Singularities of logarithmic foliations

Fernando Cukierman\textsuperscript{1}, Marcio G. Soares\textsuperscript{2} and Israel Vainsencher\textsuperscript{3}

ABSTRACT

A logarithmic 1-form on $\mathbb{CP}^n$ can be written as
\[
\omega = \left( \prod_{i=0}^{m} F_j \right) \sum_{i=0}^{m} \lambda_i \frac{dF_i}{F_i} = \lambda_0 \hat{F}_0 dF_0 + \cdots + \lambda_m \hat{F}_m dF_m
\]
with $\hat{F}_i = (\prod_{j=0}^{m} F_j) / F_i$ for some homogeneous polynomials $F_i$ of degree $d_i$ and constants $\lambda_i \in \mathbb{C}^*$ such that $\sum \lambda_i d_i = 0$. For general $F_i, \lambda_i$, the singularities of $\omega$ consist of a schematic union of the codimension 2 subvarieties $F_i = F_j = 0$ together with, possibly, finitely many isolated points. This is the case when all $F_i$’s are smooth and in general position. In this situation, we give a formula which prescribes the number of isolated singularities.

1. INTRODUCTION

The search for numerical invariants attached to algebraic foliations goes back to Poincaré [13]. He was interested in determining bounds for the degree of curves left invariant by a polynomial vector field on $\mathbb{C}^2$.

Recent work treat the question by establishing relations for the number of singularities of the foliation and certain Chern numbers and then using positivity of certain bundles. For a survey of recent results, see [4], [7], [10], [14].

A foliation of dimension $r$ on a smooth variety $X$ of dimension $n$ is a coherent subsheaf $\mathcal{F}$ of the tangent sheaf $TX$ of generic rank $r$, locally split in codimension $\geq 2$.

If $r = n - 1$ (codimension one foliations), the foliation corresponds to a global section of $\Omega^1_X \otimes L$ for some line bundle $L$.

Suppose $X = \mathbb{CP}^n$, with homogeneous coordinates $x_0, \ldots, x_n$. Recall Euler’s sequence,
\[
\Omega^1_{\mathbb{CP}^n}(1) \rightarrow \mathcal{O}^{\oplus n+1} \rightarrow \mathcal{O}(1).
\]

A global section $\omega$ of
\[
\Omega^1_{\mathbb{CP}^n}(d) \subset \mathcal{O}^{\oplus n+1}(d - 1)
\]
can be written as
\[
\omega = \sum_{i=0}^{n} F_i dx_i
\]
where $F_i$ is a homogeneous polynomial of degree $d - 1$, subject to the condition

\textsuperscript{1}Partially supported by SPU-Argentina and CAPES-Brasil.
\textsuperscript{2}Partially supported by CAPES-Brasil and CNPq-Brasil.
\textsuperscript{3}Partially supported by CNPq-Brasil. Aceito Compositio Math. 15/11/2004
\[ \sum F_i x_i = 0 \]

(contraction by the radial vector field on \( \mathbb{C}^{n+1} \)).

The degree of a codimension one foliation \( \mathcal{F} \), \( \deg \mathcal{F} \), is the number of tangencies of the leaves of \( \mathcal{F} \) with a generic one-dimensional linear subspace of \( \mathbb{C}P^n \). A simple calculation shows that \( \deg \mathcal{F} = d - 2 \) if the 1-form defining \( \mathcal{F} \) has components \( F_i \) of degree \( d - 1 \). The form \( \omega \) is integrable if \( \omega \wedge d\omega = 0 \).

Integrable 1–forms make up a Zariski closed subset of \( \mathbb{P}(H^0(\Omega^1(d))) \). We denote by \( \text{Fol}(\mathbb{C}P^n; d) \) the space of codimension one integrable holomorphic foliations of degree \( d - 2 \) of \( \mathbb{C}P^n \).

Not much is known about the dimensions nor the number of irreducible components of \( \text{Fol}(\mathbb{C}P^n; d) \) (but see [8] and [9]).

When \( \omega \) can be written as

\[
\omega = \prod_{j=0}^{m} F_j \sum_{i=0}^{m} \lambda_i \frac{dF_i}{F_i} = \lambda_0 \hat{F}_0 dF_0 + \cdots + \lambda_m \hat{F}_m dF_m
\]

for some homogeneous polynomials \( F_i \) of degree \( d_i \) and \( \lambda_i \in \mathbb{C}^* \) such that \( \sum \lambda_i d_i = 0 \), we say \( \omega \) is logarithmic of type \( d = d_0, \ldots, d_m \). Given positive integers \( d_0, \ldots, d_m \), set \( d = \sum_{i=0}^{m} d_i \) and consider the hyperplane

\[
\mathbb{C}P(m - 1, d) = \{ (\lambda_0, \ldots, \lambda_m) \in \mathbb{C}P^m \mid \sum d_i \lambda_i = 0 \}.
\]

