Degenerations and Applications to Interpolation Problems

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Algebraic Geometry, D-Modules, Foliations and Their Interactions
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all work joint with Ciro Ciliberto

Outline

Notation

Fix general points $p_1, \ldots, p_n$ in the plane, and multiplicities $m_1, \ldots, m_n$.
Let

$$\mathcal{L} = \mathcal{L}_d(m_1, \ldots, m_n)$$

be the linear system of plane curves C of degree d having multiplicity at least m_i at p_i for each i:

$$\mathrm{mult}_{p_i}(C) \geq m_i$$

We write

$$\mathcal{L}_d(m_1^{e_1}, \ldots, m_r^{e_r})$$

for repeated multiplicities.

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for repeated multiplicities. So

$$\mathcal{L}_4(2^5)$$

is the (projective) space of quartics with 5 double points.

Virtual and Expected Dimensions

The space of plane curves of degree d has dimension $d(d+3)/2$.
Imposing a point of multiplicity m is $m(m+1)/2$ conditions.
The virtual dimension of $\mathcal{L}$ is

$$v(\mathcal{L}) = d(d+3)/2 - \sum_i m_i(m_i+1)/2$$

and the expected dimension is

$$e(\mathcal{L}) = \max\{-1, v\}$$

(projective dimensions here;
-1 means an empty projective space, a zero vector space)

The Blowup of the Plane

One usually formulates this
on the blowup of the plane at the n points:
This creates a surface X, with divisor classes H, $E_1, \ldots, E_n$.
The relevant invertible sheaf is $L = \mathcal{O}_X(dH - \sum_i m_i E_i)$.

$$v = \dim H^0(X, L) - \dim H^1(X, L) - 1$$

Having the expected dimension: either $H^0 = 0$ or $H^1 = 0$.

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$$\dim(\mathcal{L}) \geq \text{expected} \geq \text{virtual}$$

Main Question: What is the actual dimension of $\mathcal{L}$?
This is equivalent to asking: what is $\dim H^0(X, L)$?

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$$\mathcal{L}_4(2^5)$$

Again

$$v = 14 - 5 \cdot 3 = -1$$

but the system consists of the double conic through the five points.

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The Segre Conjecture: If $\mathcal{L}$ contains a reduced member, then $\mathcal{L}$ has the expected dimension.

This is open.

Gimigliano-Harbourne-Hirschowitz

Easy:
If there exists a (-1)-curve C on the blowup X of the plane) (a smooth rational curve C with $C^2 = -1$) such that

$$\mathcal{L} \cdot C = -n \leq -2 \quad \text{and} \quad \dim(\mathcal{L}) \geq 0$$

then $\mathcal{L}$ <u>does not</u> have the expected dimension:

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then $\mathcal{L}$ does not have the expected dimension:

The curve C will split at least twice from the system, and the virtual dimension of the residual goes up: $\mathcal{L} = nC + \mathcal{R}$, with $v(\mathcal{R}) = v(\mathcal{L}) + n(n-1)/2$. So

$$\dim(\mathcal{L}) = \dim(\mathcal{R}) \geq v(\mathcal{R}) > v(\mathcal{L})$$

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GHH Conjecture: This is if and only if:
If no such (-1)-curve exists,
then $\mathcal{L}$ has the expected dimension.

Nagata's Conjecture

Nagata's Conjecture: (for $\mathcal{L}_d(m^n)$)
If $n > 9$ and $d^2 < nm^2$
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Ciliberto-Miranda: Segre $\iff$ Harbourne-Hirschowitz

Either one imply Nagata.

Results - not comprehensive

- GHH is true for $n \leq 9$. (Castelnuovo, 1891; Nagata, 1960; Gimigliano, Harbourne, 1986)
- GHH is true for $m_i \leq 11$ (7: S. Yang 2004; Dumnicki 2005).
- GHH for $\mathcal{L}_d(m^n)$ is true for $m \leq 37$ (20: Ciliberto-Cioffi-Miranda-Orecchia 2000; Dumnicki 2006)
- Nagata's Conjecture is true for $n = k^2$ points (Nagata 1960)
- GHH Conjecture is true for $n = k^2$ points (Evain 2005 - via Horace; Ciliberto-Miranda 2006; C-M-Dumitrescu 2007)
- If $d/m < 3.1607$ then $\mathcal{L}_d(m^{10})$ is empty (Harbourne-Roe' 2005; C-M computations) [There are many other such results...]

