EXTENSION THEOREMS FOR EXTERNAL CUSPS WITH MINIMAL REGULARITY

GABRIEL ACOSTA AND IGNACIO OJEA

ABSTRACT. Sobolev functions defined on certain simple domains with an isolated singular point (such as power type external cusps) can not be extended in standard but in appropriate weighted spaces. In this article we show that this result holds for a large class of domains that generalizes external cusps, allowing minimal boundary regularity. The construction of our extension operator is based on a modification of reflection techniques originally developed for dealing with uniform domains. The weight involved in the extension appears as a consequence of the failure of the domain to comply with basic properties of uniform domains, and it turns out to be a *quantification* of that failure. We show that weighted, rather than standard spaces, can be treated with our approach for weights which are given by a monotonic function either of the distance to the boundary or of the distance to the tip of the cusp.

1. INTRODUCTION

Let Ω be an open connected set in $\mathbb{R}^n$ and let $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a locally integrable nonnegative function. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the weighted Sobolev space $W_\omega^{k,p}(\Omega)$ ($W^{k,p}(\Omega)$ if $\omega \equiv 1$) is the space of functions defined in Ω having weak derivatives of order α, for $|\alpha| \leq k$, and satisfying

$$\|f\|_{W_\omega^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|\omega D^\alpha f\|_{L^p(\Omega)} < \infty$$

For $\Omega \subsetneq \mathbb{R}^n$, an extension operator is a linear bounded operator,

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n), \qquad \text{such that} \qquad \Lambda f|_\Omega = f. \tag{1.1}$$

If such an operator exists, then Ω is called an extension domain for Sobolev spaces (EDS).

It is well known that smooth domains are EDS. In fact, since the boundary of a smooth domain can be locally flattened by means of a regular transformation, Λ can be constructed in that case applying a simple reflection method (see [5, 14]). On the other hand, by using the so called Sobolev representation formula in a cone and singular integrals Calderón [6] showed that Lipschitz domains are also EDS for $1 < p < \infty$. This result was extended to the range $1 \leq p \leq \infty$ by Stein [5, 17] by using an appropriate averaged reflection procedure.

Reflection type techniques are a natural approach to deal with extension of functions and more complex ways of reflection are needed in order to handle more general domains. In this context Jones introduced in [11] the (broader than Lipschitz) class of (ε, δ) domains, also called *locally uniform domains* and showed, thanks to a subtle reflection method, that every (ε, δ) domain is indeed an EDS. In terms of the Whitney decompositions $\mathcal{W}$ and $\mathcal{W}^c$, of Ω and $(\Omega^c)^o$ respectively, Jones's idea hinges on the fact that (ε, δ) domains enjoy the following properties:

(a) Whitney cubes $Q \in \mathcal{W}^c$ near Ω have a "reflected" cube of similar size $Q^* \in \mathcal{W}$.

(b) reflected cubes $Q_1^*, Q_2^* \in \mathcal{W}$ of neighboring cubes $Q_1, Q_2 \in \mathcal{W}^c$ can be joined by a bounded chain of touching cubes in $\mathcal{W}$. Thank to this, an appropriate polynomial approximation of f in Q^* can be used to define $\Lambda(f)$ in Q.

Even when (ε, δ) domains allow very rough boundaries, simple domains such as

$$\Omega = \{x \in \mathbb{R}^2 : \qquad 0 < x_2 < 1, \quad |x_1| < x_2^\gamma\} \tag{1.2}$$

fail to be E.D.S. for $\gamma > 1$. This is a shared feature among general domains having outer peaks. In this regard Maz'ya and Poborchi [14] introduced:

Definition A. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain with compact boundary $\partial\Omega$. Assume that $0 \in \partial\Omega$ and that $\partial\Omega \setminus \{0\}$ is locally the graph of a Lipschitz function. We say that Ω has an external cusp at the origin if there exists a neighborhood of the origin, $U \subset \mathbb{R}^n$, such that*

$$U \cap \Omega = \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \in \varphi(z)\varpi\}$$

where $\varpi \subset \mathbb{R}^{n-1}$ is a bounded domain and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lipschitz increasing function such that $\frac{\varphi(t)}{t} \to 0$ $(t \to 0^+)$ and $\varphi(0) = 0$.

Remark 1.1. *In Definition A, φ defines the profile of the cusp, that is Ω's behavior toward the origin is depicted exactly by φ. Every "horizontal" section of Ω has the same shape as ϖ, scaled by φ.*

For the class of domains given by Maz'ya and Poborchi, Sobolev functions can not be extended in standard, but in weighted spaces. To be more precise [14, 15]:

Theorem A (Maz'ya and Poborchi [14, 15])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with an external cusp in the origin (in the sense of to Definition A),*

(a) *If $kp < n - 1$, or $k = n - 1$ and $p = 1$, and φ satisfies:*

$$\frac{\varphi(t)}{t} \text{ is a non-decreasing function} \tag{1.3}$$

Then there exists an extension operator

$$\Lambda : W^{kp}(\Omega) \to W_\sigma^{kp}(\mathbb{R})$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\varphi(|x|)}{|x|}\right)^k & x \in \Omega^c \end{cases}$$

(b) *If $kp > n - 1$, and φ is such that:*

$$\exists C_\varphi \text{ constant}: \qquad \varphi(2t) \leq C_\varphi \varphi(t) \tag{1.4}$$

then, there exists an extension operator

$$\Lambda : W^{kp}(\Omega) \to W_\sigma^{kp}(\mathbb{R})$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\varphi(|x|)}{|x|}\right)^{\frac{n-1}{p}} & x \in \Omega^c \end{cases}$$

In either case (a) or b)), assuming (1.4) holds, if $\tilde{\sigma}$ is such that there is an extension operator $\tilde{\Lambda} : W^{kp}(\Omega) \to W_{\tilde{\sigma}}^{kp}(\mathbb{R}^n)$, then

$$\tilde{\sigma}(x) \leq C\sigma(x) \quad \forall x \in U \setminus \Omega$$

Remark 1.2. *Let us mention that under the same hypotheses of Theorem A, and for the critical case $kp = n - 1$, with φ satisfying a condition similar to (1.4) (but slightly weaker), the authors show that: there exists an extension operator*

$$\Lambda : W^{kp}(\Omega) \to W_\sigma^{kp}(\mathbb{R})$$

where

$$\sigma(x) = \left(\frac{\varphi(|x|)}{|x|} \right)^k \left| \log \left(\frac{\varphi(|x|)}{|x|} \right) \right|^{1 - \frac{1}{p}}.$$

Observe that property (1.4) excludes exponential cusps, with profiles given by functions such as $\varphi(t) = a e^{-\frac{b}{t}}$. This property is not necessary for the existence of the extension operator in the case $kp < n - 1$, but it is used in the proof of the optimality of the weight σ (item c)). Throughout this work the weight given in item a) (resp. b)) is called *derivative* (resp. *dimensional*) weight.

Among several applications [3, 4], Theorem A allows us to handle convergence issues in the context of finite element approximations of elliptic partial differential equations. External cusps can not be exactly fitted by polygonal discretizations and the finite element mesh needs to be graded according to $\sigma(x)$ in order to get optimal order error estimates . On the other hand, weighted Sobolev spaces in external cusps have shown to be of interest in linear elasticity. Indeed, the classical Korn inequality, fundamental to prove existence of solutions of the linearized elasticity equations does not hold in domains with outer peaks, however a variant involving weights depending on the distance to the boundary or the distance to the tip of the cusp does hold. Similar results hold for the divergence operator for which a continuous right inverse can be defined on this kind of weighted spaces [1, 2, 9, 10]. In this context becomes clear that a version of Theorem A for weights of the type described can also be useful in applications.

In this paper we present a twofold generalization of Theorem A: a first involving a broader class of domains and a second allowing weighthed spaces. Concerning the former, and loosely speaking, we prove, in terms of Maz'ya and Poborchi's definition, that the extension can be performed even in a context in which φ does not give the precise profile of Ω, but just an approximate description of the narrowing toward the origin, as long as the boundary of Ω satisfies minimal regularity conditions. We also show that the extension is possible with either, the *derivative* or the *dimensional* weight regardless the relationship between the parameters k, p and n (although we do not treat the critical case $kp = n - 1$). On the other hand we also present some results for the weighted case. Even when Chua [7, 8] has shown that Jones's technique can also be applied to handle extensions in $W_\omega^{k,p}$ for very general weights (essentially if ω is doubling and if functions in $W_\omega^{k,p}$ can be well approximated in cubes by polynomials), in our context, for several reasons pointed out later, we need to restrict our study to weights that are mainly given by monotonic functions of the distance to the boundary or the distance to the tip of the cusp.

Our arguments can be summarized as follows. We introduce a general definition of external cusp, that includes those domains satisfying Definition A, (see Definitions 2.2 and 2.3). Our definition is given in terms of a) a *representative* chain (called the *spine* of the cusp) of central cubes belonging to $\mathcal{W}$ and b) certain *local variable uniformity* property. The latter property ensures that any cube $Q \in \mathcal{W}^c$ close enough to Ω has a reflected cube $Q^* \in \mathcal{W}$ of the type needed in Jones' arguments. Therefore, in a first stage, a local unweighted extension à la Jones is carried out. In a second stage, and for cubes $T \in \mathcal{W}^c$ which are not too close to Ω we define a reflected set T^* which is not necessarily a single, but a finite union of cubes belonging to the central Spine. We show that T^* can be essentially taken in two different ways where either the cardinal of T^* is large and the size of the cubes in T^* is much smaller than that of T or T^* has a single cube of the same size of T but the distance from T to T^* is much larger than the

size of T. In each case our arguments lead to a proper weight required in the extension process. Our reflection procedure makes the treatment of the weighted case easy, for weights of the type described before.

The paper is organized in the following way. In Section 2 we introduce some notation, recall basic aspects of uniform domains and Whitney decompositions, and finally give our definition of *normal* and *curved* cusps. Section 3 is devoted to prove some preliminary lemmas on polynomials. In Section 4 we present a detailed proof of the extension theorem for normal cusps (Theorem 4.1) whereas the adaptation for curved cusps is developed in Section 5 (Theorem 5.1). Section 6 is just an appendix to Sections 4 and 5, where some accessory results are proved. In Section 7 we treat the weighted case. Finally, in Section 8 we summarize the results presented in the paper, add some comments, and show a few examples.

2. A GENERAL DEFINITION FOR EXTERNAL CUSP

Given a rectangle $R \subset \mathbb{R}^n$ with edges parallels to the coordinate axis (this is the case for all the rectangles considered in this article), the size vector of R is denoted with $\vec{\ell}(R) = (\ell_1(R), \ell_2(R), \ldots, \ell_n(R))$, where $\ell_i(R)$ is the length of the R's i-th edge. For a cube Q we use $\ell(Q)$ to denote the length of any of its edges, and for a rectangle R, we define $\ell(R) := \max_{1 \leq i \leq n}\{\ell_i(R)\}$. A pair of rectangles R_1, R_2 are *equivalent*, and we write $R_1 \sim R_2$, if there are constants C_1, C_2 such that

$$C_1 \ell_i(R_1) \leq \ell_i(R_2) \leq C_2 \ell_i(R_1)$$

for $1 \leq i \leq n$. For a rectangle R, we denote its center with c_R. If $c_R = (c_1, \cdots, c_n)$ the upper face F_R^u of R is given by

$$F_R^u = \{(x_1, \cdots, x_n) \in R : x_n = c_n + \frac{1}{2}\ell_n(R)\}$$

and analogously is defined the lower face F_R^l.

Given a rectangle R, we denote by aR $(a > 1)$, the expanded rectangle centered at c_R with edges $\ell_i(aR) = a\ell_i(R)$. For a cube Q, centered in $c_Q = (c_1, \cdots, c_n)$ we denote $z_Q = c_n - \frac{1}{2}\ell(Q)$ (the last coordinate of points belonging to F_Q^l). We say that R_1 and R_2 are *touching rectangles* if $R_1^o \cap R_2^o = \emptyset$ and $R_1 \cap R_2 = F$ with F a face of R_1 or R_2.

For every collection of sets $\mathcal{C}$, we denote with $\cup\mathcal{C}$ the union of all the sets in $\mathcal{C}$, i.e. $\cup\mathcal{C} := \cup_{S \in \mathcal{C}} S$. Finally, throughout this article $\hat{x}_n$ stands for the x_n axis, and C denotes a generic constant that may change from line to line.

Let us recall the following definition that plays a crucial role in the sequel:

Definition 2.1. *(Locally Uniform Domains)* D *is a* (ε, δ) *domain if for all* $x, y \in D$ *with* $|x - y| < \delta$ *there is a rectifiable curve ggm joining* x *and* y *such that:*

$$\ell(\gamma) < \frac{|x - y|}{\varepsilon} \tag{2.1}$$

$$d_{\partial\Omega}(z) > \frac{\varepsilon|x - z||z - y|}{|x - y|} \qquad \forall z \in \gamma \tag{2.2}$$

If $\delta > diamD$, *we say that* D *is a* uniform *domain (note that then* D *is connected).*

Lemma 2.1 (Jones [11]). *If* D *is a* (ε, δ) *domain,* $|\partial D| = 0$.

Uniform domains include Lipschitz domains, but they form a much larger class. If fact, if Ω is uniform, $\partial\Omega$ could be very rough. On the other hand, classical examples of domains with a single singular point and that are not uniform are precisely external cusps of power type, where property (2.2) fails, and also inner cups of the same kind, where property (2.1) fails.

Below we define cusps in terms of its Whitney decomposition, so we recall:

Theorem 2.2 (Whitney). *Let $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, be an open set. Then, there is a collection $\mathcal{W} = \mathcal{W}(\Omega) := \{Q_j\}$ of (countably) infinite dyadic closed cubes such that $\Omega = \cup\mathcal{W}$, and:*

- $Q_j^o \cap Q_k^o = \emptyset \quad \forall Q_j, Q_k \in \mathcal{W}(\Omega)$
- $\ell(Q_j) \leq d(Q_j, \partial\Omega) \leq 4\sqrt{n}\ell(Q_j) \quad \forall Q_j$
- *if $Q_j \cap Q_k \neq \emptyset$ then: $\ell(Q_j) \leq 4\ell(Q_k)$*

Remark 2.3. *The proof of Theorem 2.2 can be found, for instance, in [17]. One can easily observe that for any pair of open sets A and B, with $A \subset B$, every cube $Q \in \mathcal{W}(A)$ is contained in some cube $\widetilde{Q} \in \mathcal{W}(B)$.*

Remark 2.4. *Like in Definition A, the cusps defined below are assumed to be tangential to certain fixed direction that is taken arbitrarily along $\hat{x}_n$, with the tip of the cusp placed at the origin 0. Our definition is "intrinsic" to some extent, since it is based on Whitney cubes and certain variable uniformity property.*

Let us recall that we denote $\mathcal{W}$ and $\mathcal{W}^c$ the Whitney decompositions of Ω and $(\Omega^c)^o$ respectively.

