f_{p_{K_1}} + \Delta u_N) v_\ell + \int_{K_1} (f_{p_{K_1}} - f) v_\ell \\
&- \int_{K_2} (f_{p_{K_2}} + \Delta u_N) v_\ell + \int_{K_2} (f_{p_{K_2}} - f) v_\ell \Big).
\end{aligned}$$

Then,

$$\begin{aligned}
\sum_{\ell\in\mathcal{E}}\eta_\ell^2 &= \|\sum_{\ell\in\mathcal{E}} C(\ell)p_\ell^{-1}v_\ell\|_{H^{1,-\beta}(\mathcal{T})}a\Big(e,\frac{\sum_{\ell\in\mathcal{E}}C(\ell)p_\ell^{-1}v_\ell}{\|\sum_{\ell\in\mathcal{E}}C(\ell)p_\ell^{-1}v_\ell\|_{H^{1,-\beta}(\mathcal{T})}}\Big)\\
&\quad+\sum_{\ell\in\mathcal{E}}C(\ell)\Big(-\int_{K_1}(f_{p_{K_1}}+\Delta u_N)v_\ell p_\ell^{-1}+\int_{K_1}(f_{p_{K_1}}-f)v_\ell p_\ell^{-1}\\
&\quad-\int_{K_2}(f_{p_{K_2}}+\Delta u_N)v_\ell p_\ell^{-1}+\int_{K_2}(f_{p_{K_2}}-f)v_\ell p_\ell^{-1}\Big)\\
&\quad\le |||e|||\sum_{\ell\in\mathcal{E}}C(\ell)p_\ell^{-1}\|v_\ell\|_{H^{1,-\beta}(\mathcal{T}|_{\omega_\ell})}+\sum_{\ell\in\mathcal{E}}C(\ell)\Big(\|f_{p_{K_1}}+\Delta u_N\|_{H^{0,\beta}(K_1)}\\
&\quad+\|f_{p_{K_1}}-f\|_{H^{0,\beta}(K_1)}+\|f_{p_{K_2}}+\Delta u_N\|_{H^{0,\beta}(K_2)}+\|f_{p_{K_2}}-f\|_{H^{0,\beta}(K_2)}\Big)\|v_\ell\|_{H^{0,-\beta}(\mathcal{T}|_{\omega_\ell})}p_\ell^{-1}.
\end{aligned}$$

From the estimates for v_ℓ, given in ii) and iii) in the proof of Theorem 4.3, it follows that

$$\begin{aligned}
\sum_{\ell\in\mathcal{E}}\eta_\ell^2 &\le C|||e|||\sum_{\ell\in\mathcal{E}}p_\ell^{-1/2}p_\ell^{\delta}\|R_\ell\|_{H^{0,\beta}(\ell)}+\sum_{\ell\in\mathcal{E}}\Big(\|f_{p_{K_1}}+\Delta u_N\|_{H^{0,\beta}(K_1)}\\
&\quad+\|f_{p_{K_1}}-f\|_{H^{0,\beta}(K_1)}+\|f_{p_{K_2}}+\Delta u_N\|_{H^{0,\beta}(K_2)}+\|f_{p_{K_2}}-f\|_{H^{0,\beta}(K_2)}\Big)\|R_\ell\|_{H^{0,\beta}(\ell)}p_\ell^{-1/2}p_\ell^{-1+\delta}\\
&\quad\le Cp_{max}^{\delta}|||e|||\big(\sum_{\ell\in\mathcal{E}}\eta_\ell^2\big)^{1/2}+C\sum_{\ell\in\mathcal{E}}p_\ell^{2(-1+\delta)}\big(\|f_{p_{K_1}}+\Delta u_N\|^2_{H^{0,\beta}(K_1)}+\|f_{p_{K_1}}-f\|^2_{H^{0,\beta}(K_1)}\\
&\quad+\|f_{p_{K_2}}+\Delta u_N\|^2_{H^{0,\beta}(K_2)}+\|f_{p_{K_2}}-f\|^2_{H^{0,\beta}(K_2)}\big)^{1/2}\big(\sum_{\ell\in\mathcal{E}}\eta_\ell^2\big)^{1/2}.
\end{aligned}$$