The Veronese

The Veronese embedding

$$v_d : \mathbb{P}^2 \to \mathbb{P}^{d(d+3)/2}, \quad \text{image} \quad = V_d; \quad \text{degree}(V_d) = d^2$$

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(in particular, for proving this is empty):
Degenerate the Veronese to a union of d^2 planes,
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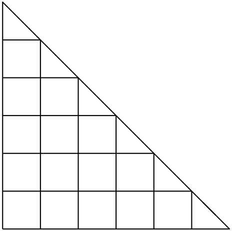
Strategy for analyzing $\mathcal{L}_d(2^n)$
(in particular, for proving this is empty):
Degenerate the Veronese to a union of d^2 planes,
and show there are no n-tangent hyperplanes.

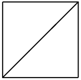
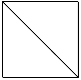
A hyperplane tangent to a plane must contain the plane.
Show: there are no hyperplanes containing some n of the planes.

A Degeneration of the Veronese

V_d degenerates to a union of d planes and $\binom{d}{2}$ quadrics:

$d = 6$:



plus:  or  for each quadric

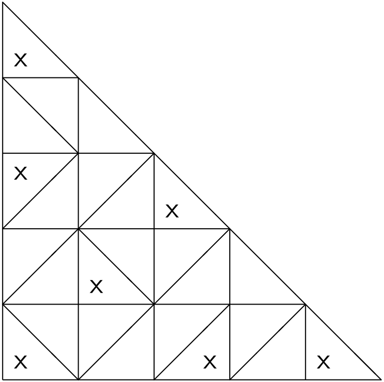
The "vertices" are the coordinate points of the ambient $\mathbb{P}^{d(d+3)/2}$.

This gives a degeneration of V_d to d^2 planes.

$\mathcal{L}_5(2^7)$ is empty

$$v = 20 - 7 \cdot 3 = -1$$

Proof:



Double Points

Theorem

$\mathcal{L}_d(2^n)$ has the expected dimension whenever $d \geq 5$.

Proof by Olivia Dumitrescu and RM, using the Veronese degeneration techniques (and induction).

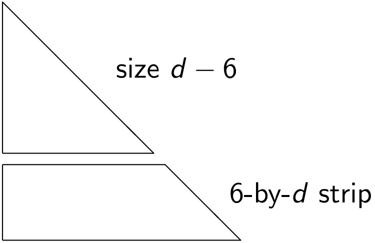
(Earlier proofs in 1980s by Arbarello-Cornalba (Severi, Castelnuovo; deformation theory) and by Hirschowitz (Horace Method).)

The induction: if the theorem is true for d, then it is true for $d+6$.

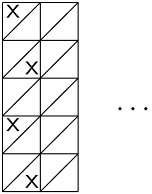
This is related to recent *tropical* constructions by Jan Draisma.

The Induction

is rather straightforward to set up:



Top triangle is handled by induction.
Bottom strip is easy:



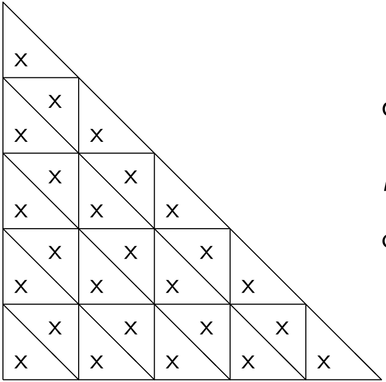
Higher Multiplicities

Theorem

The system $\mathcal{L}_{km}(m^{k^2})$ has the expected dimension; in particular it is empty for $k \geq 4$.

Stronger than Nagata, but weaker than GHH.

