Algebras of Matrix Differential Operators

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Introduction

The study of commuting pairs of ordinary differential operators dates back to the begging of the twentieth century, with the pioneering works of Schur and Burchnall and Chaundy. Let

\[ D_2 = \partial^2 + V(x), \quad D_3 = a_3(x)\partial^3 + a_2(x)\partial^2 + a_1(x)\partial + a_0(x) \]

and now impose the relation \([D_2, D_3] = 0\).

Then we can assume \(a_3(x) = 1\) up to scaling, and we also get \(a_2(x) = A_2\) and arbitrary constant. Furthermore

\[ a_1 = \frac{3}{2}V + A_1, \quad a_0 = A_2V + \frac{3}{4}V' + A_0. \]
Finally from the zero order term of \([D_2, D_3]\) we get, after two trivial integrations,

\[ V''^2 + 2V^3 + 4A_1V^2 - 2A_{-1}V + A_{-2} = 0. \]

Therefore \( V = V(x) \) is an elliptic function, that is a doubly periodic meromorphic function or a degeneration of one, such as a trigonometric or a rational function.

A degeneration yields the simplest example \( V = -\frac{2}{x^2} \) and

\[ D_2 = \partial^2 - \frac{2}{x^2}, \quad D_3 = \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3}, \]

and \( D_3^2 = D_2^3. \)
If we keep one of the periods finite we may choose \( V = -\frac{2}{\sin^2(x)} \), then

\[
D_2 = \partial^2 - \frac{2}{\sin^2 x}, \quad D_3 = \partial^3 + \left(1 - \frac{3}{\sin^2 x}\right)\partial + \frac{3\cos x}{\sin^3 x},
\]

and \( D_3^2 = D_2(D_2 + I)^2 \).

The punch line is that only very special choice of \( V(x) \) allows for the existence of a differential operator of order three that would commute with one of order two.

Burchnall and Chaundy pointed out that any commuting pair \((Q, P)\) of ordinary differential operators

\[
Q = \partial^m + u_2(x)\partial^{m-2} + \cdots + u_m(x), \quad P = \partial^n + v_2(x)\partial^{n-2} + \cdots + u_n(x)
\]

satisfy a polynomial relation \( F(Q, P) = 0 \).
Therefore the eigenvalues of the joint eigenvalue problem

\[ Q\psi = z\psi, \quad P\psi = w\psi \]

satisfy the algebraic relation \( F(z, w) = 0 \), the spectral curve.

These commuting pairs are classified by a set of algebro-geometric data. This set consist of the spectral curve \( \Gamma \) with a mark point \( \gamma_\infty \), a holomorphic vector bundle \( E \) on \( \Gamma \) and some additional data related to the local structure of \( \Gamma \) and \( E \) in a neighborhood of \( \gamma_\infty \).

Matrix Orthogonal Polynomials

Let $W = W(x)$ be a weight matrix of size $N$ on the real line. By this we mean a complex $N \times N$-matrix valued integrable function on the interval $(a, b)$ such that $W(x)$ is positive definitive almost everywhere and with finite moments $M_n$ for all $n$,

$$M_n = \int_a^b x^n W(x) \, dx$$

Let $A$ be the algebra of all $N \times N$ matrices over $\mathbb{C}$, and let $A[x]$ be the algebra of all polynomials in the undetermined $x$ with coefficients in $A$.

Stieltjes 1894, M. G. Krein 1949.
We introduce the following Hermitian sesquilinear form in $A[x]$:

$$(P, Q) = \int_a^b P(x)W(x)Q(x)^* \, dx.$$ 

$$(aP + bQ, R) = a(P, R) + b(Q, R),$$

$$(TP, Q) = T(P, Q),$$

$$(P, Q)^* = (Q, P),$$

$$(P, P) \geq 0; \quad \text{if} \ (P, P) = 0 \ \text{then} \ P = 0.$$ 

In other words we have that $A[x]$ is a left inner product A-module.
Principle of measurable choice (E. A. Azoff)

Let $X$ and $Y$ be complete separable metric spaces and $E$, a closed \( \sigma \)-compact subset of $X \times Y$. Then $\pi_1(E)$ is a Borel set in $X$ and there exists a Borel function $\phi : \pi_1(E) \to Y$ whose graph is contained in $E$.

