RAVIART-THOMAS INTERPOLATION IN FRACTIONAL WEIGHTED SOBOLEV SPACES

MARÍA LUZ ALVAREZ, MARIA GABRIELA ARMENTANO, AND RICARDO G. DURÁN

Abstract. We prove error estimates for the Raviart-Thomas interpolation in weighted L^2-norm for functions in appropriate weighted Sobolev spaces. These results allow us to obtain a priori error estimates in the fractional order case for mixed approximations of degenerate elliptic problems.

1. Introduction

Boundary value problems associated with second order elliptic operators given by $\mathcal{L}u = -div(A\nabla u)$, where $A = A(x)$ is a symmetric matrix, can be analyzed by variational methods to prove well posedness and to obtain error estimates for finite element approximations.
A fundamental tool for these analysis are the Sobolev spaces. When the operator is degenerate or non-uniformly elliptic the classic Sobolev spaces have to be replaced by weighted ones. For the theoretical variational analysis the use of weighted Sobolev spaces goes back to more than fifty years ago (we refer the reader to the classic book [14]). On the other hand, several papers have used weighted norms to analyze numerical approximations, for example [1, 9, 17, 18] for standard finite element methods and [2, 7] for mixed finite element methods.
We will work with the Muckenhoupt classes A_p. For $1 < p < \infty$, a non-negative function $w \in L^1_{loc}(\mathbb{R}^n)$ belongs to A_p if it satisfies,

$$[w]_{A_p} := \sup_B \left(\frac{1}{|B|} \int_B w \right) \left(\frac{1}{|B|} \int_B w^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.
Given $w \in A_p$, $L^p_w(\Omega)$ denotes the Lebesgue space associated with the measure $w(x)dx$. For $p = 2$ and $k \in \mathbb{N}$ we introduce the weighted Sobolev space

$$H^k_w(\Omega) = \{v \in L^2_w(\Omega) \colon D^\alpha v \in L^2_w(\Omega), \forall |\alpha| \leq k\},$$

which is a Hilbert space with the norm given by

$$\|v\|^2_{H^k_w(\Omega)} = \sum_{j=0}^{k} |v|^2_{H^j_w(\Omega)}$$

2010 *Mathematics Subject Classification.* Primary: 65N30; Secondary: 35J70.
Key words and phrases. Mixed finite elements, Degenerate elliptic problems, Weighted spaces.
This work was partially supported by Agencia Nacional de Promoción de la Investigación, el Desarrollo Tecnológico y la Innovación (Argentina) under grant PICT 2018-3017, and by Universidad de Buenos Aires under grants 20020160100144BA and 20020170100056BA.

where

$$|v|^2_{H^j_w(\Omega)} = \sum_{|\alpha|=j} \|D^\alpha v\|^2_{L^2_w(\Omega)}$$

For spaces of vector fields we will use analogous notations but with boldface type letters.
In [7] the classic theory for mixed finite element approximations of second order uniformly elliptic problems was generalized to treat degenerate elliptic equations provided that the diffusion coefficient belongs to A_2. In this way it is shown in that paper that optimal order error estimates for the Raviart-Thomas mixed finite element solution follows from appropriate interpolation error estimates in weighted norms. To recall those results we need to introduce some notation. Given $w \in A_2$, an n-dimensional simplex K, $k \in \mathbb{N}_0$, and a vector field $\boldsymbol{\sigma} \in \mathbf{H}^1_w(K)$, let $\Pi^k_K \boldsymbol{\sigma} \in \mathcal{RT}_k(K)$ be the Raviart-Thomas interpolation (in the next section we will recall the definitions for $k = 0$ which is the case we will work with. For general k we refer, for example, to [6] for the definitions).
Then, calling h_K and ρ_K the diameters of K and the biggest ball contained in K respectively, it is proved in [7], that for an integer m, $0 \le m \le k$, and $\boldsymbol{\sigma} \in \mathbf{H}^{m+1}_w(K)$, it holds

$$\|\boldsymbol{\sigma} - \Pi^k_K \boldsymbol{\sigma}\|_{L^2_w(K)} \le C h^{m+1}_K |\boldsymbol{\sigma}|_{H^{m+1}_w(K)} \tag{1.1}$$

where the constant depends only on the eccentricity h_K/ρ_K, n, k, and $[w]_{A_2}$.
However, it is well known that the scale of integer order Sobolev spaces is not enough to measure the smoothness of functions, and consequently, intermediate spaces have to be considered. For degenerate equations it can happen that a solution of a given problem belongs to the weighted Sobolev space of order m but not to that of order $m + 1$. This situation may be a consequence of the degenerate character of the equation or due to singularities in the domain or the data. Therefore, the estimates (1.1) are not enough to prove the correct order error estimates.
Our goal is to extend the error analysis of [7] by proving generalizations of (1.1) for fractional order weighted spaces that we will introduce in the next section.
In the classical unweighted case a usual way to prove fractional order error estimates is to use the intermediate spaces obtained by the real method of interpolation between Banach spaces (see, for example, [3, pag. 660]). It is interesting to observe that no explicit characterization of the real interpolation spaces is needed for that proof. This argument can be easily generalized to the weighted case, and so fractional order estimates analogous to (1.1) can be proved for $\boldsymbol{\sigma}$ in an intermediate space between $\mathbf{H}^m_w(K)$ and $\mathbf{H}^{m+1}_w(K)$ when $m \ge 1$.
The situation is very different for $\boldsymbol{\sigma}$ in a fractional order space lying between $\mathbf{L}^2_w(K)$ and $\mathbf{H}^1_w(K)$. Indeed, the Raviart-Thomas interpolation $\Pi_K \boldsymbol{\sigma}$ cannot even be defined for a general $\boldsymbol{\sigma} \in \mathbf{L}^2_w(K)$, and therefore, there are not endpoints to interpolate the inequalities. Consequently one needs to work with appropriate definitions of fractional order norms not relying on interpolation methods. In the unweighted case the useful norms to prove this kind of estimates are those of the Sobolev-Slobodeckij spaces, also called fractional Sobolev spaces. Therefore, to extend the error estimates to weighted norms the first question is what is a reasonable generalization of the fractional Sobolev spaces. We will introduce a definition useful for our purposes and motivated by weighted seminorms introduced to prove Poincaré type inequalities.
The rest of the paper is organized as follows. In Section 2 we introduce the weighted Sobolev spaces and prove some lemmas that we will use in the rest of the paper. Section 3 deals

with the error estimates for Raviart-Thomas interpolation. Then, in Section 4, we give applications to the approximation of degenerate elliptic problems. In section 5 we show how our results can be extended to smooth domains. We conclude the paper presenting in Section 6 some numerical results.

2. Weighted Fractional Spaces and preliminary results

In this section we introduce the weighted fractional Sobolev spaces and prove some useful results.

