

Exceptional planar webs and their associated (exceptional) surfaces

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1. INTRODUCTION TO WEB GEOMETRY

1.1 *webs, abelian relation, rank*

1.2 *examples : algebraic webs, exceptional webs*

1.3 *algebraization, Chern's problem*

2. TOOLS AND RESULTS

2.1 *webs with an infinitesimal symmetry*

2.2 *characterization of maximal rank webs*

2.3 *CDQL webs*

3. FROM WEBS TO PROJECTIVE SURFACES

3.1 *the « canonical map » of a web*

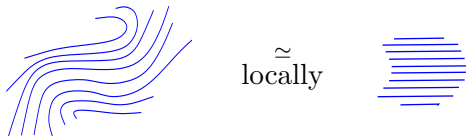
3.2 *Segre's principal directions and generalizations*

3.3 *exceptional projective surfaces*

Web geometry : first definitions

M : a \mathbb{C} -manifold ($M = (\mathbb{C}^2, 0)$)

Definition : Foliation \mathcal{F}



Definition : 1. A completely decomposable (CD) d -web is

$$W_d = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \cdots \boxtimes \mathcal{F}_d$$

where $\mathcal{F}_1, \dots, \mathcal{F}_d$ are foliations in general position on M .

2. d -web W_d : defined by gluing local CD d -webs.

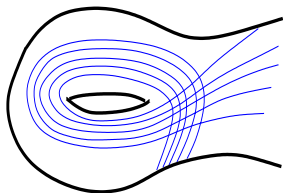


FIG.: A non decomposable 2-web

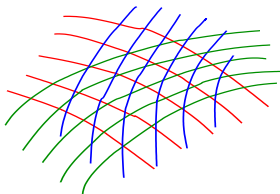


FIG.: A planar 3-web

Web geometry : a classification problem

Definition : W_d and W'_d are *equivalent* if it exists φ invertible s.t.

$$\varphi^*(W'_d) = W_d$$

Web geometry :

Classification of webs up to equivalence.

Classically : equivalence = local analytic equivalence

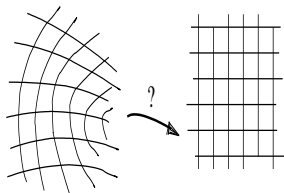


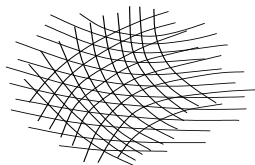
FIG.: The local geometry of a 2-web is trivial in dimension 2

Web geometry : a classification problem

Claim : The local geometry of a planar 3-web is rich

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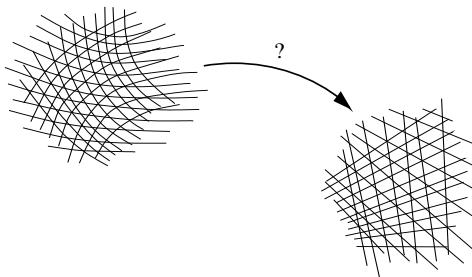
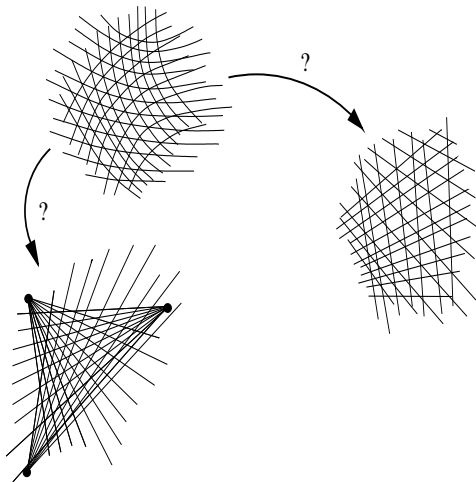