Definition 2.2 (Normal Cusp). *Let $\Omega \subset \mathbb{R}^n$ be an open set such that $0 \in \partial\Omega$. Let $\varepsilon > 0$ and $K > 1$ be given parameters. We say that Ω has a $(\varepsilon, K)-$normal external cusp (or outer peak) at the origin if it satisfies:*

(i) *There exists a chain $\mathcal{S} = \{S_i\}_{i=1}^{\infty} \subset \mathcal{W}$ of cubes increasingly numbered towards the origin (i.e.: $d(S_{i+1}, 0) \leq d(S_i, 0)$), such that*

$$S_i \cap S_{i+1} = F_{S_{i+1}}^u, \tag{2.3}$$

and

$$d(S_i, 0) \to 0 \quad (i \to \infty). \tag{2.4}$$

(ii) *Using the abbreviated notation $z_i := z_{S_i}$ for cubes $S_i \in \mathcal{S}$, denoting by $S(z)$ the cube at heigth $z > 0$ (i.e. $S(z) := S_i$ if $z_i \leq z < z_{i-1}$) and i_z the index of the cube at height z (i.e. $i_z = i$ if $S_i = S(z)$), we have that*

$$\{x \in \Omega : x_n < z\} \subset \bigcup_{i=i_z}^{\infty} \Omega_i \ \text{ for any } \ z_1 > z > 0 \tag{2.5}$$

with $\Omega_i = KS_i \cap \Omega$.

(iii) *For every pair of points $x, y \in \Omega_i \cup \Omega_{i+1}$, there is a rectifiable curve, $\gamma \subset \Omega$, joining x and y, and satisfying:*

$$\ell(\gamma) \leq \frac{1}{\varepsilon}|x - y| \tag{2.6}$$

$$d_{\partial\Omega}(z) \geq \varepsilon\frac{|x - z||z - y|}{|x - y|}. \tag{2.7}$$

(iv) *We have*

$$\frac{\ell(S_i)}{z_i} \longrightarrow 0 \qquad (i \to \infty) \tag{2.8}$$

*The set $\mathcal{S}$ is named the **spine** of Ω.*

Condition (2.3) means that the *spine* $\mathcal{S}$ is straight, and parallel to $\hat{x}_n$. It also implies that the chain is decreasing, or $\ell(S_{i+1}) \leq \ell(S_i)$. This last fact is not really necessary but is assumed for the sake of simplicity: the sizes of the cubes in $\mathcal{S}$ could oscillate, as long as its oscillation is controlled by some universal parameter, depending only on Ω.

On the other hand, conditions (2.3) and (2.4) imply that every cube S_i of $\mathcal{S}$ touches $\hat{x}_n$, while (2.5) guarantees that Ω's behavior (its narrowing toward the origin) is faithfully represented by

the behavior of the chain $\mathcal{S}$: a fixed expansion of the tails of $\mathcal{S}$ reaches the whole boundary of Ω below certain height z, and consequently $\partial\Omega$ narrows toward the origin as fast as $\ell(S(z))$. In other words, the function $\ell(S(z))$ plays the role of $\varphi(z)$ in Definition A.

Finally, conditions (2.6) and (2.7) provide some regularity to the boundary of Ω and exclude the existence of non connected components in Ω.

Condition (2.8) is not necessary at all for the extension process. We include it in order to exclude cones and other non-singular domains from our definition of cusp. However, it is important to notice that our extension theorems (see Theorems 4.1 and 5.1) stand even for domains where (2.8) is not fulfilled. In that cases the weight turns to be a constant, and a classical (unweighted) extension is obtained.
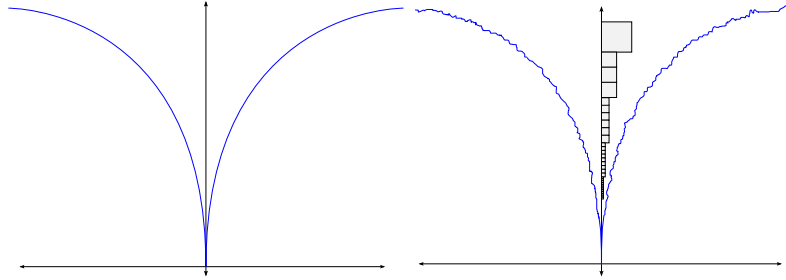

FIGURE 1. Cusp of type (1.2) vs. *Normal* cusp with its *spine*

Normal cusps are, somehow, "symmetric" with respect to $\hat{x}_n$. More precisely, *normal* cusps are those that grow around an axis, which is placed approximately at its center; see 1. The following definition includes cusps that are *tangential* to a certain axis, which is not necessarily interior to the domain:

Definition 2.3 (Curved Cusp). *Let $\Omega \subset \mathbb{R}^n$ be an open set such that $0 \in \partial\Omega$. Let $\varepsilon > 0$ and $K > 1$ be given parameters. We say that Ω has a $(\varepsilon, K)-$curved external cusp (or outer peak) at the origin if there exists a chain of cubes $\mathcal{S} = \{S_i\}_i$, $S_i \in \mathcal{W}$ increasingly numbered toward the origin 0, satisfying:*

$$S_i \cap S_{i+1} \neq \emptyset \tag{2.9}$$

$$d(S_i, \hat{x}_n) \leq C_\Omega \ell(S_i) \qquad \text{for some } C_\Omega \tag{2.10}$$

$$\ell(S_{i+1}) \leq \ell(S_i) \tag{2.11}$$

and if Ω satisfies conditions (ii), (iii) and (iv) of Definition 2.2.

Conditions (2.9) and (2.10) constitute a relaxation of condition (2.3). Here, the chain is not forced to be straight and parallel to $\hat{x}_n$, but to approximate it asymptotically. As we remarked earlier, condition (2.11) is not necessary, but comfortable. Since we abandoned property (2.3), (2.11) it is not implicit any more, and so we include it in the definition of *curved* cusps.

Remark 2.5. *For both* normal *or* curved *cusps we add two extra conditions, that are analogous to the ones requiered by Maz'ya and Poborchi in Theorem A.*

$$\frac{\ell(S_i)}{z_i} \leq C \frac{\ell(S_j)}{z_j} \quad \forall i > j \quad C \text{ constant.} \tag{2.12}$$

and

$$\ell(S_j) \leq K\ell(S_i) \quad \forall i > j \text{ such that } d(S_i, 0) > \frac{1}{2}d(S_j, 0) \tag{2.13}$$

Property (2.12) *is a generalization of* (1.3). *We use it to proof item* a) *in Theorems 4.1 and 5.1. On the other hand,* (2.13) *is a generalization of* (1.4), *and it is necessary for the proof of item* b) *in both theorems.*

Finally, let us point out some details that arise from the comparison between Definition A and Definitions 2.2 and 2.3:

Note that a set satisfying Definition A is a domain that, being cut at different heights, shows the same shape, scaled according to the height by function φ. This is clearly not the case in Definition 2.3. On the other hand, since the decreasing conditions are stated on the spine $\mathcal{S}$, and not on $\partial\Omega$, our definitions allow small oscillations in $\partial\Omega$, instead of the monotonic behavior imposed by φ. Furthermore, Definition A asks $\partial\Omega \setminus \{0\}$ to be locally Lipschitz, whereas the variable uniform properties (2.6) and (2.7) constitute a much more relaxed condition. In Section 8 we provide some examples that show in different ways the generalization implicit in Definitions 2.2 and 2.3.

3. Some preliminary lemmas

For a polynomial P, $\deg P$ stands for the degree of P.

Lemma 3.1. *Let R be a rectangle, P a polynomial with $\deg(P) \leq k$, then:*

$$\|P\|_{L^{\infty}(R)} \leq \frac{C}{|R|^{\frac{1}{p}}}\|P\|_{L^{p}(R)} \qquad 1 \leq p \leq \infty$$

with C depending only on k.

Proof. Let $\hat{Q} = [-\frac{1}{2}, \frac{1}{2}]^{n}$. Let $F : \hat{Q} \to R$ be the linear application: $F : \hat{x} \to x$, $F(\hat{x}) = \vec{\ell}(R) \cdot \hat{x}^{t} + c_{R}$. Observe that $|DF| = |R|$. We consider the polinomial $\hat{P}$ defined on $\hat{Q}$ as $\hat{P}(\hat{x}) = P(F(\hat{x}))$. Notice that $deg(\hat{P}) = deg(P)$. Changing variables, we obtain:

$$\|P\|_{L^{\infty}(R)} = \|\hat{P}\|_{L^{\infty}(\hat{Q})} \leq \hat{C}\|\hat{P}\|_{L^{p}(\hat{Q})}$$

$$= \hat{C}\left(\int_{\hat{Q}}|\hat{P}(\hat{x})|^{p}d\hat{x}\right)^{\frac{1}{p}} \leq \hat{C}\left(\int_{R}|P(x)|^{p}\frac{1}{|R|}dx\right)^{\frac{1}{p}}$$

where the first inequality follows from the equivalence of norms in the finite dimensional space of polynomials of degree $\leq k$ defined on $\hat{Q}$.

$\square$

Lemma 3.2. *Let R and Q be rectangles such that $R \subset Q$, and P a polynomial with $\deg P \leq k$. Then, there exists a constant C, depending only on k, such that:*

$$\|P\|_{L^{p}(Q)} \leq C\left(\frac{|Q|}{|R|}\right)^{\frac{1}{p}}\sum_{|\alpha| \leq k}\|D^{\alpha}P\|_{L^{p}(R)}\vec{\ell}(Q)^{\alpha}$$

Proof. We may asume $0 \in R$. Let $q \in Q$ such that $\|P\|_{L^{\infty}(Q)} = |P(q)|$, then:

$$\|P\|_{L^{p}(Q)} \leq \|P\|_{L^{\infty}(Q)}|Q|^{\frac{1}{p}} = |P(q)||Q|^{\frac{1}{p}} \leq |Q|^{\frac{1}{p}}\sum_{|\alpha| \leq k}|D^{\alpha}P(0)|\frac{|q^{\alpha}|}{\alpha!}$$

$$\leq C|Q|^{\frac{1}{p}}\sum_{|\alpha| \leq k}\|D^{\alpha}P\|_{L^{\infty}(R)}\vec{\ell}(Q)^{\alpha} \leq C\left(\frac{|Q|}{|R|}\right)^{\frac{1}{p}}\sum_{|\alpha| \leq k}\|D^{\alpha}P\|_{L^{p}(R)}\vec{\ell}(Q)^{\alpha}$$

$\square$

Corolary 3.3. *Let $R \subset Q$ rectangles such that $\frac{|Q|}{|R|} \leq C$, and P a polynomial with $\deg(P) \leq k$. Then, there exists a constant C, depending only on k such that:*

$$\|P\|_{L^p(Q)} \leq C\|P\|_{L^p(R)}$$

Remark 3.4. *A version of Corollary 3.3 is proved in [11], Lemma 2.1. In our case we need to compare polynomials in rectangles that are not of similar size (a fact that eventually leads to the weights involved in the extension) and we need the less comfortable variant given in Lemma 3.2.*

Following [11], the extension operator is built in terms of polynomials that approximate the function in certain sets. Let $f \in W^{k,p}(\Omega)$, and $S \subset \Omega$ a set of positive measure, we denote with $P_{k-1}(S)$ (or just $P(S)$ if the degree is clear from the context) the unique polynomial of degree $k-1$ such that:

$$\int_S D^\alpha(f - P_{k-1}(S)) = 0 \quad \text{for all}\alpha, \text{ with } |\alpha| \leq k-1.$$

Thanks to the Poincaré inequality one knows that $P(R)$ has good approximation properties if R is a rectangle (regardless the eccentricity of R). In the spirit of Lemma 2.2 in [11] we also need such a result for the union of two touching rectangles of similar size.

Lemma 3.5. *Let R_1, R_2 rectangles such that $R_1 \sim R_2$. Assume that either R_1 and R_2 are touching or $R_1 \subseteq R_2$ (renumbering if necessary). Then, for any $f \in W^{k,p}(R_1 \cup R_2)$:*

$$\|f - P(R_1 \cup R_2)\|_{L^p(R_1 \cup R_2)} \leq C\ell(R_1)^k \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p(R_1 \cup R_2)}.$$

Proof. Clearly it is enough to prove the result in the case $k = 1$. If $R_1 \subseteq R_2$ (or vice versa) then the result follows by the Poincaré inequality for convex domains (or from a fixed cube by scaling arguments). Let us then treat the case of touching rectangles. Define $f_{R_1 \cup R_2} = \frac{1}{|R_1 \cup R_2|} \int_{R_1 \cup R_2} f$, then $P(R_1 \cup R_2) = f_{R_1 \cup R_2}$. Write

$$\|f - P(R_1 \cup R_2)\|^p_{L^p(R_1 \cup R_2)} = \|f - P(R_1 \cup R_2)\|^p_{L^p(R_1)} + \|f - P(R_1 \cup R_2)\|^p_{L^p(R_2)}.$$

We now show how to deal with the first term (the other follows analogously). We have

$$\|f - P(R_1 \cup R_2)\|_{L^p(R_1)} \leq \frac{|R_1|}{|R_1| + |R_2|}\|f - P(R_1)\|_{L^p(R_1)} + \frac{|R_2|}{|R_1| + |R_2|}\|f - P(R_2)\|_{L^p(R_1)}$$

the first term is fine. For the other term we write

$$\|f - P(R_2)\|_{L^p(R_1)} \leq \|f - P(R_1)\|_{L^p(R_1)} + \|P(R_1) - P(R_2)\|_{L^p(R_1)}$$

and again the first term is all right. In order to treat $\|P(R_1) - P(R_2)\|_{L^p(R_1)}$ observe that, since R_1 and R_2 are touching, there exist rectangles R_3 and R_4 such that $R_3 \subset R_1 \cup R_2 \subset R_4$, $R_1 \sim R_1 \cap R_3 \sim R_2 \cap R_3 \sim R_2 \sim R_3 \sim R_4$ then (using, for instance, Corollary 3.3)

$$\|P(R_1) - P(R_2)\|_{L^p(R_1)} \leq C\|P(R_1) - P(R_3)\|_{L^p(R_1 \cap R_3)} + \|P(R_3) - P(R_2)\|_{L^p(R_4)}$$

and

$$\|P(R_1) - P(R_3)\|_{L^p(R_1 \cap R_3)} \leq \|P(R_1) - f\|_{L^p(R_1)} + \|f - P(R_3)\|_{L^p(R_3)}$$

while (using again Corollary 3.3)

$$\|P(R_3) - P(R_2)\|_{L^p(R_4)} \leq C\|P(R_3) - P(R_2)\|_{L^p(R_2 \cap R_3)} \leq C\|f - P(R_2)\|_{L^p(R_2)} + \|P(R_3) - f\|_{L^p(R_3)}.$$

The lemma follows.

$\square$

4. EXTENSION

The objective of this section is to build an extension operator for cusps in the unweighted case $W^{k,p}$, proving the main result of this paper:

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with an external normal cusp at the origin.*

   (a) *If the spine $\mathcal{S}$ satisfies (2.12), there is an extension operator*

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}_\sigma(\mathbb{R}^n)$$

     where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^k & x \in \Omega^c \end{cases}$$

   (b) *If the spine $\mathcal{S}$ satisfies (2.13), there is an extension operator*

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}_\sigma(\mathbb{R}^n)$$

     where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^{\frac{n-1}{p}} & x \in \Omega^c \end{cases}$$

   (c) *In either case (a) or b)), assuming (2.13) stands, if $\tilde{\sigma}$ is such that there is an extension operator $\tilde{\Lambda} : W^{k,p}(\Omega) \to W^{k,p}_{\tilde{\sigma}}(\mathbb{R}^n)$, then*

$$\tilde{\sigma}(x) \le C\sigma(x) \quad \forall x \in U \setminus \Omega$$

Remark 4.2. *Theorem 4.1 states the existence of a general extension operator for normal cusps. The only difference between items (a) and (b) is the requierement of properties (2.12) and (2.13) respectively. In contrast to Theorem A, no conditions on k, p or n are needed in the proof of any of these cases.*

Remark 4.3. *Condition (2.12) is not necessary if $kp \ne 1$ (see Remark 4.20). Consequently, since every domain satisfying Definition A is an external cusp (normal or curved) in our terms (see Corollary 8.5), hyphotesis (1.3) is not necessary in Theorem A when $kp \ne 1$.*

Below we provide a detailed proof for items *a)* and *b)* of Theorem 4.1. Item *c)* is discussed in the last section of this article. The case of curved cusps requires a little modification (similar to that needed in [14]) of our arguments and therefore is sketched later in Theorem 5.1.