Define a rational map \( \Psi \) by

\[
\mathbb{C}P(m - 1, d) \times \prod_{i=0}^{m} \mathbb{P}(H^0(\mathbb{C}P^n, \mathcal{O}(d_i))) \xrightarrow{\Psi} \text{Fol}(\mathbb{C}P^n; d)
\]

\[
((\lambda_0, \ldots, \lambda_m), (F_0, \ldots, F_m)) \mapsto (\prod_{j=0}^{m} F_j \sum_{i=0}^{m} \lambda_i \frac{dF_i}{F_i})
\]

The closure of the image of \( \Psi \) is the set \( \text{Log}_n(d) \) of logarithmic foliations of type \( d \) (of degree \( d - 2 \)) of \( \mathbb{C}P^n \). Recall the following result.

**Theorem.** (Calvo-Andrade [5]) For fixed \( d_i \) and \( n \geq 3 \), logarithmic foliations form an irreducible component of the space of codimension one integrable holomorphic foliations of \( \mathbb{C}P^n \) of degree \( d - 2 \) (with \( d = \sum d_i \)).

The singular scheme of the foliation defined by \( \omega \in H^0(\Omega^1(d)) \) is the scheme of zeros of \( \omega \). This is the closed subscheme with ideal sheaf given by the image of the co-section \( \omega^\vee : (\Omega^1(d))^\vee \to \mathcal{O} \).

For \( \omega \) general in \( H^0(\Omega^1(d)) \), there are just finitely many singularities, to wit (cf. Jouanolou, [12, p. 87, Th. 2.3], setting in his notation, \( m = d - 1 \), \( r = n \) ),

\[
\int_{\mathbb{C}P^n} c_n \left( \Omega^1(d) \right) = \sum_{i=0}^{n} (-1)^i \left( \begin{array}{c} n + 1 \\ 0 \end{array} \right) d^{n-1}.
\]

On the other hand of course, a general \( \omega \) is not integrable.
Theorem. (Jouanolou [12]) For integrable $\omega$, the singular set must contain a codimension 2 component.

It is easy to see that, for logarithmic (hence integrable) forms

$$\omega = \lambda_0 \hat{F}_0 dF_0 + \cdots + \lambda_m \hat{F}_m dF_m$$

the singular set contains the union of all codimension 2 subsets

$$F_i = F_j = 0, \ i \neq j.$$ 

It is worth mentioning that Jouanolou describes examples of integrable 1-forms with singular schemes containing positive dimensional components of “wrong” positive dimension. We found no hint as to the existence of isolated singularities for general enough foliations.

Let $D_i$ be the divisor associated to $F_i$. We assume the following genericity conditions to hold:

(1) \[ \left\{ \begin{array}{l} \text{the } D_i\text{'s, } i = 0, \ldots, m, \text{ are smooth and in general position.} \\
\lambda_i \neq 0, \ i = 0, \ldots, m. \end{array} \right. \]

Remark that (1) defines a Zariski open subset of

$$\mathbb{CP}(m - 1, d) \times \prod_{i=0}^{m} \mathbb{P}(H^0(\mathbb{CP}^n, O(d_i)))$$

Before stating our main result recall that the complete symmetric function $\sigma_\ell$, of degree $\ell$ in the variables $X_1, \ldots, X_k$ is defined by: $\sigma_0 = 1$ and, for $\ell \geq 1$,

$$\sigma_\ell(X_1, \ldots, X_k) = \sum_{i_1 + \cdots + i_k = \ell} X_1^{i_1} \cdots X_k^{i_k}.$$ 

We then have

Theorem. Let $\mathcal{F}$ be a logarithmic foliation on $\mathbb{CP}^n$ of type $d = d_0, \ldots, d_m$, given by

$$\omega = \lambda_0 \hat{F}_0 dF_0 + \cdots + \lambda_m \hat{F}_m dF_m$$

and satisfying (1). Then the singular scheme $S(\mathcal{F})$ of $\mathcal{F}$ can be written as a disjoint union

$$S(\mathcal{F}) = Z \cup R$$

where

$$Z = \bigcup_{i<j} D_i \cap D_j$$

and $R$ is finite, consisting of

$$N(n, d) = \sum_{i=0}^{n} (-1)^i \binom{n + 1}{i} \sigma_{n-i}(d)$$

points counted with natural multiplicities. Moreover,
(1) \( N(n, d) = 0 \) if \( n \geq m \) and \( d_i = 1 \) for all \( i \).

(2) \( N(n, d) = \binom{m}{n+1} \) if \( n < m \) and \( d_i = 1 \) for all \( i \).

(3) \( N(n, d) > 0 \) whenever \( d_i \geq 2 \) for some \( i \).