Exploits a degeneration of V_{km} into k^2 V_m's:

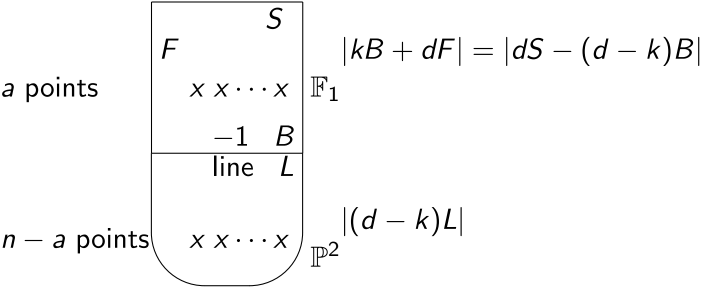


degree m on each plane

k^2 multiplicity m points,

one in each plane

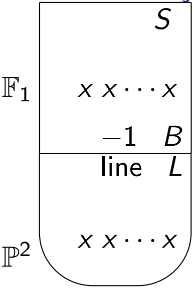
Other Degenerations



$\mathbb{F}_1$ is a ruled surface, isomorphic to $\mathbb{P}^2$ blown up once.
S and B are sections, F is the fiber
This is a degeneration of $\mathbb{P}^2$, with $\mathcal{L}_d(m^n)$.
a and k can be chosen as parameters.


one

Fiber Product Analysis



System $\cong \mathcal{L}_d(d-k, m^a)$

System $\cong \mathcal{L}_{d-k}(m^{n-a})$

Sections in the limit are sections of both systems that agree on the intersection curve $B = L$. This is a fiber product.

A transversality result (for the restrictions of the linear systems to the curve) makes the computation straightforward: If the systems on the two surfaces have the expected dimension, then so will the fiber product. Hence by semi-continuity, so will the general system.

Blow up the offending (-1)-curve

If one of the linear systems on the two surfaces *does not* have the expected dimension, then (by GHH/induction) we should locate a (-1)-curve that is a base curve for the system.

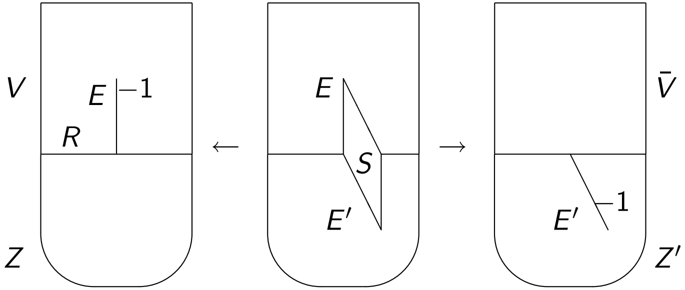
Blow it up to get another ruled surface component R to the degeneration. Pull back the line bundle, and twist appropriately by R.

Iterate, creating more components, until all linear systems have the expected dimension.

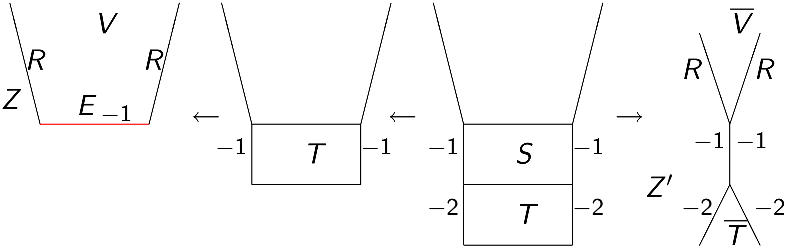
Make the (now more complicated) fiber product computation, and hope that the dimension of the system on the degeneration is the expected dimension of the system on $\mathbb{P}^2$.

Throwing a (-1)-curve

Actually, it is often more efficient to blow up an offending (-1)-curve, and if the result is the ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$, we may be able to blow it down the other way:



If the (-1)-curve hits the double curve twice, we blow up twice, and blow down the second exceptional surface the other way:



Ten Points: The Hardest Cases

The virtual dimension of $\mathcal{L}_d(m^{10})$ is

$$v = d(d+3)/2 - 10m(m+1)/2.$$

When is this equal to -1?

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When is this equal to -1?

d	m	empty
3	1	easy: cubic through ten general points
19	6	posed by Dixmier; 80s: solved by Hirschowitz
38	12	Gimigliano's thesis
174	55	?
778	246	?
1499	474	?
6663	2107	?
$\vdots$	$\vdots$	?

For these linear systems, one expects there to be no such curves ($H^0 = 0$) and because $v = -1$, this is equivalent to having $H^1 = 0$ (for the line bundle on the ten-fold blowup of $\mathbb{P}^2$)

Degree 174, Multiplicity 55

Theorem

(Ciliberto-RM)
$\mathcal{L}_{174}(55^{10})$ *is empty.*

Proof: via a degeneration of the plane.