Let us notice that thanks to item iii), in Definition 2.2, and the results of Jones for locally uniform domains it is clear that it is enough to construct an extension operator Λ for functions u such that $supp(u) \subset D_r = \{x = (x_1, \cdots, x_n) : |x_n| < r/2\}$, where $r << \sum_{i=1}^{\infty} \ell(S_i)$. Our operator Λ is defined in a set of cubes belonging to $\mathcal{W}^c$. Let us call $\mathcal{W}_2 \subset \mathcal{W}^c$ to the set of cubes belonging to $\mathcal{W}^c$ and contained in D_r. We divide $\mathcal{W}_2$ in three parts related to three different stages of the extension process.

$$\mathcal{W}_3 = \left\{ Q \in \mathcal{W}_2 : \quad z_Q > 0 \ \text{ and } \ \ell(Q) \le \left(\frac{\varepsilon}{5\sqrt{n}} \frac{K-1}{K}\right) \ell(S(z_Q)) \right\} \tag{4.1}$$

$$\mathcal{W}_4 = \left\{ Q \in \mathcal{W}_2 \setminus \mathcal{W}_3 : \quad z_Q > 0 \ \text{ and } \ \ell(Q) \le z_Q/(8\sqrt{N}) \right\} \tag{4.2}$$

$$\mathcal{W}_5 = \left\{ Q \in \mathcal{W}_2 \setminus (\mathcal{W}_3 \cup \mathcal{W}_4) \right\} \tag{4.3}$$

Furthermore, let us denote Q_j the cubes in $\mathcal{W}_3$, so: $\mathcal{W}_3 = \{Q_j\}_j$, and similarly, $\{T_j\}_j$ the cubes in $\mathcal{W}_4$, and $\{U_j\}_j$ those in $\mathcal{W}_5$. Finally, let $\{\xi_j\}_j$, $\{\phi_j\}_j$ and $\{\psi_j\}_j$ be a partition of the

unity on $\cup \mathcal{W}_2$, such that $\xi_j \in C_0^\infty$, $\phi_j \in C_0^\infty$, $\psi_j \in C_0^\infty$; $\mathrm{sop}(\xi_j) \subset \frac{17}{16}Q_j$, $\mathrm{sop}(\phi_j) \subset \frac{17}{16}T_j$, $\mathrm{sop}(\psi_j) \subset \frac{17}{16}U_j$, and:

$$\sum_j \xi_j(x) + \sum_j \phi_j(x) + \sum_j \psi_j(x) = 1 \qquad \forall x \in \cup \mathcal{W}_2$$

As usual, we may also assume that:

$$|D^\alpha \xi_j(x)| \leq \frac{C}{\ell(Q_j)^{|\alpha|}}, \qquad |D^\alpha \phi_j(x)| \leq \frac{C}{\ell(T_j)^{|\alpha|}}, \qquad |D^\alpha \psi_j(x)| \leq \frac{C}{\ell(U_j)^{|\alpha|}}$$

In each stage of the extension process we define a polynomial for each cube in each set $\mathcal{W}_i$ (in the first stage, a polynomial for each cube in $\mathcal{W}_3$, etc.). The extension operator is finally constructed by using our partition of the unity.

Following Jones' ideas we define, for each $Q \in \mathcal{W}_2$, some set $S(Q) \subset \Omega$, so the polynomial for Q will be $P(S(Q))$.

First stage. This stage follows closely the reflection method given in [11]. It is based on Lemma 4.6, where the existence of a reflected cube for every $Q \in \mathcal{W}_3$ is proved. We need to state a previous lemma:

Lemma 4.4. *Given Ω an external normal cusp, with parameters ε, K, there is a constant $\widetilde{K}$ (that could be taken $\widetilde{K} = \frac{K(K+1)}{2}$) such that, if $x \in \Omega$, and*

$$z_i - \frac{K-1}{2}\ell(S_i) \leq x_n \leq z_i + \frac{K+1}{2}\ell(S_i)$$

then, $x \in \widetilde{K}S_i$.

Proof. Let us take $j = i_{x_n}$. We suppose $j < i$ (the complementary case is analogous). Property (2.5) implies that $KS_j \ni x$. On the other hand $z_j \leq z_i + \frac{K+1}{2}\ell(S_i)$. But, since $\ell(S_j) \geq \ell(S_i)$, we have $\ell(S_j) = 2^N \ell(S_i)$ for some $N \in \mathbb{N}_0$. The largest size for S_j is obtained when the cubes in $\mathcal{S}$ grow exponentially between S_i and S_j. In that case:

$$z_j - z_i = \sum_{m=0}^{N-1} 2^m \ell(S_i) \leq \frac{K+1}{2}\ell(S_i)$$

and $2^N \leq \frac{K+1}{2}$, which leads us to conclude that

$$\ell(S_j) \leq \frac{K+1}{2}\ell(S_i).$$

But then $x \in K\frac{K+1}{2}S_i$, since $x \in KS_j$. $\qquad \square$

Remark 4.5. *It is easy to see that properties (2.6) and (2.7) hold for finite unions of sets Ω_i (and not only for $\Omega_i \cup \Omega_{i+1}$). Therefore we may apply both properties for $\widetilde{\Omega}_i \cup \widetilde{\Omega}_{i+1}$, where $\widetilde{\Omega}_i = \widetilde{K}S_i \cap \Omega$. On the other hand (2.6) implies that the curve from (2.6) is contained in a finite union of sets Ω_i (or in a universal dilation of S_i).*

Lemma 4.6. *For each $Q \in \mathcal{W}_3$ there is a cube $Q^* \in \mathcal{W}$ such that:*

$$\frac{1}{4}\ell(Q) \leq \ell(Q^*) \leq \ell(Q) \tag{4.4}$$

$$d(Q^*, Q) \leq C\ell(Q) \tag{4.5}$$

Proof. Let i be such that $z_Q \in [z_i, z_{i-1})$, and $x \in \Omega$ such that $d(Q, x) \leq 5\sqrt{n}\ell(Q)$. We may asume that $\frac{\varepsilon}{\sqrt{n}K} < \frac{1}{2}$. In this case, observe that

$$x_n \geq z_Q - 5\ell(Q) \geq z_i - 5\frac{\varepsilon(K-1)}{5\sqrt{n}K}\ell(S_i) \geq z_i - \frac{K-1}{2}\ell(S_i)$$

The right hand term of the equation is exactly the floor of the expanded cube KS_i. On the other hand:

$$x_n \leq z_{i-1} + 5\ell(Q) \leq z_{i-1} + 5\frac{\varepsilon(K-1)}{5\sqrt{n}K}\ell(S_i)$$

$$\leq z_i + \ell(S_i) + \frac{K-1}{2}\ell(S_i) = z_i + \frac{K+1}{2}\ell(S_i)$$

and the right term is the roof of the expanded cube KS_i. Consequently, $x \in \widetilde{\Omega}_i$. Let $y \in \widetilde{\Omega}_i$ be such that $|x - y| = \frac{5\sqrt{n}}{\varepsilon}\ell(Q)$. Note that this is possible because:

$$|x - y| = \frac{5\sqrt{n}}{\varepsilon}\ell(Q) \leq \frac{K-1}{K}\ell(S_i) < \text{diam}(\widetilde{\Omega}_i)$$

Let, then, γ be the curve given by properties (2.6) and (2.7). If $\xi \in \gamma$ is such that $|x - \xi|, |\xi - y| \geq \frac{|x-y|}{2}$, we have: $d_{\partial\Omega}(\xi) \geq \frac{\varepsilon}{4}|x - y| = \frac{5\sqrt{n}}{4}\ell(Q)$. If $S \in \mathcal{W}$, $S \ni \xi$, then:

$$4\sqrt{n}\ell(S) \geq d(S, \partial\Omega) \geq d_{\partial\Omega}(\xi) - \sqrt{n}\ell(S) \geq \frac{5\sqrt{n}}{4}\ell(Q) - \sqrt{n}\ell(S)$$

Therefore

$$\ell(S) \geq \frac{1}{4}\ell(Q)$$

Let us consider all the cubes $T \in \mathcal{W}$ satisfying $\ell(T) \geq \frac{1}{4}\ell(Q)$ and take Q^* to be the one that minimizes the distance to Q. Then $\ell(Q^*) \leq \ell(Q)$. On the other hand

$$d(Q^*, Q) \leq d(S, Q) \leq \frac{1}{\varepsilon}|x - y| + d(x, Q) \leq \left(\frac{5\sqrt{n}}{\varepsilon^2} + 5\sqrt{n}\right)\ell(Q)$$

This completes the proof of the lemma. $\qquad\square$

Corolary 4.7. *If $Q_1, Q_2 \in \mathcal{W}_3$, $Q_1 \cap Q_2 \neq \emptyset$, then $d(Q_1^*, Q_2^*) \leq C\ell(Q_1)$*

The following lemma is crucial. It is analogous to Lemma 2.8 in [11].

Lemma 4.8. *Given $Q_1, Q_2 \in \mathcal{W}_3$, $Q_1 \cap Q_2 \neq \emptyset$, there is a constant $C = C(\varepsilon, n, K)$ and a chain of cubes $\mathcal{F}_{1,2} = \{V_1 := Q_1^*, V_2, \ldots, V_r := Q_2^*\} \subset \mathcal{W}$ such that $r \leq C$ and $\ell(V_i) \sim \ell(Q_1)$, $\forall i$.*

Proof. Since $Q_1 \cap Q_2 \neq \emptyset$, we may assume that either $z_{Q_1}, z_{Q_2} \in [z_i, z_{i-1})$, or $z_{Q_1} \in [z_i, z_{i-1})$ and $z_{Q_2} \in [z_{i+1}, z_i)$, for some i. In any case, thanks to Remark 4.5, Q_1^* and Q_2^* are not far from $\Omega_i \cup \Omega_{i+1}$. Let us then assume, for the sake of simplicity, that $Q_1^*, Q_2^* \subset \Omega_i \cup \Omega_{i+1}$. Then there is a curve γ joining Q_1^* and Q_2^* with $\ell(\gamma) \leq Cd(Q_1^*, Q_2^*) \leq C\ell(Q_1)$. Here C denotes different constants, but all of them independent of the cubes considered. Let us consider, then, the chain

$$\mathcal{F}_{1,2} = \{V_1 = Q_1^*, V_2, \ldots, V_r = Q_2^*\} \subset \mathcal{W}$$

of cubes touching γ. We need a lower bound for the size of V_j, $j = 1, \ldots, r$. $\ell(V_2) \geq \frac{1}{4}\ell(Q_1^*) \geq \frac{1}{16}\ell(Q_1)$. Analogously, $\ell(V_{r-1}) \geq C\ell(Q_1)$. If $1 < j < r$, let us take $z \in \gamma \cap V_j$. Then:

$$d_{\partial\Omega}(z) \geq \varepsilon\frac{|x - z||z - y|}{|x - y|} \geq C\frac{\ell(Q_1)^2}{\ell(Q_1)} \geq C\ell(Q_1)$$

If follows that no more than C cubes can be placed along γ, and then $r \leq C$.

$\qquad\square$

For each $Q_j \in \mathcal{W}_3$ let us define $P_{Q_j} = P(Q_j^*)$. The first term of the extension operator will be:

$$\Lambda_1 f(x) = \sum_{Q_j \in \mathcal{W}_3} P_{Q_j}(x)\xi_j(x)$$

Thanks to Lemmas 4.6 and 4.8, and Corollary 4.7, this operator can be bounded following [11, 7]. We give the details for the sake of completeness.

Remark 4.9. *In this first stage, and in particular during the proof of the next lemma, we could invoke Corollary 3.3. However, in order to be consistent with the rest of the stages we show how to use Lemma 3.2 instead.*

Lemma 4.10. *If $Q \in \mathcal{W}_3$ far from $\mathcal{W}_4$ (i.e. $Q \in \mathcal{W}_3$ is surrounded by cubes in $\mathcal{W}_3$) , then:*

$$\|D^\alpha \Lambda_1 f\|_{L^p(Q)} \le C\Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup\mathcal{F}(Q))} + \|f\|_{W^{kp}(Q^*)}\Big\}$$

where $\mathcal{F}(Q)$ is the set of all the cubes that participate in a chain $\mathcal{F}_j(Q)$, connecting Q^* with Q_j^*, for $Q_j \cap Q \ne \emptyset$.

Proof. We have

$$\|D^\alpha \Lambda_1 f\|_{L^p(Q)} = \Big\|D^\alpha \sum_{Q_j \cap Q \ne \emptyset} P_{Q_j}\xi_j\Big\|_{L^p(Q)} \le \Big\|D^\alpha \sum_{Q_j \cap Q \ne \emptyset} (P_{Q_j} - P_Q)\xi_j\Big\|_{L^p(Q)} + \big\|D^\alpha P_Q\big\|_{L^p(Q)} = I + II$$

The second term is easily bounded by means of Lemma 3.2, taking into account that Q and Q^* can be included inside an auxiliary cube $\tilde{Q}$, $Q \sim \tilde{Q} \sim Q^*$. Alternating the derivatives of f we get

$$II \le C \sum_{|\gamma+\alpha|<k} \ell(Q)^{|\gamma|}\|D^{\gamma+\alpha}P_Q\|_{L^p(Q^*)}$$

$$\le C \sum_{|\gamma+\alpha|<k} \ell(Q)^{|\gamma|}\Big\{\|D^{\gamma+\alpha}(P_Q - f)\|_{L^p(Q^*)} + \|D^{\gamma+\alpha}f\|_{L^p(Q^*)}\Big\} \le$$

$$\le C\|\nabla^k f\|_{L^p(Q^*)}\ell(Q)^{k-|\alpha|} + \|f\|_{W^{kp}(Q^*)} \le C\|f\|_{W^{kp}(Q^*)}$$

On the other hand:

$$I \le C \sum_{Q_j \cap Q \ne \emptyset} \sum_{\beta \le \alpha} \|D^{\alpha-\beta}\xi_j D^\beta(P_{Q_j} - P_Q)\|_{L^p(Q)}$$

$$\le C \sum_{Q_j \cap Q \ne \emptyset} \sum_{\beta \le \alpha} \frac{1}{\ell(Q)^{|\alpha-\beta|}}\|D^\beta(P_{Q_j} - P_Q)\|_{L^p(Q)}$$

For each j, let us alternate the polynomials associated to the cubes of the chain between Q_j^* and Q^*, given by Lemma 4.8. We set $\mathcal{F}_j = \{T_1 = Q^*, T_2, \ldots, T_r = Q_j^*\}$ and obtain:

$$\|D^\beta(P_{Q_j} - P_Q)\|_{L^p(Q)} \le \sum_{i=1}^{r-1} \|D^\beta(P(T_{i+1}) - P(T_i))\|_{L^p(Q)} \le$$

$$\sum_{i=1}^{r-1} \Big\{\|D^\beta(P(T_{i+1}) - P(T_i \cup T_{i+1}))\|_{L^p(Q)} + \|D^\beta((P(T_i \cup T_{i+1}) - P(T_i))\|_{L^p(Q)}\Big\} \le$$

$$C\sum_{i=1}^{r-1} \Big\{\|D^\beta(P(T_{i+1}) - P(T_i \cup T_{i+1}))\|_{L^p(T_{i+1})} + \|D^\beta((P(T_i \cup T_{i+1}) - P(T_i))\|_{L^p(T_i)}\Big\} \le$$

$$C\sum_{i=1}^{r-1}\left\{\|D^{\beta}(P(T_{i+1})-f)\|_{L^p(T_{i+1})}+\|D^{\beta}(f-P(T_i\cup T_{i+1}))\|_{L^p(T_i\cup T_{i+1})}+\|D^{\beta}(f-P(T_i))\|_{L^p(T_i)}\right\}\leq$$

$$C\sum_{i=1}^{r-1}\ell(Q)^{k-|\beta|}\|\nabla^k f\|_{L^p(T_i\cup T_{i+1})}\leq\ell(Q)^{k-|\beta|}\|\nabla^k f\|_{L^p(\cup\mathcal{F}_j)}$$

And then:

$$I\leq C\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup\mathcal{F}(Q))}$$

$\square$

Finally, let us observe that from Lemmas 4.6 and 4.8 it follows that:

$$\Big\|\sum_{\substack{Q_l\in\mathcal{W}_3\\Q_l\cap Q_j\neq\emptyset}}\chi_{\cup\mathcal{F}_{jl}}\Big\|_{\infty}\leq C<\infty\qquad\text{for all}Q_j\in\mathcal{W}_3,\tag{4.6}$$

$$\Big\|\sum_{Q_j\in\mathcal{W}_3}\chi_{\cup\mathcal{F}(Q_j)}\Big\|_{\infty}\leq C<\infty\tag{4.7}$$

This means that each cube Q_j^* is used at most a fixed number of times, then

$$\|D^{\alpha}\Lambda_1 f\|_{L^p(\cup\mathcal{W}_3)}^p=\sum_{Q\in\mathcal{W}_3}\|D^{\alpha}\Lambda_1 f\|_{L^p(Q)}^p\leq C\|f\|_{W^{k,p}(\Omega)}^p$$

therefore

$$\|D^{\alpha}\Lambda_1 f\|_{L^p(\cup\mathcal{W}_3)}\leq C\|f\|_{W^{k,p}(\Omega)}\tag{4.8}$$

hence, the operator (Λ_1) is bounded *far from* $\mathcal{W}_4$.

