INTERPOLATION ERROR ESTIMATES IN $W^{1,p}$ FOR DEGENERATE Q_1 ISOPARAMETRIC ELEMENTS

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Abstract. Optimal order error estimates in H^1, for the Q_1 isoparametric interpolation, were obtained in [2] for a very general class of degenerate convex quadrilateral elements. In this work we show that the same conlusions are valid in $W^{1,p}$ for $1 \leq p < 3$ and we give a counterexample for the case $p \geq 3$, showing that the result can not be generalized for more regular functions. Despite this fact, we show that optimal order error estimates are valid for any $p \geq 1$, keeping the interior angles of the element bounded away from 0 and π, independently of the aspect ratio. We also show that the restriction on the maximum angle is sharp for $p \geq 3$.

Keywords. Lagrange interpolation, isoparametric finite elements, quadrilateral elements, anisotropic elements, maximum angle condition.

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1. Introduction

Interpolation error estimates for finite elements play an important role in most of the classical convergence arguments used in the finite element literature. For convex quadrilateral elements and $1 \leq p$, the usual $W^{1,p}$ error estimate for the Q_1 isoparametric Lagrange interpolation, called hereafter Q, reads

$$\|u - Qu\|_{L^p(K)} + h|u - Qu|_{W^{1,p}(K)} \leq Ch^2|u|_{W^{2,p}(K)} \tag{1.1}$$

where h denotes the diameter of K.

Two facts about (1.1) are well known: the convexity of K is a sufficient condition to get the estimate

$$\|u - Qu\|_{L^p(K)} \leq Ch^2|u|_{W^{2,p}(K)} \tag{1.2}$$

with C bounded independently of the shape of K, however

$$|u - Qu|_{W^{1,p}(K)} \leq Ch|u|_{W^{2,p}(K)} \tag{1.3}$$

requires extra assumptions on K to keep C uniformly bounded.

For the sake of completness, we will provide a sketched proof of (1.2) for a general convex element K, while the main purpose of this article is to deal with (1.3) looking at the behaviour of C on both p and the shape of K.

Available results for (1.3) go back to the early work by Ciarlet and Raviart [7], where the authors show that if

$$h/\underline{h} \leq \mu_1 \tag{1.4}$$

where $\underline{h}$ is the length of the shortest side of K, and

$$|\cos\theta| \leq \mu_2 < 1 \tag{1.5}$$

for each angle θ of K, then (1.3) holds with a constant C depending only on μ_1 and μ_2.

Obviously (1.5) keeps the interior angles of K bounded away from 0, and π, and it is not difficult to see that (1.5) together with (1.4) implies the so called *regularity condition*, i.e., the existence of a constant σ such that

$$h/\rho \leq \sigma \tag{1.6}$$

where ρ denotes the diameter of the maximum circle contained in K (see for instance [6]). Under this condition, the quadrilateral can degenerate into a triangle (i.e. its maximum angle might approach π) but it can not become too narrow. In [9] Jamet improved the results of [7] proving that the error estimate (1.3) holds with a constant depending only on σ.

Anisotropic finite elements, i.e., elements for which (1.6) does not hold, have deserved much attention since the works [5, 8], where it is shown that (1.6) is not neccesary to get (1.3) for triangular elements if the maximum angle of the triangle is bounded away from π. Since then, the latter property has been called the *maximum angle condition.* For quadrilateral elements we will use the same standard definition.

Definition 1.1. *A quadrilateral or a triangle K verifies the* maximum angle condition *with constant $\psi_M < \pi$, or shortly $MAC(\psi_M)$, if the interior angles of K are less than ψ_M.*

Since for quadrilateral elements the convexity is equivalent to have the maximum angle bounded by π, $MAC(\psi_M)$ can be seen as natural step in order to strengthen the convexity assumption. For $p = 2$ it is shown in [2] that $MAC(\psi_M)$ implies (1.3), moreover the same result is also proved there for the broader class of convex quadrilaterals given by the *regular decomposition property* (see [2]) which reads as follows:

Definition 1.2. *Let K be a convex quadrilateral, and let d_1 and d_2 be the diagonals of K. We say that K satisfies the* regular decomposition property *with constants $N \in \mathbb{R}$ and $0 < \psi_M < \pi$, or shortly $RDP(N, \psi_M)$, if we can divide K into two triangles along one of its diagonals, that will be called always d_1, in such a way that $|d_2|/|d_1| \leq N$ and both triangles satisfy $MAC(\psi_M)$.*

This condition is, as far as we know, the more general one under wich (1.3) holds for $p = 2$. A natural question is whether it can be generalized for different values of p. We will give an affirmative answer for p in the range $1 \leq p < 3$ showing also, by means of an appropriate counterexample, that this result is not longer true for $3 \leq p$.

A large amount of geometrical conditions on quadrilateral elements, besides those mentioned above, have been introduced in the literature. We refer the reader to [2, 4, 10] where many of them are recalled and compared.

For parallelograms it can be easily shown that $MAC(\psi_M)$ implies (1.3) for $1 \leq p$. Moreover, the following sharp version of (1.3), called "anisotropic estimate", holds for rectangles (see [4])

$$|u - Qu|_{W^{1,p}(K)} \leq C\Big\{h_1\Big\|\frac{\partial}{\partial x_1}\nabla u\Big\|_{L^p(K)} + h_2\Big\|\frac{\partial}{\partial x_2}\nabla u\Big\|_{L^p(K)}\Big\} \tag{1.7}$$

where h_1 and h_2 denote the size of the element in the directions x_1 and x_2. This estimate can be extended to parallelograms under the $MAC(\psi_M)$, taking the derivatives in (1.7) along the sides of the element. Also in [4] the same result is extended to some class of subparametric elements which can be seen as the image of rectangles under certain small perturbations of linear mappings. Let us notice the elementary fact that for parallelograms $MAC(\psi)$ implies uniform bounds also for the minimum angle of K. For this reason we introduce the following definition

Definition 1.3. *We say that a quadrilateral K satisfies the* double angle condition *with constants ψ_m, ψ_M, or shortly $DAC(\psi_m, \psi_M)$, if the interior angles θ of K verify $0 < \psi_m \leq \theta \leq \psi_M < \pi$.*

Remark 1.1. *Naturally, $DAC(\psi_m, \psi_M)$ is equivalent to (1.5). We prefer to introduce the previous definition in order to stress its geometrical meaning.*

One of our results shows that (1.3) holds for elements verifying $DAC(\psi_m, \psi_M)$ and any $1 \leq p$. To our best knowledge there are no available results stating this property in such plain geometrical terms. Moreover, we show that even when the condition on the *maximum* angle is not neccesary for $p = 2$ (see for instance [9]) it can not be relaxed for $3 \leq p$.

Note that the following elementary implications hold:

$$DAC(\psi_m, \psi_M) \Rightarrow MAC(\psi_M) \Rightarrow RDP(N, \psi_M)$$

(to prove the second implication let us notice that if K satisfies $MAC(\psi_M)$ then, dividing K by its longest diagonal, we obviously have that it also satisfies $RDP(1, \psi_M)$). The reciprocal implications are false as shown in Figure 1 a) and b) taking $s \to 0$ and $s \to 1/2$ respectively.

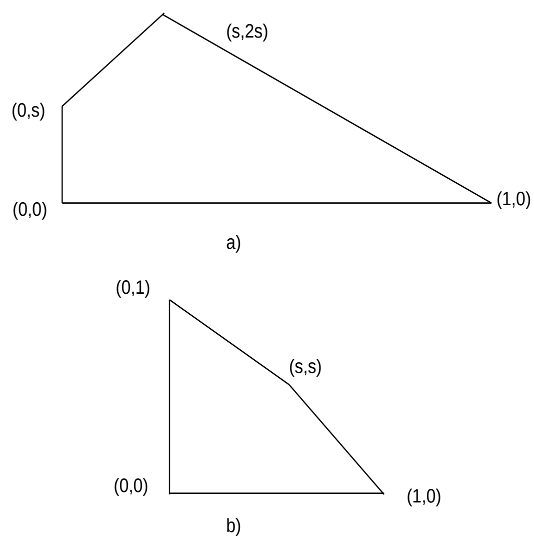

Figure 1

The condition $DAC(\psi_m, \psi_M)$ is easier to handle than $RDP(\psi_M, N)$. We have tried to keep the article more readable dealing with both conditions in separate sections. Those results which apply in both cases are treated at the same time in order to avoid repetitions.

