Uniformization by radicals.

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History

Let K be a field,

$$p(y) \in K[y]$$

an irreducible polynomial degree d.

Abel:

if $d > 4$ then there are polynomials p not solvable by radicals.

If $\lambda \in K$

$$p(\lambda) = 0$$

cannot be (for instance) written:

$$\lambda = ... \sqrt[n_{i+1}]{q_i(a) + \sqrt[n_i]{...}}$$

where $q \in K(x_1, x_2, ..., x_{d+1})$, $a = (a_1, a_2, ..., a_{d+1})$ are the coeff. of p.

Galois :

The Galois group of the splitting field of p :

$$G(p)$$

is not solvable.

Riemann:

$K = \mathbb{C}(x)$

$$p(y) = P(x, y) = 0$$

defines a plane complex algebraic curve and then a Riemann surface X.

The "roots" of p are the alg. functions,

$$G(p) \equiv M(y)$$

$M(y)$ monodromy group of the map:

$$y : X \to \mathbb{CP}^1$$

induced by the projection $(x, y) \to y$.

$M(y)$ topological invariant of the covering can be computed by arcs lifting .

Definition
Let X be a Riemann surface of genus g,

$$R(X)$$

its rational functions field. We say that X is

rationally uniformized by radicals

if there is $y \in R(X)$ such that

$$R(X) = \mathbb{C}(x)(y), \quad X = \{p(x,y) = 0\} :$$

$$M(y) \equiv G(p)$$

is solvable.

Zariski :

Solution to a question posed by Enriques:

Theorem: *If $g \geq 7$ and X has general moduli, X cannot be rationally uniformized by radicals.*

If $g \leq 6$, X has gonality ≤ 5 and hence is rationally uniformized by radicals

Proof of Zariski theorem (Sketch)
Assume $y \in R(X) : M(y)$ solvable, y indecomposable:

$y \neq f \cdot g$ maps of degree > 1 $f : X \to Z$ $g : Z \to \mathbb{CP}^1$.

Fix $x \in \mathbb{CP}^1$ not a branch point, $A = p^{-1}(x)$.

From Galois theory :

1. $d = \deg y = \#A = p^r$, p prime;
2. A is an affine space over $\mathbb{Z}_p$;
3. Galois action $G \times A \to A$ is affine.

Hence:

1. $h \in M$ gives an affine map $h : A \to A$;
2. fixed points of h form affine subspace;
3. ramification index at any branch point is

$$\geq \frac{p^r - p^{r-1}}{2}$$

ramifications are big:example $p \geq 5$ $r = 1$ the ram. index ≥ 2.

Count of moduli gives that X is not general if $g \geq 7$.

Generalization

Theorem (Friedland-Guralnick-Magaard-Neubauer....)
Let X be the general complex curve of genus $g > 3$,

$$y \in R(X)$$

indecomposable (non constant). Then

$$M(y) = A_d \text{ or } S_d$$

the symmetric or the alternating group.

Existence result: S_d is possible (for any algebraic variety);

A_d is possible:
Magaard Volklein: (general curves)

$$d \geq 2g + 1$$

(admissible coverings)

Artebani-P : any curve $d > 12g + 4$ (uses an Algebraic De Rham)

(Brivio-P. for a surface S, A_d is possible if $d >> 0$; open in higher dimension.)

Everything is open in higher dimension.

Problem
Are surfaces uniformized by radicals?
Consider the case of ruled surfaces.

Zariski conjecture

Definition

Let X be a genus g, Riemann surface $R(X)$ its rational functions field, X is

algebraically uniformized by radicals

if there is an algebraic field extension

$$R(X) \subset S = R(Y)$$

which corresponds to a dominant map $\pi : Y \to X$:

Y is rationally uniformized by radicals .

Remark

Rationally uniformized means that there is a $y : X \to \mathbb{CP}^1$: for the Galois closure $L = \overline{R(X)}^y$ of $\mathbb{C}(x) \subset R(X) = \mathbb{C}(x)(y)$ is solvable. Algebraic uniformization by radicals requires to embed

$$R(X) \subset S$$

such that some

$$\mathbb{C}(x) \subset S$$

is solvable.

Zariski wrote :

> Si potrebbe dunque pensare che si possa invece fornire per *ogni* equazione $f = 0$ una risoluzione *multipla* per radicali $x = x(t)$, $y = y(t)$, in cui ad ogni punto (x, y) della curva $f = 0$ corrispondano più valori di t. ... È poco probabile che ciò accada effettivamente, ma in ogni modo si ha qui un nuovo problema, che noi non discutiamo in questa Nota e che potrà essere oggetto di una ulteriore ricerca.
>
> (One may therefore think that for *every* equation $f = 0$ one can find a *multiple* solution by radicals $x = x(t)$, $y = y(t)$, in which several values of t correspond to every given point (x, y) of the curve $f = 0$. ... It is unlikely that this could really happen, but in any case we have here a new problem, which we do not discuss in this Note, and which might be object of further research).