It will be shown below, see formula (8) in 4.3, that

\[
N(n, d) = \text{the coefficient of } h^n \text{ in } \frac{(1-h)^{n+1}}{\Pi_{i=0}^{m}(1-d_i h)}
\]

from which we deduce:

1.0.1. Example. If \( d_i = 1 \) for all \( i \) then \( \frac{(1-h)^{n+1}}{\Pi_{i=0}^{m}(1-d_i h)} \) reduces to \( \frac{(1-h)^n}{(1-h)^m} \) and we have items (1) and (2) of theorem:

(1) \( n \geq m \). In this case \( \frac{(1-h)^n}{(1-h)^m} \) is a polynomial of degree \( n-m < n \) and hence the coefficient of \( h^n \) vanishes, so that there are no isolated zeros.

(2) \( n < m \). In this case \( \frac{(1-h)^n}{(1-h)^m} \) reads \( \frac{1}{(1-h)^{m-n}} \) and it’s easily seen that the coefficient of \( h^n \) is \( \binom{m}{n+1} \).

2. Proof of the theorem

We will show that, if a point is non isolated in \( S(\mathcal{F}) \), then it lies in \( D_i \cap D_j \) for some \( i < j \). Indeed, let \( C \) be an irreducible component of \( S(\mathcal{F}) \) of dimension \( 1 \leq \dim C \leq n-2 \). By ampleness and general position, we may pick a point \( p \in C \) lying in the intersection of precisely \( k \) of the divisors \( D_i, \ 1 \leq k \leq \min\{n, m+1\} \).

Let \( f_i \) be a local equation for \( D_i \) at \( p \). Near \( p \), the foliation \( \mathcal{F} \) is given by the 1–form

\[
\varpi = \hat{f} \sum_{i=0}^{m} \lambda_i \frac{df_i}{f_i}.
\]

Renumbering the indices we may assume \( p \in D_0 \cap \cdots \cap D_{k-1} \). The local defining equations \( f_i = 0 \) of the \( D_i \)'s, for \( i = 0, \ldots, k-1 \), are part of a regular system of parameters, i.e., \( df_0, \ldots, df_{k-1} \) are linearly independent at \( p \). Write \( \tilde{g} = f_k \cdots f_m \).

Since \( p \notin D_j, \ k \leq j \leq m \), we may assume \( \tilde{g} \) vanishes nowhere around \( p \) and write \( \varpi \) as

\[
\varpi = f_0 \cdots f_{k-1} \tilde{g} \left[ \sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + \sum_{i=k}^{m} \lambda_i \frac{df_i}{f_i} \right] = f_0 \cdots f_{k-1} \tilde{g} \left[ \sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + \eta \right],
\]

where \( \eta = \sum_{i=k}^{m} \lambda_i \frac{df_i}{f_i} \) is a holomorphic closed form near \( p \). Since \( \eta \) is closed, by the formal Poincaré lemma it is exact near \( p \), say \( \eta = d\xi \). Set \( \vartheta = \varpi/\tilde{g} \). Then \( \mathcal{F} \) is
defined around $p$ by
\[
\vartheta = f_0 \cdots f_{k-1} \left[ \sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + d\xi \right] = f_0 \cdots f_{k-1} \left[ \lambda_0 \frac{d(\exp[\xi/\lambda_0] f_0)}{\exp[\xi/\lambda_0] f_0} + \sum_{j=1}^{k-1} \lambda_j \frac{df_j}{f_j} \right].
\]

Set $z_0 = \exp[\xi/\lambda_0] f_0$ and $z_1 = f_1, \ldots, z_{k-1} = f_{k-1}$. Since $u = \exp[\xi/\lambda_0]$ is a unit, we have that also $z_0, \ldots, z_{k-1}$ are part of a regular system of parameters at $p$. Now $\vartheta$ can be written as
\[
\vartheta = \frac{z_0}{u} z_1 \cdots z_{k-1} \left[ \lambda_0 \frac{dz_0}{z_0} + \sum_{j=1}^{k-1} \lambda_j \frac{dz_j}{z_j} \right].
\]

Thus $\mathcal{F}$ is defined around $p$ by the 1–form
\[
(2) \quad \tilde{\vartheta} = z_0 z_1 \cdots z_{k-1} \left[ \lambda_0 \frac{dz_0}{z_0} + \sum_{j=1}^{k-1} \lambda_j \frac{dz_j}{z_j} \right] = \sum_{j=0}^{k-1} \lambda_j z_0 \cdots \hat{z}_j \cdots z_{k-1} dz_j.
\]

If $k = 1$, (2) shows that the foliation is defined near $p$ by $dz_0$ and then is non-singular at $p$. Hence we necessarily have $k \geq 2$. Note that the ideal of the scheme of zeros of $\tilde{\vartheta}$ (as well as of $\omega$) near $p$ is generated by the $k$ monomials $z_0 \cdots \hat{z}_j \cdots z_{k-1}$ with $0 \leq j \leq k-1$. That is just the scheme union $\cup_{i,j} D_i \cap D_j$, for $0 \leq i < j \leq k-1$. Thus $C$ must be contained in $D_i \cap D_j$, for some $i < j$, and therefore $C$ is an irreducible component of $D_i \cap D_j$ and $\dim C = n - 2$.