4.1. **Second stage.** This stage, where the extension operator is defined over $\mathcal{W}_4$, is the heart of the extension process. The first stage was essentially a translation of Jones' theorem, which extends functions to an *expanded cusp*, where no weight is needed. Second stage, on the other hand, extends functions to a *cone*: the cuspidal behaviour of Ω is compensated here by a weight. Stage three, in turn, completes the extension to a neighborhood of the origin, but does not contain any interesting idea: we detail it later for the sake of completeness.

Let us begin stating some properties of $\mathcal{W}_4$ itself. Let T be a cube in $\mathcal{W}_4$, and $S_i=S(z_T)$. Observe that from the definition of $\mathcal{W}_3$ we know that $\ell(T)>C\ell(S_i)$, with the constant $C=\frac{\varepsilon}{5\sqrt{n}}\frac{K-1}{K}$. In order to simplify notation in subsequent calculations we set $C=1$ and assume that $\ell(T)\geq\ell(S_i)$.

Let $\widetilde{\mathcal{W}_4}$ denote de Whitney decomposition of $\mathbb{R}^n\setminus\hat{x}_n$. Observe that the structure of $\widetilde{\mathcal{W}_4}$ is very simple: cubes grow exponentially as we move away from the axis. Since the positive semiaxis of $\hat{x}_n$ is contained in Ω, Remark 2.3 implies that for every cube $T\in\mathcal{W}_4$, there is a cube $\widetilde{T}\in\widetilde{\mathcal{W}_4}$, such that $T\subset\widetilde{T}$. The following lemma proves that in fact $\ell(T)\sim\ell(\widetilde{T})$, $\forall T\in\mathcal{W}_4$.

Lemma 4.11. *There is a constant C such that $d(T,\hat{x}_n)\leq C\ell(T)$, for all $T\in\mathcal{W}_4$*

Proof. Let $x^*\in\partial\Omega$ be such that $d(T,\partial\Omega)=d(T,x^*)$. Let γ be the segment joining T and x^*, and $Q\in\mathcal{W}_3$, the nearest cube to T such that $Q\cap\gamma\neq\emptyset$. It is clear that $\ell(Q)\leq d(Q,\partial\Omega)\leq d(T,\partial\Omega)\leq 4\sqrt{n}\ell(T)$. Let us denote $x^q\in\partial\Omega$ the point such that $d(Q,\partial\Omega)=d(Q,x^q)$. Then:

$$d(T,\hat{x}_n)\leq d(T,Q)+\sqrt{n}\ell(Q)+d(Q,\hat{x}_n)\leq 4\sqrt{n}\ell(T)+4n\ell(T)+d(Q,x^q)+d(x^q,\hat{x}_n)$$

$$\leq C\ell(T)+d(x^q,\hat{x}_n)\leq C\ell(T)+\widetilde{K}\ell(S(z_Q))$$

Consequently, if $\ell(S(z_Q))\leq C\ell(T)$, the result is proved.

Let us denote $I = i_{z_Q}$. Furthermore, let $T_1 \in \mathcal{W}_4$ be such that $T_1 \cap Q \neq \emptyset$ and $T_1 \cap \gamma \neq \emptyset$. Then, $\frac{1}{4}\ell(Q) \leq \ell(T_1) \leq 4\ell(Q)$. Suppose that $\ell(Q) < \frac{1}{16}\ell(S_I)$. Then

$$z_{T_1} \geq z_Q - \ell(T_1) \geq z_I - 4\ell(Q) > z_I - \frac{1}{4}\ell(S_I) \geq z_I - \ell(S_{I+1}) \geq z_{I+1}$$

But, since $T_1 \in \mathcal{W}_4$,

$$\ell(Q) \geq \frac{1}{4}\ell(T_1) \geq \frac{1}{4}\ell(S(z_{T_1})) \geq \frac{1}{4}\ell(S_{I+1}) \geq \frac{1}{16}\ell(S_I)$$

which is a contradiction. Consequently, $\ell(T) \geq C\ell(Q) \geq C\ell(S_I)$, and the result follows. $\square$

Remark 4.12. *A much simpler proof for this lemma can be provided assuming property* (2.13). *However, item* i) *in Theorem 4.1 can be proved without* (2.13), *and so we prefer to detail the general proof.*

As we stated above, Lemma 4.11 shows that $\ell(T) \sim \ell(\widetilde{T})$, $\forall T \in \mathcal{W}_4$. This fact implies that the number of cubes of a certain size in $\mathcal{W}_4$ is comparable with the number of cubes of the same size in $\widetilde{\mathcal{W}}_4$. In some passages of this stage, we estimate the number of cubes in $\mathcal{W}_4$ by the number of cubes in $\widetilde{\mathcal{W}}_4$, which is easier to count.

In this second stage a weight is needed in order to bound the norm of the extension operator: we provide two different versions for the extension to the cubes in $\mathcal{W}_4$, the first one is *horizontal* (each cube will be associated with a set at the same height), leading to the weight $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{\frac{n-1}{p}}$ corresponding to item (b) in Theorem 4.1. Property (2.13) is needed in this case. The second version is *vertical* giving another possible weight: $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{k}$ as in item (a) in Theorem 4.1). Property (2.13) is not needed for this version.

First version: dimensional-horizontal weight. For each cube $T_j \in \mathcal{W}_4$ let us define

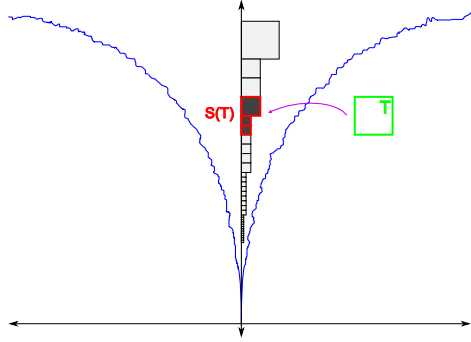$$S(T_j) = \bigcup \{S_i : z_{T_j} \leq z_i < z_{T_j} + \ell(T_j)\}$$

FIGURE 2. *Reflected* tower: second stage's first version.

Remark 4.13. $S(T_j)$ *is the* reflected *set of* T_j *as well as* Q_j^* *is the reflected cube for* Q_j *in the first stage. Observe that* $S(T_j)$ *is not a cube, nor a rectangle. However, normality property* (2.3) *implies that it is a* tower *of cubes, eventually of different sizes. Since cubes in* $\mathcal{W}_4$ *are far from* Ω, T_j *will be larger than the* S_i's *in* $S(T_j)$. *Nevertheless, the dyadic nature of cubes in Whitney decompositions implies that its height is exactly* $\ell(T_j)$. *Finally, if* $S(T_j) = \{S_{I_j}, S_{I_j+1} \ldots, S_{I_j+N_j}\}$,

property (2.13) guarantees that $\frac{\ell(S_{I_j})}{\ell(S_{I_j+N_j})} \leq C < \infty$. Therefore, for each T_j there is a pair of rectangles R_j^1 and R_j^2 such that:

$$R_j^1 \subset S(T_j) \subset R_j^2,$$

$$\vec{\ell}(R_j^1) = (\ell(S_{I_j+N_j}), \ldots, \ell(S_{I_j+N_j}), \ell(T_j))$$

,

$$\vec{\ell}(R_j^2) = (\ell(S_{I_j}), \ldots, \ell(S_{I_j}), \ell(T_j)),$$

satisfying: $\frac{\ell_i(R_j^2)}{\ell_i(R_j^1)} \leq C$ for all T_j and $i = 1, \ldots, n$; i.e.: $R_j^1 \sim R_j^2$.

Let us define, for each $T_j \in \mathcal{W}_4$, $P_{T_j} = P(R_j^1)$. Our extension operator is

$$\Lambda_2 f(x) = \sum_{T_j \in \mathcal{W}_4} P_{T_j}(x)\phi_j(x) \tag{4.9}$$

for $x \in T \in \mathcal{W}_4$.

The following lemma is equivalent to Lemma 4.10. However, since $\mathcal{W}_4$ is far from Ω, a weight is needed:

Lemma 4.14. *If $T \in \mathcal{W}_4$ (far from $\mathcal{W}_3$ and $\mathcal{W}_5$), then:*

$$\|D^\alpha \Lambda_2 f\|_{L^p(T)} \leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \|f\|_{W^{k,p}(\cup \mathcal{F}(T))}$$

where $\mathcal{F}(T)$ is the union of all the $S(T_j)$ with $T_j \cap T \neq \emptyset$, and R^1 is the rectangle in $S(T)$ provided by Remark 4.13.

Proof. As we procceded in Lemma 4.10, we alternate the polinomial corresponding to T, P_T:

$$\|D^\alpha \Lambda_2 f\|_{L^p(T)} = \Big\|D^\alpha \sum_{T_j \cap T \neq \emptyset} P_{T_j}\phi_j\Big\|_{L^p(T)} \leq \underbrace{\Big\|D^\alpha \sum_{T_j \cap T \neq \emptyset}(P_{T_j} - P_T)\phi_j\Big\|_{L^p(T)}}_{I} + \underbrace{\big\|D^\alpha P_T\big\|_{L^p(T)}}_{II}$$

Since $d(T, S(T)) \leq C\ell(T)$, the second term can be bounded by means of Lemma 3.2, by considering an auxiliary cube $\tilde{T} \sim T$, such that $T, S(T) \subset \tilde{T}$:

$$II \leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \sum_{\gamma: |\gamma+\alpha|<k} \ell(T)^{|\gamma|}\|D^{\alpha+\gamma}P_T\|_{L^p(S(T))}.$$

If we go on like in Lemma 4.10:

$$II \leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}} \|f\|_{W^{k,p}(S(T))}$$

On the other hand:

$$I \leq C \sum_{T_j \cap T \neq \emptyset} \sum_{\beta \leq \alpha} \frac{1}{\ell(T)^{|\alpha-\beta|}}\|D^\beta(P_{T_j} - P_T)\|_{L^p(T)},$$

and $T \cap T_j \neq \emptyset$, implies that $S(T) \cap S(T_j) \neq \emptyset$ and $R^1 \sim R_j^1$. In fact, $S(T) \subset S(T_j)$ or $S(T_j) \subset S(T)$ (which imply $R^1 \subset R_j^1$ or $R_j^1 \subset R^1$ resp.), or $S(T)$ and $S(T_j)$ form a new, longer tower where $S(T)$ is over $S(T_j)$, or vice versa (which implies that R^1 and R_j^1 are touching). We show only the case that leads to touching rectangles (the other cases follow similarly):

$$\|D^\beta(P_{T_j} - P_T)\|_{L^p(T)} \leq \|D^\beta(P_{T_j} - P(R^1 \cup R_j^1))\|_{L^p(T)} + \|D^\beta(P_T - P(R^1 \cup R_j^1))\|_{L^p(T)} \leq$$

$$\leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}}\sum_{\gamma:|\gamma+\beta|<k}\ell(T)^{|\gamma|}\Big\{\|D^{\beta+\gamma}(P_T - P(R^1 \cup R_j^1))\|_{L^p(R^1)}$$

$$+\|D^{\beta+\gamma}(P_{T_j} - P(R^1 \cup R_j^1))\|_{L^p(R_j^1)}\Big\} \leq$$

$$\leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}}\sum_{\gamma:|\gamma+\beta|<k}\ell(T)^{|\gamma|}\Big\{\|D^{\beta+\gamma}(P_T - f)\|_{L^p(R^1)} + \|D^{\gamma+\beta}(f - P(R^1 \cup R_j^1))\|_{L^p(R^1)}$$

$$+\|D^{\beta+\gamma}(P_{T_j} - f)\|_{L^p(R_j^1)} + \|D^{\gamma+\beta}(f - P(R^1 \cup R_j^1))\|_{L^p(R_j^1)}\Big\}$$

Applying Lemma 3.5 we obtain:

$$\|D^{\beta}(P_{T_j} - P_T)\|_{L^p(T)} \leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}}\sum_{\gamma:|\gamma+\beta|<k}\ell(T)^{|\gamma|}\ell(R^1)^{k-|\gamma|-|\beta|}\sum_{\tau:|\tau|=k}\|D^{\tau}f\|_{L^p(S(T)\cup S(T_j))}$$

And, consequently:

$$I \leq C \sum_{T_j \cap T \neq \emptyset}\sum_{|\beta|\leq|\alpha|}\frac{1}{\ell(T)^{|\alpha-\beta|}}C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}}\sum_{\gamma:|\gamma+\beta|\leq k}\ell(T)^{|\gamma|}\ell(R^1)^{k-|\gamma|-|\beta|}\sum_{\tau:|\tau|=k}\|D^{\tau}f\|_{L^p(S(T)\cup S(T_j))}$$

$$\leq C \sum_{T_j \cap T \neq \emptyset}\sum_{|\beta|\leq|\alpha|}\ell(T)^{|k|-|\alpha|}C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}}\sum_{\gamma:|\gamma+\beta|<k}\sum_{\tau:|\tau|=k}\|D^{\tau}f\|_{L^p(S(T)\cup S(T_j))}$$

$$\leq C\Big(\frac{|T|}{|R^1|}\Big)^{\frac{1}{p}}\|f\|_{W^{k,p}(\mathcal{F}(T))}$$

$\square$

Lemma 4.10 bounds the norm of the extension operator in all the cubes in $\mathcal{W}_3$ *far from* $\mathcal{W}_4$. That is, in all cubes $Q \in \mathcal{W}_3$ such that all the neighbours of Q are in $\mathcal{W}_3$. Lemma 4.14 does the same thing for cubes in $\mathcal{W}_4$, *far from* $\mathcal{W}_3$. Let us consider now cubes in the frontier of these sets: let $Q \in \mathcal{W}_3$ and $T \in \mathcal{W}_4$ be such that $Q \cap T \neq \emptyset$. Notice that $\frac{1}{4} \leq \frac{\ell(Q)}{\ell(T)} \leq 4$. Furthermore:

$$4\sqrt{n}\ell(Q) \geq d(Q,\partial\Omega) \geq d(T,\partial\Omega) - \sqrt{n}\ell(Q) \geq \ell(T) - \sqrt{n}\ell(Q)$$

and then

$$\ell(T) \leq 5\sqrt{n}\ell(Q) \leq C\ell(S_I)$$

where S_I is the cube in the *spine* of Ω such that $z_Q \in [z_I, z_{I-1})$. This implies that $\ell(Q^*) \sim \ell(T)$, and since $S_I \cap S(T) \neq \emptyset$, by means of lemma 4.8, there is a chain of cubes joining Q^* and $S(T)$. Hence, the proof for the following lemma is the same that the one for Lemmas 4.10 and 4.14.