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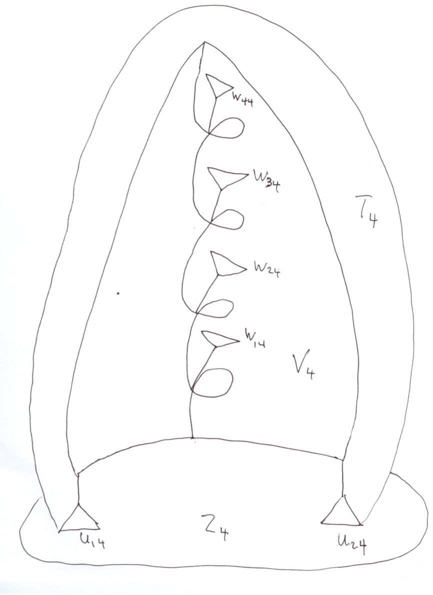
Proof: via a degeneration of the plane.
The analysis in fact gives more:

Theorem

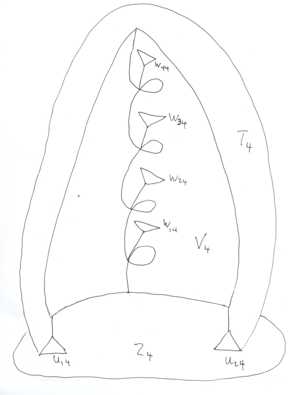
$\mathcal{L}_d(m^{10})$ *has the expected dimension if* $d \geq (174/55)m$.

Strategy:

- 1. Start with a simple degeneration (like the one with two surfaces).
- 2. Consider all possible limits of the line bundle on the degeneration.
- 3. If there exists a limit such that all linear systems on all component surfaces have the expected dimension, CONCLUDE that the enter limit has the expected dimension.
- 4. If not, there are special linear systems on components no matter what line bundle limit you have. FIND the offending (-1)-curve.
- 5. Blow it up (and contract a surface the other way if possible).
- 6. Go to step 2.

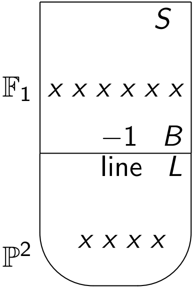


W_{44}
W_{34}
W_{24}
W_{14}
T_4
V_4
Z_4
U_{14}
U_{24}



U_{14}, U_{24}, W_{14}, W_{24}, W_{34}, W_{44} are planes
Z_4 is the plane blown up 7 times, plus two additional double blowups
T_4 is the plane with two double blowups
V_4 is the plane blown up 4 times, plus ten double blowups

Start with the $\mathbb{P}^2 \cup \mathbb{F}_1$ degeneration:

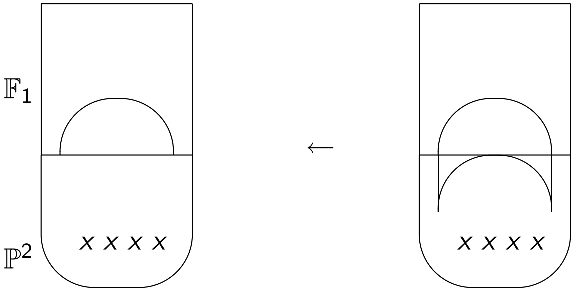


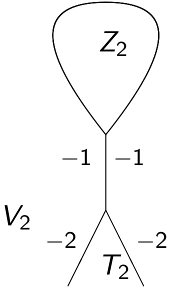
System $\cong \mathcal{L}_{174}(174 - k, 55^6)$

System $\cong \mathcal{L}_{174-k}(55^4)$

The cubic C in $\mathcal{L}_3(2, 1^6)$ is a (-1)-curve on the $B_6(\mathbb{F}_1)$.

Blow it up, twice, and contract the second surface:


$\mathbb{F}_1$
$\mathbb{P}^2$
$x\ x\ x\ x$
$\longleftarrow$
$x\ x\ x\ x$


Z_2
-1
-1
V_2
-2
-2
T_2

Now on this degeneration, there are two conics on the $\mathbb{P}^2$ passing through the four points, and one each through the two points just blown up.