The formula for the finite part is proved in the next section in a slightly more general context.

2.1. Remark. The argument above shows that the codimension two part, $Z = \bigcup D_{ij}$, of the singular scheme of a general logarithmic foliation is equal to the singular scheme of the normal crossing divisor $\bigcup D_i$. This will enable us to use Aluffi’s formula for the Segre class. We also note that, since $D_{ij}$ is smooth and connected, the component $C$ is actually equal to some $D_{ij}$.

3. Formulas

Let $\mathcal{E} \rightarrow X$ be a holomorphic vector bundle of rank $n$ over a complex projective smooth variety of dimension $n$. Let $s : X \rightarrow \mathcal{E}$ be a section. Assume (1) the scheme of zeros $W$ of $s$ is a disjoint union
\[
W = Z \cup R
\]
with $R$ finite; (2) there are Cartier divisors $D_0, \ldots, D_m$, $m \geq 1$, such that
\[
Z = \bigcup_{i<j} D_{ij}
\]
as schemes, where
\[ D_{ij} = D_i \cap D_j; \]
(3) for all choices of indices
\[ I_r = (0 \leq i_1 < \cdots < i_r \leq m), \]
the intersection \( D_{I_r} = \bigcap_{i \in I_r} D_i \) is transversal.

We are mainly interested in the case where \( X = \mathbb{CP}^n \) and the section \( s \) is a logarithmic form as in the Theorem in p. 3.

We give an expression for the number of points in \( R \), counted with natural multiplicities, in terms of the intersection numbers
\[ D^J \cdot c_j(\mathcal{E}) \]
with
\[ J = (j_0, \ldots, j_m), \ D^J = D_0^{j_0} \cdots D_m^{j_m}, \ |J| + j = n. \]

When \( Z = \bigcup_{i < j} D_{ij} \) is a disjoint union, the formula is just a simple direct application of usual excess intersection techniques as reviewed below.

Disjointness implies that \( Z \) is a local complete intersection with explicitly known normal bundle.

The ideal of \( W \) is the image \( \mathcal{I}(W) \) of the co-section
\[ s^\vee : \mathcal{E}^\vee \to \mathcal{O}. \]

It can be written as
\[ \mathcal{I}(W) = \mathcal{I}(Z) \cdot \mathcal{I}(R). \]

Locally, it is of the form \( \mathcal{I} = \langle z_0, z_1 \rangle \cdot \mathfrak{m} \), where \( z_0, z_1 \) are equations for the pair of transversal divisors cutting \( Z \), and \( \mathfrak{m} \) denotes an ideal of finite co-length corresponding to the finite part \( R \subset W \). (Note that \( \mathfrak{m} = \langle 1 \rangle \) if \( R \) is disjoint from the present coordinate chart.)

Let \( \pi : X' \to X \) be the blowup along \( Z \). Put \( E' = \pi^{-1}(Z) \), the exceptional divisor. The pullback \( \pi^*s^\vee \) of the co-section maps \( \pi^*\mathcal{E}^\vee \) onto
\[ \mathcal{O}(-E') \cdot \mathcal{I}(R'). \]

(\( R' = \pi^{-1}R \)). We get an induced map of sheaves
\[ (s')^\vee : \pi^*\mathcal{E}^\vee \otimes \mathcal{O}(E') \to \mathcal{I}(R') \subset \mathcal{O}. \]

Dualizing, we find a section \( s' \) of
\[ (3) \quad \mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(-E') \]
whose scheme of zeros is precisely \( R' \simeq R \), the finite part.