Let $H(N)$ and $U(N)$ denote respectively, the space of all Hermitian $N \times N$ matrices and the unitary group.

**Corollary.** There is a Borel function $\psi : H(N) \to U(N)$ associating with each Hermitian matrix $H$, a unitary matrix $\psi(H)$ such that $\psi(H)^* H \psi(H)$ is real diagonal.

**Proposition.** Let $P = \sum_{0 \leq j \leq n} x^j P_j$ be an $A$-polynomial of degree $n$. Then $\ker (P, P) = \bigcap_{0 \leq j \leq n} \ker(P_j^*)$. In particular $(P, P)$ is nonsingular if $P_j$ is nonsingular for some $0 \leq j \leq n$. Moreover $(P, P) = 0$ implies $P = 0$. 
**Proposition** Let $V_n = \{ F \in A[x] : \deg F \leq n \}$ for all $n \geq 0$, $V_{-1} = 0$ and $V_{n-1}^\perp = \{ H \in V_n : (H, F) = 0 \mbox{ for all } F \in V_{n-1} \}$. Then $V_{n-1}^\perp$ is a left free $A$-module of dimension one and

(i) $V_n = V_{n-1} \oplus V_{n-1}^\perp$ for all $n \geq 0$.

(ii) There is a unique monic polynomial $P_n$ in $V_{n-1}^\perp$ and it is of degree $n$ for all $n \geq 0$.

**Corollary** $\{P_n\}_{n \geq 0}$ is the unique sequence of monic orthogonal polynomials in $A[x]$. Any sequence $\{Q_n\}_{n \geq 0}$ of orthogonal polynomials in $A[x]$ is of the form $Q_n = A_n P_n$ where $A_n \in GL_N(\mathbb{C})$. Moreover the sequence $\{P_n\}_{n \geq 0}$ satisfies a three term recursion relation of the form

$$xP_n(x) = A_n P_{n-1}(x) + B_n P_n(x) + P_{n+1}(x)$$
Matrix Differential Operators

A differential operator $D$ could be made to act either on the left or on the right on $A[x]$. If one wants to have matrix weights $W$ that are not direct sums of scalar one and that have matrix polynomials as their eigenfunctions, one should settle for right-hand-side differential operators. We agree now to say that $D$ given by

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \quad \partial = \frac{d}{dx},$$

acts on $P(x)$ by means of

$$PD = \sum_{i=0}^{s} \partial^i(P)(x)F_i(x).$$
Proposition. Let $W = W(x)$ be a weight matrix of size $N$ and let $\{P_n\}_{n \geq 0}$ be the sequence of monic orthogonal polynomials in $A[x]$. If

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \quad \partial = \frac{d}{dx},$$

is a linear right-hand side ordinary differential operator of order $s$ such that

$$P_n D = \Lambda_n P_n \quad \text{for all} \quad n \geq 0$$

with $\Lambda_n \in A$, then $F_i = F_i(x) \in A[x]$ and $\deg F_i \leq i$. Moreover $D$ is determined by the sequence $\{\Lambda_n\}_{n \geq 0}$. 
If

\[ F_i(x) = \sum_{j=0}^{i} x^j F^i_j(D), \]

then

\[ \Lambda_n = \sum_{i=0}^{s} [n]_i F^i_i(D) \quad \text{for all} \quad n \geq 0, \]

where

\[ [\nu]_i = \nu(\nu - 1) \cdots (\nu - i + 1), \quad [\nu]_0 = 1. \]

Let

\[ \mathcal{D} = \left\{ D = \sum_{i=0}^{s} \partial^i F_i(x) : F_i \in A[x], \deg F_i \leq i \right\}. \]
We are ready to introduce the main character of our tale.