The article is organized as follows: in Section 2, and following closely [2], we present a general family of "reference" elements introducing some useful bounds for the subsequent sections and showing how to decompose the error by means of the linear Lagrange interpolation, in Section 3 and Section 4 we construct the changes of variable which allow us to work in the family of reference for elements verifying $RDP(\psi_M, N)$ and $DAC(\psi_m, \psi_M)$ respectively. Section 5 is devoted to bound some terms given by the error decomposition and finally in the last section we present the main results and some counterexamples.

2. The family of reference elements $K(a, b, \tilde{a}, \tilde{b})$

We will use the following notation. For a general convex quadrilateral K, M_i with $1 \leq i \leq 4$, will denote its vertices in anticlockwise order. If one vertex is placed at the origin we will use M_1 to denote it. Given $a, b, \tilde{a}, \tilde{b} > 0$, $K(a, b, \tilde{a}, \tilde{b})$ will represent a convex quadrilateral with vertices $M_1 = (0, 0)$, $M_2 = (a, 0)$, $M_3 = (\tilde{a}, \tilde{b})$ and $M_4 = (0, b)$. In particular $\hat{K} = K(1, 1, 1, 1)$ is the reference unit square and its vertices will be denoted with $\hat{M}_i$. For a general convex $K(a, b, \tilde{a}, \tilde{b})$

we will denote with d_1 the diagonal joining the vertices M_2 and M_4, and with d_2 the remaining diagonal. We will call T_1 and T_2 the triangles which lie respectively above and below d_1, and α the angle between d_1 and the segment l joining M_3 and M_4.

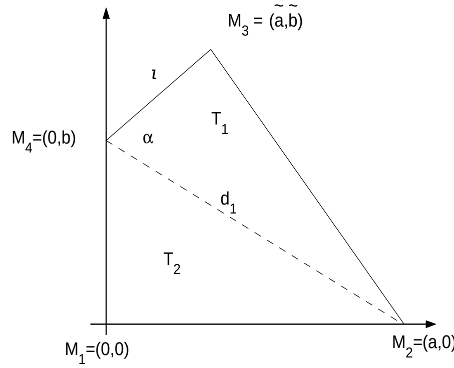

Figure 2

We will use the variable $\hat{x}$ on $\hat{K}$ and x on K, and when there is no danger of confusion we will also use $(x, y) \in K$, $(\hat{x}, \hat{y}) \in \hat{K}$. In order to define the isoparametric elements on K, let $F_K : \hat{K} \to K$ be the transformation $F_K(\hat{x}) = \sum_{i=1}^{4} M_i \hat{\phi}_i(\hat{x})$, where $\hat{\phi}_i$ is the bilinear basis function associated with the vertex $\hat{M}_i$, i.e., $\hat{\phi}_i(\hat{M}_j) = \delta_i^j$. Now, the basis functions on K, no longer bilinear in general, are defined by $\phi_i(x) = \hat{\phi}_i(F_K^{-1}(x))$ and the Q_1 isoparametric interpolation operator Q on K is defined by

$$Qu(x) = \hat{Q}\hat{u}(\hat{x})$$

where $x = F_K(\hat{x})$ and $\hat{Q}$ is the bilinear Lagrange interpolation of $\hat{u} = u \circ F_K$ on $\hat{K}$.

In order to bound the interpolation error for a given K, we will construct, in the following sections, an affine transformation L taking K into one element of the type $K(a, b, \tilde{a}, \tilde{b})$. The following elementary lemma provides some conditions on L which allow us to compare the error on both elements.

Lemma 2.1. *Let $K, \overline{K}$ be two quadrilateral elements, and let $L : K \to \overline{K}$ be an* **affine** *transformation $L(x) = Bx + p$. Assume that $L(K) = \overline{K}$, $\kappa(B) \le C$ and $det(B) = 1$, where κ is the condition number of B. If $\overline{Q}$ is the isoparametric interpolation on $\overline{K}$ and $\overline{u} = u \circ L$ then there are positive constants C_1 and C_2 depending only on C such that for any $1 \le p$*

$$C_1|\overline{u} - \overline{Q}\overline{u}|_{W^{1,p}(\overline{K})} \le |u - Qu|_{W^{1,p}(K)} \le C_2|\overline{u} - \overline{Q}\overline{u}|_{W^{1,p}(\overline{K})}$$

and,

$$C_1|\overline{u}|_{W^{2,p}(\overline{K})} \le |u|_{W^{2,p}(K)} \le C_2|\overline{u}|_{W^{2,p}(\overline{K})}.$$

Proof. By definition of the isoparametric interpolation we have

$$Qu(x) = \hat{Q}\hat{u}(F_K^{-1}(x)) \qquad \text{and,} \qquad \overline{Q}\overline{u}(\overline{x}) = \hat{Q}\hat{\overline{u}}(F_{\overline{K}}^{-1}(\overline{x})).$$

Where $\overline{x}$ denotes the variable on $\overline{K}$. Since L is an affine transformation, $F_K = L \circ F_{\overline{K}}$ and so $\overline{Q}\overline{u}(\overline{x}) = Qu(x)$. Then, the lemma follows easily by observing that $\|B\|, \|B^{-1}\| < C$ and the fact that $\det B = 1$. □

Assuming that the previous transformation can be performed (a fact that will be proved later) we can suppose that our element belongs to the class $K = K(a, b, \tilde{a}, \tilde{b})$ and following [2, 9], we can decompose the error in the following way:

$$|u - Qu|_{W^{1,p}(K)} \leq |u - \Pi u|_{W^{1,p}(K)} + |\Pi u - Qu|_{W^{1,p}(K)}$$

where Π is the P_1-Lagrange interpolation operator associated with the vertices $M_1 = (0,0)$, $M_2 = (a, 0)$ and $M_4 = (0, b)$ (i.e., Πu is an affine function which agrees with u on these three vertices). Moreover, since $\Pi u - Qu$ belongs to the finite element isoparametric space and vanishes at M_1, M_2 and M_4, it follows that

$$(\Pi u - Qu)(x) = (\Pi u - u)(M_3)\phi_3(x)$$

(recall that ϕ_3 is the basis function corresponding to M_3), therefore,

$$|u - Qu|_{W^{1,p}(K)} \leq |u - \Pi u|_{W^{1,p}(K)} + |(\Pi u - u)(M_3)||\phi_3|_{W^{1,p}(K)} \tag{2.1}$$

and so, it is enough to estimate the two terms on the right hand side. This fact, toghether with the construction of the affine transformation L, is the object of the rest of the present paper.

Before we finish this section, we present some general bounds for $|\phi_3|_{W^{1,p}(K)}$.