A key tool to prove error estimates are Poincaré type inequalities. Several papers introduced weighted fractional seminorms to prove this kind of estimates (see for example [8, 10, 13, 15]). The results obtained in those papers are stronger than what we need, and so, we will work with a simple seminorm for which the Poincaré inequality is very simple to prove.

Given a domain $D \subset \mathbb{R}^n$, $1 < p < \infty$, and $w \in A_p$, we define, for $0 < s < 1$, the seminorm

$$|v|^p_{W^{s,p}_w(D)} = \int_D \int_D \frac{|v(x) - v(y)|^p}{|x-y|^{n+ps}} w(x) dx dy$$

and the associated space

$$W^{s,p}_w(D) = \left\{ v \in L^p_w(D) \colon |v|_{W^{s,p}_w(D)} < \infty \right\},$$

which is a Banach space with the norm given by

$$\|v\|^p_{W^{s,p}_w(D)} = \|v\|^p_{L^p_w(D)} + |v|^p_{W^{s,p}_w(D)}.$$

For $p = 2$ we denote with $H^s_w(D)$ the space $W^{s,2}_w(D)$.

Also we drop the subscript when $w(x) \equiv 1$, thus recovering the standard notation for the classical fractional Sobolev spaces $W^{s,p}(D)$.

The interpolation error in an n-dimensional polytope Ω will be bounded locally on each element K. Let $\{\mathcal{T}_h\}$ be a family of conforming triangulations of Ω. Let h stand for the mesh-size; namely $h = \max_{K \in \mathcal{T}_h} h_K$. We assume that the family of triangulations is regular, i.e., there exists a positive constant γ such that, for all $K \in \mathcal{T}_h$ and all h,

$$\frac{h_K}{\rho_K} \le \gamma.$$

Throughout the paper, we use C to denote a generic constant, which can be different at each occurrence and may depend only on n, w, and γ, unless otherwise specified. Given two quantities A and B the notation $A \lesssim B$ means that $A \le CB$. We also write $A \simeq B$ when $A \lesssim B$ and $B \lesssim A$.

To estimate the interpolation error on each element $K \in \mathcal{T}_h$ we use a well known trace theorem, a Poincaré type inequality, and an embedding between a weighted Sobolev space into an unweighted one. These results are the object of the next three lemmas. We denote with ℓ any of the $n-1$-dimensional simplices that form ∂K (edges in 2d, faces in 3d, hyperfaces in general).

Lemma 2.1. Given $p \in (1, \infty)$ and $v \in W^{s,p}(K)$, if $s \in (1/p, 1)$ we have

$$\|v\|_{L^p(\ell)} \lesssim \left(\frac{|\ell|}{|K|} \right)^{1/p} \left(\|v\|_{L^p(K)} + h^s_K |v|_{W^{s,p}(K)} \right)$$

Proof. It follows from a known trace theorem for Lipschitz domains (see [12, Theorem 1.5.1.2] applied on a reference element combined with standard scaling arguments. □

Lemma 2.2. Let $w \in L^1_{loc}(\mathbb{R}^n)$ be any non negative function. Given $0 < s < 1$, and $v \in H^s_w(K)$, we have

$$\|v - \overline{v}\|_{L^2_w(K)} \lesssim h_K^s |v|_{H^s_w(K)} \tag{2.1}$$

where $\overline{v} = \frac{1}{|K|}\int_K v$, and the hidden constant depends only on γ.

Proof. Observe that

$$v(x) - \overline{v} = \frac{1}{|K|}\int_K (v(x) - v(y))dy$$

and then,

$$\begin{aligned}
\int_K |v(x) - \overline{v}|^2 w(x)dx &= \frac{1}{|K|^2}\int_K \left(\int_K (v(x) - v(y))dy\right)^2 w(x)dx \\
&\leq \frac{1}{|K|}\int_K\int_K |v(x) - v(y)|^2 w(x)dydx \\
&= \frac{1}{|K|}\int_K\int_K \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}|x-y|^{n+2s} w(x)dydx
\end{aligned}$$

thus,

$$\int_K |v(x) - \overline{v}|^2 w(x)dx \leq \frac{h_K^{n+2s}}{|K|}\int_K\int_K \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} w(x)dydx \lesssim h_K^{2s}|v|^2_{H^s_w(K)}$$

□

Lemma 2.3. For $0 < \varepsilon < s < 1$, $1 < p < 2$ and $w \in A_q$, with $q = 2/p$, we have

$$H^s_w(K) \subset W^{s-\varepsilon,p}(K)$$

and moreover, for $v \in H^s_w(K)$,

$$\|v\|_{L^p(K)} \leq \|v\|_{L^2_w(K)}\Big(\int_K w(x)^{-q'/q}\,dx\Big)^{1/pq'} \tag{2.2}$$

and

$$|v|_{W^{s-\varepsilon,p}(K)} \lesssim h_K^{\varepsilon}|v|_{H^s_w(K)}\Big(\int_K w(x)^{-q'/q}\,dx\Big)^{1/pq'} \tag{2.3}$$

where q' is the dual exponent of q and the hidden constant depends on ε and p.

Proof. Writing

$$\int_K |v(x)|^p dx = \int_K |v(x)|^p w(x)^{1/q} w(x)^{-1/q} dx$$

the first estimate follows immediately from the Hölder inequality with q and q'.
To prove the second estimate, applying the Hölder inequality we have,

$$\begin{aligned}
|v|^p_{W^{s-\varepsilon,p}(K)} &= \int_K\int_K \frac{|v(y) - v(x)|^p}{|y-x|^{\frac{n}{q}+ps}}\frac{1}{|y-x|^{\frac{n}{q'}-\varepsilon p}}\,dydx \\
&\leq \int_K \Big(\int_K \frac{|v(y) - v(x)|^2}{|y-x|^{n+2s}}\,dy\Big)^{1/q}\Big(\int_K \frac{1}{|y-x|^{n-\varepsilon pq'}}\,dy\Big)^{1/q'} dx.
\end{aligned}$$

But, it is easy to show that

$$\Big(\int_K \frac{1}{|y-x|^{n-\varepsilon pq'}}\,dy\Big)^{1/q'} \le Ch_K^{\varepsilon p},$$

with C depending on ε and p. So,

$$\begin{aligned}|v|^p_{W^{s-\varepsilon,p}(K)} &\le Ch_K^{\varepsilon p}\int_K\Big(\int_K \frac{|v(y)-v(x)|^2}{|y-x|^{n+2s}}\,dy\Big)^{1/q}dx\\ &= Ch_K^{\varepsilon p}\int_K\Big(\int_K \frac{|v(y)-v(x)|^2}{|y-x|^{n+2s}}w(x)\,dy\Big)^{1/q}w(x)^{-1/q}\,dx.\end{aligned}$$

Then, using again the Hölder inequality we obtain (2.3). □

3. Error estimates for the Raviart-Thomas interpolation

In this section we prove our main result. As we have mentioned in the introduction we are interested in error estimates for non smooth functions and so we restrict the analysis to the lowest order Raviart-Thomas space.