Given a sequence of orthogonal polynomials \( \{Q_n\}_{n \geq 0} \) we are interested in the following algebra

\[
D(W) = \{ D : Q_nD = \Gamma_n(D)Q_n, \ \Gamma_n(D) \in A \text{ for all } n \geq 0 \}.
\]

**Proposition** Given a sequence \( \{Q_n\}_{n \geq 0} \) of orthogonal polynomials and \( D \in D(W) \) let \( \Gamma(D, n) = \Gamma_n(D) \). Then \( D \mapsto \Gamma(D, n) \) is a representation of \( D(W) \) into \( A \), for each \( n \geq 0 \). Moreover the sequence of representations \( \{\Gamma_n\}_{n \geq 0} \) separates the elements of \( D(W) \).
The adjoint operation

**Proposition.** If $D \in \mathcal{D}$ is a right-hand side linear differential operator which satisfies the symmetry condition $(PD, Q) = (P, QD)$ for all $P, Q \in A[x]$, then $D \in \mathcal{D}(W)$.

**Theorem.** Let $\{P_n\}_{n \geq 0}$ be the sequence of monic orthogonal polynomials associated to the weight matrix $W = W(x)$. Given $D = \sum_{i=0}^{s} \partial^i F_i \in \mathcal{D}(W)$ let $\tilde{D} = \sum_{i=0}^{s} \partial^i G_i \in \mathcal{D}$, where the $G_i$ are defined inductively by

(i) $G_0 = (P_0, P_0) \Lambda_0(D)^*(P_0, P_0)^{-1}$, and

(ii) $j!G_j = (P_j, P_j) \Lambda_j(D)^*(P_j, P_j)^{-1} P_j - \sum_{i=0}^{j-1} \partial^i (P_j) G_i$ for $1 \leq i \leq s$.

Then $(PD, Q) = (P, Q\tilde{D})$ for all $P, Q \in A[x]$. 
Corollary. For any $D \in \mathcal{D}(W)$ there exists a unique differential operator $D^* \in \mathcal{D}(W)$ such that $(PD, Q) = (P, QD^*)$ for all $P, Q \in A[x]$. We shall refer to $D^*$ as the adjoint of $D$. The map $D \mapsto D^*$ is a *-operation in the algebra $\mathcal{D}(W)$, and the orders of $D$ and $D^*$ coincide. Moreover $S(W)$ is a real form of the space $\mathcal{D}(W)$, i.e.

$$\mathcal{D}(W) = S(W) \oplus iS(W).$$

If $\{Q_n\}_{n \geq 0}$ is a sequence of orthogonal polynomials and $\{\Gamma_n\}_{n \geq 0}$ is the corresponding sequence of representations of $\mathcal{D}(W)$, then

$$\Gamma_n(D^*) = (Q_n, Q_n)\Gamma_n(D)^*(Q_n, Q_n)^{-1}$$

for all $D \in \mathcal{D}(W)$. In particular if $\{Q_n\}_{n \geq 0}$ is a sequence of orthonormal polynomials then $D$ is symmetric if and only if $\Gamma_n(D)$ is Hermitian for all $n \geq 0$. 

Corollary. The representations $\Lambda_n$ of $\mathcal{D}(W)$ are completely reducible.