Since the polynomial degrees of neighboring elements are comparable we have that

$$\big(\sum_{\ell\in\mathcal{E}}\eta_\ell^2\big)^{1/2}\le C\Big\{p_{max}^{\delta}|||e|||+\Big(\sum_{K\in\mathcal{T}}p_K^{2\delta}\eta_K^2+p_K^{2(-1+\delta)}\|f_{p_K}-f\|^2_{H^{0,\beta}(K)}\Big)^{1/2}\Big\}.$$

Then,

$$\big(\sum_{\ell\in\mathcal{E}}\eta_\ell^2\big)^{1/2}\le Cp_{max}^{\delta}\Big\{|||e|||+\big(\sum_{K\in\mathcal{T}}\eta_K^2\big)^{1/2}+\big(\sum_{K\in\mathcal{T}}p_K^{-2}\|f_{p_K}-f\|^2_{H^{0,\beta}(K)}\big)^{1/2}\Big\},$$

and by (32)

$$\big(\sum_{\ell\in\mathcal{E}}\eta_\ell^2\big)^{1/2}\le Cp_{max}^{\delta}\Big\{|||e|||+\big(\sum_{K\in\mathcal{T}}p_K^{-2}\|f_{p_K}-f\|^2_{H^{0,\beta}(K)}\big)^{1/2}\Big\}.$$

Hence, we are in conditions to compute η.

$$\eta^2=\sum_{K\in\mathcal{T}}(\eta_{B_K}^2+\eta_{E_K}^2)\le C\big(\sum_{K\in\mathcal{T}}\eta_{B_K}^2+\sum_{\ell\in\mathcal{E}}\eta_\ell^2\big)\quad\le C\max\{p_{max}^{2\delta},1\}\Big\{|||e|||+\big(\sum_{K\in\mathcal{T}}p_K^{-2}\|f_{p_K}-f\|^2_{H^{0,\beta}(K)}\big)^{1/2}\Big\}^2,$$

as claimed. □

Remark 4.2. *Our results can be extended to the hp version of the finite elements methods as follows:*

Let $\mathcal{T}$ be an admissible partition of Ω in parallelograms satisfying (14). Let $h_K := diamK$ and $h := max\{h_K | K \in \mathcal{T}\}$, we assume that there exists a constant γ such that:

$$h_K \leq \gamma h_{K'} \quad \forall K, K' \in \mathcal{T} \text{ con } K \cap K' \neq \emptyset. \tag{33}$$

Then, for each element $K \in \mathcal{T}$, we can define the local error indicator as:

$$\eta_K^2 = \eta_{B_K}^2 + \eta_{E_K}^2,$$

where

$$\eta_{B_K}^2 = \frac{h_K^4}{p_K^2} \|f_{p_K} + \Delta u_N\|_{H^{0,\beta}(K)}^2 \quad \text{and} \quad \eta_{E_K}^2 = \frac{1}{4} \sum_{\ell \subset \partial K \cap \mathcal{E}^\circ} \eta_\ell^2,$$

with

$$\eta_\ell^2 = \frac{h_\ell}{p_\ell} \|R_\ell\|_{H^{0,\beta}(l)}^2, \quad R_\ell = \left[\!\!\left[\frac{\partial u_N}{\partial n} \right]\!\!\right]_\ell,$$

and the global estimator is given by

$$\eta^2 = \sum_{K \in \mathcal{T}} \eta_K^2.$$

Therefore, assuming that $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ with $s \geq \frac{1-\beta}{2}$, there exists a constant C independent of $\mathbf{p}$, $\mathcal{T}$ and δ such that

$$\begin{aligned} |||e||| &\leq \frac{C}{\delta} p_{max}^\delta \Big\{ \eta + \Big(\sum_{K \in \mathcal{T}} \frac{h_K^4}{p_K^2} \|f - f_{p_K}\|_{L_\beta^2(K)}^2 \Big)^{1/2} \Big\}, \\ \eta &\leq C p_{max}^\delta \Big\{ |||e||| + \Big(\sum_{K \in \mathcal{T}} \frac{h_K^4}{p_K^2} \|f_{p_K} - f\|_{H^{0,\beta}(K)}^2 \Big)^{1/2} \Big\} \end{aligned}$$

Although these estimates are also quasi-optimal, for the hp version of FEM, we want to recall that the involved weighted norms depend on the mesh $\mathcal{T}$.

We end the paper by emphasizing that, as far as we know, the quasi-optimal estimates reached in Theorems 4.1 and 4.4 are the best results that can be obtained for error estimators of the residual type for the two dimensional case.

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