We start by analyzing the Jacobian of the the transformation $F_K : [0, 1]^2 = \hat{K} \to K$ defined as

$$F_K(\hat{x}, \hat{y}) = (a\hat{x}(1 - \hat{y}) + \tilde{a}\hat{x}\hat{y}\, , b\hat{y}(1 - \hat{x}) + \tilde{b}\hat{x}\hat{y}) = (x, y). \tag{2.2}$$

We have

$$DF_K(\hat{x}, \hat{y}) = \begin{pmatrix} a + \hat{y}(\tilde{a} - a) & \hat{x}(\tilde{a} - a) \\ \hat{y}(\tilde{b} - b) & b + \hat{x}(\tilde{b} - b) \end{pmatrix}$$

and,

$$J_K := \det DF_K(\hat{x}, \hat{y}) = ab(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1)). \tag{2.3}$$

Observe that since K is convex, we have $J_K > 0$ in the interior of $\hat{K}$. Indeed, since J_K is an affine function it is enough to verify that it is positive at some vertex of $\hat{K}$ and non negative at the remaining ones. The positivity at $\hat{M}_1 = (0, 0)$ is trivial, as well as the non negativity at $\hat{M}_2$ and $\hat{M}_4$. On the other hand, since K is convex, $(\tilde{a}, \tilde{b})$ lies above the segment joining M_2 and M_4 and therefore, it follows that

$$J_K(1, 1) = ab(\tilde{b}/b + \tilde{a}/a - 1) \geq 0. \tag{2.4}$$

Now, since $\frac{\partial \hat{\phi}_3}{\partial \hat{x}} = \hat{y}$ and $\frac{\partial \hat{\phi}_3}{\partial \hat{y}} = \hat{x}$, a simple computation yields

$$\left(\frac{\partial \phi_3}{\partial x} \circ F_K\right)(\hat{x}, \hat{y}) = b\hat{y}/J_K(\hat{x}, \hat{y})$$

and,

$$\left(\frac{\partial \phi_3}{\partial y} \circ F_K\right)(\hat{x}, \hat{y}) = a\hat{x}/J_K(\hat{x}, \hat{y}).$$

Therefore, using (2.3), and making a change of variables, we have

$$\left\|\frac{\partial \phi_3}{\partial x}\right\|^p_{L^p(K)} = \int_0^1 \int_0^1 \frac{b\hat{y}^p}{a^{p-1}(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}}\, d\hat{x}\, d\hat{y} \tag{2.5}$$

and,

$$\left\|\frac{\partial\phi_3}{\partial y}\right\|^p_{L^p(K)} = \int_0^1\int_0^1 \frac{a\hat{x}^p}{b^{p-1}(1+\hat{x}(\tilde{b}/b-1)+\hat{y}(\tilde{a}/a-1))^{p-1}}\,d\hat{x}\,d\hat{y}. \tag{2.6}$$

Hence, calling

$$I_p(a,b,\tilde{a},\tilde{b}) := \int_0^1\int_0^1 \frac{1}{(1+\hat{x}(\tilde{b}/b-1)+\hat{y}(\tilde{a}/a-1))^{p-1}}\,d\hat{x}\,d\hat{y} \tag{2.7}$$

we get

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \le \frac{b}{a^{p-1}} I_p(a,b,\tilde{a},\tilde{b}) \tag{2.8}$$

and,

$$\left\|\frac{\partial\phi_3}{\partial y}\right\|^p_{L^p(K)} \le \frac{a}{b^{p-1}} I_p(a,b,\tilde{a},\tilde{b}). \tag{2.9}$$

Thus, to control $|\phi_3|_{W^{1,p}(K)}$ we need to bound $I_p(a,b,\tilde{a},\tilde{b})$ for positive a, b, $\tilde{a}$ and $\tilde{b}$ satisfying (2.4). This will be achieved later taking into account the geometrical conditions of K.

3. Implications of the $RDP(N,\psi_M)$

Error estimates in H^1 for elements K verifying the $RDP(N,\psi_M)$ are handled in [2] by constructing an **affine** transformation $L(\overline{x}) = B\overline{x}+P$ and a "reference" element $K(a,b,\tilde{a},\tilde{b})$ in such a way that $L(K(a,b,\tilde{a},\tilde{b})) = K$. In the next Lemma, we summarize the main results involving L with the notation given in Figure 2, and refer the reader to [2] for the proof.

Lemma 3.1. *Let K be a convex quadrilateral verifying the $RDP(N,\psi_M)$. Then, there exist an affine transformation $L(\overline{x}) = B\overline{x}+P$, a convex $K(a,b,\tilde{a},\tilde{b})$ given by positive numbers $a,b,\tilde{a},\tilde{b}$, and positive constants $C = C(\psi_M,N)$, $\overline{N} = \overline{N}(\psi_M,N)$, $\overline{\delta} = \overline{\delta}(\psi_M,N)$ and $\overline{\psi_M} = \overline{\psi_M}(\psi_M,N) < \pi$ such that*

a) $L(K(a,b,\tilde{a},\tilde{b})) = K$;
b) $\|B\|, \|B^{-1}\| < C$, $det(B) = 1$;
c) *The diameter of both elements are comparable, i.e.,*

$$C^{-1}diam(K) \le diam(K(a,b,\tilde{a},\tilde{b})) \le C diam(K);$$

d) $K(a,b,\tilde{a},\tilde{b})$, *satisfies* $RDP(\overline{\psi_M},\overline{N})$ *taking* $d_1 = \overline{M_2M_4}$ *as the dividing diagonal;*
e) *The length of the side* $l = \overline{M_3M_4}$ *is comparable to the length of the shortest side* s *of* $K(a,b,\tilde{a},\tilde{b})$*. Moreover*

$$|s| \le |l| \le C^2|s|;$$

f) *The angle* α *is bounded away from* 0 *and* π*, indeed,* $0 < \overline{\delta} < \alpha < \overline{\psi_M} < \pi$.

Proof. See Lemma 3.3 of [2]. □

Remark 3.1. *In view of the preceding lemma, and Lemma 2.1, whenever we deal with a convex quadrilateral K of diameter h satisfying the $RDP(N,\psi_M)$ we will assume that it is of the type $K(a,b,\tilde{a},\tilde{b})$. Moreover, we will assume the existence of positive constants $N_1(\psi_M,N)$, $N_2(\psi_M,N)$, $N_3(\psi_M,N)$ and $\overline{\psi} < \pi$ such that:*

$(H1)$ $|d_2|/|d_1| \le N_1$;

$(H2)$ $1/\sin\alpha \le N_2$;

$(H3)$ $|l| \le N_3|s|$ *where s is the shortest side of $K(a,b,\tilde{a},\tilde{b})$;*

(H4) $\theta \le \overline{\psi}$, *for all angle* θ *of* T_1 *and* T_2.

Indeed, $(H1)$ *and* $(H4)$ *follow immediately from d) of Lemma 3.1,* $(H2)$ *follows from f), and* $(H3)$ *from e).*

Some useful bounds can be derived from $(H1)$, $(H2)$, $(H3)$ and $(H4)$.

Lemma 3.2. *Let* $K(a, b, \tilde{a}, \tilde{b})$ *be a quadrilateral satisfying (H1), (H2), (H3) and (H4), then*

$$\tilde{a}/a \le N_3 < N_4 \qquad \text{and,} \qquad \tilde{b}/b \le N_4 \tag{3.1}$$

where $N_4 = N_3 + 1$

$$h \le N_3(1 + N_1)|d_1| \tag{3.2}$$

and,

$$\frac{|K|}{|T_2|} \le N_5, \tag{3.3}$$

where $N_5 = 2N_4 + 1$.

Proof. We have, $\tilde{a}/a \le |l|/a$ and $|\tilde{b} - b|/b \le |l|/b$ and so (3.1) follows from (H3).

In order to prove (3.2), let us observe that if h is the length of one of the diagonals, then, it follows from (H1) that $h \le \max\{1, N_1\}|d_1|$ and so (3.2) holds because $N_3 \ge 1$. Otherwise, h agrees with the length of one of the sides of K and so $h = |l|$ or $h = |l_{23}|$, where l_{23} is the side joining M_2 and M_3, because the lengths of the other two sides are bounded by $|d_1|$. Now, in view of H3, $|l| \le N_3|l_{23}|$ and therefore, it is enough to see that $|l_{23}| \le (1 + N_1)|d_1|$. But, from the triangle inequality, $|l_{23}| \le a + |d_2| \le |d_1| + |d_2| \le (1 + N_1)|d_1|$ and therefore (3.2) holds. Finally, (3.3) follows easily from (3.1). □

The following lemma can also be found in [[2], Lemma 4.5]. We reproduce its proof for the sake of completness. To simplify notation we introduce $N_6 = \max\{N_2, 1/\sin((\pi + \overline{\psi})/2)\}$.