We observe that given a weight matrix $W(x)$ the algebra $\mathcal{D}(W)$ is most likely going to be trivial. By integration by parts one finds necessary and sufficient conditions on smooth weights $W$ to have a symmetric second order differential operator. A similar result holds for a symmetric differential operator of any order. Therefore one has, modulo the difficult task of explicitly solving the corresponding system of differential equations, a way of getting $S(W)$ and hence $\mathcal{D}(W)$. 
The ad-conditions

We have a sequence of representations \( \{ \Lambda_n \}_{n \geq 0} \) of \( \mathcal{D}(W) \) into \( A \). In other words we have a homomorphism \( \Lambda \) of \( \mathcal{D}(W) \) into the direct product of \( \mathbb{N}_0 \) copies of \( A \). Moreover \( \Lambda \) is injective. To give a precise description of the range of this homomorphism, recall that our starting point is a weight matrix \( W(x) \) on the real line and its unique sequence of monic orthogonal polynomials \( \{ P_n \}_{n \geq 0} \), together with the three-term recursion relation

\[
x P_n(x) = A_n P_{n-1}(x) + B_n P_n(x) + P_{n+1}(x), \quad n \geq 0,
\]

where we put \( P_{-1}(x) = 0 \).
It is convenient to introduce the block tridiagonal matrix $L$

\[
L = \begin{pmatrix}
B_0 & I \\
A_1 & B_1 & I \\
& \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

The recursion relation now takes the form

\[
LP = xP
\]

where $P$ stands for the vector

\[
P(x) = \begin{pmatrix}
P_0(x) \\
P_1(x) \\
P_2(x) \\
\vdots
\end{pmatrix}
\]
Assume that \( D \in \mathcal{D}(W) \), i.e.,

\[
P_n D = \Lambda_n P_n \quad n \geq 0.
\]

If \( \Lambda \) denotes the block diagonal matrix

\[
\Lambda = \begin{pmatrix}
\Lambda_0 & & \\
& \Lambda_1 & \\
& & \ddots
\end{pmatrix}
\]

we observe that from (1) we get, for any integer \( m \geq 0 \),

\[
(adL)^m(\Lambda)P = (L - xI)^m \Lambda P.
\]
**Theorem** If $D \in \mathcal{D}(W)$ and $\Lambda$ is the block diagonal matrix with $\Lambda_n = \Lambda_n(D)$ we have

$$(ad L)^{m+1}(\Lambda) = 0$$

for some $m$. Conversely, if $\Lambda$ is a block diagonal matrix satisfying this condition for some $m \geq 0$, then there is a unique differential operator $D$ in $\mathcal{D}(W)$ such that $\Lambda_n = \Lambda_n(D)$ for all $n \geq 0$. Moreover the order of $D$ is equal to the minimum $m$ satisfying (2).

Scalar examples

Classical Orthogonal Polynomials

Jacobi:

\[ w(x) = (1 - x)^\alpha (1 + x)^\beta, \quad -1 < x < 1; \quad \alpha, \beta > -1 \]

\( \alpha = \beta \) Gegenbauer; \( \alpha = \beta = \frac{1}{2} \) Chebyshev first kind;

\( \alpha = \beta = -\frac{1}{2} \) Chebyshev second kind; \( \alpha = \beta = 0 \) Legendre

Laguerre:

\[ w(x) = x^\alpha e^{-x}, \quad x > 0; \quad \alpha > -1 \]

Hermite:

\[ w(x) = e^{-x^2}, \quad -\infty < x < \infty \]
\[ D(p_n)(x) = a_2(x)p''_n(x) + a_1(x)p'_n(x) = \lambda np_n(x) \]

Jacobi:

\[ (1 - x^2)p''_n + (\beta - \alpha - (\alpha + \beta + 2)x)p'_n = -n(n + \alpha + \beta + 1)p_n \]

changing variables: \( x = 1 - 2z \)

\[ z(1 - z)p''_n + (\alpha + 1 - (\alpha + \beta + 2)z)p'_n = -n(n + \alpha + \beta + 1)p_n \]

Laguerre:

\[ xp''_n + (\alpha + 1 - x)p'_n = -np_n \]

Hermite:

\[ p''_n - 2xp'_n = -2np_n \]

A matrix instructive example

\[ W_a(x) = e^{-x^2} e^{Ax} e^{A^* x}, \quad -\infty < x < \infty \]

\[ A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad a \in \mathbb{C}^x \]

