Invariants of Hypersurfaces with an Isolated Singularity constructed with Vector Fields

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D-modules, Algebraic Geometry and Foliations

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Vá
por
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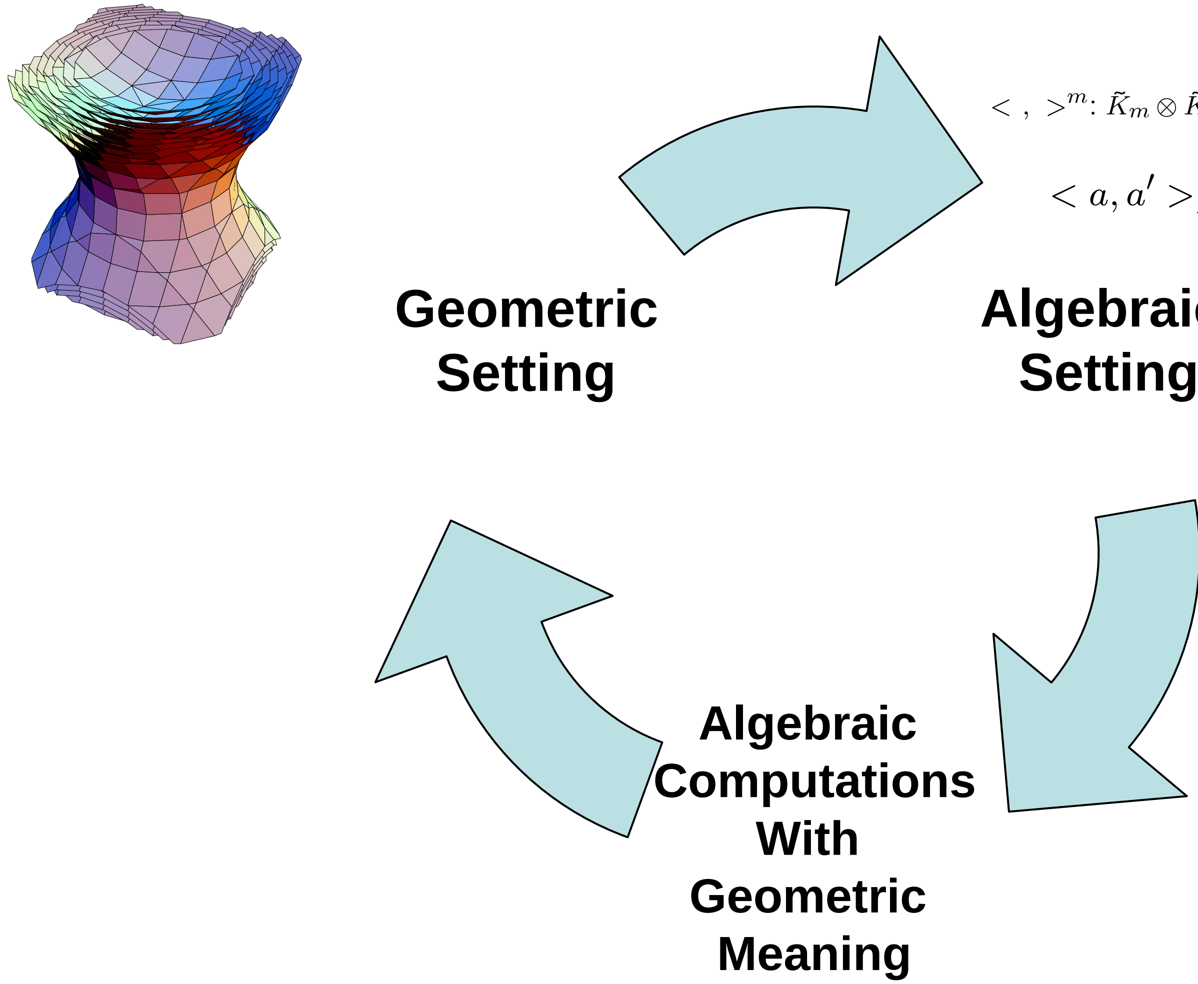
$$(f^m : f_1) \oplus (f^m : f_1) \xrightarrow{\frac{f_1}{f^m}\cdot} (f^m : f_1) \xrightarrow{\tilde{\pi}_0} \frac{\mathcal{B}}{(f)} = \mathbf{B}_0 \xrightarrow{L_{\mathbf{B}_0}} \mathbb{R}$$

$$< \, , \, >^m : \tilde{K}_m \otimes \tilde{K}_m \longrightarrow \mathbb{R} \qquad < \, , \, >^m = < \frac{f_1 \cdot}{f^m}, \cdot >_0$$

$$< a, a' >_{L_{\mathbf{A}}, f, m} = < \frac{a}{f^{m-1}}, a' >_{L_{\mathbf{A}}},$$