Given a simplex K we recall that

$$\mathcal{RT}_0(K) = \mathcal{P}_0(K)^n + x\mathcal{P}_0(K),$$

where $\mathcal{P}_0(K)$ denotes the space of constant functions restricted to K.

For $i = 1, \cdots, n+1$, denote by ℓ_i the hyperfaces of K, and by $\boldsymbol{n}_i$ the corresponding unit exterior normal vectors.

Then, for a vector field $\boldsymbol{\tau}$ the Raviart-Thomas interpolation $\Pi_K\boldsymbol{\tau}$ is defined by the following degrees of freedom (see for example [6]),

$$\int_{\ell_i}\Pi_K\boldsymbol{\tau}\cdot\boldsymbol{n}_i = \int_{\ell_i}\boldsymbol{\tau}\cdot\boldsymbol{n}_i, \qquad i=1,\cdots,n+1.$$

Observe that such $\Pi_K\boldsymbol{\tau}$ exists whenever $\boldsymbol{\tau}$ is such that , for $i=1,\cdots,n+1$, $\boldsymbol{\tau}\cdot\boldsymbol{n}_i$ is well defined and belongs to $L^1(\ell_i)$. In view of Lemma 2.1 this is the case if $\boldsymbol{\tau}\in\mathbf{W}^{s,p}(K)$ for some $s\in(1/p,1)$ with $p\in(1,\infty)$.

We can now state and prove our main result. In the proof we will use the following fact that can be easily proved. For $p>1$ and $w\in A_p$,

$$\Big(\frac{1}{|K|}\int_K w(x)dx\Big)\Big(\frac{1}{|K|}\int_K w(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} \le C[w]_{A_p} \tag{3.1}$$

with a constant depending only on n and γ.

Theorem 3.1. Given $s>1/2$ and $w\in A_{2s}$, for $\boldsymbol{\tau}\in\mathbf{H}^s_w(K)$ we have

$$\|\boldsymbol{\tau}-\Pi_K\,\boldsymbol{\tau}\|_{\mathbf{L}^2_w(K)} \lesssim h_K^s|\boldsymbol{\tau}|_{\mathbf{H}^s_w(K)} \tag{3.2}$$

Proof. We use a fundamental property of the A_p classes: since $w\in A_{2s}$ there exists q depending on w, $1<q<2s$, such that $w\in A_q$ (see for example the books [11, 19]). Then, we can write $q=2/p$ with $p\in(1,2)$ and choose $\varepsilon>0$ such that $1/p<s-\varepsilon$.

From Lemma 2.3 we know that $\mathbf{H}^s_w(K) \subset \mathbf{W}^{s-\varepsilon,p}(K)$ and so $\Pi_K \boldsymbol{\tau}$ is well defined. Moreoever, if P_i denotes the opposite vertex to ℓ_i, we have

$$\Pi_K \boldsymbol{\tau} = \sum_{i=1}^{n+1} \Big(\int_{\ell_i} \boldsymbol{\tau} \cdot \boldsymbol{n}_i \Big) \psi_{\ell_i}, \tag{3.3}$$

where the basis functions are defined as

$$\psi_{\ell_i}(x) = \frac{(x - P_i)}{n|K|} \tag{3.4}$$

Using Lemma 2.1 for $\mathbf{W}^{s-\varepsilon,p}(K)$ on each ℓ_i we have

$$\begin{aligned} \Big| \int_{\ell_i} \boldsymbol{\tau} \cdot \boldsymbol{n}_i \Big| &\le \|\boldsymbol{\tau}\|_{\mathbf{L}^1(\ell_i)} \le |\ell_i|^{1/p'} \|\boldsymbol{\tau}\|_{\mathbf{L}^p(\ell_i)} \\ &\lesssim \left(\frac{|\ell_i|^{1/p'} |\ell_i|^{1/p}}{|K|^{1/p}} \right) (\|\boldsymbol{\tau}\|_{\mathbf{L}^p(K)} + h_K^{s-\varepsilon} |\boldsymbol{\tau}|_{\mathbf{W}^{s-\varepsilon,p}(K)}), \end{aligned} \tag{3.5}$$

where p' denotes as usual the dual exponent of p.
On the other hand, it follows from (3.4) that $\|\psi_{\ell_i}\|_{L^\infty(K)} \le h_K/n|K|$, and using this estimate together with (3.5) in (3.3) we get

$$\begin{aligned} \|\Pi_K \boldsymbol{\tau}\|_{\mathbf{L}^\infty(K)} &\lesssim \left(\frac{h_K}{|K|} \frac{|\ell_i|}{|K|^{1/p}} \right) (\|\boldsymbol{\tau}\|_{\mathbf{L}^p(K)} + h_K^{s-\varepsilon} |\boldsymbol{\tau}|_{\mathbf{W}^{s-\varepsilon,p}(K)}) \\ &\lesssim \frac{1}{|K|^{1/p}} (\|\boldsymbol{\tau}\|_{\mathbf{L}^p(K)} + h_K^{s-\varepsilon} |\boldsymbol{\tau}|_{\mathbf{W}^{s-\varepsilon,p}(K)}). \end{aligned}$$

Consequently, from (2.2) and (2.3), we obtain

$$\|\Pi_K \boldsymbol{\tau}\|_{\mathbf{L}^\infty(K)} \lesssim \frac{1}{|K|^{1/p}} (\|\boldsymbol{\tau}\|_{\mathbf{L}^2_w(K)} + h_K^s |\boldsymbol{\tau}|_{\mathbf{H}^s_w(K)}) \Big(\int_K w^{-q'/q} \Big)^{1/pq'}$$

Then,

$$\begin{aligned} \|\Pi_K \boldsymbol{\tau}\|_{\mathbf{L}^2_w(K)} &\le \|\Pi_K \boldsymbol{\tau}\|_{\mathbf{L}^\infty(K)} \Big(\int_K w \Big)^{1/2} \\ &\lesssim \frac{1}{|K|^{1/p}} \Big(\int_K w \Big)^{1/2} \Big(\int_K w^{-q'/q} \Big)^{1/pq'} (\|\boldsymbol{\tau}\|_{\mathbf{L}^2_w(K)} + h_K^s |\boldsymbol{\tau}|_{\mathbf{H}^s_w(K)}) \\ &\lesssim \Big(\frac{1}{|K|} \int_K w \Big)^{1/2} \Big(\frac{1}{|K|} \int_K w^{-q'/q} \Big)^{1/pq'} (\|\boldsymbol{\tau}\|_{\mathbf{L}^2_w(K)} + h_K^s |\boldsymbol{\tau}|_{\mathbf{H}^s_w(K)}) \end{aligned}$$