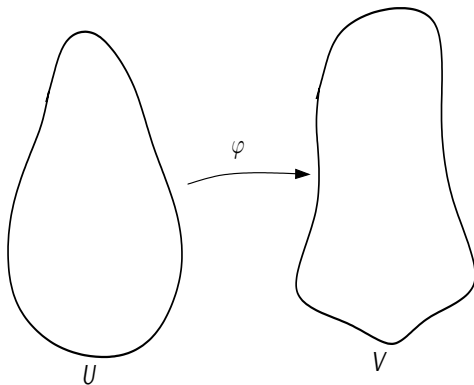
FIG.: Problem of linearization

Web geometry : a classification problem

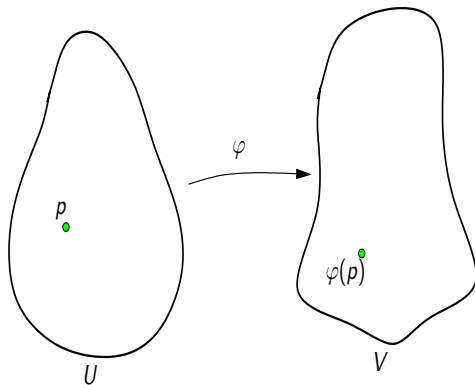
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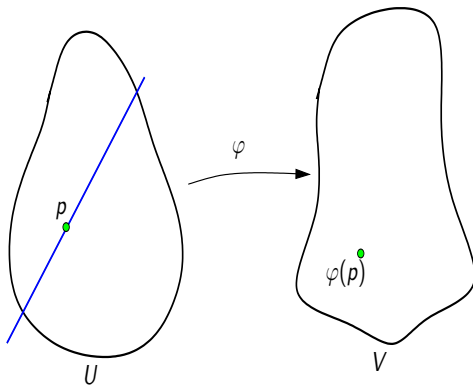
Example 1 : planar 3-web associated to $\varphi \in \text{Diff}(\mathbb{C}^2, 0)$



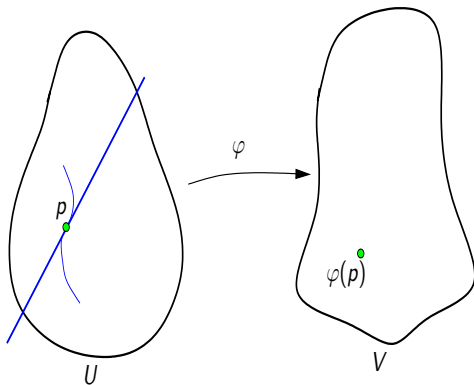
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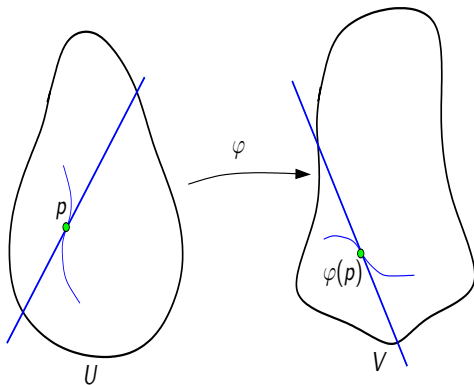
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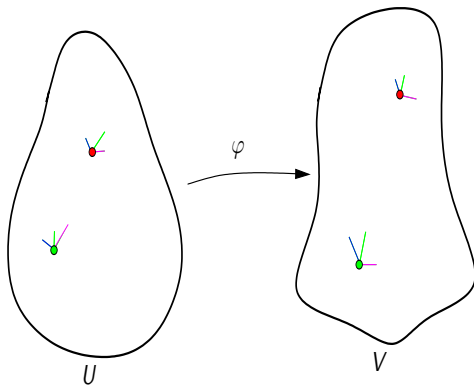
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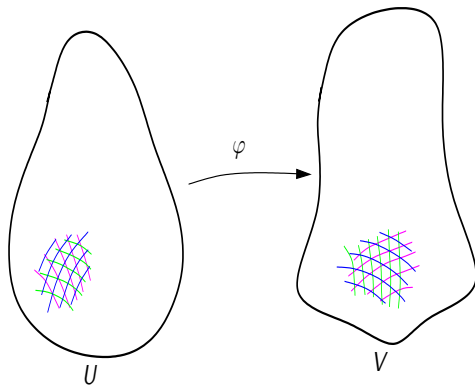
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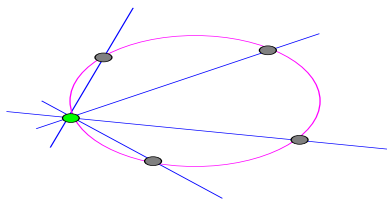


Example 2 : Bol's web \mathcal{B}

1.

$$\mathcal{B} = M_{0,5} \xrightarrow{\cong} M_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

2.



3.