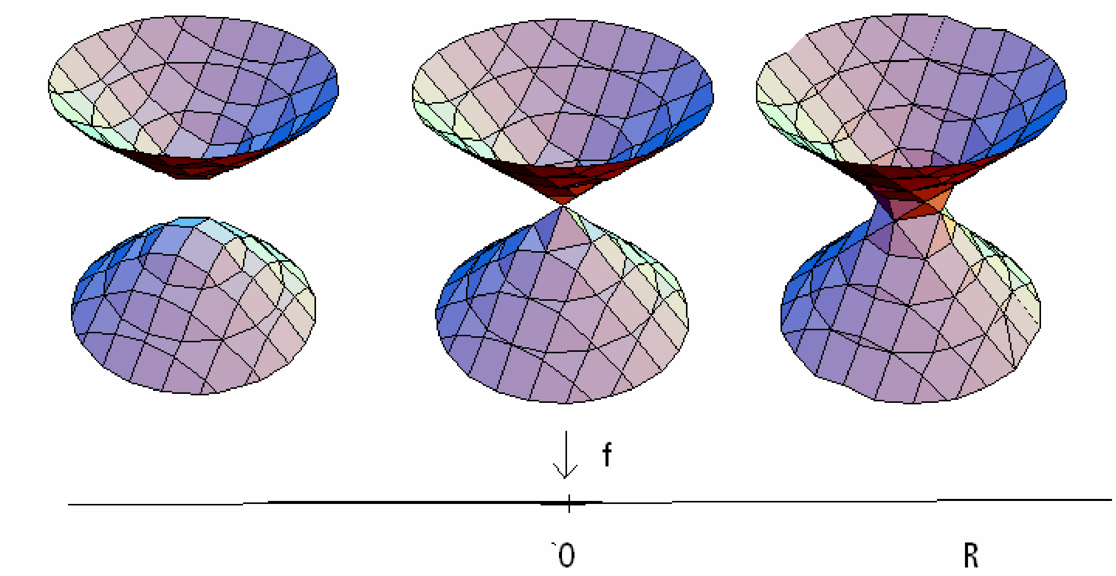
$$\tau_+ = \tau_0 + \sum_{m=1}^{\ell+1} \sigma_{\mathbf{A}, f, m} \qquad , \qquad \tau_- = \tau_0 + \sum_{m=1}^{\ell+1} (-1)^m \sigma_{\mathbf{A}, f, m}.$$