Therefore, using (3.1) we conclude that

$$\|\Pi_K \boldsymbol{\tau}\|_{\mathbf{L}^2_w(K)} \lesssim [w]_{A_q}^{1/2} (\|\boldsymbol{\tau}\|_{\mathbf{L}^2_w(K)} + h_K^s |\boldsymbol{\tau}|_{\mathbf{H}^s_w(K)})$$

To finish the proof we use this estimate and standard argument. Indeed, since Π_K is the identity on constant vector fields we have

$$\begin{aligned} \|\boldsymbol{\tau} - \Pi_K \boldsymbol{\tau}\|_{\mathbf{L}^2_w(K)} &= \|\boldsymbol{\tau} - \overline{\boldsymbol{\tau}} - \Pi_K(\boldsymbol{\tau} - \overline{\boldsymbol{\tau}})\|_{\mathbf{L}^2_w(K)} \\ &\lesssim (\|\boldsymbol{\tau} - \overline{\boldsymbol{\tau}}\|_{\mathbf{L}^2_w(K)} + h_K^s |\boldsymbol{\tau}|_{\mathbf{H}^s_w(K)}) \end{aligned}$$

and therefore (3.2) follows by the Poincaré inequality (2.1). □

4. Application to degenerate elliptic problems

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $w \in A_2$ we define $H^1_{w,0}(\Omega)$ as the closure in $H^1_w(\Omega)$ of $C^\infty_0(\Omega)$. The following weighted Poincaré inequality is well known [16], for $v \in H^1_{w,0}(\Omega)$,

$$\|v\|_{L^2_w(\Omega)} \leq C\|\nabla v\|_{\mathbf{L}^2_w(\Omega)} \tag{4.1}$$

where the constant depends only on w and Ω.
We consider the problem

$$\begin{cases} -div(a\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

where $a = a(x)$ belongs to the class A_2.
Using (4.1) and the classic Lax-Milgram lemma it follows that, for f in $(H^1_{a,0}(\Omega))^*$, the dual space of $H^1_{a,0}(\Omega)$, there exists a unique solution $u \in H^1_{a,0}(\Omega)$ of Problem (4.2).
Introducing $\boldsymbol{\sigma} = -a\nabla u \in \mathbf{L}^2_{a^{-1}}(\Omega)$ and the space

$$\mathbf{H}_{a^{-1}}(div, \Omega) = \{\boldsymbol{\tau} \in \mathbf{L}^2_{a^{-1}}(\Omega) \colon div\, \boldsymbol{\tau} \in L^2_{a^{-1}}(\Omega)\}$$

$(\boldsymbol{\sigma}, u) \in \mathbf{L}^2_{a^{-1}}(\Omega) \times L^2_a(\Omega)$ satisfy,

$$\begin{cases} \displaystyle\int_\Omega a^{-1}\, \boldsymbol{\sigma} \cdot \boldsymbol{\tau} - \int_\Omega u\, div\, \boldsymbol{\tau} = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_{a^{-1}}(div, \Omega) \\ div\, \boldsymbol{\sigma} = f \end{cases}$$

which is the usual formulation for mixed finite element approximations.
Assuming that $f \in L^2_{a^{-1}}(\Omega)$ the classic analysis for uniform elliptic problems was extended in [7] to obtain optimal order a priori error estimates for Raviart-Thomas approximations of degenerate problems as (4.2).
The global approximation spaces, for vector and scalar variable respectively, associated with a partition $\mathcal{T}_h$ are

$$\boldsymbol{S}_h = \{\boldsymbol{\tau} \in \mathbf{H}_{a^{-1}}(div, \Omega) \colon \boldsymbol{\tau}|_K \in \mathcal{RT}_0(K), \ \ \forall K \in \mathcal{T}_h\}, \tag{4.3}$$

and

$$V_h = \{v \in L^2_a(\Omega) \colon v|_K \in \mathcal{P}_0(K) \ \ \forall K \in \mathcal{T}_h\}. \tag{4.4}$$

If P_h is the orthogonal L_2 projection over V_h, the approximate solution $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{S}_h \times V_h$ is defined by

$$\begin{cases} \displaystyle\int_\Omega a^{-1}\, \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} - \int_\Omega u_h\, div\, \boldsymbol{\tau} = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{S}_h \\ div\, \boldsymbol{\sigma}_h = P_h f \end{cases} \tag{4.5}$$

Although the original continuous problem is well posed for $f \in (H^1_{a,0}(\Omega))^*$, the definition of the Raviart-Thomas mixed finite element solution requires more regularity on f. Indeed, P_h cannot be defined for general $(H^1_{a,0}(\Omega))^*$ because $V_h \not\subset H^1_{a,0}(\Omega)$. However, in many cases the hypothesis $f \in L^2_{a^{-1}}(\Omega)$ can be relaxed. Indeed, the Raviart-Thomas mixed approximation is well defined whenever $P_h f$ makes sense, for example when $f \in L^1(\Omega)$, which will be the case in our numerical examples.

Now, applying the analysis of [7] together with the results of the previous sections we obtain the following error estimates.

Theorem 4.1. Assume that the right hand side f is such that $P_h f$, and consequently the mixed approximation, are well defined. Let $\{\mathcal{T}_h\}$ be a regular family of triangulations of Ω. If the solution of the continuous problem satisfies $(\boldsymbol{\sigma}, u) \in (\mathbf{H}^s_{a^{-1}}(\Omega), H^s_a(\Omega))$, for some $1/2 < s < 1$, and $a^{-1} \in A_{2s}$, then we have

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{L}^2_{a^{-1}}(\Omega)} \lesssim h^s |\boldsymbol{\sigma}|_{\mathbf{H}^s_{a^{-1}}(\Omega)} \tag{4.6}$$

and

$$\|u - u_h\|_{L^2_a(\Omega)} \lesssim h^s \Big\{ |\boldsymbol{\sigma}|_{\mathbf{H}^s_{a^{-1}}(\Omega)} + |u|_{H^s_a(\Omega)} \Big\} \tag{4.7}$$

Proof. We have to check that the arguments given for the error estimates in [7] can be applied under our hypotheses.
Let $\Pi_h \boldsymbol{\sigma}$ be the global Raviart-Thomas interpolation defined by $\Pi_h \boldsymbol{\sigma}|_K = \Pi_K \boldsymbol{\sigma}$. Since $\boldsymbol{\sigma} \in \mathbf{H}^s_{a^{-1}}(\Omega)$ with $s > 1/2$, it follows that $\Pi_h \boldsymbol{\sigma}$ is well defined. On the other hand,

$$\int_\Omega P_h f\, v = \int_\Omega f\, v \qquad \forall v \in V_h$$

and so, the key point in the proof of [7, Lemma 2.4], which is the commutativity property $div\, \Pi_h \boldsymbol{\sigma} = P_h f$, holds in our case. Consequently we have