Zariski conjecture. *The general curve of genus $g \geq 7$(??) cannot be algebraically uniformized by radicals.*

The question is to embed $R(X) \subset S$ the rational field of X in S; S obtained by a series of abelian covering of $\mathbb{C}(x)$.

The Zariski conjecture/problem seems very difficult.

We consider a related problem:

Problem
Find a curve algebraic uniformized by radicals
but not rationally uniformized by radicals.

Result: Two examples of curves alg. but not rat. uniformized by radicals:

1. P-Schlesinger: g=7 (Debarre -Fahlaoui) counterexample to a conjecture of Abramovich-Harris conjecture
2. P-Schlesinger-Rizzi g=9.

Remark: If $Y \to X$ is dominant and the gonality of Y is k the gonality of X is $\leq k$. Hence if $k < 5$ both Y and X are rationally uniformized by radicals.

Construction of curves algebraically uniformized by radicals

1. C smooth curve of genus p;
2. $C^{(k)} =$ k^{th}-symmetric power of C;
3. H hyperplane of $C^{(k)}$:
$$H \equiv \{x + C^{(k-1)} \subset C^{(k)}\}.$$
4. X curve, $f : X \to C^{(k)}$ birational onto its image.

Assume

1. C rationally uniform. by rad.
2. $H \cdot f(X) \leq 4$;

Define the correspondence:

$$Y' \in C \times X = \{(p, y) : f(p) = y\}$$

Y normalization of Y'.

Second projection gives map

$$Y \to X;$$

First projection gives $g : Y \to C$ $\deg g \leq 4$

If $y : C \to \mathbb{CP}^1$ has solvable monodromy, $g \circ y$ has solvable monodromy.

Y is rat. uniform. and X is alg. uniform.

For $k = 2$ we find curves in $S = C^{(2)}$ using Riemann Roch for divisor L; $X \in |L|$.

1. Debarre Fahlaoui ($\Delta =$ diagonal in $S = C^{(2)}$)
$$g(C) = 1, \quad L = 3H - K_S = 3H + \frac{\Delta}{2}$$
2. P.R.S.
$$g(C) = 2, \quad L = 3H + K_S = 5H - \frac{\Delta}{2}$$

One proves that the general curve is not rat. unif. by radicals

step 1. $y \in R(X)$ $\deg(y) \geq 5$, $M(y)$ not solvable.

step 2. The gonality of $X\ > 4$.

step 1. follows the proof of Zariski with some refinement on group theory.

Proof that the gonality of $X > 4$
(it is the geometric part).

Two methods:

1. Lazarsfeld : Vector bundle: used by Debarre. Some complications. The vector bundles are not numerically unstable. One cannot argue using Bogomolov theorem etc. .
2. Mumford Tyurin : when $g(C) = 2$, $C^{(2)}$ is the blow up of the Jacobian $J(C)$ of C. $J(C)$ is symplectic.

Consider the second case $g(C) = 2$. Assume by contradiction that any curve $X \in |L|$ has gonality 4 (other cases are easier). Let

$$Z = hilb^4(C^{(2)})$$

for any $X \in |L|$, $X^{(4)} \subset Z$

$$M = \{D \in Z : \ D \in X^{(4)}, \ h^0(X, \mathcal{O}_X(D) > 1\}.$$

One considers the albanese map:

$$alb : hilb^4(C^{(2)}) \to J(C)$$

Following Beauville (Mumford Tyurin) the fibres K^4 of alb outside the exceptional divisor of

$$C^{(2)} \to J(C)$$

are symplectic variety of dimension 6 with respect to a natural form Ω

One proves (part 2 needs some extra work)

1. $M \subset K^4$
2. M is Lagrangian with respect to Ω

Consequence $\quad \dim M \leq 3$

Next translate into projective geometry:

$$|L| = \mathbb{P}^5$$

consider the map:

$$\rho : S = C^{(2)} \longrightarrow |L| = \mathbb{P}^5$$

Look at the incidence correspondence

$$\mathcal{I} \subset M \times \mathbb{P}^5 = \{(D, [X]) : D \subset X\}.$$

The fibers of the projection

$$\pi_2 : \mathcal{I} \longrightarrow \mathbb{P}^5$$

have dimension 1, hence

$$\dim \mathcal{I} = 6$$

The general fibre of

$$\pi_1 : \mathcal{I} \longrightarrow M$$

is a linear space of dimension ≥ 3 :

The point D of M impose only 2 conditions on L :

It follows that the 4 points of $D \in M$ lie on a 4-secant line of $\rho(S)$.

This is impossible by a standard argument.