Uniformization by radicals.

Pirola Rizzi Schlesinger

History

Let $K$ be a field,

$$p(y) \in K[y]$$

an irreducible polynomial degree $d$.

**Abel:**
if $d > 4$ then there are polynomials $p$ not solvable by radicals.

If $\lambda \in K$

$$p(\lambda) = 0$$
cannot be (for instance) written:

$$\lambda = \ldots \sqrt[n_i+1]{q_i(a)} + \sqrt[n_i\ldots]{\ldots}$$

where $q \in K(x_1, x_2, \ldots, x_{d+1})$, $a = (a_1, a_2, \ldots, a_{d+1})$ are the coeff. of $p$.

**Galois:**

The Galois group of the splitting field of $p$:

$$G(p)$$

is not solvable.
Riemann:

\[ K = \mathbb{C}(x) \]

\[ p(y) = P(x, y) = 0 \]

defines a plane complex algebraic curve and then a Riemann surface \( X \).

The ”roots” of \( p \) are the alg. functions,

\[ G(p) \equiv M(y) \]

\( M(y) \) monodromy group of the map:

\[ y : X \to \mathbb{CP}^1 \]

induced by the projection \((x, y) \to y\).

\( M(y) \) topological invariant of the covering can be computed by arcs lifting.
Definition
Let $X$ be a Riemann surface of genus $g$,

$$R(X)$$

its rational functions field. We say that $X$ is

rationally uniformized by radicals

if there is $y \in R(X)$ such that

$$R(X) = \mathbb{C}(x)(y), \quad X = \{p(x, y) = 0\} :$$

$$M(y) \equiv G(p)$$

is solvable.

Zariski :
Solution to a question posed by Enriques:

Theorem: If $g \geq 7$ and $X$ has general moduli, $X$ cannot be rationally uniformized by radicals.

If $g \leq 6$, $X$ has gonality $\leq 5$ and hence is rationally uniformized by radicals.
Proof of Zariski theorem (Sketch)
Assume $y \in R(X) : M(y)$ solvable, $y$ indecomposable:

\[ y \neq f \cdot g \text{ maps of degree } > 1 \quad f : X \to Z \quad g : Z \to \mathbb{CP}^1. \]

Fix $x \in \mathbb{CP}^1$ not a branch point, $A = p^{-1}(x)$.

From Galois theory:
1. $d = \deg y = \# A = p^r$, $p$ prime;
2. $A$ is an affine space over $\mathbb{Z}_p$;
3. Galois action $G \times A \to A$ is affine.

Hence:
1. $h \in M$ gives an affine map $h : A \to A$;
2. fixed points of $h$ form affine subspace;
3. ramification index at any branch point is
   \[ \geq \frac{p^r - p^{r-1}}{2} \]
   \text{ramifications are big: example } p \geq 5 \quad r = 1 \text{ the ram. index } \geq 2.

Count of moduli gives that $X$ is not general if $g \geq 7$. 
**Generalization**

**Theorem** (Friedland-Guralnick-Magaard-Neubauer....)

Let $X$ be the general complex curve of genus $g > 3$,

$$ y \in R(X) $$

indecomposable (non constant). Then

$$ M(y) = A_d \text{ or } S_d $$

the symmetric or the alternating group.
Existence result: \( S_d \) is possible (for any algebraic variety);

\( A_d \) is possible:
Magaard Volklein: (general curves)
\[ d \geq 2g + 1 \]
(admissible coverings)
Artebani-P: any curve \( d > 12g + 4 \) (uses an Algebraic De Rham)
(Brivio-P. for a surface \( S \), \( A_d \) is possible if \( d >> 0 \); open in higher dimension.)

Everything is open in higher dimension.

Problem
Are surfaces uniformized by radicals?
Consider the case of ruled surfaces.
Zariski conjecture

Definition

Let \( X \) be a genus \( g \), Riemann surface \( R(X) \) its rational functions field, \( X \) is

algebraically uniformized by radicals

if there is an algebraic field extension

\[
R(X) \subset S = R(Y)
\]

which corresponds to a dominant map \( \pi : Y \to X \):

\( Y \) is rationally uniformized by radicals.

Remark

Rationally uniformized means that there is a \( y : X \to \mathbb{CP}^1 \) : for the Galois closure \( L = \overline{R(X)^y} \) of \( \mathbb{C}(x) \subset R(X) = \mathbb{C}(x)(y) \) is solvable. Algebraic uniformization by radicals requires to embed

\[
R(X) \subset S
\]

such that some

\[
\mathbb{C}(x) \subset S
\]

is solvable.
Zariski wrote:

Si potrebbe dunque pensare che si possa invece fornire per ogni equazione \( f = 0 \) una risoluzione *multipla* per radicali \( x = x(t), \ y = y(t) \), in cui ad ogni punto \( (x, y) \) della curva \( f = 0 \) corrispondano più valori di \( t \).

... È poco probabile che ciò accada effettivamente, ma in ogni modo si ha qui un nuovo problema, che noi non discutiamo in questa Nota e che potrà essere oggetto di una ulteriore ricerca.