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{L}^2_{a^{-1}}(\Omega)} \le \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{\mathbf{L}^2_{a^{-1}}(\Omega)} \tag{4.8}$$

But, applying Theorem 3.1 for $w(x) = a^{-1}(x)$ we obtain,

$$\begin{aligned} \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|^2_{\mathbf{L}^2_{a^{-1}}(\Omega)} &= \sum_{K \in \mathcal{T}_h} \|\boldsymbol{\sigma} - \Pi_K \boldsymbol{\sigma}\|^2_{\mathbf{L}^2_{a^{-1}}(K)} \lesssim \sum_{K \in \mathcal{T}_h} h_K^{2s} |\boldsymbol{\sigma}|^2_{\mathbf{H}^s_{a^{-1}}(K)} \\ &\lesssim h^{2s} |\boldsymbol{\sigma}|^2_{\mathbf{H}^s_{a^{-1}}(\Omega)} \end{aligned}$$

which combined with (4.8) gives (4.6).
Analogously we can apply [7, Lemma 2.6] to obtain

$$\|u - u_h\|_{L^2_a(\Omega)} \lesssim \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{L}^2_{a^{-1}}(\Omega)} + \|u - P_h u\|_{L^2_a(\Omega)}$$

Then, (4.7) follows from (4.6) and applying Lemma 2.2. □

5. Curved domains

To apply Raviart-Thomas methods in curved domains one possibility is to approximate the boundary by using isoparametric elements as was done, for example, in [4, 5]. However, since we are considering homogeneous Dirichlet conditions (i.e., natural conditions for the mixed formulation), the results obtained in the previous sections can be extended to domains with smooth boundary using curved triangles. For the sake of simplicity we restrict our analysis to planar domains. We assume that, $\partial\Omega$ can be divided into a finite number of arcs such that each one has a parametric representation with a C^1 function.
Allowing generalized triangles with one curved edge, our domain Ω can be covered exactly by triangulations $\widetilde{\mathcal{T}}_h$ such that all triangles without edges on $\partial\Omega$ are standard triangles, while

those with an edge on $\partial\Omega$ are generalized triangles, i.e., they have two straight edges and one curved one.
The Raviart-Thomas space can be defined exactly in the same way for a generalized triangle $\widetilde{K}$, namely,

$$\mathcal{RT}_0(\widetilde{K}) = \mathcal{P}_0(\widetilde{K})^2 + x\mathcal{P}_0(\widetilde{K}),$$

and then, the global spaces and the mixed finite element approximation are defined as in (4.3), (4.4) and (4.5) but using now $\widetilde{\mathcal{T}}_h$.
Our present goal is to extend the definition of Π_K to elements in $\widetilde{\mathcal{T}}_h$ and to generalize the error estimate given in Theorem 3.1 to this case.
Let us first introduce some notation: given $\widetilde{K}$ intersecting $\partial\Omega$ we call ℓ_1, ℓ_2 and $\tilde{\ell}_3$ the two straight edges and the curved one respectively, and P_i the corresponding opposite vertices. Finally, let ℓ_3 be the segment $\overline{P_1P_2}$, and $K \subset \widetilde{K}$ be the triangle having P_i as its vertices, and we call α_i the interior angle of K at each vertex P_i, $i = 1, 2, 3$. We also denote by $\boldsymbol{n}_i$, $i = 1, 2, 3$, and $\tilde{\boldsymbol{n}}_3$ the exterior unit vector normals at ℓ_i and $\tilde{\ell}_3$ respectively.
In the next lemma we construct a basis for $\mathcal{RT}_0(\widetilde{K})$ that will allow us to define the Raviart-Thomas interpolation and to extend the error estimates for generalized triangles. Actually we will show that the basis functions have an expression similar to that given in (3.4). For simplicity we explain in details the case $K \subset \tilde{K}$, other situations can be treated by similar arguments.

Lemma 5.1. Define

$$\psi_{\tilde{\ell}_3}(x) = \frac{(x - P_3)}{2|\widetilde{K}|}, \quad \psi_{\ell_1}(x) = \frac{(x - Q_1)}{2|\widetilde{K}|}, \quad \text{and} \quad \psi_{\ell_2}(x) = \frac{(x - Q_2)}{2|\widetilde{K}|},$$

where Q_1 and Q_2 are the points defined by

$$|Q_1 - P_1| = |\ell_2| \frac{|\widetilde{K} \setminus K|}{|K|}$$

and the angle between $P_1 - Q_1$ and $\boldsymbol{n}_3$ is $\pi/2 + \alpha_1$, and

$$|Q_2 - P_2| = |\ell_1| \frac{|\widetilde{K} \setminus K|}{|K|}$$

and the angle between $P_2 - Q_2$ and $\boldsymbol{n}_3$ is $\pi/2 + \alpha_2$ (see Figure 1). Then,

$$\int_{\tilde{\ell}_3} \psi_{\tilde{\ell}_3} \cdot \tilde{\boldsymbol{n}}_3 = 1, \quad \int_{\ell_i} \psi_{\tilde{\ell}_3} \cdot \boldsymbol{n}_i = 0 \quad \text{for} \quad i = 1, 2, \tag{5.1}$$

and

$$\int_{\tilde{\ell}_3} \psi_{\ell_i} \cdot \tilde{\boldsymbol{n}}_3 = 0, \quad \int_{\ell_j} \psi_{\ell_i} \cdot \boldsymbol{n}_j = \delta_{ij} \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad j = 1, 2 \tag{5.2}$$

Proof. That $\int_{\ell_i} \psi_{\tilde{\ell}_3} \cdot \boldsymbol{n}_i = 0$, for $i = 1, 2$, follows immediately from the definition of $\psi_{\tilde{\ell}_3}$. Now, observe that $\int_{\widetilde{K}} div\, \psi_{\tilde{\ell}_3} = 1$. and so, the first condition in (5.1) follows by the divergence theorem.