Indeed, since \( R \) is disjoint from the blowup center, \( \pi : X' \to X \) is an isomorphism in a neighborhood of \( R' \). Hence, the length of \( \mathcal{O}_{X'}/\mathcal{I}(R') \) is the same as for \( R \). This implies the formula for the degree of the zero cycle,
\[ (4) \quad \deg [R] = \deg [R'] = \int_{X'} c_n(\mathcal{E}'). \]
To compute it explicitly, recall that the exceptional divisor $E'$ is the projective bundle $\mathbb{P}(\mathcal{N}_{Z/X})$ of the normal bundle of $Z$ in $X$. The restriction of $\mathcal{N}_{Z/X}$ to each $D_{ij}$ is the restriction of $\mathcal{O}(D_i) \oplus \mathcal{O}(D_j)$. Let $\iota : E' \hookrightarrow X'$ be the inclusion. We recall from [11, B.6, p. 435] a couple of facts that follow from the construction of the blowup as $	ext{Proj}(\oplus I_k)$ of the Rees algebra of the ideal sheaf $I$. The natural relatively ample line bundle $\mathcal{O}_{X'}(1)$ is presently the image of $\pi^* \mathcal{I} \to \pi^* \mathcal{O}_X = \mathcal{O}_{X'}$. The exceptional divisor $E' \subset X'$ is identified to the projectivization of the normal cone, $\text{Proj}(\oplus I_k/I_{k+1})$. Accordingly, we have the identification $\iota^* \pi^* \mathcal{I} = \mathcal{I}/\mathcal{I}^2 \to \iota^* \mathcal{O}_{X'}(1)$. The latter is but the hyperplane bundle $\mathcal{O}_{E'}(1)$ of the $\mathbb{C}P^1$–bundle $E' = \mathbb{P}(\mathcal{N}_{Z/X}) \to Z$. We may compute the self-intersection (cf. [11, 2.6, p. 44]),

$$
(E')^2 = \iota_* (\iota^* E') = \iota_* (\iota^* c_1(\mathcal{O}_{X'}(E')) \cap [X'])
$$

$$
= \iota_* (\iota^* c_1(\mathcal{O}_{X'}(-1)) \cap [X'])
$$

$$
= -\iota_* (\xi \cap [E'])
$$

with

$$
\xi = c_1(\mathcal{O}_{E'}(1)).
$$

Recall that the push-forward of powers of the hyperplane class $\xi$ of the $\mathbb{C}P^1$–bundle $E' = \mathbb{P}(\mathcal{N}_{Z/X}) \to Z$ are expressed (cf. [11, p. 47]) by Segre classes:

$$
\pi_* (\xi^j) = s_j(\mathcal{N}_{Z/X}) \forall j \in \mathbb{Z}.
$$

Writing $[D_{ij}]$ for the cycle class of $D_i \cap D_j$ in the Chow (or homology) group $A_* X$, we have, for $r \geq 0$,

$$
(E')^r = \iota_* (\iota^* (E')^r) = \iota_* ((-\xi)^r \cap [E']).
$$

We may write

$$
\pi_* ((E')^r) = \pi_* \iota_* ((-\xi)^r \cap [E'])
$$

$$
= (-1)^r \sum_{s \leq j} s_{r-1} (\mathcal{O}(D_i) \oplus \mathcal{O}(D_j)) \cap [D_{ij}]
$$

in the group $A_m X$ of cycles of dimension $m = n - 2 - k$.

Put

$$
s_{kij} = s_k (\mathcal{O}(D_i) \oplus \mathcal{O}(D_j)) \cap [D_{ij}]
$$

$$
= (-1)^k D_i \cdot D_j \cdot \sum_{u=0}^k D_i^u D_j^{k-u}.
$$

Since $s_j = 0$ for $j < 0$, we also have

$$
\pi_* ((E')) = 0.
$$
It follows from (4) and (3) that
\[
\deg [R] = \int_X \pi_* c_n(E')
\]
\[
= \int_X \sum_{r=0}^n c_{n-r}(E) \cdot \pi_* ((-E')^r)
\]
\[
= \int_X c_n(E) + \sum_{r=1}^{n-1} (-1)^{r+1} c_{n-1-r}(E) \cdot \pi_* ((E')^{r+1})
\]
\[
= \int_X c_n(E) - \sum_{r=1}^{n-1} \sum_{i<j} c_{n-1-r}(E) \cdot s_{(r-1)ij}
\]
\[
= \int_X c_n(E) - \sum_{r=1}^{n-1} (-1)^{r-1} c_{n-1-r}(E) \sum_{i<j}^{r-1} D_i^{u+1} D_j^{r-u}.
\]

The idea now is to reduce the general case to the above situation. This will be done by a sequence of blowups along smooth centers with known normal bundles.

We explain how the reduction works, say in the case when all 4-fold intersections are empty, for the sake of simplicity. The general case is entirely similar. Thus assume
\[
\forall I_4 = (0 \leq i_0 < i_2 < i_3 < i_4 \leq m),
\]
we have
\[
D_{I_4} := \bigcap_{i \in I_4} D_i = \emptyset.
\]
(This is the case if, for instance, \( \dim X = 3 \).) It follows that for all choices of triple indices,
\[
I_3 = (i < j < k) \neq I_3' = (i' < j' < k'),
\]
we must have
\[
D_{I_3} \cap D_{I_3'} = \emptyset.
\]

Now the union \( T \) of all triple intersections \( D_{I_4} \) is smooth.