Lemma 4.15. *Let $Q \in \mathcal{W}_3$ and $T \in \mathcal{W}_4$ be such that $Q \cap T \neq \emptyset$, then:*

$$\|D^{\alpha}(\Lambda_1 + \Lambda_2)f\|_{L^p(Q)} \leq C\|f\|_{W^{k,p}(\mathcal{F}(Q))},$$

$$\|D^{\alpha}(\Lambda_1 + \Lambda_2)f\|_{L^p(T)} \leq C\|f\|_{W^{k,p}(\mathcal{F}(T))}$$

We need to prove that the norm of the extension is bounded as in Lemma 4.14 all over $\mathcal{W}_4$ and not only in a particular cube. Let us pick, then, a cube $S_i \in \mathcal{S}$. A simple comparison with $\widetilde{\mathcal{W}_4}$ implies that the number of cubes T_j, with $\ell(T_j) = 2^m\ell(S_i)$, such that $S_i \subset S(T_j)$ is bounded by a constant independent of S_i. Furthermore, such a comparison allows us to bound the possible values of m, for each i: $0 \leq m \leq \log\big(\frac{z_i}{\ell(S_i)}\big)$, where $\log = \log_2$.

Proposition 4.16. *If we denote* $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|} \right)^{\frac{n-1}{p}}$, *then:*

$$\|\sigma D^{\alpha} f\|_{L^p(\cup \mathcal{W}_4)} \leq C \|f\|_{W^{k,p}(\cup \mathcal{S})}$$

Proof. We can take σ as constant in each cube $T \in \mathcal{W}_4$: $\sigma_T \sim \left(\frac{\ell(S(z_T))}{z_T} \right)^{\frac{n-1}{p}}$. Then

$$\|\sigma D^{\alpha} f\|_{L^p(\cup \mathcal{W}_4)}^p = \sum_{T : T \in \mathcal{W}_4} \|\sigma D^{\alpha} f\|_{L^p(T)}^p \leq C \sum_{T : T \in \mathcal{W}_4} \left(\frac{\ell(S(z_T))}{z_T} \right)^{n-1} \|D^{\alpha} f\|_{L^p(T)}^p$$

$$\leq C \sum_{T : T \in \mathcal{W}_4} \left(\frac{\ell(S(z_T))}{z_T} \right)^{n-1} \frac{|T|}{|R^1|} \|f\|_{W^{k,p}(\cup \mathcal{F}(T))}^p$$

Now, since $\ell_i(R^1) \sim \ell(S(z_T))$ for $i = 1, \dots, n-1$, and $\ell_n(R^1) = \ell(T)$, we have:

$$\|\sigma D^{\alpha} f\|_{L^p(\cup \mathcal{W}_4)}^p \leq C \sum_{T : T \in \mathcal{W}_4} \left(\frac{\ell(T)}{z_T} \right)^{n-1} \|f\|_{W^{k,p}(\cup \mathcal{F}(T))}^p$$

$$= C \sum_{T : T \in \mathcal{W}_4} \sum_{S : S \in \mathcal{F}(T)} \left(\frac{\ell(T)}{z_T} \right)^{n-1} \|f\|_{W^{k,p}(S)}^p$$

$$= C \sum_{S : S \in \mathcal{S}} \sum_{T : \mathcal{F}(T) \ni S} \left(\frac{\ell(T)}{z_T} \right)^{n-1} \|f\|_{W^{k,p}(S)}^p$$

Given a fixed cube $S \in \mathcal{S}$, the cubes $T \in \mathcal{W}_4$ can be classified by their sizes: $\ell(T) = 2^m \ell(S)$, where $0 \leq m \leq M = \log(\frac{z_S}{\ell(S)})$. Furthermore, $z_T \sim z_S$ for every $T \in \mathcal{W}_4$ such that $S \in \mathcal{F}(T)$. Finally, the comparison between cubes in $\mathcal{W}_4$ and cubes in $\widetilde{\mathcal{W}_4}$, guarantees that, given a cube $S \in \mathcal{S}$, there is a bound C, depending only on the dimension n, such that

$$\#\{ T \in \mathcal{W}_4 : \quad S \in \mathcal{F}(T) \quad \ell(T) = 2^m \ell(S) \} \leq C.$$

Then:

$$\|\sigma D^{\alpha} f\|_{L^p(\cup \mathcal{W}_4)}^p \leq C \sum_{S : S \in \mathcal{S}} \sum_{m=1}^{M-1} \sum_{\substack{T : \mathcal{F}(T) \ni S \\ \ell(T) = 2^m \ell(S)}} \left(\frac{\ell(T)}{z_T} \right)^{n-1} \|f\|_{W^{k,p}(S)}^p$$

$$\leq C \sum_{S : S \in \mathcal{S}} z_S^{1-n} \ell(S)^{n-1} \left(\sum_{m=1}^{M-1} \left(2^{n-1} \right)^m \right) \|f\|_{W^{k,p}(S)}^p,$$

and the result follows by recalling that $M = \log(\frac{z_S}{\ell(S)})$. $\square$

This result concludes the first version of the second stage of the extension.

Second version: derivative-vertical weight. This version of the extension is based on a different construction of the *reflected* set of a cube in $\mathcal{W}_4$. For each T, we find some $T^* \in \mathcal{S}$ such that $\ell(T^*) \sim \ell(T)$, but T^* is *far* above T. The weight in this case is due to the distance between T and T^*.

Let us consider $\tilde{T}$ a cube belonging to $\widetilde{\mathcal{W}_4}$ (the Whitney decomposition of $\mathbb{R}^n \setminus \hat{x}_n$) such that $\tilde{T} \cap T \neq \emptyset$, for some $T \in \mathcal{W}_4$. Thanks to Lemma 4.11 only a finite number (the number does not depend on $\tilde{T}$) of cubes belonging to $\mathcal{W}_4$ are contained in $\tilde{T}$. We can now pack the elements of $\mathcal{W}_4$ in cylinders of the form $\eta(\tilde{T}) = Q' \times \mathbb{R}$, where $Q' \subset \mathbb{R}^{n-1}$ is the projected face $F_{\tilde{T}}^u$ of $\tilde{T}$ into $\mathbb{R}^{n-1}$. We identify cylinders given by cubes $\tilde{T}$ sharing the projection Q'. In this way each cube

$T \in \mathcal{W}_4$ belongs to only one cylinder. Moreover, cubes inside the cylinder $\eta(\tilde{T})$ are equivalent, i.e $T_1, T_2 \in \eta(\tilde{T})$ implies that $T_1 \sim \tilde{T} \sim T_2$. For each $T_j \in \mathcal{W}_4$, we denote with $\tau(T_j)$ the set of cubes in $\mathcal{W}_4$ that share the cylinder with T_j. The set τ is called a tower.

Let us consider T^1 one of the upper cubes in $\tau(T^1)$. We define $T^* = S(z_{T^1})$ for every $T \in \tau(T^1)$. This situation is represented in Figure 3.
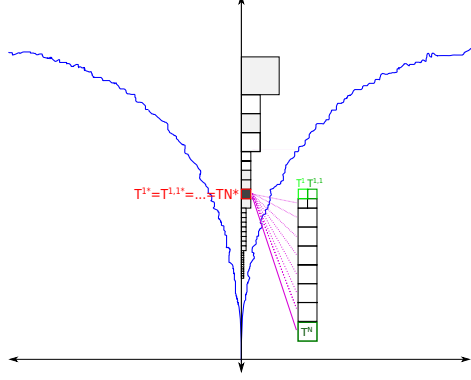

FIGURE 3. *Reflected* cubes: second stage's second version.

It is important that, with this definition, for every $T \in \mathcal{W}_4$ we have $T^* \in \mathcal{S}$, and $T^* \sim T$. However, the distance between T and T^* could be *large*, particularly in the x_n direction. In fact, since $d(T^1, T^N) \sim z_{T^1}$, we have $d(T^1, T^{1*}) \sim \ell(T^1)$, but $d(T^N, T^{N*}) \sim z_{T^1}$ (where T^1 and T^N are upper and lower cubes in a certain tower τ).

As in the first version of the second stage extension, we define $P_{T_j} = P(T_j^*)$ and

$$\Lambda_2 f(x) = \sum_{T_j \in \mathcal{W}_4} P_{T_j}(x) \phi_j(x)$$

Observe that, if $T_j, T \in \mathcal{W}_4$, and $T_j \cap T \neq \emptyset$, the tops of the towers $\tau(T)$ and $\tau(T_j)$ could be setted at very different heights (and so would be the heights of the reflected cubes T_j^* and T^*). This is so because of the following fact:

Remark 4.17. *Suppose $S \in \mathcal{S}$ is the higher cube of a certain size. Let us denote $c(S)$ the number of cubes with edges of lengh exactly $\ell(S)$. Then, since $0 \in \partial\Omega$, $z_S - c(S)\ell(S) > 0$, and consequently, $c(S) \leq \frac{z_S}{\ell(S)}$. However, no better estimate can be provided (in fact, it is easy to see that for cusps with profile $\varphi(z) = z^\nu$, there are $\sim z^{1-\nu}$ cubes with edges z^ν), so the worst case, that there could be $\sim \frac{z_S}{\ell(S)}$ cubes in $\mathcal{S}$ with side $\ell(S)$, should be assumed to hold.*

Consequently, the shape of $\mathcal{W}_4$ could show *long steps*. When two towers touching each other are in the edge of a long step, their heights are very different. This situation is represented in Figure 4. In this figure, two touching towers are shown, where reflected cubes are far from each other. Therefore, the chain (in $\mathcal{S}$) joining the reflected cubes for each tower is large.

Lemma 4.18. *For every cube $T \in \mathcal{W}_4$:*

$$\|D^\alpha \Lambda_2 f\|_{L^p(T)} \leq C\ell(T)^{k-|\alpha|} \left(\frac{\ell_n(\tau(T))}{\ell(T)} \right)^{k-\frac{1}{p}} \|f\|_{W^{k,p}(\cup \mathcal{F}(T))}$$

where $\mathcal{F}(T)$ is the union of the cubes in all the chains connecting T^* and T_j^* for $T_j \cap T \neq \emptyset$.
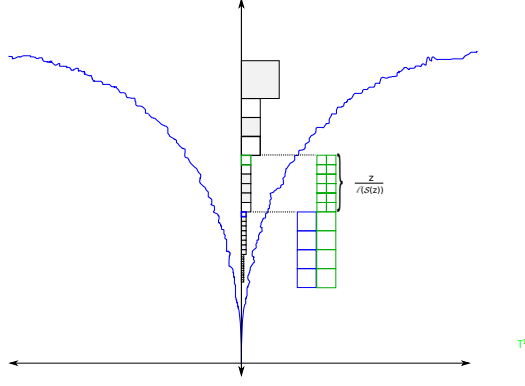
FIGURE 4. *Long steps* imply long chains.

Proof.

$$\|D^\alpha \Lambda_2 f\|_{L^p(T)} \leq \underbrace{\left\|D^\alpha \sum_{T_j \cap T \neq \emptyset}(P_{T_j} - P_T)\phi_j\right\|_{L^p(T)}}_{I} + \underbrace{\left\|D^\alpha P_T\right\|_{L^p(T)}}_{II}$$

As usual:

$$I \leq C \sum_{\beta \leq \alpha} \frac{1}{\ell(T)^{|\alpha|-|\beta|}} \sum_{T_j \cap T \neq \emptyset} \|D^\beta(P_{T_j} - P_T)\|_{L^p(T)}$$

Let us denote $\mathcal{F}_j(T) = \{T^* = S^1, S^2 \dots, T_j^* = S^M\}$ the chain of cubes joining T^* and T_j^*, then:

$$\|D^\beta(P_{T_j} - P_T)\|_{L^p(T)} \leq \sum_{l=1}^{M-1} \|D^\beta(P(S^{l+1}) - P(S^l))\|_{L^p(T)}$$

Now, if we denote R_l the minimal rectangle containing T and S^l, we have $\ell_i(R_l) \sim \ell(T)$, $i = 1, \dots, n-1$, and $\ell_n(R) \leq \ell_n(\tau(T))$.

$$\|D^\beta(P(S^{l+1}) - P(S^l))\|_{L^p(T)} \leq C\Big\{\|D^\beta(P(S^{l+1}) - P(S^{l+1} \cup S^l))\|_{L^p(T)} +$$
$$\|P(S^{l+1} \cup S^l) - P(S^l))\|_{L^p(T)}\Big\}$$

$$\leq C|T|^{\frac{1}{p}}\Big\{\|D^\beta(P(S^{l+1}) - P(S^{l+1} \cup S^l))\|_{L^\infty(R_{l+1})}$$
$$+ \|D^\beta(P(S^{l+1} \cup S^l) - P(S^l))\|_{L^\infty(R_l)}\Big\}$$

$$\leq C|T|^{\frac{1}{p}} \sum_{|\gamma+\beta|<k}\left\{\frac{\ell(R^l)^{|\gamma|}}{|S^{l+1}|^{\frac{1}{p}}}\|D^{\beta+\gamma}(P(S^{l+1}) - P(S^{l+1} \cup S^l))\|_{L^p(S^{l+1})}\right.$$
$$\left.+ \frac{\ell(R^l)^{|\gamma|}}{|S^l|^{\frac{1}{p}}}\|D^{\beta+\gamma}(P(S^l) - P(S^{l+1} \cup S^l))\|_{L^p(S^l)}\right\}$$

$$\leq C \sum_{|\gamma+\beta|<k} \ell_n(\tau(T))^{|\gamma|}\ell(S^l)^{k-|\beta|-|\gamma|}\|\nabla^k f\|_{L^p(S^l \cup S^{l+1})}$$

Consequently:

$$I \leq C \sum_{T_j \cap T \neq \emptyset} \sum_{\beta \leq \alpha} \frac{1}{\ell(T)^{|\alpha|-|\beta|}} \sum_{l=1}^{M-1} \sum_{|\gamma+\beta|<k} \ell_n(\tau(T))^{|\gamma|}\ell(S^l)^{k-|\beta|-|\gamma|}\|\nabla^k f\|_{L^p(S^l \cup S^{l+1})}$$

$$\leq C\ell(T)^{k-|\alpha|}\left(\frac{\ell_n(\tau(T))}{\ell(T)}\right)^{k-1} \sum_{l=1}^{M-1} \|\nabla^k f\|_{L^p(S^l \cup S^{l+1})}$$

Applying the Hölder inequality gives

$$I \leq C\ell(T)^{k-|\alpha|}\left(\frac{\ell_n(\tau(T))}{\ell(T)}\right)^{k-1} M^{\frac{1}{p'}}\|\nabla^k f\|_{L^p(\cup \mathcal{F}(T))}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. But M is the number of cubes in the chain joining T^* and T_j^*, which we saw that could be as large as $\frac{\ell_n(\tau(T))}{\ell(T)}$, and then:

$$I \leq C\ell(T)^{k-|\alpha|}\left(\frac{\ell_n(\tau(T))}{\ell(T)}\right)^{k-\frac{1}{p}}\|\nabla^k f\|_{L^p(\cup \mathcal{F}(T))}$$

II could be bounded by means of the same ideas.