Rodrigues’ formula

\[ P_n(x) = (-2)^{-n} e^{x^2} \begin{pmatrix} 1 & 0 \\ 0 & 2 \frac{1}{|a|^2 n + 2} \end{pmatrix} \left[ e^{-x^2} \begin{pmatrix} 1 + |a|^2 x^2 + \frac{|a|^2 n}{2} \\ \bar{a} x \\ \bar{a} x \end{pmatrix} \right]^{(n)} \]

\[ \times \begin{pmatrix} 1 & -ax \\ -\bar{a} x & 1 + |a|^2 x^2 \end{pmatrix} \]

(Grünbaum, Durán, IMRN 2004)
If $a = |a|e^{2i\theta}$, then

$$
\begin{pmatrix}
  e^{-i\theta} & 0 \\
  0 & e^{i\theta}
\end{pmatrix} W_a(t) 
\begin{pmatrix}
  e^{i\theta} & 0 \\
  0 & e^{-i\theta}
\end{pmatrix} = W_{|a|}(t).
$$

This implies that $D(W_a)$ and $D(W_{|a|})$ are conjugated. Therefore we will assume that $a > 0$. M. Castro and F. A. Grünbaum, (see IMRN, 2006) experimentally, with assistance from symbolic computation, conjectured that $D(W)$ was generated by the following differential operators:

$$
D_1 = -\frac{1}{2} \partial^2 + \partial(xI - aE_{12}) + E_{11}
$$

$$
D_2 = \partial^2 \frac{a^2}{2}(axE_{12} - E_{11}) + \partial a(axE_{22} + E_{12} - E_{21}) + 2E_{22}
$$
\[
D_3 = \partial^2 a^2 \left( a^2 x^2 E_{12} + ax(E_{22} - E_{11}) - E_{21} \right) \\
+ \partial 2a \left( a(a^2 + 2)xE_{12} + E_{22} - (a^2 + 1)E_{11} \right) + 2(a^2 + 2)E_{12}
\]

\[
D_4 = -\partial^2 \frac{a^2}{4} E_{12} + \partial \frac{a}{2} (E_{11} - E_{22}) + E_{21}
\]

\[
\Lambda_1 = nI + E_{11}, \quad \Lambda_2 = fE_{22}, \quad \Lambda_3 = f(f + a^2)E_{12}, \quad \Lambda_4 = E_{21},
\]

where \( f = a^2n + 2 \).
The set \( \{D^i_1, D^i_1D_2, D^i_1D_3, D^i_1D_4 : i \geq 0 \} \) is a basis of \( \mathcal{D}(W) \), and the multiplication table is

\[
D_2D_1 = D_1D_2, \quad D_2D_2 = (a^2D_1 + 2I)D_2, \quad D_2D_3 = 0 \\
D_2D_4 = (a^2D_1 + 2I)D_4, \quad D_3D_1 = (D_1 - I)D_3 \\
D_3D_2 = (a^2D_1 + (2 - a^2)I)D_3, \quad D_3D_3 = 0 \\
D_3D_4 = a^4D_1^2 - a^2D_1D_2 + a^2(4 - a^2)D_1 + (a^2 - 2)D_2 + 2(2 - a^2)I \\
D_4D_1 = (D_1 + I)D_4, \quad D_4D_2 = 0 \\
D_4D_3 = (a^2D_1 + (2 + a^2)I)D_2, \quad D_4D_4 = 0
\]

Moreover \( D_1 \) and \( D_{-1} = D_3 + 4D_4 \) generates \( \mathcal{D}(W) \).

\((T, 2007)\)
The proof of the proposition starts with the following

**Proposition** Let \( P_n = \sum_{i=0}^{n} x^i B^n_i \) and \( D = \sum_{j=0}^{s} \partial^j F_j(t) \), with \( F_j(t) = \sum_{i=0}^{j} t^i F^j_i \). Then \( D \in D(W) \) if and only if

\[
\sum_{r=0}^{n-m} B^n_{m+r} \left( \sum_{i=0}^{s-r} [m + r]_{i+r} F^{i+r}_i \right) - \left( \sum_{i=0}^{s} [n]_i F^i_i \right) B^n_m = 0, \tag{3}
\]

for all \( 0 \leq m \leq n \), \( 0 \leq n \).