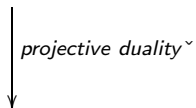
$$\mathcal{B} = W\left(x, y, \frac{x}{y}, \frac{1-x}{1-y}, \frac{x(1-y)}{y(1-x)}\right)$$

Example 3 : Algebraic web

Reduced algebraic curve $C \subset \mathbb{P}^2$, $\deg(C) = d$

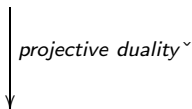
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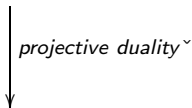
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Linear *algebraic* d -web W_C on $\check{\mathbb{P}}^2$

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Linear algebraic d -web W_C on $\check{\mathbb{P}}^2$

Remark :

$V^r \subset \mathbb{P}^{n+r}$ algebraic \checkmark
 $\deg(V^r) = d$ \longrightarrow algebraic d -web of codim r on $G_n(\mathbb{P}^{n+r})$

Algebraic webs

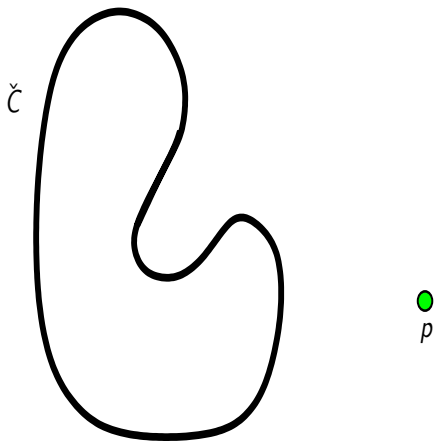
$C \subset \mathbb{P}^2$: reduced algebraic curve

$$\deg(C) = d$$

Algebraic webs

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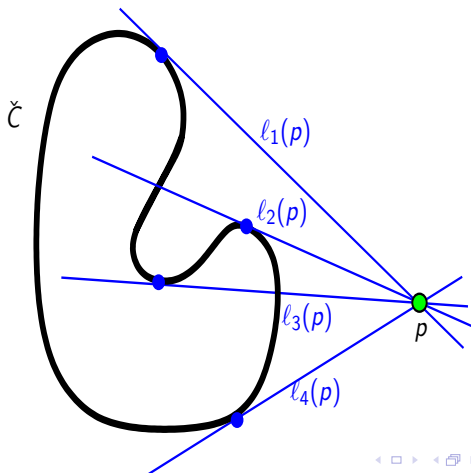
$\deg(C) = d \iff \check{C} \subset \check{\mathbb{P}}^2$ is of class d



Algebraic webs

$C \subset \mathbb{P}^2$: reduced algebraic curve

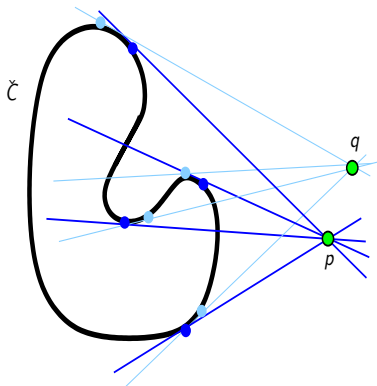
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Algebraic webs

$C \subset \mathbb{P}^2$: reduced algebraic curve

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Definition : the *algebraic web* W_C is the linear d -web on $\check{\mathbb{P}}^2$ formed by the tangents lines to \check{C} .

A real picture of an algebraic web

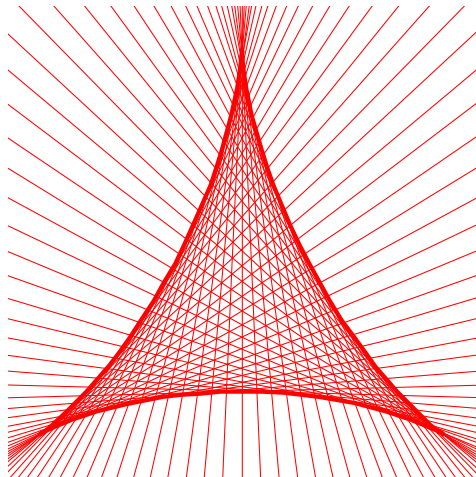


FIG.: Algebraic 3-web formed by the tangents of a hypocycloid

Another picture of an algebraic web

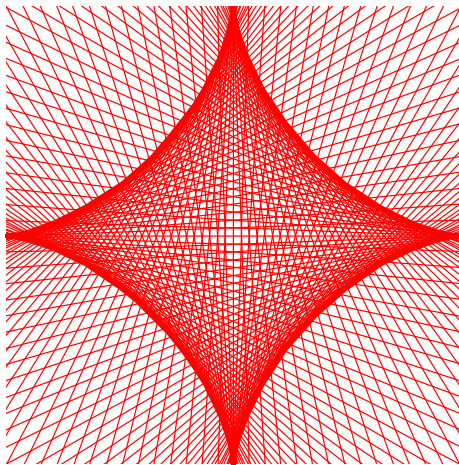
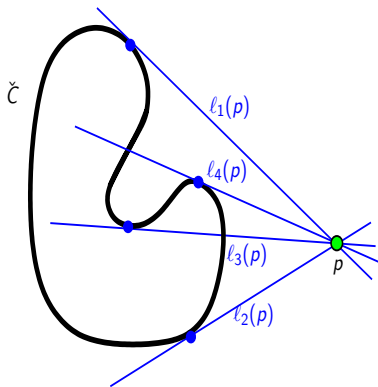


FIG.: Algebraic 4-web formed by the tangents of an astroid



Remark : the maps

$$\mathbb{P}^2 \ni p \longmapsto \ell_i(p) \in C$$

are (local) first integral for the algebraic web W_C .