$\square$

A proposition equivalent to 4.16 can now be easily proved:

Proposition 4.19. *If we denote $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^k$, then:*

$$\|\sigma D^\alpha \Lambda_2 f\|_{L^p(\cup \mathcal{W}_4)} \leq C\|f\|_{W^{k,p}(\cup \mathcal{S})}$$

Proof. As we did in Proposition 4.16, let us observe that the weight σ could be considered constant in each cube $T \in \mathcal{W}_4$, $\sigma_T \sim \left(\frac{\ell(S(z_T))}{z_T}\right)^k$. Then:

$$\|\sigma D^\alpha \Lambda_2 f\|_{L^p(\cup \mathcal{W}_4)}^p = \sum_{T \in \mathcal{W}_4} \|\sigma D^\alpha \Lambda_2 f\|_{L^p(T)}^p \leq C \sum_{T \in \mathcal{W}_4} \left(\frac{\ell(S(z_T))}{z_T}\right)^{kp} \|D^\alpha \Lambda_2 f\|_{L^p(T)}^p$$

$$\leq C \sum_{T \in \mathcal{W}_4} \left(\frac{\ell(S(z_T))}{z_T}\right)^{kp} \ell(T)^{(k-|\alpha|)p}\left(\frac{\ell_n(\tau(T))}{\ell(T)}\right)^{kp-1}\|f\|_{W^{k,p}(\cup \mathcal{F}(T))}^p$$

$$= C \sum_{S:S \in \mathcal{S}} \sum_{T:\mathcal{F}(T) \ni S} \left(\frac{\ell(S(z_T))}{z_T}\right)^{kp} \ell(T)^{(k-|\alpha|)p}\left(\frac{\ell_n(\tau(T))}{\ell(T)}\right)^{kp-1}\|f\|_{W^{k,p}(S)}^p$$

Now, observe that if we fix a cube $S \in \mathcal{S}$, every cube $T \in \mathcal{W}_4$ such that $S \in \mathcal{F}(T)$ satisfies: $\ell(T) \sim \ell(S)$ and $\ell_n(\tau(T)) \leq z_S$. By considering this and $|\alpha| \leq k$, we obtain:

$$\|\sigma D^\alpha \Lambda_2 f\|_{L^p(\cup \mathcal{W}_4)}^p \leq C \sum_{S:S \in \mathcal{S}} \sum_{T:\mathcal{F}(T) \ni S} \left(\frac{\ell(S(z_T))}{z_T}\right)^{kp}\left(\frac{z_S}{\ell(T)}\right)^{kp-1}\|f\|_{W^{k,p}(S)}^p$$

Now, using property (2.12) we obtain:

$$\leq C \sum_{S:S \in \mathcal{S}} \sum_{T:\mathcal{F}(T) \ni S} \left(\frac{\ell(S)}{z_S}\right)^{kp}\left(\frac{z_S}{\ell(T)}\right)^{kp-1}\|f\|_{W^{k,p}(S)}^p \leq C \sum_{S:S \in \mathcal{S}} \sum_{T:\mathcal{F}(T) \ni S} \frac{\ell(S)}{z_S}\|f\|_{W^{k,p}(S)}^p$$

But, for a fixed $S \in \mathcal{S}$, the number of cubes T such that $S \in \mathcal{F}(T)$ is at most $C \frac{z_S}{\ell(S)}$ and then:

$$\leq C \sum_{S:S\in\mathcal{S}} \frac{z_S}{\ell(S)} \frac{\ell(S)}{z_S} \|f\|_{W^{k,p}(S)}^p \leq C \sum_{S:S\in\mathcal{S}} \|f\|_{W^{k,p}(S)}^p = \|f\|_{W^{k,p}(\cup\mathcal{S})}^p$$

$\square$

Remark 4.20. *If $kp \neq 1$, Property (2.12) is not necessary. Using $\ell(S(z_T)) \leq \ell(T)$, an argument similar to the one given in Proposition 4.16 can be applied: at the end of the proof of Proposition 4.16 the edges of all cubes T such that $S \subset S(T)$ are classified in terms of $\ell(S)$. In the last part of the proof of Proposition 4.19, the same idea can be used in order to classify z_T for all cubes T such that $S \in \mathcal{F}(T)$. When $kp \neq 1$ the summation is bounded and the result follows.*

Third Stage. This stage is devoted to define our extension operator in the cubes of $\mathcal{W}_5$. We explain the construction of the reflected sets for each version of the extension, but we do not enter into details since the ideas are exactly the same given in Lemmas 4.14 and 4.18 and Propositions 4.16 and 4.19, according to the case.

For the first (dimensional) version of the extension, let us define:

$$S(U) = \bigcup\{S_i : \ell(U) \leq z_i < 2\ell(U)\}$$

It is clear that $d(U, S(U)) \leq C\ell(U)$. On the other hand, $S(U)$ is a tower of cubes that admits an interior rectangle R^1, with $\ell_i(R^1) \sim \ell(S(\ell(U)))$ for $i = 1, \ldots, n-1$ and $\ell_n(R^1) \sim \ell(U)$. Because of property (2.13), there is an exterior rectangle $R^2 \supset S(U)$ such that $R^2 \sim R^1$. Hence, Remark 4.13 holds for cubes in $\mathcal{W}_5$, and so do Lemma 4.14 and Proposition 4.16. As we did earlier, we define $P_{U_j} = P(R_j^1)$. The last thing to notice is that if $T \in \mathcal{W}_4$ and $U \in \mathcal{W}_5$ are such that $T \cap U \neq \emptyset$, then $d(S(U), S(T)) \leq C\ell(T)$, and then there is a finite chain of towers that join $S(U)$ and $S(T)$. This guarantees that the transition between $\mathcal{W}_4$ and $\mathcal{W}_5$ is smooth.

For the second (derivative) version, let us define $U^* = S_i$ the cube in $\mathcal{S}$ with i the maximum index such that $\ell(S_i) \geq \ell(U)$. This implies $\ell(U^*) \sim \ell(U)$, which is the essential property of the reflected cube in this case. On the other hand, $d(U, U^*) \leq Cz_{U^*}$. Once again, we define $P_{U_j} = P(U_j^*)$. It is clear that if $T \in \mathcal{W}_4$ and $U \in \mathcal{W}_5$ are such that $T \cap U \neq \emptyset$ and $U^* \sim T^*$, then $d(U^*, T^*) \leq Cz_{T^*}$, and so the norm of the extension can be bounded in the frontier between $\mathcal{W}_4$ and $\mathcal{W}_5$ as we did in $\mathcal{W}_4$.

As we did in the previous sections, let us define, for both versions:

$$\Lambda_3 f(x) = \sum_{U_j \in \mathcal{W}_5} P_{U_j}(x) \psi_j(x)$$

The last matter that we need to deal with is the superposition induced by this definitions of reflected sets. In the previous stage we introduced $\widetilde{\mathcal{W}_4}$ in order to help us *counting* some sets of cubes in $\mathcal{W}_4$. Similarly, let us introduce now $\widetilde{\mathcal{W}_5} = \mathcal{W}(\mathbb{R}^n \setminus \{0\})$. Thanks to Remark 2.3 we may define, for every $U \in \mathcal{W}_5$, $\widetilde{U}$ the cube in $\widetilde{\mathcal{W}_5}$ such that $U \subset \widetilde{U}$. On the other hand the ideas exposed earlier (see Lemma 4.11) lead us to conclude that $U \sim \widetilde{U}$. The number of cubes in $\widetilde{\mathcal{W}_5}$ with edges of a certain length 2^{-l} are bounded by a constant depending only on the dimension n. The same holds for cubes in $\mathcal{W}_5$. Consequently, after this third stage, every cube in $\mathcal{S}$ is loaded with at most a bounded quantity of cubes of the exterior of Ω.

Our complete extension operator is, then,

$$\Lambda f(x) = \Lambda_1 f(x) + \Lambda_2 f(x) + \Lambda_3 f(x)$$

For every $x = (x', x_n) \in \mathcal{W}_4$, $x_n \sim |x|$. We used this fact to write the weight in terms of $|x|$ instead of x_n. Since the third term of the extension is also radial, the weight can be taken

$$\left(\frac{\ell(S(|x|))}{|x|}\right)^\gamma$$

for every $x \in \cup \mathcal{W}_2$, where γ is the exponent corresponding to the case.

Except for a few technical details that are treated in Section 6, the proof of Theorem 4.1 is concluded.

5. CURVED CUSPS

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with an external curved cusp at the origin.*

(a) *If the spine $\mathcal{S}$ satisfies (2.12), there is an extension operator*

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}_\sigma(\mathbb{R}^n)$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^k & x \in \Omega^c \end{cases}$$

(b) *If the spine $\mathcal{S}$ satisfies (2.13), there is an extension operator*

$$\Lambda : W^{k,p}(\Omega) \to W^{k,p}_\sigma(\mathbb{R}^n)$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^{\frac{n-1}{p}} & x \in \Omega^c \end{cases}$$

(c) *In either case (a) or b)), assuming (2.13) stands, if $\tilde{\sigma}$ is such that there is an extension operator $\tilde{\Lambda} : W^{k,p}(\Omega) \to W^{k,p}_{\tilde{\sigma}}(\mathbb{R}^n)$, then*

$$\tilde{\sigma}(x) \leq C\sigma(x) \quad \forall x \in U \setminus \Omega$$

We only sketch a proof of Theorem 5.1. We introduce a *stage zero*, consisting of an extension of a curved cusp Ω to a larger domain that includes a *normal* cusp $\widehat{\Omega}$. Functions defined on $\widehat{\Omega}$ will be extended as in Theorem 4.1. The most important fact to mention is that after the stage one, the distance of cubes in $\mathcal{W}_4$ and $\mathcal{W}_5$ to Ω are comparable with the distance of them to $\widehat{\Omega}$, and so will be the weights.

5.1. **Stage *zero*.** Let Ω be a domain satisfying Definition 2.3, and $\mathcal{S} = \{S_i\}_{i=1}^\infty$ its spine. Then $d(S_i, \hat{x}_n) \leq C_\Omega \ell(S_i)$, and we may take $C_\Omega \geq K$. Assuming $\ell(S_i) \leq 1$, let us consider:

$$\widetilde{\Omega} = \bigcup_i 4(C_\Omega + 1)S_i$$

Clearly, $\Omega \subset \bigcup_i C_\Omega S_i$. Even more: let us take S_i' the traslation of S_i to $\hat{x}_n$, so $S_i' \cap S_{i+1}' = F^u_{S_{i+1}'}$, and $z_{S_i'} = z_{S_i}$. Then, if we denote $\widehat{\Omega} = \bigcup_i 2C_\Omega S_i'$, we have:

$$\Omega \subset \widehat{\Omega} \subset \widetilde{\Omega}$$

Lemma 4.6 can be reproduced in order to find a *reflected* cube for every $Q \in \mathcal{W}_2$ such that $Q \subset \widetilde{\Omega}$, by just changing K for $4(C_\Omega + 1)$. Consequently, a first (unweighted) extension can be performed as in stage one. Let us denote $\Lambda_0 f$ the extension of f to $\widetilde{\Omega}$, and let us take $\widehat{f} : \widehat{\Omega} \to \mathbb{R}$, $\widehat{f} = \Lambda_0 f|_{\widehat{\Omega}}$. Observe that $\widehat{\Omega}$ is a *normal* cusp. Then, we can extend $\widehat{f}$ as in Theorem 4.1. Let us denote $\widehat{\mathcal{W}}_3, \widehat{\mathcal{W}}_4, \widehat{\mathcal{W}}_5$, the subsets of the Whitney decomposition of the exterior of $\widehat{\Omega}$ corresponding to stage one, two and three respectively. If we denote $\widehat{\mathcal{S}} = \{\widehat{S}_i\}_i$ the spine of $\widehat{\Omega}$

(observe that $\widehat{S}_i$ is not necessaily S_i', but they are equivalent), we have $\widehat{S}_i \sim S_i$. Now, if we take $T \in \widehat{\mathcal{W}}_4$, such that $z_i \leq z_T < z_{i-1}$, then

$$\ell(T) \geq C\ell(\widehat{S}_i) \geq C\ell(S_i) \tag{5.1}$$

Furthermore,

$$d(T, S_i) \leq Cd(T, \widehat{S}_i) \leq C\ell(T) \tag{5.2}$$

and

$$d(T, \partial\widehat{\Omega}) \leq d(T, \partial\Omega) \leq Cd(T, \widehat{S}_i) \leq Cd(T, \partial\widehat{\Omega}) \tag{5.3}$$

The weight of the extension operator based on $\widehat{\Omega}$ is expressed in terms of $\widehat{S}_i$, but these inequalities allow us to change it for S_i, and then Theorem 5.1 is proved by using the results given in the next section.

6. DENSITY OF C^∞ FUNCTIONS

Following Jones we prove that certain set of regular functions is dense in $W^{k,p}(\Omega)$.

Proposition 6.1 ([11])**.** *Let D be a ε-uniform domain, and $f \in W^{k,p}(D)$. For every $\eta > 0$ there is a function $g \in C^\infty(\mathbb{R}^n)$ such that $\|f - g\|_{W^{k,p}(D)} < \eta$.*

Remark 6.2. *The proof of this result relies on the existence of certain chain of dyadic cubes constructed in [11] by means of properties (2.1) and (2.2). Similar arguments can be carried out using properties (2.6) and (2.7) instead of (2.1) and (2.2). In this way Proposition 6.1 can be proved for certain sets $\check{\Omega}_i \subset \Omega_i \cup \Omega_{i+1}$ defined below.*

Theorem 6.3. *Let Ω be a cusp, and $f \in W^{k,p}(\Omega)$. Given $\eta > 0$, there is a function $g \in C^\infty(\mathbb{R}_+^n)$ such that $\|f - g\|_{W^{k,p}(\Omega)} < C\eta$*

Proof. Let us define

$$\check{\Omega}_i = \Omega \cap \left\{ x = (x', x_n) : \quad z_i - \frac{\ell(S_{i+1})}{2} \leq x_n < z_{i-1} + \frac{\ell(S_{i-1})}{2} \right\},$$

$$\check{\Omega}_i' = \Omega \cap \left\{ x = (x', x_n) : \quad z_i - \frac{\ell(S_{i+1})}{2} \leq x_n < z_i + \frac{\ell(S_i)}{2} \right\}$$

Let us consider $g_i \in C^\infty$ such that $\|f - g_i\|_{W^{k,p}(\check{\Omega}_i)} < \frac{\eta}{2^i}\ell(S_i)^k$. Observe that the existence of such a function is guaranteed by Proposition 6.1 and Remark 6.2.