Geometric Setting:
Isolated Hypersurface Singularity

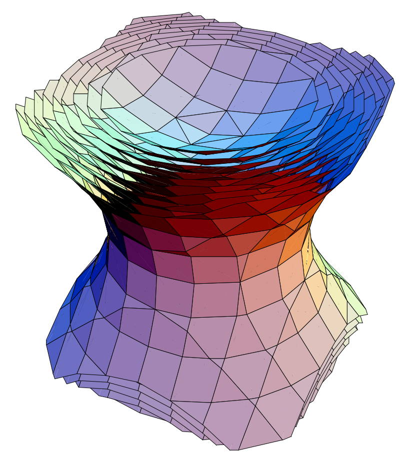
$$\mathbb{R}^{n+1}$$

$$\downarrow f$$

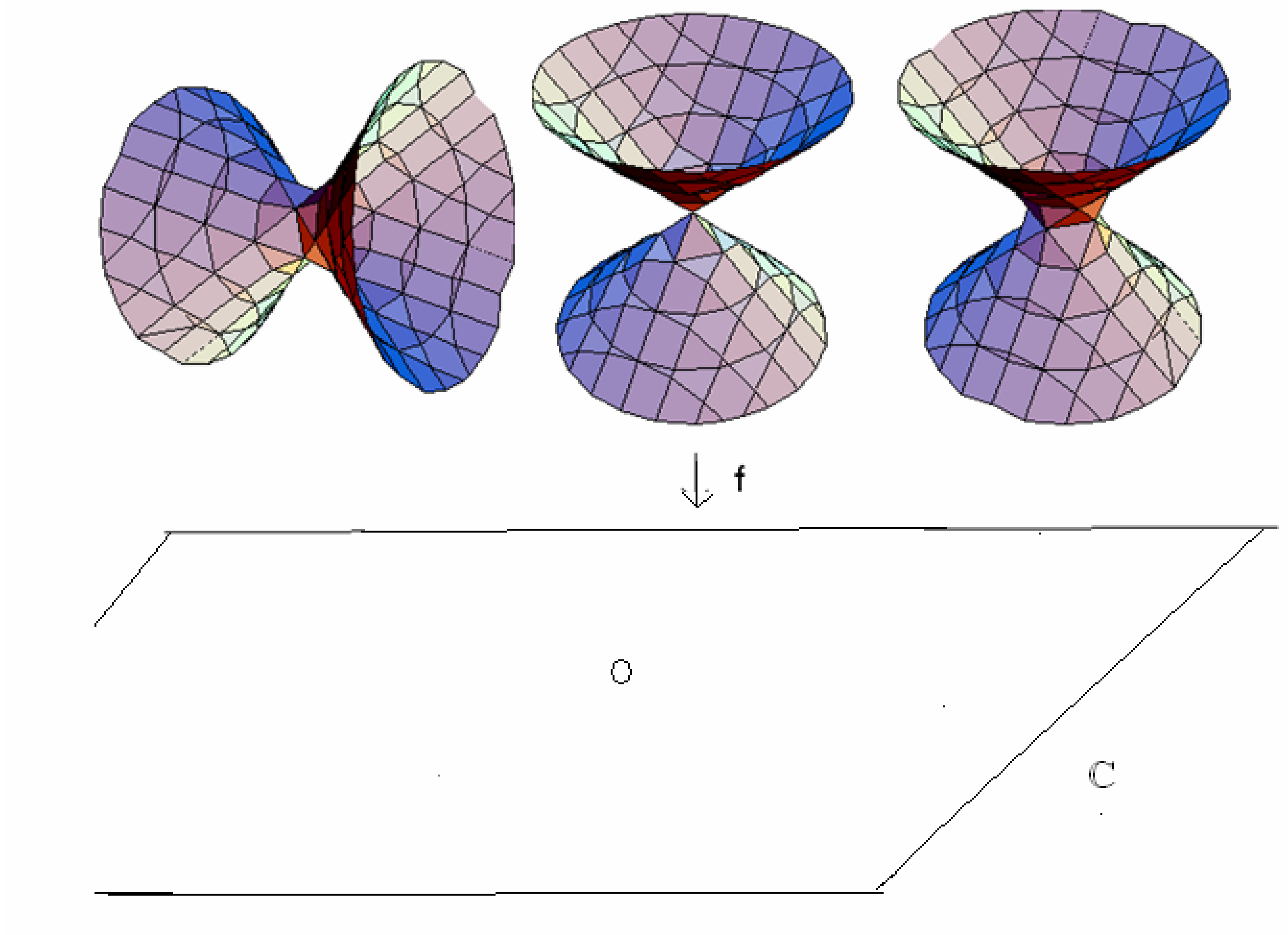
$$\mathbb{R}$$



$$\chi_- \neq \chi_+$$



f is a fibration outside of 0



$$\mathbb{C}^{n+1}$$
$$\downarrow f$$
$$\mathbb{C}$$

Monodromy

The Milnor fibre $V = f^{-1}(t)$ is a bouquet of μ spheres, where μ may be computed algebraically

$$\mu = \mathbf{dim}_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{(f_0, \ldots, f_n)} \qquad f_j := \frac{\partial f}{\partial z_j}$$

$$H^n(V, \mathbb{Q}) = \mathbb{Q}^\mu \ , \quad H^j(V, \mathbb{Q}) = 0 \ , \ j = 1, \ldots, n-1$$

When $t \to 0$ all the homology in V_t collapses to 0, but at different speeds! (Langevine, Teissier, Varchenko,...)

The easiest way to see this is with a Morsification of f:

$$F_s := f(z) + sz_0$$

The critical points of F_s

$$f_0 - s = f_1 = \ldots = f_n = 0$$

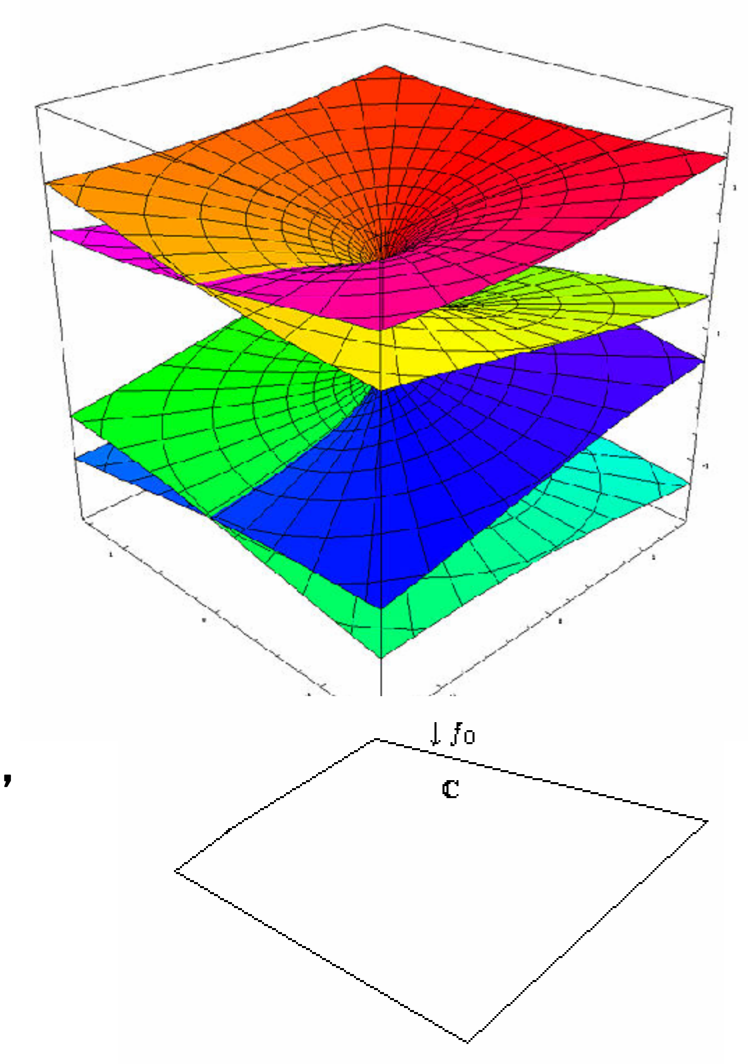
arrive through different components of

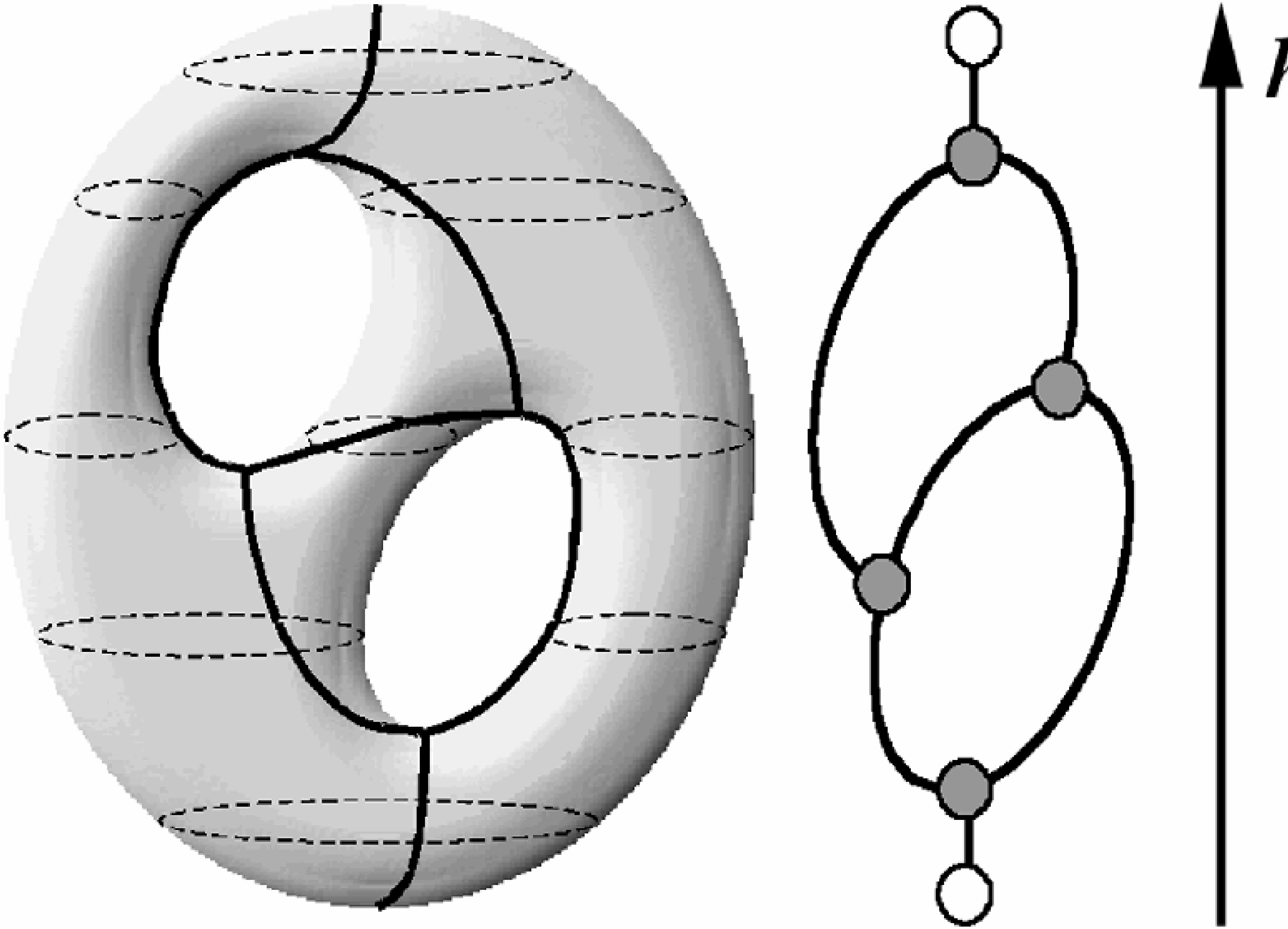
$$\Gamma = \{f_1 = \ldots = f_n = 0\} = \cup \Gamma_j$$

$$\downarrow f_0$$

$$\mathbb{C}$$

Each critical point has a 'vanishing cycle', giving generators of $H^n(V_{t_0}, \mathbb{Q})$.