Abelian relations and rank

$C \subset \mathbb{P}^2$ algebraic curve of degree d : $W_C = W(\ell_1, \dots, \ell_d)$.

Abel's Theorem :

$$\left\| \sum_{i=1}^d F_i(\ell_i(p)) = 0 \iff \exists \omega \in H^0(C, \omega_C^1) \text{ s.t. } F_i(\bullet) = \int^\bullet \omega. \right.$$

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Generalization to webs :

Let $U_i : (\mathbb{C}^2, 0) \rightarrow \mathbb{C}$ are first integrals of $W_d = W(U_1, \dots, U_d)$.

Definitions :

1. Abelian relation (AR)

$$F_1(U_1) + \dots + F_d(U_d) = 0$$

2. \mathbb{C} -vector space $\mathcal{A}(W_d) = \langle \text{AR's of } W_d \rangle$

3. Rank of W_d : $rk(W_d) = \dim \mathcal{A}(W_d)$

Properties of abelian relations and of the rank

Proposition : If $W_d = W_C$ with $\deg(C) = d$, then

1. $\mathcal{A}(W_C) \simeq H^0(C, \omega_C^1)$
2. $rk(W_C) = h^0(\omega_C^1) = p_a(C) = \frac{(d-1)(d-2)}{2}$

Bol's Theorem : For any d -web W_d

$$rk(W_d) \leq \frac{(d-1)(d-2)}{2} =: \pi_d \quad (\text{Bol's bound})$$

Definition : W_d is of *maximal rank* if $rk(W_d) = \pi_d$

Example : Algebraic webs are examples of webs of maximal rank

Theorem (Lie, Poincaré) :

$$\| \| \quad rk(W_4) = \pi_4 = 3 \quad \implies \quad W_4 \text{ is algebraizable}$$

Theorem (Blaschke-Howe, Griffiths) : For any $n \geq 2$:

$$\| \| \quad \begin{array}{l} W_d : 1 - \text{codimensional} \\ \text{linear } d\text{-web on } U \subset \mathbb{C}^n \implies W_d \text{ is algebraizable} \\ \text{with one complete AR} \end{array}$$

Webs of maximal rank : algebraization

Theorem (Bol, (Chern-Griffiths), Trépreau) : For $n > 2$:

||| W_d : 1 – codimensional
 d -web on $U \subset \mathbb{C}^n$ \implies W_d is algebraizable
of maximal rank $\pi(d, n)$

Work in progress (Pirio-Trépreau) : Algebraization of maximal rank webs of codimension $r > 1$

Philosophy : Sufficiently many AR \implies algebraization

Webs of maximal rank : exceptional webs

Fact : for a planar 5-web W_5

$$\left\| \right. \quad rk(W_5) = \pi_5 = 6 \quad \not\Rightarrow \quad W_5 \text{ is algebraizable}$$

Bol's counterexample :

$$\left\| \right. \quad \mathcal{B} = W\left(x, y, \frac{x}{y}, \frac{1-x}{1-y}, \frac{x(1-y)}{y(1-x)}\right) \quad \text{is} \quad \begin{array}{l} \underline{1.} \text{ not linearizable} \\ \underline{2.} \text{ of maximal rank } \pi_5 = 6 \end{array}$$

Definition :

$$\text{Exceptional web} = \left\{ \begin{array}{l} \text{non-algebraizable web} \\ \text{of maximal rank} \end{array} \right.$$

Abelian relations of Bol's web

Rogers dilogarithm

$$\mathbf{D}(x) := \mathbf{Li}_2(x) + \frac{1}{2} \log(x) \log(1-x) - \frac{\pi^2}{6}$$

satisfies Abel's 5-terms relation **(Ab)** :

$$\mathbf{D}(x) - \mathbf{D}(y) - \mathbf{D}\left(\frac{x}{y}\right) - \mathbf{D}\left(\frac{1-y}{1-x}\right) + \mathbf{D}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

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Thus **(Ab)** is an abelian relation for

$$\mathcal{B} = W\left(x, y, \frac{x}{y}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right)$$

\implies Bol's web is exceptional (Bol, 1936).