Lemma 3.3. *If* $K = K(a, b, \tilde{a}, \tilde{b})$ *is convex and satisfies the hypotheses* (H1), (H2), (H3), (H4) *then,*

1. $\max\{|l|/a, |l|/b\} \le N_2(\tilde{b}/b + \tilde{a}/a - 1)$;
2. *If* $\tilde{b}/b \le 1$ *then* $a/b \le N_2$;
3. *If* $\tilde{b}/b \le 1/2$ *and* $\tilde{a}/a > 1$ *then* $b/a \le 2N_3$;
4. *If* $\tilde{b}/b > 1$ *and* $a/b \le tg((\pi - \overline{\psi})/2)$ *then* $|l| \le N_6\tilde{a}$;
5. *If* $\tilde{b}/b > 1$ *and* $b/a \le tg(\alpha/2)$ *then* $|l| \le 2N_2(\tilde{b} - b)$;
6. $\min\{1/|\tilde{a} - a|, 1/\tilde{b}\} \le \sqrt{2}N_3/|l|$;
7. $\min\{1/|\tilde{b} - b|, 1/\tilde{a}\} \le \sqrt{2}/|l|$.

Proof. *1)* Calling $y(x) = -(b/a)(x - a)$ the equation of the straight line passing through M_2 and M_4 and calling y^{-1} its inverse, we have

$$\frac{\tilde{b} - y(\tilde{a})}{b} = \frac{\tilde{a} - y^{-1}(\tilde{b})}{a} = \frac{\tilde{b}}{b} + \frac{\tilde{a}}{a} - 1. \tag{3.4}$$

Now, an elementary geometrical analysis yields $|l| \sin\alpha \le \tilde{b} - y(\tilde{a})$ and $|l| \sin\alpha \le \tilde{a} - y^{-1}(\tilde{b})$ and therefore, *1)* follows from (3.4) and (H2).

2) Calling β the angle between d_1 and the segment joining M_4 and (a, b), we have that, since $\tilde{b} \le b$, then $\alpha \le \beta$. So, $b/a = \text{tg}\beta \ge \text{tg}\alpha$ and therefore, using again (H2) we obtain *2)*.

3) Under these hypotheses we have that $|l| \ge b - \tilde{b} \ge b/2$. Then $b/a \le 2|l|/a$ and therefore, *3)* follows from (H3).

4) Calling γ the angle between d_1 and the segment joining M_1 and M_4 it is easy to see that, under these hypotheses, $\gamma \leq (\pi - \overline{\psi})/2$. Now, using (H4) we have

$$\alpha \leq \alpha + \gamma \leq (\pi + \overline{\psi})/2$$

and, since $|l|/\tilde{a} = 1/\sin(\alpha+\gamma)$, *4)* follows from (H2) and the fact that $\overline{\psi} < \pi$.

5) With β as in *2)*, we have in this case that $\beta \leq \alpha$. Then, $\mathrm{tg}\beta = b/a \leq \mathrm{tg}(\alpha/2)$ and so, $\beta \leq \alpha/2$. Therefore, $\alpha - \beta \geq \alpha/2$ and *5)* follows by observing that

$$\tilde{b} - b = |l|\sin(\alpha - \beta) \geq |l|\sin(\alpha/2) \geq |l|(\sin\alpha)/2.$$

6) Let us call η the interior angle of K at the vertex M_2 and l_{23} the side joining M_2 and M_3. Then, we have

$$|\tilde{a} - a|/|l_{23}| = \cos\eta \qquad \text{and,} \qquad \tilde{b}/|l_{23}| = \sin\eta$$

and so, *6)* follows from (H3).

7) Follows by an analysis similar to that given in *6)*. □

Before finishing this section we focus on some partial bounds for $|\phi_3|_{W^{1,p}(K)}$, for $1 \leq p < 3$. We start with the following remark which tells us that bounds for $1 \leq p < 2$ can be derived from the case $p = 2$ already obtained in [2].

Remark 3.2. *In Lemma 4.6 of* [2] *it is shown that*

$$|\phi_3|^2_{W^{1,2}(K)} \leq C(\psi_M, N)\frac{h}{l} \tag{3.5}$$

for any convex K satisfying $(H_1), (H_2), (H_3)$, and (H_4) (this result also follows as a particular case of Lemma 3.5 below).

This inequality allows us to bound $|\phi_3|_{W^{1,p}(K)}$ easily if $1 \leq p < 2$. Indeed, consider for instance $\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)}$. Taking $\hat{p} = \frac{1}{p-1} > 1$ and applying Hölder's inequality in the rigth hand side of (2.5), we have

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \leq \left(\int_0^1\int_0^1 \frac{b^{\frac{1}{p-1}}\hat{y}^{\frac{p}{p-1}}}{a(1+\hat{x}(\tilde{b}/b-1)+\hat{y}(\tilde{a}/a-1))}\,d\hat{x}\,d\hat{y}\right)^{p-1}$$

and, taking into account that $\hat{y} \leq 1$ and $2 \leq \frac{p}{p-1}$,

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \leq b^{2-p}\left\|\frac{\partial\phi_3}{\partial x}\right\|^{2(p-1)}_{L^2(K)}.$$

Using now (3.5), and the facts that $p < 2$ and $b \leq h$, we get

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \leq C(\psi, N)^{p-1}\frac{h}{l^{p-1}}.$$

Analougous arguments for $\left\|\frac{\partial\phi_3}{\partial y}\right\|^p_{L^p(K)}$ show, for any $1 \leq p \leq 2$, that

$$|\phi_3|_{W^{1,p}(K)} \leq C(\psi_M, N)^{1/q}\frac{h^{1/p}}{l^{1/q}} \tag{3.6}$$

where q is the dual exponent of p. We will show that, under the same hypotheses on K, (3.6) holds for $1 \leq p < 3$, even when it can not be deduced from (3.5).

Remark 3.3. *For any* $2 < p < 3$, *and* $0 < t$ *we define* $\eta_p(t) = \frac{t^{3-p}-t}{(p-2)(1-t)}$. *Taking the limit as* $p \to 2$ *we will write* $\eta_2(t) = \frac{t\ln(t)}{t-1}$.

An easy computation shows that η_p *is an increasing function of* t *for any* $2 \le p < 3$, *and*

$$0 \le \eta_p(t) \le \max\{1, t\}. \tag{3.7}$$

Since the bound of $|\phi_3|_{W^{1,p}(K)}$ relies on I_p (see (2.8), (2.9)), we devote the next Lemma to the study of I_p. In view of Remark 3.2, we will assume $2 \le p$.

Lemma 3.4. *Let* $2 \le p < 3$ *and* $a, b, \tilde{a}, \tilde{b} > 0$ *such that* $\tilde{b}/b + \tilde{a}/a - 1 > 0$. *Then, the integral* $I_p = I_p(a, b, \tilde{a}, \tilde{b})$ *defined in (2.7) satisfies*

(1) *If* $\tilde{b}/b \le 1$ *and* $\tilde{a}/a \le 1$ *then,* $I_p \le \frac{1}{(\tilde{b}/b+\tilde{a}/a-1)^{p-1}}$;

(2) *If* $\tilde{b}/b \le 1$ *and* $\tilde{a}/a > 1$ *then,* $I_p \le \frac{1}{3-p} \min\{\frac{\mu_p(\tilde{a}/a)}{(1-\tilde{b}/b)(\tilde{a}/a-1)}, \frac{b}{\tilde{b}}\}$;

(3) *If* $\tilde{b}/b > 1$ *and* $\tilde{a}/a \le 1$ *then,* $I_p \le \frac{1}{3-p} \min\{\frac{\mu_p(\tilde{b}/b)}{(1-\tilde{b}/b)(\tilde{a}/a-1)}, \frac{a}{\tilde{a}}\}$;

(4) *If* $\tilde{b}/b > 1$ *and* $\tilde{a}/a > 1$ *then,* $I_p \le 1$;

where in (2) *it is understood that the minimum is* $b/\tilde{b}$ *when* $\tilde{b}/b = 1$ *and analogously in* (3). *Finally,* $\mu_p(t) = (t-1)\eta_p(t) + 1$, *with* $1 \le t$ *and* η_p *defined in Remark 3.3.*

Observe that, thanks to (3.7), μ_p *can be easily bounded in terms of* t.