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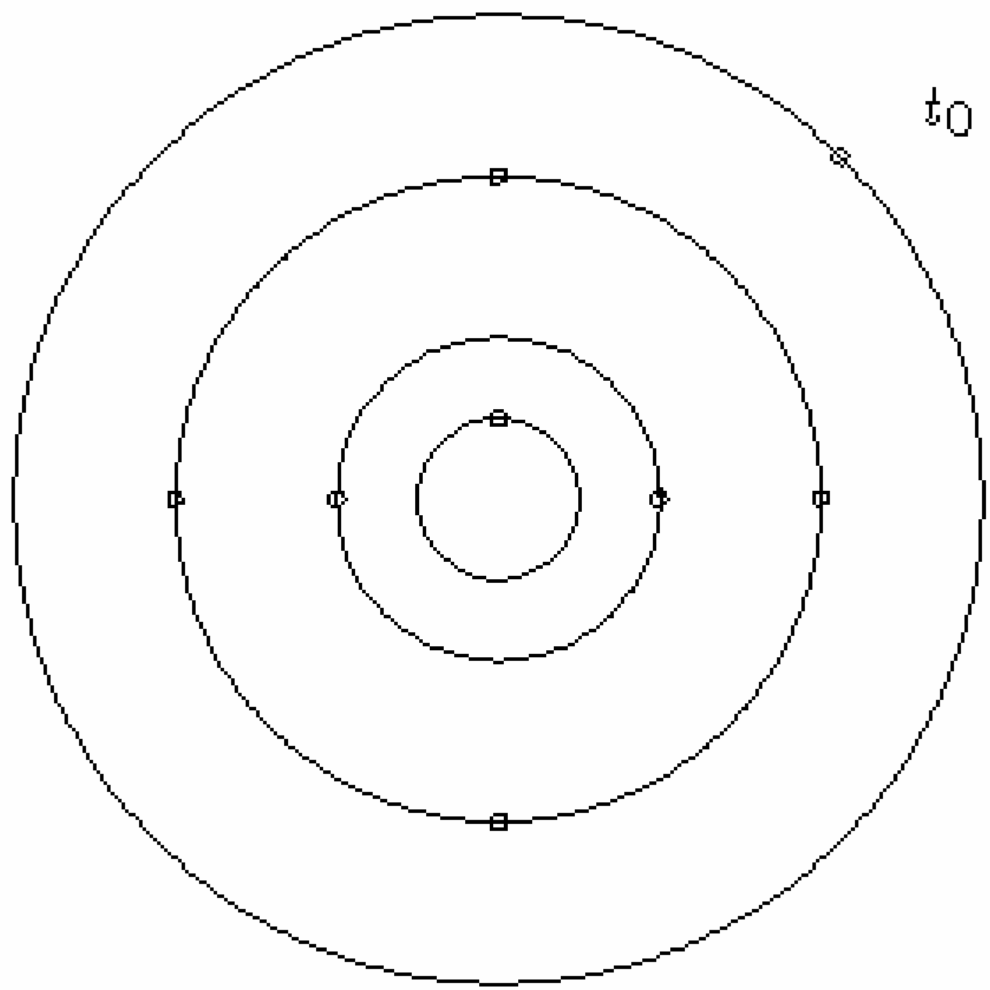
The critical values of $|s| << 1$ will appear on "circles" $r_1 \geq r_2 > \dots$