Abelian relations of Bol's web

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Chern's Problem : Classify all exceptional webs

Why is Chern's problem interesting

Reason 1 : Chern-Griffiths (1981)

(...) we cannot refrain from mentioning what we consider to be the fundamental problem on the subject, which is to determine the maximum rank non-linearizable webs. The strong conditions must imply that there are not many.

Reason 2 : Abel's 5-terms relation **(Ab)** appears in

- ▶ hyperbolic geometry
- ▶ algebraic K-theory
- ▶ etc...

⇒ Abelian relations of exceptional webs could be related to others domains in mathematics.

Credo :

Exceptional planar webs are analogs of plane algebraic curves.

Such webs exist :

- ▶ First new example : Pirio, Robert (2001)
- ▶ A continuous family : Pirio-Trépreau (2003)
- ▶ ...

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2.2 *characterization of maximal rank webs*

2.3 *CDQL webs*

Webs with an infinitesimal symmetry

Joint work with D. Marìn and J.-V. Pereira

W_d with an infinitesimal symmetry $X \Rightarrow \mathcal{L}_X \in \text{End}(\mathcal{A}(W_d))$

Theorem :

$X =$ infinitesimal symmetry of W_d s.t. $\mathcal{F}_X \notin W_d$

$$\text{rk}(W_d \boxtimes \mathcal{F}_X) = \text{rk}(W_d) + (d - 1)$$

Corollary :

For any $d \geq 5$, it exists a continuous family of exceptional d -webs on \mathbb{P}^2 of the form

$$W_C \boxtimes \mathcal{F}_X$$

« Proof » : for any d , there are continuous families of algebraic curves $C \subset \mathbb{P}^2$ invariant by a linear action of \mathbb{C}^* on \mathbb{P}^2

Characterization of webs of maximal rank

Blaschke-Dubourdieu :

$$W_3 \text{ on } U \subset \mathbb{C}^2 \rightsquigarrow \text{curvature } K(W_3) \in \Omega_U^2$$

Remark : for φ invertible

$$K(\varphi^* W_3) = \varphi^*(K(W_3))$$

Theorem :

W_3 has maximal rank

i.e. $rk(W_3) = \pi_3 = 1$



W_3 is « flat »

i.e. $K(W_3) \equiv 0$

Characterization of webs of maximal rank

Generalization to planar d -webs (Pantazi-Mihăileanu)
(Hénaut-Ripoll)

W_d on $U \subset \mathbb{C}^2$ \rightsquigarrow (E, ∇_{W_d}) fiber bundle on U
of rank $\pi_d = \frac{1}{2}(d-1)(d-2)$
such that $\mathcal{A}(W_d) \hookrightarrow \text{Ker}(\nabla_{W_d})$

Theorem (Pantazi-Hénaut) :

W_d has maximal rank $\iff (E, \nabla_{W_d})$ is flat
i.e. $rk(W_d) = \pi_d$ \iff *i.e.* $K(\nabla_{W_d}) \equiv 0$

Characterization of webs of maximal rank

Definition : the curvature of W_d is

$$K(W_d) := \sum_{W_3 < W_d} K(W_3) \in \Omega_U^2$$

Corollary (Mihăileanu) :

$$\left\| \begin{array}{l} W_d \text{ has maximal rank} \\ \text{i.e. } rk(W_d) = \pi_d \end{array} \right. \implies \begin{array}{l} W_d \text{ is « flat »} \\ \text{i.e. } K(W_d) \equiv 0 \end{array}$$

Remark : if W_d is a global d -web on a surface S with singular set $\Delta = \Delta(W_d) \subset S$, one can also define its curvature

$$K(W_d) \in \Omega_S^2(\star\Delta)$$

Joint work with J.-V. Pereira

Remark : Bol's web \mathcal{B} = 4 pencils of lines + a pencil of conics

Definition : *CDQL web* = Completely Decomposable QuasiLinear web

- CDQL web on \mathbb{P}^2 = pencils of lines + a non-linear foliation
- CDQL web on T = pencils of lines + a non-linear foliation

CDQL version of Chern's Pbm : Classify exceptional CDQL webs

Theorem (Pereira-Pirio) :

Up to projective equivalence, there are
4 infinite families and
13 sporadic examples
of exceptional CDQL webs on \mathbb{P}^2

The four infinite families are (with $k \geq 1$)