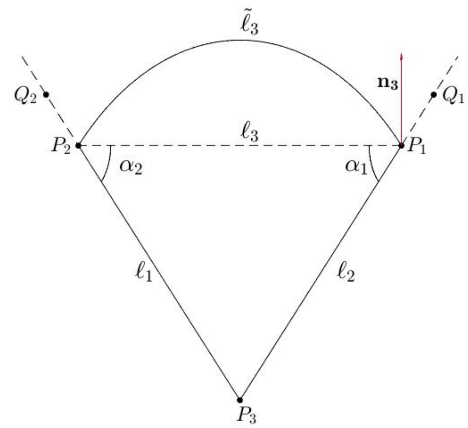

FIGURE 1. Curved triangle and data for the proof of Lemma 5.1

To prove (5.2) consider, for example, $i = 1$. Since Q_1 belongs to the straight line containing ℓ_2 it follows that $\int_{\ell_2} \psi_{\ell_1} \cdot \boldsymbol{n}_2 = 0$. On the other hand applying the divergence theorem in $\widetilde{K} \setminus K$ we have

$$\int_{\tilde{\ell}_3} \psi_{\ell_1} \cdot \tilde{\boldsymbol{n}}_3 = \int_{\ell_3} \psi_{\ell_1} \cdot \boldsymbol{n}_3 + \int_{\widetilde{K}\setminus K} div\, \psi_{\ell_1} = \frac{|\ell_3|}{2|\widetilde{K}|}(P_1 - Q_1) \cdot \boldsymbol{n}_3 + \frac{|\widetilde{K} \setminus K|}{|\widetilde{K}|} \tag{5.3}$$

but,

$$\begin{aligned} \frac{|\ell_3|}{2|\widetilde{K}|}(P_1 - Q_1) \cdot \boldsymbol{n}_3 &= \frac{|\ell_3|}{2|\widetilde{K}|}|P_1 - Q_1| \cos(\pi/2 + \alpha_1) \\ &= -\frac{|\ell_3|}{2|\widetilde{K}|}|\ell_2|\frac{|\widetilde{K} \setminus K|}{|K|} \sin \alpha_1 = -\frac{|\widetilde{K} \setminus K|}{|\widetilde{K}|}, \end{aligned} \tag{5.4}$$

where we have used that $\sin \alpha_1 = \frac{2|K|}{|\ell_2||\ell_3|}$. Then, (5.4) combined with (5.3) yields $\int_{\tilde{\ell}_3} \psi_{\ell_1} \cdot \tilde{\boldsymbol{n}}_3 = 0$. Finally, that $\int_{\ell_1} \psi_{\ell_1} \cdot \boldsymbol{n}_1 = 1$ follows by the divergence theorem. □

An immediate consequence of this lemma is the existence of the Raviart-Thomas interpolation on generalized triangles. Indeed, given $\boldsymbol{\tau} \in \mathbf{H}^s_w(\widetilde{K})$, for some $s > 1/2$ and $w \in A_{2s}$, we define

$$\Pi_{\widetilde{K}}\boldsymbol{\tau} = \sum_{i=1}^{2} \Big(\int_{\ell_i} \boldsymbol{\tau} \cdot \boldsymbol{n}_i \Big) \psi_{\ell_i} + \Big(\int_{\tilde{\ell}_3} \boldsymbol{\tau} \cdot \tilde{\boldsymbol{n}}_3 \Big) \psi_{\tilde{\ell}_3}$$

and, it follows easily from (5.1) and (5.2), that $\Pi_{\widetilde{K}}$ satisfies the commutative diagram property also in this case, i.e., if Π_h is the global projection associated with $\widetilde{\mathcal{T}}_h$ then $div\, \Pi_h \boldsymbol{\tau} = P_h(div\, \boldsymbol{\tau})$. Moreover, the error estimates can be proved in this case with the same arguments used in Theorem 3.1. In fact we have

Theorem 5.2. Given $s > 1/2$, $w \in A_{2s}$, and $\widetilde{K} \in \widetilde{\mathcal{T}}_h$, for $\boldsymbol{\tau} \in \mathbf{H}^s_w(\widetilde{K})$ we have

$$\|\boldsymbol{\tau} - \Pi_{\widetilde{K}}\, \boldsymbol{\tau}\|_{\mathbf{L}^2_w(\widetilde{K})} \lesssim h^s_{\widetilde{K}} |\boldsymbol{\tau}|_{\mathbf{H}^s_w(\widetilde{K})}$$

Proof. The key ingredients in the proof of Theorem 3.1 are the three lemmas given in Section 2 and the bounds for the basis functions. All these results hold also for $\widetilde{K}$.
First, the trace theorem recalled in Lemma 2.1 can be proved by a change of variables to a reference triangle. Indeed, we can define an invertible transformation $F\colon \hat{K} \to \widetilde{K}$, where $\hat{K}$ is a reference element of order one such that the mapping as well as its inverse are of class C^1, $|J_F| \sim h^2_{\widetilde{K}}$, where J_F is the Jacobian of F, and the first derivatives of F and F^{-1} are $O(h)$ and $O(h^{-1})$ respectively (see [20, Theorem 1]).
On the other hand, Lemmas 2.2 and 2.3 hold for $\widetilde{K}$, indeed we have only used in the proofs the shape regularity of K but not that it was a simplex.
Finally the bounds for the basis functions used in the case of triangles follows also in this case from Lemma 5.1 observing that, from the definition of Q_i, $i = 1, 2$ we obtain that $|P_i - Q_i| \lesssim h_{\widetilde{K}}$. □

Summing up we conclude that Theorem 4.1 also holds for curved domains.

6. Numerical results

We finish the paper presenting some numerical approximations for a simple model problem by the lowest order Raviart-Thomas elements in two dimensions.
We consider problem (4.2) with $\Omega = \{x \in \mathbb{R}^2 \colon |x| < 1\}$ and $a(x) = |x|^\alpha$, which belongs to A_2 when $\alpha \in (-2, 2)$, and choose f such that the exact solution is $u(x) = |x|^\beta - 1$ for some β.
The following Lemma will allow us to determine conditions on s such that $\boldsymbol{\sigma} \in \mathbf{H}^s_{a^{-1}}(\Omega)$.

Lemma 6.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and $w(x) = |x|^\delta$ with $\delta > -n$. For $0 < s < 1$, if $s < \frac{n}{2} + \gamma + \frac{\delta}{2}$ and $\delta < 2s$ then, $v_j := x_j |x|^{\xi-1} \in H^s_w(\Omega)$, for $j = 1, \cdots, n$.

Proof. Observe first that, since $2\xi + \delta > -n$ we have that $v_j \in L^2_w(\Omega)$. Therefore, it only rests to check that $|v_j|_{H^s_w(\Omega)}$ is finite. We divide the proof in two cases.