This rate of convergence of critical points will induce an ascending filtration

$$\{\omega \in H^n(V,\mathbb{Q})\textbf{collapsing rate} \geq r_1\}$$

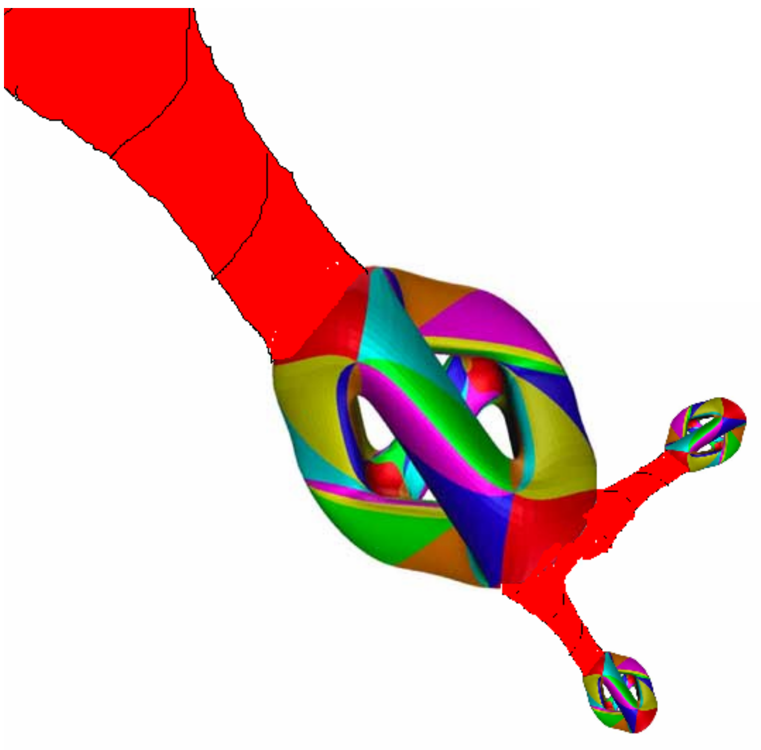
$$\cap$$

$$\{\omega \in H^n(V,\mathbb{Q})\textbf{collapsing rate} \geq r_2\}$$

$$\cap$$

$$H^n(V,\mathbb{Q})$$

invariant under monodromy.



Problem: Over the real numbers, compute the homology groups

$$V_{\pm\varepsilon}^{\mathbb{R}} = f^{-1}(\pm\varepsilon) \subset B \subset R^{n+1}$$

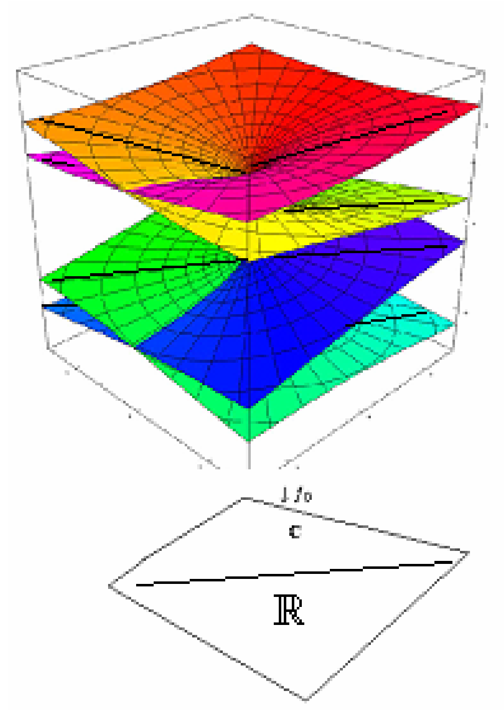
in terms of

$$f \ , \ \frac{\partial f}{\partial x_j} \ , \ \frac{\partial^2 f}{\partial x_j \partial x_k}, \dots$$

with the corresponding filtration,

and the "filtered Euler Characteristic"

Over $\mathbb{R}$, we also have the critical points of F_s converging to 0 with s, and each ’branch’ has an index, which is telling us the dimension of the ’vanishing cycle’ and hence to which group $H^k(V_{t_0}^{\mathbb{R}}, \mathbb{Q})$ it contributes.



$$\mathbf{A} := \frac{\mathcal{A}_{\mathbb{R}^{n+1},0}}{(f_0, \ldots, f_n)}$$

finite dimensional algebra The class of the Hessian

$$Hess_A = [det\left(\frac{\partial^2 f}{\partial x_j \partial x_k}\right)_{j,k=0,\ldots,n}]_{\mathbf{A}} \in \mathbf{A}$$

generates the socle (the unique minimal non-zero ideal) of the algebra $\mathbb{A}$**.**

A symmetric bilinear form:

$$< \ , >_{L_{\mathbf{A}}}: \mathbf{A} \times \mathbf{A} \stackrel{\cdot}{\rightarrow} \mathbf{A} \stackrel{L_{\mathbf{A}}}{\rightarrow} \mathbb{R}$$

composing multiplication with any linear map $L_{\mathbf{A}} : \mathbf{A} \longrightarrow \mathbb{R}\ ,\ L_{\mathbf{A}}(Hess_{\mathbf{A}}) > 0$**.** **Eisenbud-Levine and Khimshiashvili say this bilinear form is nondegenerate and its signature** $\sigma_{\mathbf{A}}$ **is independent of the choice of** $L_{\mathbf{A}}$**. Arnold says**