- ▶ $A_I^k = [(dx^k - dy^k) d(xy)]$
- ▶ $A_{II}^k = [dx dy (dx^k - dy^k) d(xy)]$
- ▶ $A_{III}^k = [(xdy - ydx)(dx^k - dy^k) d(xy)]$
- ▶ $A_{IV}^k = [dx dy (xdy - ydx) (dx^k - dy^k) d(xy)]$

Exceptional CDQL webs on \mathbb{P}^2 : classification

Among the sporadic examples, 7 are invariant by homotheties.
These are (with $\xi = \exp(2i\pi/3)$)

$$\blacktriangleright A_5^a = [dx dy d(x/y) d(x+y)] \boxtimes [d(xy(x+y))]$$

$$\blacktriangleright A_5^b = [dx dy d(x/y) d(x+y)] \boxtimes [d(xy/(x+y))]$$

$$\blacktriangleright A_5^c = [dx dy d(x/y) d(x+y)] \boxtimes [d(\frac{x^2+xy+y^2}{xy(x+y)})]$$

$$\blacktriangleright A_5^d = [dx dy d(x+y) d(x+\xi y)] \boxtimes [d(xy(x+y)(x+\xi y))]$$

$$\blacktriangleright A_6^a = [dx dy d(x+y) d(x+\xi y) d(\frac{x}{y})] \boxtimes [d(xy(x+y)(x+\xi y))]$$

$$\blacktriangleright A_6^b = [dx dy (dx^3 + dy^3)] \boxtimes [d(x^3 + y^3)]$$

$$\blacktriangleright A_7^a = [dx dy (dx^3 + dy^3) d(x/y)] \boxtimes [d(x^3 + y^3)].$$

Exceptional CDQL webs on \mathbb{P}^2 : classification

Among the last sporadic examples, there are Bol's web and his brothers

$$\blacktriangleright \mathcal{B} = [dx \, dy \, d(\frac{x}{1-y}) \, d(\frac{y}{1-x})] \boxtimes \left[d\left(\frac{xy}{(1-x)(1-y)}\right) \right]$$

$$\blacktriangleright \mathcal{B}_6 = \mathcal{B} \boxtimes [d(x+y)]$$

$$\blacktriangleright \mathcal{B}_7 = \mathcal{B}_6 \boxtimes [d(x/y)]$$

$$\blacktriangleright \mathcal{B}_8 = \mathcal{B}_7 \boxtimes [d(\frac{1-x}{1-y})]$$

Exceptional CDQL webs on \mathbb{P}^2 : classification

The last two sporadic examples are

$$\blacktriangleright H_5 = [(dx^3 + dy^3) d(x/y)] \boxtimes \mathcal{H}_{esse}$$

$$\blacktriangleright H_{10} = [(dx^3 + dy^3) \prod_{i=0}^2 d(\frac{y-\xi^i}{x}) \prod_{i=0}^2 d(\frac{x-\xi^i}{y})] \boxtimes \mathcal{H}_{esse}$$

where \mathcal{H}_{esse} designates Hesse's pencil of cubics

$$\mathcal{H}_{esse} = \left[d\left(\frac{xy}{1+x^3+y^3}\right) \right]$$

2.3 Exceptional CDQL webs on tori : classification

Corollary (Pereira-Pirio) :

Up to isogenies, there are

— 1 continuous family (for $d = 5$)

— 3 sporadic examples (one for each $d \in \{5, 6, 7\}$)

of exceptional CDQL d -webs on 2-dimensional complex tori

Classification on \mathbb{P}^2 : « proof »

- 0.
- $W_{d+1} = L_{p_1} \boxtimes \cdots \boxtimes L_{p_d} \boxtimes \mathcal{F}$ with $\begin{cases} p_1, \dots, p_d \in \mathbb{P}^2 \\ \mathcal{F} \in \text{Fol}(\mathbb{P}^2) \end{cases}$
 - singular set of $W_{d+1} =: \Delta \subset \mathbb{P}^2$

1. $\left[\begin{array}{l} W_{d+1} \text{ has} \\ \text{max. rank} \end{array} \right] \xrightarrow{\text{Mihaileanu}} K(W_{d+1}) \equiv 0 \Leftrightarrow \left[\begin{array}{l} K(W_{d+1}) \\ \text{holom. on } \mathbb{P}^2 \end{array} \right]$

————— $\Rightarrow \left[\begin{array}{l} K(W_{d+1}) \text{ is holom. over} \\ \text{the generic point of } C \subset \Delta \end{array} \right] = (1)$