Proof. *1)* Since $0 \le \hat{x}, \hat{y} \le 1$, it follows that

$$\frac{1}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} \le \frac{1}{(\tilde{b}/b + \tilde{a}/a - 1)^{p-1}}$$

and so, *1)* holds.

2) For $2 < p < 3$ a direct computation shows

$$I_p = \frac{1}{(3-p)(p-2)} \frac{1 + (\frac{\tilde{a}}{a} + \frac{\tilde{b}}{b} - 1)^{3-p} - (\frac{\tilde{a}}{a})^{3-p} - (\frac{\tilde{b}}{b})^{3-p}}{(\frac{\tilde{a}}{a} - 1)(1 - \frac{\tilde{b}}{b})}. \tag{3.8}$$

Clearly, the numerator can be bounded in terms of $\tilde{a}/a$ since $\tilde{b}/b \le 1$ and $p < 3$, however this would provide a poor bound as p approaches 2. In order to avoid this problem we define

$$II_p = \frac{1 + (\frac{\tilde{a}}{a} + \frac{\tilde{b}}{b} - 1)^{3-p} - (\frac{\tilde{a}}{a})^{3-p} - (\frac{\tilde{b}}{b})^{3-p}}{p-2} \tag{3.9}$$

and notice that it can be written as

$$II_p = (1 - (\frac{\tilde{a}}{a} + \frac{\tilde{b}}{b} - 1))\eta_p(\frac{\tilde{a}}{a} + \frac{\tilde{b}}{b} - 1) + (\frac{\tilde{a}}{a} - 1)\eta_p(\frac{\tilde{a}}{a}) + (\frac{\tilde{b}}{b} - 1)\eta_p(\frac{\tilde{b}}{b})$$

where η_p has been defined in Remark 3.3. Now, since $0 \le \eta_p(t)$ and $\frac{\tilde{b}}{b} \le 1$, we know that the last term of II_p is negative, so

$$II_p \le (1 - (\frac{\tilde{a}}{a} + \frac{\tilde{b}}{b} - 1))\eta_p(\frac{\tilde{a}}{a} + \frac{\tilde{b}}{b} - 1) + (\frac{\tilde{a}}{a} - 1)\eta_p(\frac{\tilde{a}}{a})$$

and by (3.7) we easily deduce that $(1-t)\eta_p(t) \le 1$ for any $0 \le t$. Hence,

$$II_p \le 1 + (\frac{\tilde{a}}{a} - 1)\eta_p(\frac{\tilde{a}}{a}) = \mu_p(\frac{\tilde{a}}{a}).$$

Thus, from (3.8) and (3.9) and the previous inequality we get

$$I_p = \frac{II_p}{(3-p)(\frac{\tilde{a}}{a}-1)(1-\frac{\tilde{b}}{b})} \leq \frac{\mu_p(\frac{\tilde{a}}{a})}{(3-p)(\frac{\tilde{a}}{a}-1)(1-\frac{\tilde{b}}{b})}. \tag{3.10}$$

On the other hand, since $\frac{\tilde{a}}{a} \geq 1$, $\frac{\tilde{b}}{b} \leq 1$, and $\eta(t) \leq 1$ for $0 \leq t \leq 1$,

$$I_p \leq \int_0^1 \int_0^1 \frac{1}{(1+\hat{x}(\tilde{b}/b-1))^{p-1}}\, d\hat{x}\, d\hat{y} = \frac{(\frac{\tilde{b}}{b})^{2-p}-1}{(2-p)(\frac{\tilde{b}}{b}-1)} = \frac{b}{\tilde{b}}\eta_p(\frac{\tilde{b}}{b}) \leq \frac{b}{\tilde{b}}. \tag{3.11}$$

Hence, for $2 < p < 3$, item *2)* follows from (3.10) and (3.11) by observing that $1 \leq \frac{1}{3-p}$.

The case $p = 2$ follows by taking $p \to 2$ in (3.10) and (3.11), yielding the same bound as that given in [2].

3) Follows like part *2)* after changing the roles of a and $\tilde{a}$ with b and $\tilde{b}$ respectively.

4) Under these hypotheses it follows immediately that $1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1) \geq 1$ and so *4)* holds. □

Now we are ready to get bounds for $|\phi_3|_{W^{1,p}(K)}$ in terms of geometric properties of the reference configuration for elements verifying the $RDP(\psi_M, N)$.

Lemma 3.5. *If $K = K(a, b, \tilde{a}, \tilde{b})$ is convex and satisfies* (H1), (H2), (H3), and (H4) *then, for any $1 \leq p < 3$ there exists a constant C depending only on $\overline{\psi}$, p, and N_i, $i = 1, 2, 3$ such that*

$$|\phi_3|_{W^{1,p}(K)} \leq C\frac{h^{1/p}}{|l|^{1/q}}. \tag{3.12}$$

Proof. We only need to deal with the case $2 \leq p < 3$, since $1 \leq p < 2$ was derived from the case $p = 2$ in Remark 3.2.

Let us consider four cases as in Lemma 3.4.

1) If $\tilde{b}/b \leq 1$ and $\tilde{a}/a \leq 1$ then, from (2.8), part *1)* of Lemma 3.4 and part *1)* of Lemma 3.3, it follows that

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \leq \frac{b}{a^{p-1}}\frac{1}{(\tilde{b}/b + \tilde{a}/a - 1)^{p-1}} \leq N_2^{p-1}\frac{b}{|l|^{p-1}} \leq N_2^{p-1}\frac{h}{|l|^{p-1}}.$$

Analogously, using now (2.9), we obtain

$$\left\|\frac{\partial\phi_3}{\partial y}\right\|^p_{L^p(K)} \leq N_2^{p-1}\frac{h}{|l|^{p-1}}.$$

2) Assume now that $\tilde{b}/b \leq 1$ and $\tilde{a}/a > 1$. Using again (2.8) and (2.9) but combined now with part *2)* of Lemma 3.4 and (3.1) we obtain

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \leq \frac{b}{a^{p-1}(3-p)}\min\left\{\frac{\mu_p(N_4)}{(1-\tilde{b}/b)(\tilde{a}/a-1)}, \frac{b}{\tilde{b}}\right\} \tag{3.13}$$

and,

$$\left\|\frac{\partial\phi_3}{\partial y}\right\|^p_{L^p(K)} \leq \frac{a}{b^{p-1}(3-p)}\min\left\{\frac{\mu_p(N_4)}{(1-\tilde{b}/b)(\tilde{a}/a-1)}, \frac{b}{\tilde{b}}\right\}. \tag{3.14}$$

Now, if $b/\tilde{b} \leq 2$, the proof concludes by using (3.13), (3.14) and (H3). Otherwise, $\tilde{b}/b < 1/2$ and so $1/(1 - \tilde{b}/b) < 2$. On the other hand, from part *3)* of Lemma 3.3 we know that $b/a \leq 2N_3$ and therefore, using again (3.13) combined with part *6)* of Lemma 3.3 we obtain

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \le \frac{2N_3}{a^{p-2}(3-p)}\min\left\{\frac{2\mu_p(N_4)}{\tilde a/a-1},\frac{b}{\tilde b}\right\}$$

$$\le \frac{4N_3}{(3-p)a^{p-2}}(\mu_p(N_4)+1)h\min\left\{\frac{1}{\tilde a-a},\frac{1}{\tilde b}\right\} \le \frac{4\sqrt{2}N_3^2}{(3-p)a^{p-2}}(\mu_p(N_4)+1)\frac{h}{|l|}.$$

and from (H3), $|l|\le N_3 a$, and the fact that $p-2\ge 0$

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \le \frac{4\sqrt{2}N_3^p}{(3-p)}(\mu_p(N_4)+1)\frac{h}{|l|^{p-1}}.$$

The bound for $\|\frac{\partial\phi_3}{\partial y}\|^p_{L^p(K)}$ follows in a similar way from (3.14) and part *2)* of Lemma 3.3.