Let \( \pi : X' \to X \) be the blowup along \( T \). The strict transform \( D'_{ij} \) is equal to the blowup of \( D_{ij} \) along the disjoint union of Cartier divisors \( D_{ijk} \), hence \( D'_{ij} \simeq D_{ij} \) holds. Moreover, since \( D_{ij} \cap D_{ij} \) is a union of connected components of the blowup center, it follows that \( D'_{ij} \cap D'_{jk} = \emptyset \). We also have that the \( D'_{ij} \) meet transversally.

Look at the pullback \( \pi^{-1} W \) of the zero scheme of the section \( s \). We will take coordinates on \( X \) in a neighborhood of a point \( 0 \in D_{123} \), say. Near 0, \( W \) is equal to the union \( D_{12} \cup D_{13} \cup D_{23} \). Let \( z_i = 0 \) be a local equation of \( D_i \). Then the ideal of \( W \) near 0 is equal to the intersection
\[
\langle z_1, z_2 \rangle \cap \langle z_1, z_3 \rangle \cap \langle z_2, z_3 \rangle = \langle z_1 z_2, z_1 z_3, z_2 z_3 \rangle.
\]
The blowup center, \( T \), is locally given by \( \langle z_1, z_2, z_3 \rangle \). The restriction of \( X' \) over the present affine neighborhood of the point 0 is covered by three affine open subsets, one for each choice of \( z_i \) as a generator of the exceptional ideal \( O(-E') \).
Say we take $z_1$ as a local generator. We may write $z_i = z_1 z_i'$, $i = 1, 2$. Here $z_i'$ is a local equation of the strict transform of $D_i$.

The pullback of $W$ is given by the ideal

$$\mathcal{I}(\pi^{-1}W) = z_1^2 (z_2', z_3', z_2' z_3') = z_1^2 (z_2', z_3').$$

This is twice the exceptional ideal, times the ideal of the strict transform of $D_{23}$.

Note that the strict transforms of $D_{13}$ and of $D_{12}$ are empty in the present neighborhood of $X'$. Thus the $D'_{ij}$ are presently disjoint.

The local expression shows that the image $\mathcal{I}(W)\mathcal{O}_{X'}$ of the co-section

$$\pi^* s' : \mathcal{E}' \to \mathcal{O}_{X'},$$

is of the form

$$\mathcal{I}(W)\mathcal{O}_{X'} = \mathcal{O}(-2E') \cdot \mathcal{I}(Z') \cdot \mathcal{I}(R'),$$

where the finite piece $R' = \pi^{-1}(R) \simeq R$ and $Z' = \bigcup D'_{ij}$ is the disjoint union of pairwise transversal intersections of Cartier divisors $D'_{ij}$.

Hence, we may apply the previous case to the section $s' = s \otimes \mathcal{O}(-2E')$ of $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(-2E')$. We find

$$\deg [R] = \deg [R']$$

$$= \int_{X'} c_n(\mathcal{E}') - \sum_{r=0}^{n-1} (-1)^{r-1} c_{n-1-r}(\mathcal{E}') \sum_{i<j} (D'_{ij})^{u+1} \cdot (D'_{ij})^{r-u}.$$

Let $E'_i$ denote the sum of the (disjoint) exceptional divisors over all $D_i$ with $i \in I_3$. Using the formulas $D'_{ij} = \pi^* D_i - E'_i$ and universal formulas for $c(\mathcal{E} \otimes \mathcal{O}(-2E'))$ and applying $\pi_*$, the above expression can be written in terms of the intersection numbers $D^J \cdot c_j(\mathcal{E})$.

In general, let $r$ be the smallest integer such that for all possible choices of indices

$$I_{r+2} = (0 \leq i_0 < i_1 < \cdots < i_{r+1} \leq m),$$

we have

$$D_{i_{r+2}} := \bigcap_{i \in I_{r+2}} D_i = \emptyset.$$

If $m \geq 2$, we have $r \leq \min(n-1, m-1)$ because $\dim X = n$ and the divisors are in general position. Of course if $r \geq m$ no $I_{r+2}$ exists! If $m = 1$, set $r = 1$.

We then have that the union

$$Z_{r+1} = \bigcup_{I_{r+1}} D_{I_{r+1}}$$

of all $(r+1)$-fold intersections among $D_i$’s is smooth. Let $\pi^1 : X^1 \to X$ be the blowup along $Z_{r+1}$. A local analysis as performed above shows that the strict transforms $D^1_i$ are in general position and the intersections $D_{I_{r+1}}^1$ are empty. Moreover, there is a section $s^1$ of $\mathcal{E}^1 = \mathcal{E} \otimes \mathcal{O}(-rE^1)$ with zeroes scheme $W^1$ equal to the disjoint union $Z^1 \cup R^1$, with $R^1 = (\pi^1)^{-1}(R) \simeq R$. Here $Z^1$ is the scheme union
of the pairwise intersections $D^1_{ij}$. Continuing this way, we construct a sequence of blowups,