$$\chi_{\pm} = 1 \pm \sigma_{\mathbf{A}}$$

$$f : \mathbf{A} \longrightarrow \mathbf{A}$$

is a nilpotent map so

$$K_m := Ann_{\mathbf{A}}(f) \cap (f^{m-1}),$$

$$0 \subset K_{\ell+1} \subset \cdots \subset K_1 \subset K_0 := \mathbf{A}$$

and a family of bilinear forms

$$< \ , \ >_{L_{\mathbf{A}},f,m} : K_m \times K_m \to \mathbb{R}$$

$$< a, a' >_{L_{\mathbf{A}},f,m} = < \frac{a}{f^{m-1}}, a' >_{L_{\mathbf{A}}},$$

defined for $m = 0, \ldots, \ell + 1.$

For $m = 0, \ldots, \ell + 1$ **the order** m **bilinear form** $<, >_{L_{\mathbf{A}},f,m}$ **on** K_m **induces a non-degenerate bilinear form**

$$< \ , \ >_{L_{\mathbf{A}},f,m} : \frac{K_m}{K_{m+1}} \times \frac{K_m}{K_{m+1}} \longrightarrow \mathbb{R},$$

whose signature $\sigma_{\mathbf{A},f,m}$ **is independent of the linear map** $L_{\mathbf{A}}$ **chosen.**

Let $X = \sum_{i=0}^{n} X^i \frac{\partial}{\partial x_i}$ be a real analytic vector field with an algebraically isolated zero at 0 in $\mathbb{R}^{n+1}$, then the (Poincaré-Hopf) index of X at 0 is:

Topologically: The degree of the map

$$\frac{X}{|X|}|_{S^n_\varepsilon} : S^n_\varepsilon \longrightarrow S^n$$

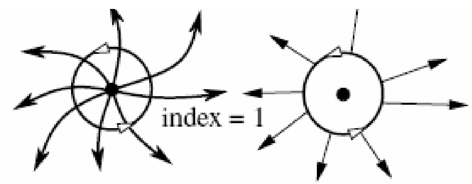
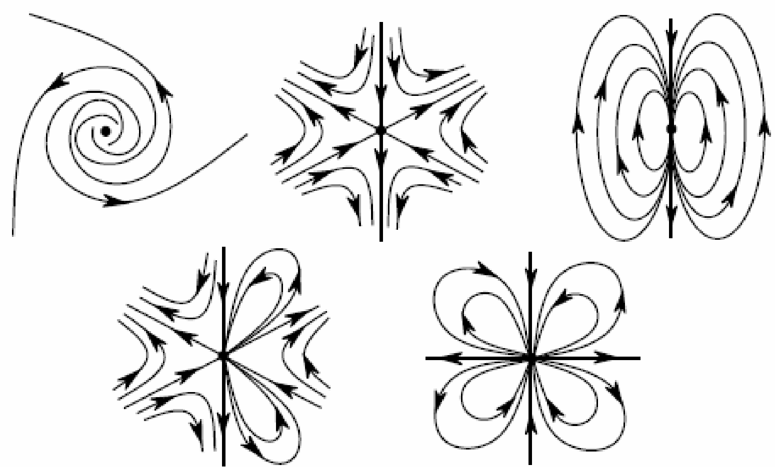


Figure 1: Index of a critical point.



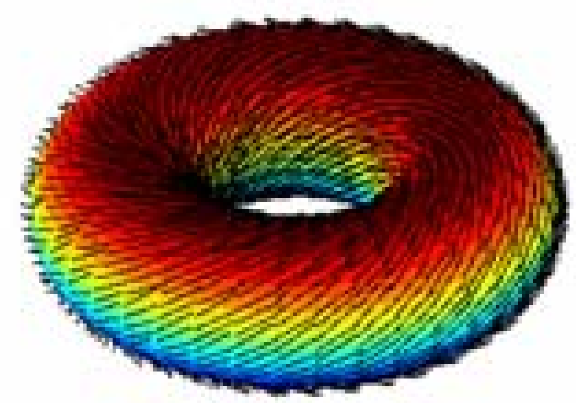
Additivity Property of the Index: If X_s is an analytic family of real analytic vector fields wit X_0 having an isolated singularity at 0 then for $|s| << 1$ and U a neighbourhood of 0

$$Ind(X_0, 0) = \sum_{p \in U, X_s(p) = 0} Ind(X_s, p)$$

This makes the index a topological number.

Poincaré-Hopf index Theorem: If M is a compact manifold and X is a vector field on M with isolated singularities, then the Euler-Poincaré characteristic $\chi(M)$ of M is

$$\chi(M) = \sum_{p \in M, X(p)=0} Ind(X, p)$$



Algebraically the Poincaré-Hopf index may be computed as the signature of the bilinear form constructed for the finite dimensional algebra

$$\mathrm{B} := \frac{\mathcal{A}_{\mathbb{R}^{n+1},0}}{(X^0,\ldots,X^n)} \quad , \quad <\,,\,>_{L_{\mathrm{B}}} : \mathrm{B}\times\mathrm{B} \xrightarrow{\cdot} \mathrm{B} \xrightarrow{L_{\mathrm{B}}} \mathbb{R}$$

where $L_{\mathrm{B}} : \mathrm{B} \to \mathbb{R}$ is a linear map with

$$L_{\mathrm{B}}(J_X) > 0$$

(Eisenbud-Levine-Khimiashishvili).