(One may therefore think that for *every* equation \( f = 0 \) one can find a *multiple* solution by radicals \( x = x(t), \ y = y(t) \), in which several values of \( t \) correspond to every given point \( (x, y) \) of the curve \( f = 0 \). ... It is unlikely that this could really happen, but in any case we have here a new problem, which we do not discuss in this Note, and which might be object of further research).

**Zariski conjecture.** *The general curve of genus \( g \geq 7 \) cannot be algebraically uniformized by radicals.*
The question is to embed $R(X) \subset S$ the rational field of $X$ in $S$; $S$ obtained by a series of abelian covering of $\mathbb{C}(x)$.

The Zariski conjecture/problem seems very difficult.

We consider a related problem:

**Problem**
Find a curve algebraic uniformized by radicals but not rationally uniformized by radicals.

**Result**: Two examples of curves alg. but not rat. uniformized by radicals:

1. P-Schlesinger: $g=7$ (Debarre-Fahlaoui) counterexample to a conjecture of Abramovich-Harris conjecture
2. P-Schlesinger-Rizzi $g=9$.

**Remark**: If $Y \to X$ is dominant and the gonality of $Y$ is $k$ the gonality of $X$ is $\leq k$. Hence if $k < 5$ both $Y$ and $X$ are rationally uniformized by radicals.
Construction of curves algebraically uniformized by radicals

1. $C$ smooth curve of genus $p$;
2. $C^{(k)} = k^{th}$-symmetric power of $C$;
3. $H$ hyperplane of $C^{(k)}$:
   \[ H \equiv \{ x + C^{(k-1)} \subset C^{(k)} \} \]
4. $X$ curve, $f : X \to C^{(k)}$ birational onto its image.

Assume
1. $C$ rationally uniform by rad.
2. $H \cdot f(X) \leq 4$;

Define the correspondence:
\[ Y' \in C \times X = \{(p, y) : f(p) = y\} \]
$Y$ normalization of $Y'$.
Second projection gives map
\[ Y \to X; \]
First projection gives $g : Y \to C$ \deg $g \leq 4$

If $y : C \to \mathbb{CP}^1$ has solvable monodromy, $g \circ y$
has solvable monodromy.

$Y$ is rat. uniform. and $X$ is alg. uniform.
For \( k = 2 \) we find curves in \( S = C^{(2)} \) using Riemann Roch for divisor \( L; X \in |L| \).

1. Debarre Fahlaoui (\( \Delta = \) diagonal in \( S = C^{(2)} \))

\[
g(C) = 1, \quad L = 3H - K_S = 3H + \frac{\Delta}{2}
\]

2. P.R.S.

\[
g(C) = 2, \quad L = 3H + K_S = 5H - \frac{\Delta}{2}
\]

One proves that the general curve is not rat. unif. by radicals

step 1. \( y \in R(X) \ deg(y) \geq 5, \ M(y) \) not solvable.

step 2. The gonality of \( X \) > 4.

step 1. follows the proof of Zariski with some refinement on group theory.
Proof that the gonality of $X > 4$
(it is the geometric part).

Two methods:

1. Lazarsfeld: Vector bundle: used by Debarre. Some complications. The vector bundles are not numerically unstable. One cannot argue using Bogomolov theorem etc.

2. Mumford Tyurin: when $g(C) = 2$, $C^{(2)}$ is the blow up of the Jacobian $J(C)$ of $C$. $J(C)$ is symplectic.
Consider the second case $g(C) = 2$. Assume by contradiction that any curve $X \in |L|$ has gonality 4 (other cases are easier). Let

$$Z = \text{hilb}^4(C^{(2)})$$

for any $X \in |L|$, $X^{(4)} \subset Z$

$$M = \{ D \in Z : D \in X^{(4)}, h^0(X, \mathcal{O}_X(D) > 1\}.$$

One considers the albanese map:

$$alb : \text{hilb}^4(C^{(2)}) \to J(C)$$

Following Beauville (Mumford Tyurin) the fibres $K^4$ of alb outside the exceptional divisor of

$$C^{(2)} \to J(C)$$

are symplectic variety of dimension 6 with respect to a natural form $\Omega$
One proves (part 2 needs some extra work)

1. $M \subset K^4$
2. $M$ is Lagrangian with respect to $\Omega$

Consequence $\dim M \leq 3$

Next translate into projective geometry:

$|L| = \mathbb{P}^5$

consider the map:

$\rho : S = C^{(2)} \rightarrow |L| = \mathbb{P}^5$

Look at the incidence correspondence

$\mathcal{I} \subset M \times \mathbb{P}^5 = \{(D, [X]) : D \subset X\}$.

The fibers of the projection

$\pi_2 : \mathcal{I} \rightarrow \mathbb{P}^5$

have dimension 1, hence

$\dim \mathcal{I} = 6$
The general fibre of

\[ \pi_1 : \mathcal{I} \to M \]

is a linear space of dimension \( \geq 3 \):

The point \( D \) of \( M \) impose only 2 conditions on \( L \):

It follows that the 4 points of \( D \in M \) lie on a 4-secant line of \( \rho(S) \).

This is impossible by a standard argument.