$$X^r \xrightarrow{\pi^r} \cdots \xrightarrow{\pi^2} X^1 \xrightarrow{\pi^1} X$$

such that ultimately the bundle

$$\mathcal{E}^r = \mathcal{E} \otimes \mathcal{O}(-rE^1 - (r - 1)E^2 - \cdots - E^r)$$

is endowed with a section $s^r$ whose scheme of zeros is exactly

$$R^r = (\pi^r)^{-1} \cdots (\pi^2)^{-1}(\pi^1)^{-1}(R) \simeq R.$$ 

Thus, we get the formula

$$\deg(R) = \int_X \pi^1 \cdots \pi^r (c_n(\mathcal{E}^r)).$$

The right hand side may clearly be written in terms of the intersection numbers $D^j \cdot c_j(\mathcal{E})$.

4. Examples

Set for short $c_i = c_i \mathcal{E}$. Let

$$\sigma_i = \sigma_i(D) = \sum_{i_0 + \cdots + i_m = i} D_{i_0}^0 \cdots D_{i_m}^m$$

denote the sum of all monomials of degree $i$ in the classes of the $D_i$.

4.1. $m = 1$. We find

$n = 3 : \quad \deg(R) = c_3 - D_0D_1c_1 + D_0^2D_1 + D_0D_1^2.$

$n = 4 : \quad \deg(R) = c_4 - D_0D_1c_2 + (D_0^2D_1 + D_0D_1^2)c_1 - (D_0^3D_1 + D_0^2D_1^2 + D_0D_1^3).$

These first few cases suggest the formula for general $n$, still with $m = 1$,

$$\deg(R) = c_n - \sum_{i=1}^{n-2} (-1)^{n-i} \sigma_{n-i}(D)c_i - (-1)^n \sigma_n(D).$$

which will be generalized in the sequel.
4.2. **Aluffi’s formula.** This was explained to us by P. Aluffi. In fact, nearly closed formula can be achieved using Fulton’s residual intersection formula (RIF) [11, 9.2.3, p. 163], instead of the above blowup sequence. It requires the knowledge of the Segre class of the excess locus $Z = \bigcup D_{ij}$. This is rendered feasible thanks to Aluffi’s formula for the Segre class of the singular scheme of a normal crossing divisor $D = \sum D_i$, (cf. [2], proof of Lemma II.2). The formula reads

$$s(Z, X) = \left[ \left( 1 - \frac{1 - D}{\prod_{i=0}^{m} (1 - D_i) / (1 - D_i)} \right) \cap [X] \right] \otimes_X O(D).$$

The right hand side uses Aluffi’s $\cdot \otimes L$ operation on the Chow group introduced in [1]: if $a_i$ is a class of codimension $i$ in the Chow group, and $L$ is a line bundle, then

$$a_i \otimes L = \frac{a_i}{c(L)^i}.$$

We have

$$s(Z, X) = [X] - \left[ \left( 1 - \frac{1 - D}{\prod_{i=0}^{m} (1 - D_i) / (1 - D_i)} \right) \cap [X] \right] \otimes_X O(D)$$

The operation $\cdot \otimes L$ behaves well with respect to Chern classes of ‘rank 0 bundles’(!). That is: if $E, F$ are bundles of the same rank, then

$$((c(E)/c(F)) \cap a) \otimes L = (c(E \otimes L)/c(F \otimes L)) \cap (a \otimes L).$$

We have to pretend that the fraction in (6) is the quotient of the Chern classes of two bundles of the same rank, so regard the second piece as

$$\left( \left( 1 - D \right) \cdot 1^m \right) \cap [X] \otimes_X O(D)$$

that is, view the numerator as the Chern class of the bundle $O(-D) \oplus O^{\pm m}$. Tensoring by $O(D)$, the numerator turns from

$$(1 - D) \cdot 1^m, \text{ into } (1 - D + D)(1 + D)^m = (1 + D)^m;$$

the denominator goes from $\prod (1 - D_i)$ to $\prod (1 + D - D_i)$; and again nothing happens to the term $[X]$, because it is of codimension 0. Bottom line,

$$s(Z, X) = [X] - \frac{(1 + D)^m}{\prod_{i=0}^{m} (1 + D - D_i)} \cap [X].$$

We apply Fulton’s RIF, in his notation, to the regular embedding $i : X \to Y$ with $X$ as above, and $i$ equal to the zero section of $Y := \mathcal{E}$; we take for $f : V = X \to Y = \mathcal{E}$ the given section $s$ as in the beginning of §3. Now we have, in one hand, $X \cdot V = c_n(\mathcal{E})$ by [11, Ex. 3.3.2, p. 67 or 6.3.4, p. 105]. Presently, the residual intersection class $\mathcal{R}$ is equal to the class of the finite part $R$ since the latter is disjoint from $Z$. Hence we may write

$$[\mathcal{R}] = c_n(\mathcal{E}) \cap [X] - [c(\mathcal{E}) \cap s(Z, V)]_n,$$
where \([\cdot]_n\) denotes the \(n\)–codimensional part of a cycle. We get,

\[
[R] = [c(\mathcal{E}) \cap ([X] - s(Z, V))]_n
\]

\[
= c(\mathcal{E}) \cap \left[ \frac{(1 + D)^m}{\prod_{i=0}^{m} (1 + D - D_i)} \right]_n.
\]