Case $\xi \leq 0$:

We decompose the domain of integration $\Omega \times \Omega$ as $A \cup B$ with

$$A := \{(x, y) \in \Omega \times \Omega \colon |x|/2 < |x - y|\}, \text{and} \quad B := (\Omega \times \Omega) \setminus A$$

Further we split the integral over A into two parts:
First, since $\xi \leq 0$, for $|y| < |x|$, we have $|y_j|y|^{\xi-1} - x_j|x|^{\xi-1}|^2 \lesssim |y|^{2\xi}$, and since $|x| < 2|x - y|$ in A, we obtain

$$\iint\limits_{A\cap\{|y|<|x|\}} \frac{|y_j|y|^{\xi-1} - x_j|x|^{\xi-1}|^2}{|x-y|^{n+2s}} |x|^\delta dy dx \lesssim \iint\limits_{A\cap\{|y|<|x|\}} \frac{|y|^{2\xi}}{|x|^{n+2s}} |x|^\delta dy dx$$
$$\lesssim \int_\Omega \Big(\int_{\{|y|<|x|\}} |y|^{2\xi} dy \Big) |x|^{\delta-n-2s} dx$$

but, from the hypotheses we have $2\xi > -n + 2s - \delta \geq -n$, and so

$$\iint\limits_{A\cap\{|y|<|x|\}} \frac{|y_j|y|^{\xi-1} - x_j|x|^{\xi-1}|^2}{|x-y|^{n+2s}} |x|^\delta dy dx \lesssim \int_\Omega |x|^{2\xi+\delta-2s} < \infty$$

because $2\xi + \delta - 2s > -n$.
On the other hand, for $|x| \leq |y|$, we use the fact that in A, $|y| \leq |y - x| + |x| < 3|x - y|$, which implies

$$\iint\limits_{A\cap\{|x|\leq|y|\}} \frac{|x_j|x|^{\xi-1} - y_j|y|^{\xi-1}|^2}{|x-y|^{n+2s}}|x|^\delta dydx \lesssim \iint\limits_{A\cap\{|x|\leq|y|\}} \frac{|x|^{2\xi+\delta}}{|x-y|^{n+2s}}dydx$$
$$\lesssim \int_\Omega |y|^{-n-2s} \int_{\{|x|\leq|y|\}} |x|^{2\xi+\delta}dx\, dy \lesssim \int_\Omega |y|^{2\xi+\delta-2s} < \infty$$

because $2\xi + \delta - 2s > -n$.
Let us now consider the integral over B. In this case we have $|x| - |y| \leq |x - y| \leq |x|/2$ and consequently $|x| \leq 2|y|$. On the other hand $|y| \leq |x - y| + |x| \leq 3|x|/2$. Thus, in B we have $|x| \simeq |y|$.
Therefore, by the mean value theorem we obtain

$$|x_j|x|^{\xi-1} - y_j|y|^{\xi-1}|^2 \lesssim |x|^{2\xi-2}|x-y|^2$$

Hence,

$$\iint\limits_{B} \frac{|x_j|x|^{\xi-1} - y_j|y|^{\xi-1}|^2}{|x-y|^{n+2s}}|x|^\delta dydx \lesssim \iint\limits_{B} \frac{|x|^{2\xi-2}}{|x-y|^{n+2s-2}}|x|^\delta dydx$$
$$\lesssim \int_\Omega |x|^{2\xi-2+\delta} \int_{\{|x-y|\leq|x|/2\}} |x-y|^{-n-2s+2}dydx$$
$$\lesssim \int_\Omega |x|^{2\xi+\delta-2s}dx < \infty$$

because $2\xi + \delta - 2s > -n$.

Case $\xi > 0$:

We decompose the domain of integration as in the case $\xi \leq 0$. Now we have

$$\iint\limits_{A\cap\{|y|<|x|\}} \frac{|x_j|x|^{\xi-1} - y_j|y|^{\xi-1}|^2}{|x-y|^{n+2s}}|x|^\delta dydx \lesssim \iint\limits_{A\cap\{|y|<|x|\}} \frac{|x|^{2\xi}}{|x-y|^{n+2s}}|x|^\delta dydx$$
$$\lesssim \int_\Omega \int_{\{|y-x|>|x|/2\}} \frac{|x|^{2\xi+\delta}}{|x-y|^{n+2s}}dy\, dx \lesssim \int_\Omega |x|^{2\xi+\delta-2s}dx < \infty$$

because $2\xi + \delta - 2s > -n$.
On the other hand,

$$\iint\limits_{A\cap\{|x|\leq|y|\}} \frac{|x_j|x|^{\xi-1} - y_j|y|^{\xi-1}|^2}{|x-y|^{n+2s}}|x|^\delta dydx \lesssim \iint\limits_{A\cap\{|x|\leq|y|\}} \frac{|y|^{2\xi}}{|x-y|^{n+2s}}|x|^\delta dydx$$

and using that in A we have $|y| < 3|x-y|$, we obtain

$$\iint\limits_{A\cap\{|x|\leq|y|\}} \frac{|x_j|x|^{\xi-1} - y_j|y|^{\xi-1}|^2}{|x-y|^{n+2s}} |x|^{\delta} dydx \lesssim \int_\Omega \int_{\{|x|\leq|y|\}} |y|^{2\xi-n-2s}|x|^{\delta} dxdy$$

$$\lesssim \int_\Omega |y|^{2\xi+\delta-2s} dy < \infty$$

where we have used that $\delta > -n$ and that $2\xi + \delta - 2s > -n$.
Finally, the integral over B can be bounded exactly as in the case $\xi \leq 0$. □

Next, we show the numerical results obtained, with quasi-uniform meshes, for two different combinations of α and β. Observe that $\boldsymbol{\sigma}(x) = -\beta x\,|x|^{\alpha+\beta-2}$ and $f = -\beta(\alpha+\beta)|x|^{\alpha+\beta-2}$. Thus, $f \in L^1(\Omega)$ whenever $\alpha + \beta > 0$, and so, the mixed finite element approximation is well defined in that case.
For each case, we use Lemma 6.1, with $\xi = \alpha + \beta - 1$ and $\delta = -\alpha$ to determine the order of smoothness of the vectorial solution $\boldsymbol{\sigma}$.
First, we choose $\alpha = 3/4$ and $\beta = 1/4$, and second $\alpha = 3/2$ and $\beta = -1/8$. Then we have $\boldsymbol{\sigma} = -\frac{1}{4}x|x|^{-1}$ for the first example and $\boldsymbol{\sigma} = \frac{1}{8}x|x|^{-5/8}$ for the second one.
Since $\delta < 0$, the hypothesis $\delta < 2s$ is trivial, and so, it follows from Lemma 6.1 that $\boldsymbol{\sigma} \in \mathbf{H}^s_{a^{-1}}(\Omega)$ for $s < 5/8$ in both examples.
Now we can find the theoretical order of convergence applying Theorem 5.2. In order to do that we need to check that $a^{-1}(x) = |x|^{\delta} \in A_{2s}$, which is known to be equivalent to $-2 < \delta < 2(2s-1)$ (see for example [11]). Again, since in both examples $-2 < \delta < 0$, this condition holds for any $s > 1/2$.