2. (1) $\Rightarrow \left[\begin{array}{l} \text{constraints on} \\ \mathcal{P} = \{p_1, \dots, p_d\} \\ \mathcal{F} \in \text{Fol}(\mathbb{P}^2) \end{array} \right] \Rightarrow \begin{cases} \deg(\mathcal{F}) \leq 4 \\ \mathcal{P} \subset \ell_{\mathcal{F}} \cup \text{Sing}(\mathcal{F}) \\ \dots\dots\dots \end{cases}$

3. Case by case classification depending on $\deg(\mathcal{F}) \in \{1, 2, 3, 4\}$ \square

Plan of the talk

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3.2 *Segre's principal directions and generalizations*

3.3 *exceptional projective surfaces*

Canonical map : algebraic curves

C : smooth algebraic curve of genus g

$$\langle \omega^\lambda \mid \lambda = 1, \dots, g \rangle = H^0(C, \Omega_C^1)$$

Let z be a local coordinate on C . Thus

$$\omega^\lambda(z) = f^\lambda(z) dz$$

Definition : the canonical map of C is

$$\begin{aligned} \varphi_{|\Omega_C^1|} : C &\longrightarrow \mathbb{P}^{g-1} \\ z &\longmapsto [f^1(z) : \dots : f^g(z)] \end{aligned}$$

Canonical map : webs

$W_d = W(U_1, \dots, U_d)$ d -web with FI $U_i : (\mathbb{C}^2, 0) \rightarrow C_i = \text{Im}(U_i)$.

Assume $\pi = \text{rk}(W_d) > 0$

$$\begin{aligned} \left\langle \sum_{i=1}^d F_i^\lambda(U_i) = 0 \mid \lambda = 1, \dots, \pi \right\rangle &= \mathcal{A}(W_d) \\ \Updownarrow & \\ \left\langle \sum_{i=1}^d f_i^\lambda(U_i) dU_i = 0 \mid \lambda = 1, \dots, \pi \right\rangle &= \mathcal{A}(W_d) \end{aligned}$$

Definition : 1. the i th canonical map of W_d is

$$\begin{aligned} \varphi_i : C_i &\longrightarrow \mathbb{P}^{\pi-1} \\ u_i &\longmapsto [f_i^1(u_i) : \dots : f_i^\pi(u_i)] \end{aligned}$$

2. the canonical map of W_d is $\varphi_{W_d} = (\varphi_i) : \sqcup_i C_i \rightarrow \mathbb{P}^{\pi-1}$

Canonical curve of webs of maximal rank

Algebraic curves

$$C' = \text{Im } \varphi|_{\Omega_C^1} \subset \mathbb{P}^{\pi_d-1}$$

C' non-degenerate

C' algebraic

$$\text{deg}(C) = 2g - 2$$

Exceptional webs

$$C'_{W_d} = \sqcup_i C'_i \subset \mathbb{P}^{\pi_d-1}$$

$$C'_i = \text{Im } \varphi_i$$

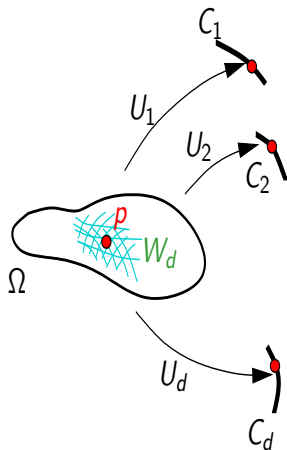
C'_{W_d} non-degenerate

C'_{W_d} algebraic or transcendent

?

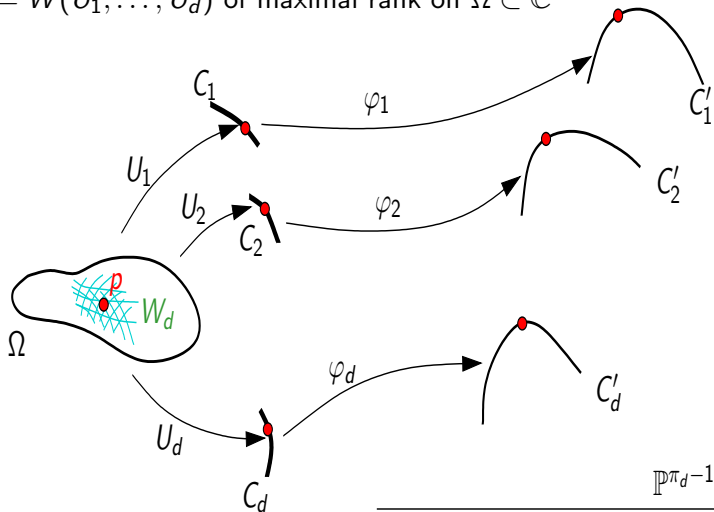
Canonical curves of webs of maximal rank