This is an infinite system of linear equations where the unknowns are the matrices \( F^j_i \). In order to simplify this system we take advantage of an involutive automorphism that the algebra \( D(W) \) possesses.
Let
\[ T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
then
\[ W(-t) = TW(t)T. \]

Given \( D = \sum_{0 \leq i \leq s} \partial^i F_i \in \mathcal{D}(W) \) let \( \tilde{D} = \sum_{0 \leq i \leq s} \partial^i (-1)^i T\tilde{F}_i T \), where \( \tilde{F}_i(t) = F_i(-t) \). Then \( \tilde{D} \in \mathcal{D}(W) \) and the map \( D \mapsto \tilde{D} \) is an involutive automorphism of the algebra \( \mathcal{D}(W) \). Let
\[
\mathcal{D}_1(W) = \{ D \in \mathcal{D}(W) : \tilde{D} = D \}, \quad \mathcal{D}_{-1}(W) = \{ D \in \mathcal{D}(W) : \tilde{D} = -D \}.
\]

Then
\[
\mathcal{D}(W) = \mathcal{D}_1(W) \oplus \mathcal{D}_{-1}(W).
\]
Then we are able to find the space of all differential operators in $\mathcal{D}_1(W)$ of order less or equal to two, which turns out to be of dimension three and generated by $I, D_1$ and $D_2$. Similarly we determine the space of all differential operators in $\mathcal{D}_{-1}(W)$ of order less or equal to two, which turns out to be of dimension two and generated by $D_3$ and $D_4$.

Let $\mathcal{A}(W)$ be the subalgebra of $\mathcal{D}(W)$ generated by all $D \in \mathcal{D}(W)$ of order less or equal to two, and let

$$\mathcal{A}_1(W) = \mathcal{A}(W) \cap \mathcal{D}_1(W) \quad \text{and} \quad \mathcal{A}_{-1}(W) = \mathcal{A}(W) \cap \mathcal{D}_{-1}(W).$$

In order to prove that $\mathcal{A}(W) = \mathcal{D}(W)$, let $\mathcal{C}_1$ be equal to the linear space generated by the leading coefficients of all $D \in \mathcal{D}_1(W)$ of order less or equal to two.
The linear space $C_1$ is a two dimensional subalgebra of $A[t]$. Moreover, for any $F \in C_1$ and $r \in \mathbb{N}$ there exists $D \in A_1(W)$ of order $2r$, with leading coefficient $F$. Then we establish that there is no $D \in D_1(W)$ of odd order, and that if $F$ is the leading coefficient of a differential operator $D \in D_1(W)$ then $F \in C_1$, from where it follows that $A_1(W) = D_1(W)$.

Let $D_n$, $D_{1,n}$ and $D_{-1,n}$ be, respectively, the subspaces of $D(W)$, $D_1(W)$ and $D_{-1}(W)$ of all differential operators of order less or equal to $n$. Similarly let $A_{1,n}$ and $A_{-1,n}$ be, respectively, the subspaces of $A_1(W)$ and $A_{-1}(W)$ of all differential operators of order less or equal to $n$. 


Then

**Theorem** For any $r \geq 1$ we have,

(i) $\dim(\mathcal{D}_{1,2r}/\mathcal{D}_{1,2(r-1)}) = 2$,

(ii) $\dim(\mathcal{D}_{-1,2r}/\mathcal{D}_{-1,2(r-1)}) = 2$,

(iii) $\dim(\mathcal{D}_{2r}/\mathcal{D}_{2(r-1)}) = 4$,

(iv) $\dim(\mathcal{A}_{-1,2r}/\mathcal{A}_{-1,2(r-1)}) = 2$,

(v) $\mathcal{A}_{-1}(W) = \mathcal{D}_{-1}(W)$.