Hence

(7) \quad \deg R = \int_X \left[ \frac{c(\mathcal{E})(1 + D)^m}{\prod_{i=0}^{m} (1 + D - D_i)} \right].

4.2.1. Remark. Let us recall a nice observation in [3] to the effect that, if \(F\) is a virtual sheaf of rank \(n - 1\) then \(c_n(F \otimes L) = c_n(F)\) for any line bundle \(L\). We may write

\[
\frac{c(\mathcal{E})(1 + D)^m}{\prod_{i=0}^{m} (1 + D - D_i)} = c \left( \mathcal{E} + \mathcal{O}(D)^{\otimes m} - \bigoplus_{i=0}^{m} \mathcal{O}(D - D_i) \right).
\]

Thus, in degree \(n\) we find

\[
\left[ \frac{c(\mathcal{E})(1 + D)^m}{\prod_{i=0}^{m} (1 + D - D_i)} \right]_n = \left[ \frac{c(\mathcal{E} \otimes \mathcal{O}(-D))}{\prod_{i=0}^{m} (1 - D)} \right]_n.
\]

This can be expanded as

\[
\sum_{i=0}^{n} c_i(\mathcal{E} \otimes \mathcal{O}(-D))\sigma_{n-i}(D) = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n-j}{i-j} c_j(\mathcal{E})(-D)^{i-j} \sigma_{n-i}(D).
\]

4.2.2. Remark. The preprint by F. Catanese, S. Hoşten, A. Khetan and B. Sturmfels [6] also contains a similar formula, deduced by different methods and in the context of another subject, namely, algebraic statistics.

4.3. Foliations on \(\mathbb{CP}^n\). For \(\mathcal{E} = \Omega^1_{\mathbb{CP}^n}(d)\), the above reduces to

\[
\deg R = \text{coefficient of } h^n \text{ in } \left[ \frac{(1 - h)^{n+1}}{\prod_{i=0}^{m} (1 - d_i h)} \right]
\]

(8) \quad = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i} \sigma_{n-i}(d)

with \(\sigma_{n-i}\) the complete symmetric function of degree \(n - i\) in \(d_0, \ldots, d_m\).

One further application of Remark 4.2.1 yields the following positivity result.
4.3.1. **Proposition.** Assume at least one $d_i \geq 2$ (and of course all $d_i \geq 1$). Then we have $\deg Z > 0$.

**Proof.** We show that, under the change of variables $d_i = e_i + 1$, the formula (8) becomes

$$\deg R = \sum_{0}^{n} \left( m - 1 \right) \sigma_{n-i}(e).$$

The latter is obviously $> 0$ if some $e_i > 0$. To show the last equality, we use 4.2.1 to write

$$c_n \left( \mathcal{O}(-h)^{\oplus n+1} - \bigoplus_{0}^{m} \mathcal{O}(-d_i h) + \mathcal{O}^{\oplus m-1} \right)$$

$$= c_n \left( \mathcal{O}^{\oplus n+1} - \bigoplus_{0}^{m} \mathcal{O}(h - d_i h) + \mathcal{O}(h)^{\oplus m-1} \right)$$

$$= \left[ c \left( \mathcal{O}(h)^{\oplus m-1} - \bigoplus_{0}^{m} \mathcal{O}(h - d_i h) \right) \right]_{n}$$

$$= \left[ \frac{(1 + h)^{m-1}}{\prod_{0}^{m} (1 - e_i h)} \right]_{n}$$

$$\sigma_n(e) + (m-1)\sigma_{n-1}(e) + \binom{m-1}{2} \sigma_{n-2}(e) + \cdots$$

\[\square\]

**Acknowledgments.** We are grateful to J.V.Pereira and to P. Aluffi for very helpful conversations. We also thank the referee for his suggestions.

**References**


Fernando Cukierman Marcio G. Soares Israel Vainsencher
UBA - Ciudad Universitaria Av. Antonio Carlos 6627 Av. Antonio Carlos 6627
1428 Ciudad de Buenos Aires 31 270-901 Belo Horizonte 31 270-901 Belo Horizonte
Argentina Brasil Brasil
fcukier@dm.uba.ar msoares@mat.ufmg.br israel@mat.ufmg.br