3) Consider now the case $\tilde b/b>1$ and $\tilde a/a\le 1$. Once again we use (2.8) and (2.9) combined now with part *3)* of Lemma 3.4 to obtain

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \le \frac{b}{a^{p-1}(3-p)}\min\left\{\frac{\mu_p(N_4)}{(\tilde b/b-1)(1-\tilde a/a)},\frac{a}{\tilde a}\right\} \tag{3.15}$$

and,

$$\left\|\frac{\partial\phi_3}{\partial y}\right\|^p_{L^p(K)} \le \frac{a}{b^{p-1}(3-p)}\min\left\{\frac{\mu_p(N_4)}{(\tilde b/b-1)(1-\tilde a/a)},\frac{a}{\tilde a}\right\}. \tag{3.16}$$

However, we can not proceed exactly as in the previous case because now we do not know, as before, that a/b is bounded from above and below.

Assume first that $b/a<1/\mathrm{tg}((\pi-\overline{\psi})/2)$. Then, as in part *2)*, we can assume that $\tilde a/a\le 1/2$ (otherwise, the estimate follows easily from (3.15) and (3.16)). In this case we have

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \le \frac{2}{a^{p-2}(3-p)\mathrm{tg}((\pi-\overline{\psi})/2)}(\mu_p(N_4)+1)\min\left\{\frac{b}{\tilde b-b},\frac{a}{\tilde a}\right\}$$

$$\le \frac{2}{a^{p-2}(3-p)\mathrm{tg}((\pi-\overline{\psi})/2)}(\mu_p(N_4)+1)h\min\left\{\frac{1}{\tilde b-b},\frac{1}{\tilde a}\right\}$$

and so, the result follows from part *7)* of Lemma 3.3.

On the other hand, if $b/a\ge 1/\mathrm{tg}((\pi-\overline{\psi})/2)$, we can use part *4)* of Lemma 3.3 and (3.15) to get

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \le \frac{b}{a^{p-2}(3-p)\tilde a} \le N_6N_3^{p-2}\frac{b}{(3-p)|l|^{p-1}} \le N_6N_3^{p-2}\frac{h}{(3-p)|l|^{p-1}}.$$

Now, in order to bound the derivative with respect to y, we consider again two cases: $a/b<1/\mathrm{tg}(\alpha/2)$ and $a/b\ge 1/\mathrm{tg}(\alpha/2)$. In the first case, we bound $\left\|\frac{\partial\phi_3}{\partial y}\right\|^p_{L^p(K)}$ proceeding as before by using part *7)* of Lemma 3.3.

In the second case, we use (3.16) to obtain

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \le 2\frac{a}{(3-p)b^{p-2}}(\mu_p(N_4)+1)\frac{1}{\tilde b-b}$$

and the proof concludes by using part *5)* of Lemma 3.3.

4) Finally, in the case $\tilde b/b>1$ and $\tilde a/a>1$ the result follows trivially from part *4)* of Lemma 3.4 and (H3) by using again (2.8) and (2.9). □

4. Implications of the $DAC(\psi_m, \psi_M)$

Before we proceed, we need the following lemma, which says how the angles are transformed under an affine mapping.

Lemma 4.1. *Let L an affine transformation associated with a matrix B. Given two vectors v_1 and v_2, let α_1 and α_2 be the angles between them and between $L(v_1)$ and $L(v_2)$, respectively. Then*

$$\frac{2}{\kappa(B)\pi}\alpha_1 \leq \alpha_2 \leq \pi(1-\frac{2}{\kappa(B)\pi}) + \alpha_1\frac{2}{\kappa(B)\pi}.$$

Proof. The proof is elementary and can be found in [1]. □

The next lemma is in the same spirit of Lemma 3.1 for elements verifying $DAC(\psi_m, \psi_M)$. Since

$$DAC(\psi_m, \psi_M) \Rightarrow RDP(\psi_M, 1)$$

we could use the same L given in Lemma 3.1. However, with this choice of L, some conditions which are important to bound the error for $3 \leq p$, do not neccesarily hold. With this in mind, we will construct a different affine mapping $\tilde{L}$.

Lemma 4.2. *Let K be a quadrilateral verifying the $DAC(\psi_m, \psi_M)$. Then there exist an affine transformation $\tilde{L}(\overline{x}) = B\overline{x} + P$, a convex $K(a, b, \tilde{a}, \tilde{b})$ given by positive numbers $a, b, \tilde{a}, \tilde{b}$, and positive constants $C = C(\psi_m, \psi_M)$, $\overline{\psi_m} = \overline{\psi_m}(\psi_m, \psi_M)$, $\overline{\psi_M} = \overline{\psi_M}(\psi_m, \psi_M) < \pi$ such that*

- a) $\frac{\tilde{a}}{a}, \frac{\tilde{b}}{b} \leq 1$ *and* $\tilde{L}(K(a, b, \tilde{a}, \tilde{b})) = K$;
- b) $\|B\|, \|B^{-1}\| < C$, $det(B) = 1$;
- c) *The diameter of both elements are comparable, i.e.,*

$$C^{-1} diam(K) \leq diam(K(a, b, \tilde{a}, \tilde{b})) \leq C diam(K);$$

- d) $K(a, b, \tilde{a}, \tilde{b})$ *satisfies* $DAC(\overline{\psi_m}, \overline{\psi_M})$;
- e) *The angle α is bounded away from 0 and π, indeed,* $\frac{\pi - \overline{\psi_M}}{2} \leq \alpha \leq \pi - \overline{\psi_m}$.

Proof. It is always possible to choose two adjacent sides of K, l_1 and l_2, such that K is contained in the parallelogram defined by these two sides. Observe that the diameter of this parallelogram has the same order of K. Call M_1 the vertex where l_1 and l_2 intersect, and β the angle at M_1 . After a rigid movement we may asumme that M_1 is placed at the origin and that the side l_2 lies on the x axis. Let us call a the length of l_2 (see Figure 3). Now, let l_{14} be the side with vertices M_1 and M_4 (in anticlockwise vertex order), and define $b = |l_{14}| sin(\beta)$. Thus, $M_4 = (b\, cotg(\beta), b)$. We define the linear mapping $\tilde{L}$ associated with the matrix

$$\begin{pmatrix} 1 & cotg(\beta) \\ 0 & 1 \end{pmatrix}$$

and $\tilde{a}, \tilde{b}$ such that $\tilde{L}(K(a, b, \tilde{a}, \tilde{b})) = K$. It is easy to check that $\tilde{a} \leq a$ and $\tilde{b} \leq b$, and hence *a)* is proved. Item *b)* follows easily taking into account that $\|B\|, \|B^{-1}\| \leq \frac{\sqrt{2}}{sin(\beta)}$ and the fact that K verifies $DAC(\psi_m, \psi_M)$. Item *c)* is evident from *b)*, and item *d)* follows again from *b)* together with Lemma 4.1. Finally, the last item follows from *d)*, indeed, since $K(a, b, \tilde{a}, \tilde{b})$ verifies $DAC(\overline{\psi_m}, \overline{\psi_M})$ the angle θ at M_3 verifies $\overline{\psi_m} \leq \theta \leq \overline{\psi_M}$. Thus, one of the remaining angles of T_1, which we may assume to be α (if this is not the case, we can perform a rigid movement keeping the properties *a), b), c)* and *d)* unchanged), verifies $\frac{\pi - \overline{\psi_M}}{2} \leq \alpha \leq \pi - \overline{\psi_m}$. □

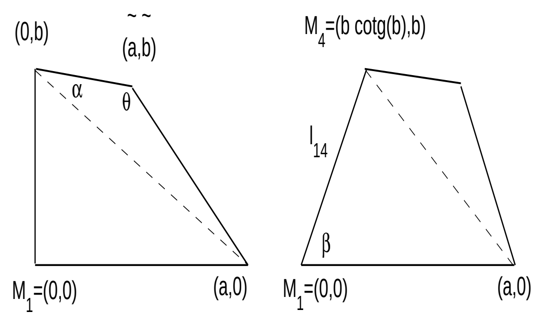

Figure 3

Remark 4.1. *In view of the preceding lemma, and of Lemma 2.1, whenever we deal with a convex quadrilateral K of diameter h satisfying the $DAC(\psi_m, \psi_M)$ we will assume that it is of the type $K(a, b, \tilde{a}, \tilde{b})$. Moreover, we will assume the existence of a positive constant $\tilde{N}(\psi_m, \psi_M)$ such that:*

($\tilde{H}1$) $\frac{\tilde{a}}{a}, \frac{\tilde{b}}{b} \leq 1$;

($\tilde{H}2$) $1/\sin\alpha \leq \tilde{N}$.