Statement (iii) was conjectured by Castro and Grünbaum. At this point since $\mathcal{A}_{1}(W) = \mathcal{D}_{1}(W)$ and $\mathcal{A}_{-1}(W) = \mathcal{D}_{-1}(W)$ it follows that the algebras $\mathcal{D}(W)$ and $\mathcal{A}(W)$ coincide. In other words the algebra $\mathcal{D}(W)$ is generated by the subspace $\mathcal{D}_{2}(W)$.
Now it follows easily that

\[ \{ D_1^i, D_1^i D_2, D_1^i D_3, D_1^i D_4 : i \geq 0 \} \]

is a basis over \( \mathbb{C} \) of \( \mathcal{D}(W) \).

The element \( D_{-1} = D_3 + 4D_4 \in \mathcal{D}_{-1}(W) \), has two nice properties: the set \( \{ D_1, D_{-1} \} \) generates the algebra \( \mathcal{D}(W) \) and \( D_{-1}^2 \) is a central element. Let \( \mathcal{Z} = \mathbb{C}[D_{-1}^2] \) be the polynomial subalgebra of \( \mathcal{D}(W) \) generated by the algebraically independent element \( D_{-1}^2 \). Then we establish that \( \mathcal{D}(W) \) is a free module over \( \mathcal{Z} \) of dimension eight. More precisely the set

\[ \{ I, D_1, D_1^2, D_1^3 \} \cup \{ D_{-1}, D_1 D_{-1}, D_1^2 D_{-1}, D_{-1} D_1 \} \]

is a \( \mathcal{Z} \)-basis of \( \mathcal{D}(W) \).
The algebra $\mathcal{D}(W)$ is also presented by generators and relations: it is generated by two elements $E$ and $F$, and the relations are

\[
F^2 E - EF^2 = 0,
\]
\[
F^4 - 2a^4 F^2 E^2 - 8a^2 F^2 E - 8F^2 + a^8 E^4 + 8a^6 E^3
\]
\[- a^4(a^4 - 24)E^2 - 4a^2(a^4 - 8)E - 4(a^4 - 4)I = 0,
\]
\[- 4a^6 E^3 + 2a^2 EF^2 - a^2 FEF - 2a^4 E^2 + 2F^2 + 4a^2(a^4 - 12)E
\]
\[+ 8(a^4 - 4)I = 0,
\]
\[
E^3 + E^2[E, F] - a^4 E^2 F - a^4 E^2[ E, F ] + a^2(a^2 - 4)EF
\]
\[+ a^2(a^2 - 4)E[E, F ] + 2(a^2 - 2)F + 2(a^2 - 2)[E, F ] = 0,
\]
\[
F^3 - 4a^4 EFE - 8a^2( EF + FE ) - 16F = 0,
\]
\[
FE^2 + E^2 F - 2 EFE - F = 0.
\]
Then we compute the center of $\mathcal{D}(W)$. Set

$$Z = \left(\frac{3}{4}D_{-1}^2 + (a^4 - 12)I\right)D_1 - 6a^2 - a^4D_1^3.$$

We establish that the center $\mathcal{Z}(W)$ of the algebra $\mathcal{D}(W)$ is generated by $D_{-1}^2$ and $Z$, and that it is isomorphic to the affine algebra of the elliptic curve

$$4x^3 - y^2 - 12xy + (a^4 - 36)x^2 - 4(a^4 - 4)y - 24(a^4 - 4)x - 4(a^4 - 4)^2 = 0.$$

A big, and rather blurry, challenge is that of finding the appropriate algebro-geometric objects associated to $\mathcal{D}(W)$ for any weight matrix $W$, that reduce in the abelian case to a curve and a vector bundle on it. Further study of this example may be instructive in this respect.
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