TABLE 1. Errors for the first example: $\alpha = 3/4$ and $\beta = 1/4$.

h	$\Vert\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\Vert_{\mathbf{L}^2_{a^{-1}}(\Omega)}$
1.9e-01	1.5834e-01
9.5e-02	1.0507e-01
4.7e-02	6.9053e-02
2.3e-02	4.5130e-02

TABLE 2. Errors for the second example: $\alpha = 3/2$ and $\beta = -1/8$.

h	$\Vert\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\Vert_{\mathbf{L}^2_{a^{-1}}(\Omega)}$
1.9e-01	5.5075e-02
9.5e-02	3.6015e-02
4.7e-02	2.3457e-02
2.3e-02	1.5247e-02

In Tables 1-2 we report, for each example, the numerical errors. Figures 2 and 3 show the graphics of $log(h)$ vs $log(\Vert\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\Vert_{\mathbf{L}^2_{a^{-1}}(\Omega)})$ for the two mentioned values of α and β and the experimental order of convergence (eoc) with respect to h, obtained by least-squares fitting.

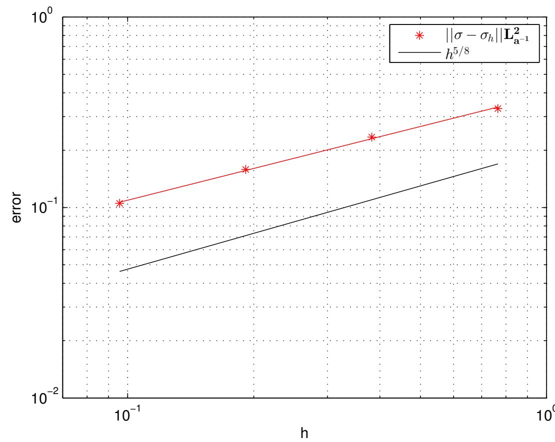

FIGURE 2. eoc = 0.55 obtained for $\alpha = 3/4$ and $\beta = 1/4$.

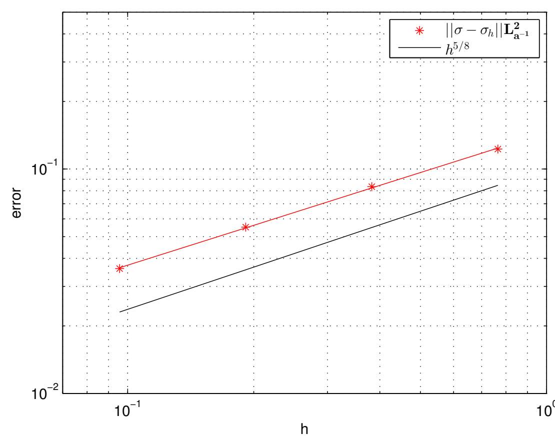

FIGURE 3. eoc = 0.59 obtained for $\alpha = 3/2$ and $\beta = -1/8$.

REFERENCES

[1] J. P. Agnelli, E. M. Garau, and P. Morin. A posteriori error estimates for elliptic problems with Dirac measure terms in weighted spaces. *ESAIM Math. Model. Numer. Anal.*, 48:1557–1581, 2014.
[2] M. L. Alvarez and R. G. Durán. A posteriori error estimates for mixed approximations of degenerate elliptic problems. *Applied Numerical Mathematics*, 188(1):146–159, 2023.
[3] I. Babuska and J. Osborn. Eigenvalue problems. In *Handbook of Numerical Analysis, Part 1*, volume 2, pages 641–787. Elsevier, 1992.
[4] F. Bertrand, S. Münzenmaier, and G Starke. First-order system least squares on curved boundaries: higher-order Raviart-Thomas elements. *SIAM J. Numer. Anal.*, 52(6):3165–3180, 2014.
[5] F. Bertrand and G. Starke. Parametric Raviart-Thomas elements for mixed methods on domains with curved surfaces. *SIAM J. Numer. Anal.*, 54(6):3648–3667, 2016.
[6] F. Brezzi, D. Boffi, L. Demkowicz, R. G. Durán, R. S. Falk, and M. Fortin. *Mixed finite elements, compatibility conditions, and applications*. Springer, 2008.
[7] M. E. Cejas, R. G. Durán, and M. Prieto. Mixed methods for degenerate elliptic problems and application to fractional laplacian. *ESAIM: M2AN*, 55:S993–S1019, 2021.

[8] M. E. Cejas, C. Mosquera, C. Pérez, and E. Rela. Self-improving Poincaré-Sobolev type functionals in product spaces. *J. Anal. Math.*, 149(1):1–48, 2023.
[9] C. D'Angelo and A. Quarteroni. On the coupling of 1d and 3d diffusion-reaction equations. Application to tissue perfusion problems. *Math. Models Methods Appl. Sci.*, 18(8):1481–1504, 2008.
[10] I. Drelichman and R.G. Durán. Improved Poincaré inequalities in fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.*, 43:885–903, 2018.
[11] J. Duoandikoetxea. *Fourier Analysis*, volume 29 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
[12] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman, 1985.
[13] R. Hurri-Syrjanen and F. López-García. On the weughted fractional Poincaré-type inequalities. *Colloq. Math.*, 157(2):213–230, 2019.
[14] A. Kufner. *Weighted Sobolev spaces*, volume 31 of *Teubner-Texte Math.* Teubner, Leipzig, 1980.
[15] F. López-García and I. Ojea. Some inequalities on weighted Sobolev spaces, distance weights and the Assouad dimension. *arxiv -eprint 2210.12322*, 2023.
[16] B. Muckenhoupt and R. L. Wheeden. Weighted norm inequalities for fractional integrals. *Transactions of the American Mathematical Society*, 192:261–274, 1974.
[17] R. H. Nochetto, E. Otárola, and A. J. Salgado. A pde approach to fractional diffusion in general domains: a priori error analysis. *Foundations of Computational Mathematics*, 15(3):733–791, 2015.
[18] R. H. Nochetto, E. Otárola, and A. J. Salgado. Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces and applications. *Numer. Math.*, 132(1):85–130, 2016.
[19] E. M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.*, volume 43 of *Princeton Mathematical Series*. Princeton University Press. Princeton, NJ, 1993.
[20] M Zlamal. Curved elements in the finite element method I. *SIAM J. Numer. Anal.*, 10(1):229–240, 1973.

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, (1428) Ciudad Autónoma de Buenos Aires, Argentina.
Email address: mlalvarez@dm.uba.ar

IMAS (UBA-CONICET) and Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, (1428) Ciudad Autónoma de Buenos Aires, Argentina.
Email address: garmenta@dm.uba.ar

IMAS (UBA-CONICET) and Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, (1428) Ciudad Autónoma de Buenos Aires, Argentina.
Email address: rduran@dm.uba.ar