Let $\{\psi_i\}_i$ a partition of the unity such that

$$\psi_i \in C_0^\infty\left(\left[z_i - \frac{\ell(S_{i+1})}{2}, z_{i-1} + \frac{\ell(S_{i-1})}{2} \right] \right),$$

$\sum \psi_i(t) \equiv 1 \; \forall t \in (0, z_1]$, and $|D^r\psi_i| \leq \frac{C}{\ell(S_i)^r}$.

Let us define

$$g(x) = \sum_{i=2}^\infty g_i(x)\psi_i(x_n)$$

Observe that, in $\check{\Omega}_i'$, $\psi_i + \psi_{i+1} \equiv 1$. And then:

$$\|D^\alpha(f - g)\|_{L^p(\check{\Omega}_i')} \leq \|D^\alpha(f - (\psi_i g_i + \psi_{i+1}g_{i+1}))\|_{L^p(\check{\Omega}_i')}$$
$$\leq \|D^\alpha(\psi_i(f - g_i))\|_{L^p(\check{\Omega}_i')} + \|D^\alpha(\psi_{i+1}(f - g_{i+1}))\|_{L^p(\check{\Omega}_i')}$$

But,

$$\|D^\alpha(\psi_i(f - g_i))\|_{L^p(\check{\Omega}'_i)} \leq \|\sum_{\beta \leq \alpha} D^{\alpha - \beta}\psi_i D^\beta(f - g_i)\|_{L^p(\check{\Omega}'_i)}$$

$$\leq \sum_{\beta \leq \alpha} \frac{C}{\ell(S_i)^{|\alpha| - |\beta|}}\|D^\beta(f - g_i)\|_{L^p(\check{\Omega}'_i)}$$

$$\leq \frac{C}{\ell(S_i)^{|\alpha| - |\beta|}}\frac{\eta}{2^i}\ell(S_i)^k \leq C\frac{\eta}{2^i}$$

Consequently:

$$\|f - g\|^p_{W^{k,p}(\Omega)} = \sum_{i=1}^\infty \|f - g\|^p_{W^{k,p}(\check{\Omega}'_i)} \leq \sum_{i=1}^\infty C\frac{\eta^p}{2^{pi}} \leq C\eta^p$$

$$\square$$

Following Jones's arguments one can show that $D^\alpha\Lambda f = D^\alpha f\chi_{\bar{\Omega}} + D^\alpha\Lambda f\chi_{\bar{\Omega}^c}$ is in $Lip_{loc}(\mathbb{R}^n \setminus \{0\})$ for all α, $|\alpha| \leq k - 1$ and for any $f \in C^\infty(\mathbb{R}^n_+)$. Indeed, taking into account that the first stage of our extension process follows the Jones's reflection method we can reproduce the ideas of Lemma 3.5 in [11]. The boundedness of f in the arguments given in that reference is crucial: here we should use the fact that f is bounded in any compact set K such that $0 \notin K$.

Recalling that $|\partial\Omega| = 0$ we see that Theorems 4.1 and 5.1 are proved by using density arguments and Theorem 6.3.

7. THE WEIGHTED CASE

For all measurable set $S \subset \mathbb{R}^n$, let $\omega(S)$ be the measure induced by the weight ω:

$$\omega(S) = \int_S \omega$$

We say ω is doubling if for every cube $Q \in \mathbb{R}^n$, $\omega(2Q) \leq C\omega(Q)$ with C independent of Q.

Chua [7, 8] adapts Jones's techniques for proving an extension theorem for locally uniform domains in the weighted case. We state a version of this results.

Theorem B. *Let Ω be an (ε, δ) connected domain, $1 \leq p < \infty$. Suppose that ω is doubling, $\omega^{-\frac{1}{p-1}}$ is locally integrable and $Lip_{loc}^{k-1}(\mathbb{R}^n)$ is dense on $W_\omega^{k,p}(\Omega)$. Finally, suppose that for every cube Q and every $f \in Lip_{loc}(\mathbb{R}^n)$*

$$\|f - f_{Q,\omega}\|_{L^p_\omega(Q)} \leq C\ell(Q)\|\nabla f\|_{L^p_\omega(Q)} \tag{7.1}$$

where $f_{Q,\omega} = \int f d\omega/\omega(Q)$. Then an extension operator $\Lambda : W_\omega^{k,p}(\Omega) \to W_\omega^{k,p}(\mathbb{R}^n)$ exists.

Property (7.1) is just a weighted Poincaré inequality. A simpler but stronger hyphothesis, that implies all the requirements on the weight, is that $\omega \in A_p$, the class of weights satisfying Muckenhoupt's condition:

$$\sup_{Q \subset \mathbb{R}^n \text{cube}} \frac{1}{|Q|}\left(\int_Q \omega(x)dx\right)\left(\int_Q \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq C < \infty.$$

Chua's extension operator is constructed as Jones' one: For each cube $Q_j \in \mathcal{W}^c$ *near* the domain, a *reflected* cube $Q_j^* \in \mathcal{W}$ is found (as in Lemma 4.6). Given $f \in W_\omega^{k,p}(\Omega)$, a suitable polynomial $P_j = P(Q_j^*, \omega)$, that can be constructed thanks to (7.1), is associated to Q_j. Thence, the operator is the smooth summation of all the $\{P_j\}_j$.

The doubling condition is crucial for Chua's arguments to hold. Indeed, since $d(Q, Q^*) \leq C\ell(Q)$, a bounded expansion of Q, $\widetilde{Q} = cQ$, contains both cubes Q and Q^*. But ω being

doubling, $\omega(\widetilde{Q}) \le C\omega(Q)$. This allows the comparison between the values of the weight ω over Q and over Q^*. Therefore, the *weighted* norm of the extension in Q can be bounded by the *weighted* norm of the function in Q^* just as in Lemma 3.2 in [11] or Lemma 4.10 in this paper.

Since the first stage of our extension process agrees with that of uniform domains, Chua's techniques could be applied. However, second stage presents a very different situation. Reflected sets for cubes in $\mathcal{W}_4$ do not fulfill properties (4.4) and (4.5) of Jones's reflected cube: (4.4) and (4.5). In the *dimensional-horizontal* version, the reflected set of Q is a *rectangle* $S(Q)$, not a cube, and whereas $d(Q, S(Q)) \sim \ell(Q)$, the edges $\ell_i(S(Q))$ are not equivalent to $\ell(Q)$, so (4.4) fails. Consequently, the values of the weight ω over Q cannot be estimated by its values over $S(Q)$. On the other hand: in the *derivative-vertical* version, the reflected set of Q is a cube Q^*, with $\ell(Q^*) \sim \ell(Q)$, but it may happen that $d(Q, Q^*) >> \ell(Q)$, so (4.5) fails. In this case, no bounded fixed expansion of Q could reach Q^*, and the doubling property of ω is useless.

However, it is noteworthy that some particular weights can be easily integrated into our extension process. We present here two examples involved in several applications: weights depending on the distance to the boundary of Ω, that fit easily with the *Derivative Version* of the extension, and weights depending on the distance to 0 (the tip of the cusp), that are naturally adapted for the *Dimensional Version*.

We analyze each type of weight separately.

7.1. **Weights depending on** $d(x, 0) = |x|$ **- Dimensional Case.** Observe that near the origin we have that $|x| \sim x_n$, $\forall x \in \Omega$. Moreover, the same thing holds close enough to Ω, and in particular in the sets $\cup \mathcal{W}_3$ and $\cup \mathcal{W}_4$. Let $\omega : \mathbb{R}^n \to \mathbb{R}$, $\omega \ge 0$, a radial weight. We denote $\omega(x) = \hat{\omega}(|x|)$, and we assume that $\hat{\omega} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a monotonic function that satisfies $\hat{\omega}(2t) \sim \hat{\omega}(t)$. Notice that the only interesting case is that either $\hat{\omega}(t) \to 0$ or $\hat{\omega}(t) \to \infty$ when $t \to 0$, since otherwise the weighted space agrees with the already treated case of $W^{k,p}$. Let us mention that $W_\omega^{k,p}$ is a Banach space [12] for any open set Ω.

Remark 7.1. *Observe that for every* $x, \tilde{x} \in Q \in \mathcal{W}_3 \cup \mathcal{W}_4$, *we have* $\omega(x) \sim \omega(\tilde{x})$, *therefore we can pick a constant value* ω_Q *such that* $\omega(x) \sim \omega_Q \; \forall x \in Q \in \mathcal{W}_3 \cup \mathcal{W}_4$.

This fact is the key tool for our weighted extension process. We detail the proof of the first stage: Lemma 4.6 guarantees that $d(Q, Q^*) \le C\ell(Q)$, $\forall Q \in \mathcal{W}_3$. This implies $z_{Q^*} \sim z_Q$, and consequently $\omega_Q \sim \omega_{Q^*}$. Furthermore, it is easy to see that $\omega_Q \sim \omega_S$ for every $S \in \mathcal{F}(Q)$.

Lemma 7.2. *If* $Q \in \mathcal{W}_3$ *if far from* $\mathcal{W}_4$, *then:*

$$\|D^\alpha \Lambda f\|_{L_\omega^p(Q)} \le C\Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L_\omega^p(\mathcal{F}(Q))} + \|f\|_{W_\omega^{k,p}(Q^*)}\Big\}$$

Proof. Just applying the constant approximation of the weight and Lemma 4.10:

$$\|D^\alpha \Lambda f\|_{L_\omega^p(Q)} = \|\omega D^\alpha \Lambda f\|_{L_p(Q)} \le C\omega_Q \|D^\alpha \Lambda f\|_{L^p(Q)}$$

$$\le C\omega_Q \Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p(\mathcal{F}(Q))} + \|f\|_{W^{kp}(Q^*)}\Big\}$$

$$\le C\Big\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L_\omega^p(\mathcal{F}(Q))} + \|f\|_{W_\omega^{kp}(Q^*)}\Big\}$$

$\square$

For the second stage we use essentially the same idea: the weight, being approximately constant over every cube, can be pulled in or out integrals, so the weighted norm can be estimated using the non-weighted lemmas proved in Section 4.

It is clear that $\omega_T \sim \omega_{S(T)}$, $\forall T \in \mathcal{W}_4$. The weighted form of Lemma 4.14 can be proved exactly as Lemma 7.2, so the next proposition follows, completing the second stage for this version:

Proposition 7.3. *If we denote* $\sigma(x) = \left(\frac{\ell(S(|x|))}{|x|}\right)^{\frac{n-1}{p}}$, *then:*

$$\|\sigma D^\alpha f\|_{L^p_\omega(\mathcal{W}_4)} \leq C\|f\|_{W^{k,p}_\omega(\mathcal{S})}$$

Exactly the same ideas can be used for the third stage.

Weights depending on $d_{\partial\Omega}$ - the derivative case. We obviously have:

$$d(x, \partial\Omega) \sim \ell(Q) \quad \forall x \in Q, \quad \forall Q \in \mathcal{W} \cup \mathcal{W}^c. \tag{7.2}$$

Let us set $\hat\omega : \mathbb{R}_+ \to \mathbb{R}_+$ a monotonic function such that $\hat\omega(2t) \sim \hat\omega(t)$. And let $\omega : \mathbb{R}^n \to \mathbb{R}^n$, be the weight $\omega(x) = \hat\omega(d_{\partial\Omega}(x))$. In this case this implies that ω can be taken as a constant ω_Q over every cube $Q \in \mathcal{W} \cup \mathcal{W}^c$.

This leads us to the following corollary of Lemma 4.10 (which proof is exactly as the one of Lemma 7.2):

Lemma 7.4. *Let Ω be a domain satisfying Definition 2.2, then:*

$$\|D\Lambda_1 f\|_{L^p_\omega(Q)} \leq C\left\{\ell(Q)^{k-|\alpha|}\|\nabla^k f\|_{L^p_\omega(\mathcal{F}(Q))} + \|D^\alpha f\|_{L^p_\omega(Q^*)}\right\}$$

For the second stage, let us recall that in the *derivative version*, $\ell(Q) \sim \ell(Q^*)$, and then $d_{\partial\Omega}(Q) \sim d_{\partial\Omega}(Q^*)$. This fact is enough to complete the second stage.

Once again, the third stage follows in the same way.

Theorem 7.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain with an external normal cusp at the origin. Let $\hat\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotonic function satisfying $\hat\omega(2t) \sim \hat\omega(t)$, and $1 \leq p < \infty$.*

(a) *Consider the weighted Sobolev space $W^{k,p}_\omega(\Omega)$, where $\omega(x) = \hat\omega(d_{\partial\Omega}(x))$, $\forall x$. Suppose that $C^\infty(\bar\Omega \cap \mathbb{R}^n_+) \cap W^{k,p}_\omega(\Omega)$ is dense in $W^{k,p}_\omega(\Omega)$ (see Remark 7.6 below) and that $\omega^{-\frac{1}{p-1}}$ is locally integrable. If the spine $\mathcal{S}$ satisfies (2.12), there exists an extension operator*

$$\Lambda : W^{k,p}_\omega(\Omega) \to W^{k,p}_{\omega\sigma}(\mathbb{R}^n)$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^k & x \in \Omega^c \end{cases}$$

(b) *Consider the weighted Sobolev space $W^{k,p}_\omega(\Omega)$, where $\omega(x) = \hat\omega(|x|)$, $\forall x$. If the spine $\mathcal{S}$ satisfies (2.13), there exists an extension operator*

$$\Lambda : W^{k,p}_\omega(\Omega) \to W^{k,p}_{\omega\sigma}(\mathbb{R}^n)$$

where

$$\sigma(x) = \begin{cases} 1 & x \in \Omega \\ \left(\frac{\ell(S(|x|))}{|x|}\right)^{\frac{n-1}{p}} & x \in \Omega^c \end{cases}$$

Definition 7.1. *For $0 \leq m \leq n$, a set F is called $m-regular$, if there exists a positive constant C such that*

$$C^{-1}r^m < \mathcal{H}^m(B(x,r)) < Cr^m,$$

for all $x \in F$ and $0 < r \leq diam(F)$. Where $\mathcal{H}^m$ stands for the m dimensional Hausdorff measure and the restriction $0 < r \leq diam(F)$ is eliminated if F is a set with only one point.

Let us mention that some self similiar fractals such as the well known Koch curve are $m-regular$ with $m \notin \mathbb{N}$ (in fact $m = \log(4)/\log(3)$ in the Koch example).

Remark 7.6. *As we stated above, for a uniform domain D, a general and simple condition that guarantees the density of $C^\infty(\bar{D})$ in $W^{k,p}_\omega(D)$, is that $\omega \in A_p$ (see [7]). Under extra assumptions on the boundary of D it is possible to find conditions for which weights of the type $d(\cdot, \partial D)^\mu$ belong to A_p. Indeed that holds for $-(n - m) < \mu < (n - m)(p - 1)$ provided that ∂D is a compact set contained in an $m - regular$ set (see Durán and López García [9]). In such a case we can replicate Theorem 6.3 by using a weighted version of Proposition 6.1 and arguing along the lines given in Remark 6.2 (using Chua's results). Therefore the density assumption in item (a) of Theorem 7.5 can be removed.*

Let us observe that for a "good" domain D, one expects $m = n - 1$, therefore the range $-1 < \mu < p - 1$ is precisely the one for which the extension problem makes sense and it is non-trivial. Indeed, on the one hand if $\mu \geq p - 1$, then $\omega^{-\frac{1}{p-1}} \notin L^1_{loc}(\mathbb{R}^n)$ and the weighted global space can not be defined in the standard way. On the other, taking for instance D Lipschitz and $\mu \leq -1$ it can be shown that $C^\infty_0(D)$ is dense in $W^{k,p}_\omega(D)$ [12], and therefore functions in that space can be extended by 0.