These are (-1)-curves now: blow them up (twice!).

There are also four quartics: each double at 3 of the 4 blown up points, and each tangent to the line L at the two points where the cubic on the other surface hits.

C_1: in $\mathcal{L}_2(1,1,1,1,[1,0],[0,0])$

C_2: in $\mathcal{L}_2(1,1,1,1,[0,0],[1,0])$

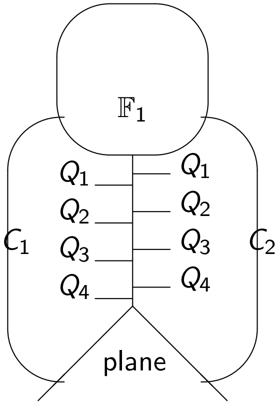
Q_1: in $\mathcal{L}_4(1,2,2,2,[1,1],[1,1])$

Q_2: in $\mathcal{L}_4(2,1,2,2,[1,1],[1,1])$

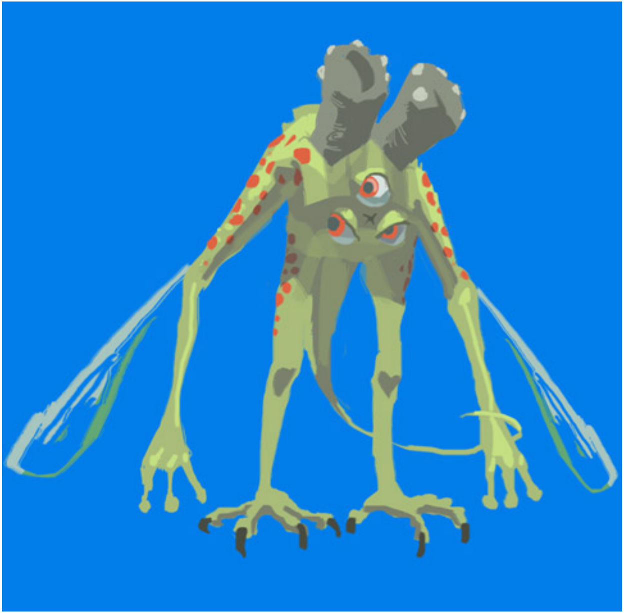
Q_3: in $\mathcal{L}_4(2,2,1,2,[1,1],[1,1])$

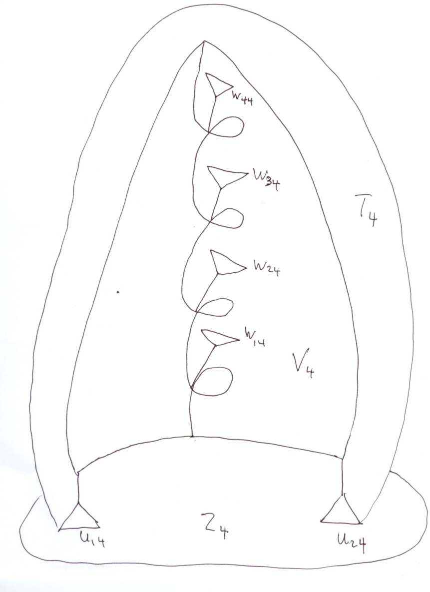
Q_4: in $\mathcal{L}_4(2,2,2,1,[1,1],[1,1])$

These are six disjoint (-1)-curves!



Blow them all up (twice, and blow surfaces down).
One gets:





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T_4
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u_{14}
u_{24}

Recall Nagata's Conjecture (for ten points):
If $d/m < \sqrt{10} = 3.16227766....$, then $\mathcal{L}_d(m^{10})$ is empty.

Strategy for approaching this: Fix a ratio $r < \sqrt{10}$.
Find a degeneration such that, if $d/m < r$, for every limit line bundle, at least one of the surfaces has an empty linear system.
CONCLUDE: the general system is also empty.

Why: If the general system is non-empty, there will be limit curves in the degeneration, and these limit curves will consist of a curve in each surface. So the limit line bundle containing this limit curve will have the property that each surface has a non-empty linear system.

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Theorem

(Olivia Dumitrescu)
If $d/m < 411/130 = 3.16153846....$, then $\mathcal{L}_d(m^{10})$ is empty.

I believe this is the world's record now for ten points!