$W_d = W(U_1, \dots, U_d)$ of maximal rank on $\Omega \subset \mathbb{C}^2$

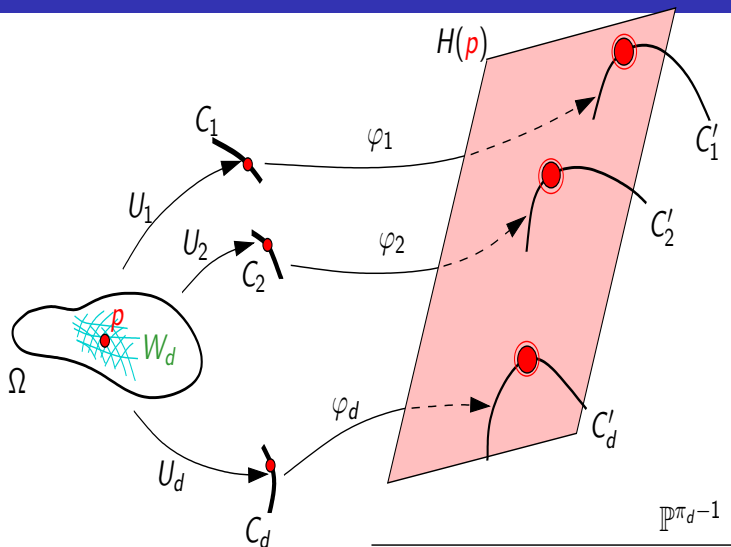


Canonical curves of webs of maximal rank

$W_d = W(U_1, \dots, U_d)$ of maximal rank on $\Omega \subset \mathbb{C}^2$



Canonical curves of webs of maximal rank



Canonical curves of webs of maximal rank

Proposition : Let $p \in \Omega$: the osculating spaces

$$\text{Osc}_{C_i}^{d-3}(\varphi_1(U_1(p))) \quad i = 1, \dots, d$$

span a hyperplane $H(p) \in \check{\mathbb{P}}^{\pi_d-1}$

Proposition :

1. : The map $H_{W_d} : \begin{array}{ccc} \Omega & \rightarrow & \check{\mathbb{P}}^{\pi_d-1} \\ p & \mapsto & H(p) \end{array}$ parametrizes a surface

$$S_{W_d} \subset \check{\mathbb{P}}^{\pi_d-1}$$

with regular osculation of order d

2. : W_d is algebraizable $\iff S_{W_d} = \mathcal{V}_d$ (Veronese surface)

3. : $H_{W_d*}(W_d)$ is canonically defined on S_{W_d}

Principal directions on projective surfaces

Projective surface $S \subset \mathbb{P}^{\pi_d-1}$ with regular osculation of order d

Proposition : (Segre $d = 5$)

Let $s \in S$. Counted with multiplicities, there are exactly 5 **principal hyperplanes** $K_1, \dots, K_5 \in \mathbb{P}^5$ such that

$$K_i \cap S = \text{tacnode in } s$$



\implies 5 **principal directions** tangent to S at s

\implies Segre's 5-web on S (invariantly attached to S)

Proposition (Segre $d=5$) :

S totally umbilic $\iff S \subset$ Veronese surface $\mathcal{V}_2 \subset \mathbb{P}^5$

Principal directions on projective surfaces

$$\text{Exceptional 5-web } W_5 \Rightarrow \begin{cases} H_{W_5} : \Omega \rightarrow \check{\mathbb{P}}^5 \\ S_{W_5} = \text{Im} (H_{W_5}) \end{cases}$$

Proposition :

$$\text{Segre's 5-web of } S_{W_5} = H_{W_5*}(W_5)$$

$\implies S_{W_5}$ determines W_5

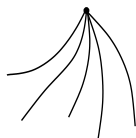
Definition : Notion of exceptional surface $S \subset \mathbb{P}^{\pi_d-1}$

Examples of exceptional surfaces

Proposition : the following 5-web is exceptional

$$W(x, y, x + y, x - y, x^2 + y^2)$$

Canonical curve :



Associated exceptional surface :

$$H = [1 : x^2 + y^2 : x^2 y^2 : (x^2 + y^2)^2 : x^3(5y^2 - x^2) : y^3(5x^2 - y^2)]$$

Examples of exceptional surfaces

Proposition : the following 5-web is exceptional

$$W(x, y, x + y, x - y, x^2 - y^2)$$

Canonical curve :



Associated exceptional surface :

$$H = [1 : x^2 : y^2 : x^3 : y^3 : (x^2 - y^2)^2]$$