Remark 7.7. *Let us notice that the density is not needed in item b) since it can be obtained by using the same arguments given in Theorem 6.3 and taking into account that the weight is essentially constant over every $\check{\Omega}_i$.*

Theorem 7.5 can be reproduced for *curved* cusps, by means of an easy adaptation of stage zero.

8. Examples and Concluding Remarks

Below we show that every domain satisfying Definition A is an external cusp in terms of Definition 2.3 (or Definition 2.2). On the other hand our results can be understood in the following way: the role of the "profile" function ϕ given in Definition A can be relaxed in the sense that it can just describe the speed of the narrowing of Ω towards the origin (i.e. if the *spine* of Ω decreases as ϕ: $\ell(S(z)) \sim \phi(z)$) provided that $\partial\Omega \setminus \{0\}$ remains smooth enough. Consequently, the weight σ in Theorem 5.1 can be expressed:

$$\sigma(x) = \left(\frac{\phi(|x|)}{|x|}\right)^\gamma$$

where $\gamma = k$ or $\frac{n-1}{p}$, depending on the case.

Let us notice that both extension procedures for Theorems 4.1 and 5.1 –in contrast to what happens in Theorem A– are *independent* of the relationship between k, p and n. Consequently, we proved that hypothesis (1.3) (or its generalization, (2.12)) is not necessary *in any case* as long as (1.4) (or (2.13)) stands, and vice-versa.

First, we present some domains that do not satisfy Definition A but are included in Definitions 2.2 or 2.3.

The first simple example is given by:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \quad y^3 < x < y^2\}.$$

This domain does not satisfy Definition A, however, it is easy to see that it is an *external curved cusp*.

The second example is general, and we devote a few lines to it:

Definition 8.1. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a domain with compact boundary $\partial\Omega$. Assume that $0 \in \partial\Omega$. We say that Ω has a restricted external cusp at the origin if there exists a neighborhood of 0, $U \subset \mathbb{R}^n$ such that*

$$U \cap \Omega = \{(x, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : x \in \varphi(z)\varpi\}$$

where $\varpi \subset \mathbb{R}^{n-1}$ is a bounded uniform *domain and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a Lipschitz increasing function such that* $\varphi(0) = 0$ *and* $\frac{\varphi(t)}{t} \to 0$ *($t \to 0^+$).*

It is clear that every outer peak satisfying Definition A is a *restricted* external cusp.

Claim 8.1. *Every* restricted *cusp satisfies Definition 2.3 (or 2.2).*

We sketch the proof of this claim through a series of observations:

Given Ω a restricted cusp, let us define Ω_z the set of points of Ω at height z and the boundary of this set $\partial \Omega_z := \{(x, z) \in \mathbb{R}^n : x \in \varphi(z)\partial\varpi\}$.

Observe that the distance from a point $(x, z) \in \Omega$ to $\partial\Omega$ is equivalent to its distance to $\partial\Omega_z$. Indeed, it is clear that $d((x, z), \partial\Omega) \le d((x, z), \partial\Omega_z)$. On the other hand, let us denote $x = \varphi(z)\zeta$, for some $\zeta \in \varpi$. Let $(x_0, z_0) = (\phi(z_0)\zeta_0, z_0) \in \partial\Omega$ be such that $d((x, z), \partial\Omega) = d((x, z), (x_0, z_0))$. Naturally, $\widetilde{x}_0 = (\varphi(z)\zeta_0, z)$ is in $\partial\Omega_z$. Then

$$
\begin{aligned}
d((x, z), \partial\Omega_z) \le |x - \widetilde{x}_0| = |\varphi(z)\zeta - \varphi(z)\zeta_0| &\le |\varphi(z)\zeta - \varphi(z_0)\zeta_0| + |\varphi(z_0) - \varphi(z)||\zeta_0| \\
&\le |\varphi(z)\zeta - \varphi(z_0)\zeta_0| + C_\varphi C_\varpi |z_0 - z| \le C(|\varphi(z)\zeta - \varphi(z_0)\zeta_0| + |z_0 - z|) \\
&\le Cd((x, z), (x_0, z_0)) = Cd((x, z), \partial\Omega)
\end{aligned}
$$

where C_φ is the Lipschitz constant of φ and $C_\varpi = \sup\{\|\xi\| : \xi \in \varpi\}$

Let r_ϖ be the inner radius of ϖ:

$$
r_\varpi = \sup_{x \in \varpi} \inf_{y \in \partial\varpi} d(x, y)
$$

and $c_\varpi \in \varpi$ a point such that $B(c_\varpi, r_\varpi) \subset \varpi$.

Let us consider the curve $\Gamma : \mathbb{R}_+ \to \mathbb{R}^n$, $\Gamma(t) = (\varphi(t)c_\varpi, t)$ that describes the "center" of Ω. Let $\widetilde{\mathcal{S}}$ be the set of all cubes $S \in \mathcal{W} = \mathcal{W}(\Omega)$ such that $S \cap \Gamma(t) \ne \emptyset$. Let $\mathcal{S} = \{S_i\}_{i=1}^\infty$ be a subset of $\widetilde{\mathcal{S}}$ such that $S_i \cap S_{i+1} \ne \emptyset$ and $z_{S_{i+1}} < z_{S_i}$ (this is posible because $\frac{\varphi(t)}{t} \to 0$). $\mathcal{S}$ is the spine of Ω.

Since φ is Lipschitz, we have:

$$
\varphi(z + C\varphi(z)) - \varphi(z) \le C_\varphi(z + C\varphi(z) - z) = C\varphi(z)
$$

Then:

$$
\varphi(z + C\varphi(z)) \le C\varphi(z) \tag{8.1}
$$

On the other hand $d(\Gamma(t), \partial\Omega_t) = r_\varpi \varphi(t)$, and consequently $d(\Gamma(t), \partial\Omega) \sim \varphi(t)$. Taking this into account, (8.1) implies that $\ell(S_i) \sim \varphi(z_i)$.

Properties (2.10) and (2.11) (as well as (2.3) and (2.4) when c_ϖ can be taken equal to 0) follow easily from the definition of $\mathcal{S}$. The covering property (2.5) is a consequence of (8.1). Since ϖ is a fixed bounded domain, there is a radius R_ϖ such that $\varpi \subset B(c_\varpi, R_\varpi)$. This radius scaled to the section Ω_z is $\varphi(z)R_\varpi$, but $\varphi(z)$ is essentially the length $\ell(S(z))$. Taking (8.1) into consideration, this implies that there is a constant K (depending on r_ϖ, R_ϖ and n), such that KS_i covers the slice of Ω between heights z_i and $z_i + \ell(S_i)$, $\forall i$. Thence, (2.5) follows.

The last thing to prove, then, is that uniformity properties (2.6) and (2.7) stand for every restricted cusp. We use the following result stated by Smith, Stanoyevitch and Stegenga in [16]:

Lemma 8.2. *Let Ω_1 and Ω_2 be uniform domains with finite diameters. Then $\Omega_1 \times \Omega_2$ is a uniform domain.*

Remark 8.3. *The definition of* uniform domain *used in [16] (for the proof of this lemma) is slightly different than that stated here. For the equivalence between both see [18] and [13].*

In Definition 2.2, Properties (2.6) and (2.7) are requiered for points in $\Omega_i \cup \Omega_{i+1}$. We prove that they stand in every slice between heights $z - C\varphi(z)$ and $z + C\varphi(z)$, for every fixed constant

C. Our proof is based on the following idea: since φ is Lipschitz $\Omega \cap \{(x, z) \in \mathbb{R}^n : z \in (z_0 - C\varphi(z_0), z_0 + C\varphi(z_0))\}$ is almost the cylinder:

$$\widehat{\Omega}_0 := \varphi(z_0)\varpi \times (z_0 - C\varphi(z_0), z_0 + C\varphi(z_0)) \tag{8.2}$$

which is uniform thanks to Lemma 8.2. In that lemma, the ε parameter of $\Omega_1 \times \Omega_2$ depends on the respective values of the parameters of Ω_1 and Ω_2 and on the quotient $\frac{diam(\Omega_1)}{diam(\Omega_2)}$. Since in (8.2) $diam(\Omega_1) \sim diam(\Omega_2)$, we may assume that the same ε stands for the cylinder for every z_0.

Let $z_0 > 0$ be a fixed number and C_0 a constant such that $C_0 < \frac{z_0}{\varphi(z_0)}$. Observe that since $\frac{t}{\varphi(t)} \to \infty$ as $t \to 0$, the constant C_0 chosen for a certain z_0 remains useful for every $z < z_0$. Let us denote:

$$\Omega_0 = \Omega \cap \{(x, z) \in \Omega : z_0 - C_0\varphi(z_0) < z < z_0 + C_0\varphi(z_0)\}.$$

We want to prove that Ω_0 is uniform. We associate points in $\widehat{\Omega}_0$ with points in Ω_0 at the same heights, so we denote $(\widehat{x}, z)$ the points in $\widehat{\Omega}_0$ and (x, z) those in Ω_0. Let $F : \widehat{\Omega}_0 \to \Omega_0$ be the function:

$$F(\widehat{x}, z) = \left(\frac{\varphi(z)}{\varphi(z_0)}\widehat{x}, z\right).$$

Suposse that $\zeta \in \varpi$ is such that $\varphi(z_0)\zeta = \widehat{x}$. Then $x = \frac{\varphi(z)}{\varphi(z_0)}\widehat{x} = \varphi(z)\zeta$, and $F(\widehat{x}, z) = (x, z) \in \Omega_0$. F is obviously bijective, with

$$F^{-1}(x, z) = \left(\frac{\varphi(z_0)}{\varphi(z)}x, z\right).$$

Now we prove that both F and F^{-1} are Lipschitz with constants independent of z_0 (this, in turn, shows that Ω_0 is uniform). We show only the case F^{-1} since proof for F is similar. Let us consider $(x, z), (y, w) \in \Omega_0$, $x = \varphi(z)\zeta$, $y = \varphi(w)\xi$ for some $\zeta, \xi \in \varpi$.

$$|F^{-1}(x, z) - F^{-1}(y, w)| = |(\varphi(z_0)\zeta - \varphi(z_0)\xi, z - w)| \leq \underbrace{|\varphi(z_0)\zeta - \varphi(z_0)\xi|}_{I} + \underbrace{|x - w|}_{II}$$

And

$$I \leq \varphi(z_0)\left|\frac{\varphi(z)}{\varphi(z)}\zeta - \frac{\varphi(w)}{\varphi(w)}\xi\right| = \varphi(z_0)\left|\frac{\varphi(w)x - \varphi(z)y}{\varphi(z)\varphi(w)}\right|$$

Since $z, w \in (z_0 - C\varphi(z_0), z_0 + C\varphi(z_0))$, Equation (8.1) implies that $\varphi(z_0) \sim \varphi(z)$ so:

$$I \leq C\left|\frac{\varphi(w)x - \varphi(z)y}{\varphi(w)}\right|$$

On the other hand:

$$|\varphi(w)x - \varphi(z)y| \leq |\varphi(w)x - \varphi(w)y| + |\varphi(w)y - \varphi(z)y| \leq \varphi(w)|x - y| + |\varphi(w) - \varphi(z)||y|$$
$$\leq \varphi(w)|x - y| + C_\varphi|w - z||\varphi(w)\xi| \leq C_\varphi C_\varpi\varphi(w)\{|x - y| + |w - z|\}$$

Hence: $I \leq C\{|x - y| + |w - z|\}$, and consequently

$$|F^{-1}(x, z) - F^{-1}(y, w)| \leq C\{|x - y| + |w - z|\} \leq C|(x, z) - (y, w)|$$

So, F^{-1} is Lipschitz with a Lipschitz constant depending only on the constants C_0, C_φ and C_ϖ.

Remark 8.4. *We do not really need Ω_0 to be* uniform *as a separate domain (with its* floor *and its* roof *as parts of the boundary): we just need to prove that the curve joining two points in Ω_0 satisfy property (2.7), which is given in terms of the distance to the boundary of Ω. But $d_{\partial\Omega}(x, z) \geq d_{\partial\Omega_0}(x, z)$, $\forall(x, z) \in \Omega_0$, so (2.7) stands.*

This complete the proof of Claim 8.1. Since the class of domains given by Definition 8.1 is broader than that of Definition A, we can state the following:

Corolary 8.5. *Every domain satisfying Definition A is an external cusp in terms of Definition 2.3 (or Definition 2.2).*

Finally, let us observe that items (c) in Theorems 4.1 and 5.1 follow by using the same arguments given in [14] p. 274 and p. 295, with $\ell(S(z))$ in the role of $\varphi(z)$.

References

[1] G. Acosta, R. Durán, A. Lombardi, *Weighted Poincaré and Korn inequalities for Höder α domains*, Math. Meth. Appl. Sci. (MMAS) 29 (4), pp. 387-400, 2006.

[2] G. Acosta, R. Durán, F. López García, *Korn inequality and divergence operator: counterexamples and optimality of weighted estimates.* Proc. Amer. Math. Soc., posted on May 22, 2012, PII S 0002 9939(2012) 11408 X, (to appear in print).

[3] G. Acosta, R.G. Durán, G. Armentano, A. Lombardi, *Finite element approximations in a non-Lipschitz domain.* SIAM J. Numer. Anal. 45, 277-295, 2007.

[4] G. Acosta, G. Armentano, *Finite element approximations in a non-Lipschitz domain: Part II* Math. Comp., 80, 1949-1978, 2011.

[5] R. A. Adams, J.F. Fournier, *Sobolev Spaces*, 2nd. edition, Academic Press, 2003.

[6] A.P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Symposia in Pure Mathematics, Vol. 4, Amer. Math. Soc., Providence, R.I., 1968.

[7] S.K. Chua, *Extension Theorems on Weighted Sobolev Spaces*, Indiana University Mathematics Journal, Vol 41 No. 4 (1992)

[8] S.K. Chua, *Some remarks on extension theorems for weighted Sobolev spaces*, Illinois Journal of Mathematics, Vol. 38, No. 1, Spring 1994. 95-126.

[9] R. Durán, F. López García, *Solutions of the divergence and analysis of the Stokes equations in planar domains*, Math. Mod. Meth. Appl. Sci. bf 20(1), pp. 95-120, 2010.

[10] R. Durán, F. López García, *Solutions of the divergence and Korn inequalities on domains with an external cusp*, Ann. Acad. Sci. Fen. Math. 35:2 (2010), 421-438.

[11] P. W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. 147 (1981), 71-88.

[12] A. Kufner, *Weighted Sobolev Spaces*, John Wiley & Sons, First Ed. 1985

[13] O. Martio, *Definitions for uniform domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. 4, (1980), 197-205.

[14] V. Maz'ya, S. Poborchi, *Differentiable functions on bad domains*, World Scientific Publishing 1998

[15] V. Maz'ya, *Sobolev Spaces: with Aplications to Elliptic Partial Differential Equations*, 2nd. augmented edition, Springer, 2011.

[16] W. Smith, A. Stanoyevitch, D. Stegenga, *Smooth approximation of Sobolev funcions on planar domains*, J. London Math. Soc. (1994) 49 (2): 309-330.

[17] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton 1970.

[18] J. Väisälä, *Uniform domains*, Tôhoku Math. J. 40,1, 1988, 101-118.

Departamento de Matemática Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires,, Pabellón I, Ciudad Universitaria, (1428) Buenos Aires, Argentina

E-mail address: gacosta@dm.uba.ar

E-mail address: iojea@dm.uba.ar