Let us notice that ($\tilde{H}2$) follows immediatly from item e) of Lemma 4.2.

The following elementary facts are deduced from ($\tilde{H}1$):

$$h = |d_1|; \tag{4.1}$$

$$\frac{|K|}{|T_2|} \leq 2. \tag{4.2}$$

Remark 4.2. *Trying to replicate the construction of $\tilde{L}$ carried out in Lemma 4.2, for an element verifiyng the $RDP(N, \psi_M)$, will eventually lead to the fact that it is not possible to bound $\kappa(B)$ in terms of ψ_M and N, since the angle β might approach 0. The transformation L constructed in Lemma 3.1 overcomes this difficulty relaxing the bounds on $\frac{\tilde{a}}{a}, \frac{\tilde{b}}{b}$ (see (3.1)). As we shall see, the condition ($\tilde{H}1$), valid for elements verifying $DAC(\psi_m, \psi_M)$, simplifies the study of the interpolation error.*

Lemma 4.3. *If $K = K(a, b, \tilde{a}, \tilde{b})$ is convex and satisfies the hypotheses ($\tilde{H}1$) and ($\tilde{H}2$), then,*

$$\max\{|l|/a, |l|/b\} \leq \tilde{N}(\tilde{b}/b + \tilde{a}/a - 1).$$

Proof. The proof is the same as that given for item *1)* of Lemma 3.3 changing (H2) by ($\tilde{\text{H}}$2). □.

For K verifying $DAC(\psi_m, \psi_M)$ the bounds for $|\phi_3|_{W^{1,p}(K)}$ can be handled easier than in the case of the RDP. This fact allows us to present directly the next lemma:

Lemma 4.4. *If $K = K(a,b,\tilde{a},\tilde{b})$ is convex and satisfies $(\tilde{H}1)$ and $(\tilde{H}2)$, then, for any $1 \le p$,*

$$|\phi_3|_{W^{1,p}(K)} \le C\tilde{N}^{1/q}\frac{h^{1/p}}{|l|^{1/q}} \tag{4.3}$$

where q is the dual exponent of p, and with C independent of K.

Proof. We will bound only $\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)}$, since the other derivative can be treated in the same way. First, let us notice that the fact that $\frac{\tilde{a}}{a}, \frac{\tilde{b}}{b} \le 1$ allows us to bound I_p like we did in item *1)* of Lemma 3.4, to get

$$I_p \le \frac{1}{(\tilde{b}/b + \tilde{a}/a - 1)^{p-1}}.$$

Hence, from (2.8) and Lemma 4.3 we get

$$\left\|\frac{\partial\phi_3}{\partial x}\right\|^p_{L^p(K)} \le \frac{b}{a^{p-1}}\frac{1}{(\tilde{b}/b + \tilde{a}/a - 1)^{p-1}} \le \tilde{N}^{p-1}\frac{b}{l^{p-1}}$$

and the proof concludes by taking into account that $b \le h$. ◻

5. Bounding $|u - \Pi u|_{W^{1,p}(K)}$ and $|(\Pi u - u)(M_3)|$

In order to handle $|u - \Pi u|_{W^{1,p}(K)}$ and $|(\Pi u - u)(M_3)|$ (see (2.1)) we will require a sharp form of the trace theorem on a triangle. The L^2 version of the following lemma can be found in [12], we state it in L^p and omit the proof since it follows step by step the one given in [12].

Lemma 5.1. *Let T be a triangle with diameter h_T and e be any of its sides. For any $1 \le p$ we have*

$$\|u\|_{L^p(e)} \le 2^{\frac{1}{q}}\left(\frac{|e|}{|T|}\right)^{1/p}\{\|u\|_{L^p(T)} + h_T|u|_{W1,p(T)}\}$$

where q is the dual exponent of p.

In the next lemma we give an estimate for $|(u - \Pi u)(M_3)|$ in terms of $|u - \Pi u|_{W^{1,p}(K)}$. We will use the notation of Figure 2.

Lemma 5.2. *If $K = K(a,b,\tilde{a},\tilde{b})$ is convex and verifies either*

a) $(H1), (H2), (H3), (H4)$ *or* *b)* $(\tilde{H}1), (\tilde{H}2)$

then, for any $1 \le p$,

$$|(u - \Pi u)(M_3)| \le C\frac{|l|^{1/q}}{h^{1/p}}\{|u - \Pi u|_{W^{1,p}(T_1)} + h|u|_{W^{2,p}(T_1)}\} \tag{5.1}$$

where q is the dual exponent of p, and $C = 2(N_2N_3(1+N_1))^{1/p}$ in case a), or $C = 2\tilde{N}^{1/p}$ in case b).

Proof. Let us denote with ∂_l the derivative in the direction of l. Using that $(u - \Pi u)(M_4) = 0$, Hölder's inequality, and Lemma 5.1 we have

$$\begin{aligned}|(u - \Pi u)(M_3)| &= \left|\int_l \partial_l(u - \Pi u)\right| \le |l|^{1/q}\|\partial_l(u - \Pi u)\|_{L^p(l)} \\ &\le 2^{1-\frac{1}{p}}\frac{|l|}{|T_1|^{1/p}}\{|u - \Pi u|_{W^{1,p}(T_1)} + h_{T_1}|u|_{W^{2,p}(T_1)}\}\end{aligned}$$

where q is the dual exponent of p.

Writing $|T_1| = |l||d_1|\sin\alpha/2$, it follows that

$$\frac{|l|}{|T_1|^{1/p}} = 2^{1/p}\frac{|l|^{1/q}}{(sin(\alpha)|d_1|)^{1/p}}$$

and, in case *a)*, we get from (3.2) and (H2) that

$$\frac{|l|}{|T_1|^{1/p}} \leq (2N_2N_3(1+N_1))^{1/p}\frac{|l|^{1/q}}{h^{1/p}}$$

while in case *b)*, we get from ($\tilde{H}2$) and (4.1) that

$$\frac{|l|}{|T_1|^{1/p}} \leq (2\tilde{N})^{1/p}\frac{|l|^{1/q}}{h^{1/p}}$$

and the proof concludes in both cases by observing that $h_{T_1} \leq h$. □

Remark 5.1. *Since K is convex, it is well known that, for any $1 \leq p$, there exists a constant C_p depending only on p such that*

$$\|w\|_{L^p(K)} \leq C_p h|w|_{W^{1,p}(K)} \tag{5.2}$$

for any w with vanishing average on K. For $p = 1$ and $p = 2$, and general convex domains, the optimal constants are known to be $C_1 = \frac{1}{2}$ and $C_2 = \frac{1}{\pi}$ (see [3, 11]).

The following lemma gives an estimate for the error $|u - \Pi u|_{W^{1,p}(K)}$ of the linear interpolant.

Lemma 5.3. *If $K = K(a, b, \tilde{a}, \tilde{b})$ is convex and verifies either*

$$a)\ \ (H1), (H2), (H3), (H4) \qquad or \qquad b)\ \ (\tilde{H}1), (\tilde{H}2)$$

then, for any $1 \leq p$,

$$|u - \Pi u|_{W^{1,p}(K)} \leq Ch|u|_{W^{2,p}(K)} \tag{5.3}$$

where $C = 2\Big(C_p(1 + 2^{\frac{1}{q}}N_5^{\frac{1}{p}}) + 2^{\frac{1}{q}}N_5^{\frac{1}{p}}\Big)$ in case a), and $C = 2(3C_p + 2)$ in case b).

