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ALGEBRAIC GEOMETRY, D-MODULES, FOLIATIONS AND THEIR INTERACTIONS - Buenos Aires july 21/26, 2008

Overview

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We will say that Y is terminal extendable, l.c.i. extendable, etc. if, in addition, X has terminal, locally complete intersection, etc. singularities.

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and such that there is given a section of π not passing through the singular point of any fiber.

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In the case (ii), which turns out to be exactly the $K3$-Weierstrass case, we can be a little bit more precise

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(i) $g = 0$ and $a \geq 6$ with $(a, b, n) \neq (6, 7, 1)$, or
(ii) $g \geq 1$ and $a \equiv 0 \pmod 3$, $a \geq 6$, or
(iii) $g \geq 1$, S is linearly normal and either $a \geq 7$, $b \geq an + 5g - 1$ or $a = 5$, $b \geq 6n + 7g - 3$.

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Now a little work using results of adjunction theory insures that π extends to a morphism $\overline{\pi} : X \to B$ if

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(iii) there exists a very ample line bundle $\mathcal{L}$ on B such that $H^1(S, \pi^*\mathcal{L} - tL_{|S}) = 0$ for every $t \geq 1$.

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$(F, L_{|F}) \cong (\mathbb{P}G, \xi)$, where G is a rank two vector bundle on an elliptic curve and ξ is the tautological line bundle.

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Now $K_F^2 = L_{|F}^2 = L^2.F = L_{|S}.f = aC.f$

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Since $\kappa(S) = 1$ we have that $L_{|S}$ and $K_{X|S}$ are numerically independent.
and since $\rho(S) = 2$, for any $A \in Pic(S)$ there are $c \geq 1, d, e$ such that $cA \equiv dL_{|S} + eK_{X|S}$.

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we have proved that $\overline{\pi} : X \to B$ is a Mori fiber space!

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Since $H_S \sim aC + bf$ is very ample we get $a \geq 3$ and $b \geq 2a + 1$ whence

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The other cases are done using some adjunction theory and the theorem of Namikawa that assures that if X is terminal then it is smoothable. Then one can use the classification of smooth Fano threefolds.

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Then Y is nonextendable.

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We now pick $D_0 = 3C + 3nf$ if $g(B) = 0$ and $D_0 = 2C + (2n + 2g)f$ if $g(B) \geq 1$.

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