Proof. Consider, for example, $v = \frac{\partial}{\partial x}(u - \Pi u)$. We want to show that

$$\|v\|_{L^p(K)} \leq Ch|v|_{W^{1,p}(K)}.$$

Let v_K be the mean value of v on K, from (5.2) we have

$$\|v - v_K\|_{L^p(K)} \leq C_p h|v|_{W^{1,p}(K)}. \tag{5.4}$$

Therefore, it remains to bound $\|v_K\|_{L^p(K)}$. Using the fact that the integral between 0 and a of $v(x, 0)$ vanishes, Hölder's inequality, and Lemma 5.1 , we obtain

$$\|v_K\|_{L^p(K)} = |v_K||K|^{1/p} = \frac{|K|^{1/p}}{a}\Big|\int_0^a (v - v_K)(x, 0)dx\Big|$$

$$\leq 2^{1-1/p}\left(\frac{|K|}{|T_2|}\right)^{1/p}\{\|v - v_K\|_{L^p(K)} + h|v|_{W^{1,p}(K)}\}$$

and the bound for $\frac{\partial}{\partial x}(u - \Pi u)$ is obtained by means of the triangle inequality, and using again (5.4) toghether with (3.3) in case a) and (4.2) in case b). Finally, the result follows by observing that the derivative with respect to y can be bounded in a similar fashion. □

6. Main Results

In this section we prove the interpolation theorem and by means of counterexamples we show that some of our results are sharp.

We begin with the following elementary Lemma, and include a sketch of its proof for the sake of completness:

Lemma 6.1. *Let $K = K(1, 1, \tilde{a}, \tilde{b})$, K convex, and $\tilde{a}, \tilde{b} \leq 1$. Then for any $1 \leq p$ we have*

$$\|u - Qu\|_{L^p(K)} \leq C|u|_{W^{2,p}(K)} \tag{6.1}$$

for a constant C independent of K.

Proof. We obviously have

$$\|u - Q(u)\|_{L^p(K)} \leq \|u - P(u)\|_{L^p(K)} + \|Q(P(u) - u)\|_{L^p(K)}$$

where P is the linear Taylor polinomial of u averaged over a *fixed* ball (see [6], Chapter 4) contained in the triangle of vertices $(0, 0), (1, 0)$ and $(0, 1)$.

Bramble-Hilbert's lemma, and Sobolev's inequality as stated in [6], yield on one hand

$$\|u - P(u)\|_{W^{2,p}(K)} \leq C|u|_{W^{2,p}(K)}$$

and, on the other

$$\|P(u) - u\|_{L^\infty(K)} \leq C\|P(u) - u\|_{W^{2,p}(K)}$$

with C independent of K in both cases (recall that $\tilde{a}, \tilde{b} \leq 1$). We conclude the proof just taking into account that

$$\|Q(P(u) - u)\|_{L^p(K)} \leq \|Q(P(u) - u)\|_{L^\infty(K)} \leq C\|P(u) - u\|_{L^\infty(K)}. \square$$

Collecting the previous lemmas, we obtain our main theorem which gives the optimal error estimate for convex quadrilaterals.

Theorem 6.1. *Let K be a convex quadrilateral with diameter h, and $1 \leq p$. There exists a constant C_0 independent of K such that*

$$\|u - Qu\|_{L^p(K)} \leq C_0 h^2 |u|_{W^{2,p}(K)}. \tag{6.2}$$

For $1 \leq p < 3$ and K satisfying $RDP(N, \psi_M)$ we have

$$|u - Qu|_{W^{1,p}(K)} \leq Ch|u|_{W^{2,p}(K)} \tag{6.3}$$

with $C = C(N, \psi_M, p)$, and the restriction on p can not be removed.

Finally, for any $1 \leq p$ and K verifying $DAC(\psi_m, \psi_M)$ we have

$$|u - Qu|_{W^{1,p}(K)} \leq Ch|u|_{W^{2,p}(K)} \tag{6.4}$$

with $C = C(\psi_m, \psi_M, p)$, and for $3 \leq p$ the condition on the maximum *angle can not be relaxed.*

Proof. It is always possible to choose two adjacent sides of K, l_1 and l_2, such that K is contained in the parallelogram defined by these two sides. Observe that this parallelogram has a diameter of the same order as that of K. Now, let L be the affine transformation taking l_1 into the segment joining $(0, 0)$ and $(1, 0)$ and l_2 into the segment joining $(0, 0)$ and $(0, 1)$ and call $\tilde{K} = L(K)$ (this transformation was also used in [13]). It is easy to see that $\tilde{K} = K(1, 1, \tilde{a}, \tilde{b})$ with $\tilde{a}, \tilde{b} \leq 1$, and standard arguments show that (6.2) follows from (6.1) by changing variables.

In order to prove (6.3), we observe that Lemma 3.1 says that K can be transformed into a convex $K(a, b, \tilde{a}, \tilde{b})$ satisfying (H1), (H2), (H3), (H4) with constants $\overline{\psi}$ and N_i, $i = 1, 2, 3$ depending only on N and ψ_M and with diameter h equivalent to that of K. Moreover, from Lemma 2.1 we know that the error estimate on K follows from that on $K(a,b,\tilde{a},\tilde{b})$. Therefore, it is enough to prove the error estimate for these reference configurations with a constant depending only on $\overline{\psi}$ and N_i.

Now, inequality (6.3) follows from (2.1) combined with (5.3), (5.1) and (3.12). Counterexample 6.1 shows that $p < 3$ is neccesary.

Finally, the proof of (6.4) follows like that of (6.3) replacing Lemma 3.1 by Lemma 4.2, and using again Lemma 2.1 and equations (2.1) combined with (5.3), (5.1) and (4.3). Counterexample 6.1 also shows that the condition on the maximum angle is neccesary if $3 \leq p$. □

Counterexample 6.1. *We will show that the assumption $1 \leq p < 3$ is not removable in the last theorem if K verifies the RDP condition. Take $K = K(1,1,s,s)$ (see Figure 1 b)) with $\frac{1}{2} < s < 1$, the idea is to take $s \to \frac{1}{2}$. Clearly, K verifies $RDP(\frac{\pi}{2}, 2)$ independently of s, moreover, K is regular for s in that range. Consider now $u(x,y) = xy$ (observe that this function is not in the Q_1 space if $s < 1$). On the one hand $\left\|\frac{\partial u}{\partial x}\right\|_{L^p(K)}, |u|_{W^{2,p}(K)} \leq 1$, and on the other, we have $Qu = s^2\phi_3$, so*

$$\left\|\frac{\partial Qu}{\partial x}\right\|_{L^p(K)} = s^2\left\|\frac{\partial \phi_3}{\partial x}\right\|_{L^p(K)}$$

and from (2.5) we can write

$$\left\|\frac{\partial \phi_3}{\partial x}\right\|^p_{L^p(K)} \geq \frac{1}{2^p}\int_{\frac{1}{2}}^1\int_0^1 \frac{1}{(1+(s-1)\hat{x}+(s-1)\hat{y})^{p-1}}d\hat{x}d\hat{y}.$$

Integrating explicitly for $p > 3$ we get

$$\left\|\frac{\partial \phi_3}{\partial x}\right\|^p_{L^p(K)} \geq \frac{\left((2s-1)^{3-p} - (\frac{3}{2}s - \frac{1}{2})^{3-p} - s^{3-p} + (\frac{1}{2} + \frac{s}{2})^{3-p}\right)}{2^p(3-p)(2-p)(s-1)^2}$$

and hence $\left\|\frac{\partial \phi_3}{\partial x}\right\|^p_{L^p(K)} \to \infty$ if $s \to \frac{1}{2}$ (due to the first term inside the brackets), thus showing that (6.3) can not hold independently of s. The case $p = 3$ is similar but a logarithmic term of the type $ln(2s-1)$ is responsible for the fact $\left\|\frac{\partial \phi_3}{\partial x}\right\|^p_{L^p(K)} \to \infty$, leading us to the same conclusion. Finally, let us observe that the same counterexample implies that the restriction on the maximum *angle can not be relaxed if $p \geq 3$.*

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