Now assume further that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is a real analytic function, that X is tangent to the fiber $V_0 := f^{-1}(0)$, giving the relation

$$df(X) = hf$$

with h a real analytic function called the cofactor. By restricting $X|_{V_0}$ and using the implicit function theorem we can define

$$Ind_{V_0}(X, p)$$

At a smooth point of f, say $f = z_n$, then $df(X) = X^n$ so $X^n = z_n h$ and

$$Jac(X) = \begin{vmatrix} \frac{\partial X^0}{\partial x_0} & \cdots & \frac{\partial X^0}{\partial x_n} \\ \frac{\partial X^2}{\partial x_0} & \cdots & \frac{\partial X^2}{\partial x_n} \\ . & \cdots & . \\ \frac{\partial X^n}{\partial x_0} & \cdots & \frac{\partial X^n}{\partial x_n} \end{vmatrix}.$$

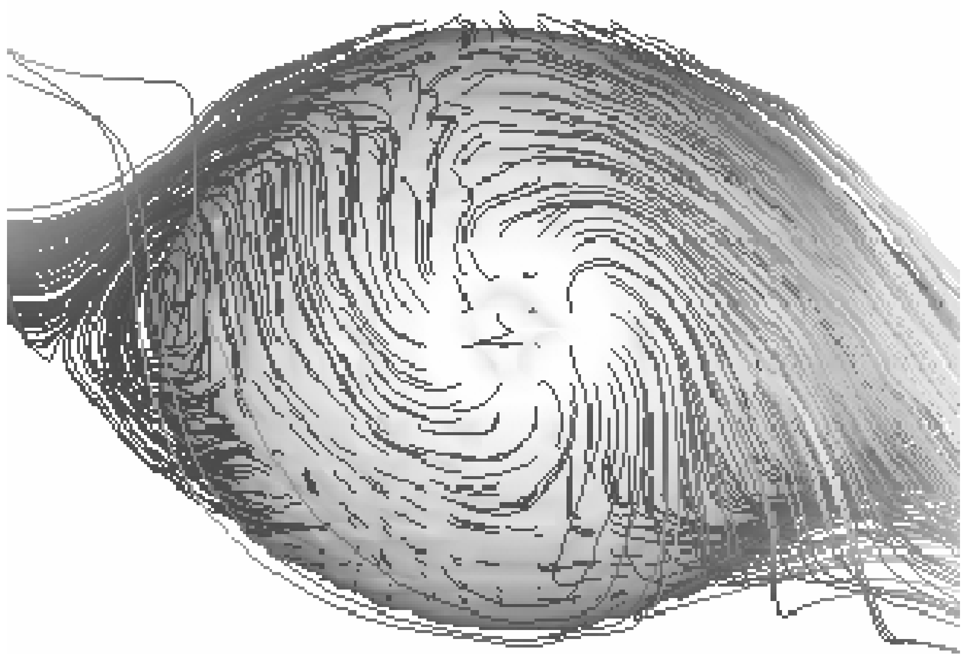
$$Jac(X)(0) = \begin{vmatrix} \frac{\partial X^0}{\partial x_0} & \cdots & \frac{\partial X^0}{\partial x_n} \\ \frac{\partial X^2}{\partial x_0} & \cdots & \frac{\partial X^2}{\partial x_n} \\ . & \cdots & . \\ 0 & 0 \cdots 0 & h(0) \end{vmatrix}$$

If 0 is a point of multiplicity 1 for X, then

$$= h(0) \begin{vmatrix} \frac{\partial X^0}{\partial x_0} & \cdots & \frac{\partial X^0}{\partial x_{n-1}} \\ \frac{\partial X^2}{\partial x_0} & \cdots & \frac{\partial X^2}{\partial x_{n-1}} \\ . & \cdots & . \\ \frac{\partial X^{n-1}}{\partial x_0} & \cdots & \frac{\partial X^{n-1}}{\partial x_{n-1}} \end{vmatrix}$$

If the singular points of X have multiplicity 1 at smooth points of f then

$$Ind_{\mathbb{R}^{n+1}}(X,p) = sign(h(p))Ind_{f=0}(X,p)$$



Additivity Property of the Index: If X_t is an analytic family of real analytic vector fields tangent to V_t, having isolated singularities, then for $|t| << 1$ and U a neighbourhood of q

$$Ind_{V_0}(X_0, q) = \sum_{p \in U, X_t(p)=0} Ind_{V_t}(X_t, p)$$

If 0 is a smooth point of V_0 then the Poincaré-Hopf index at 0 of the vector field $X|_{V_0}$ at 0 is the signature $\sigma_{\mathbf{B},h,0}$ of the bilinear form

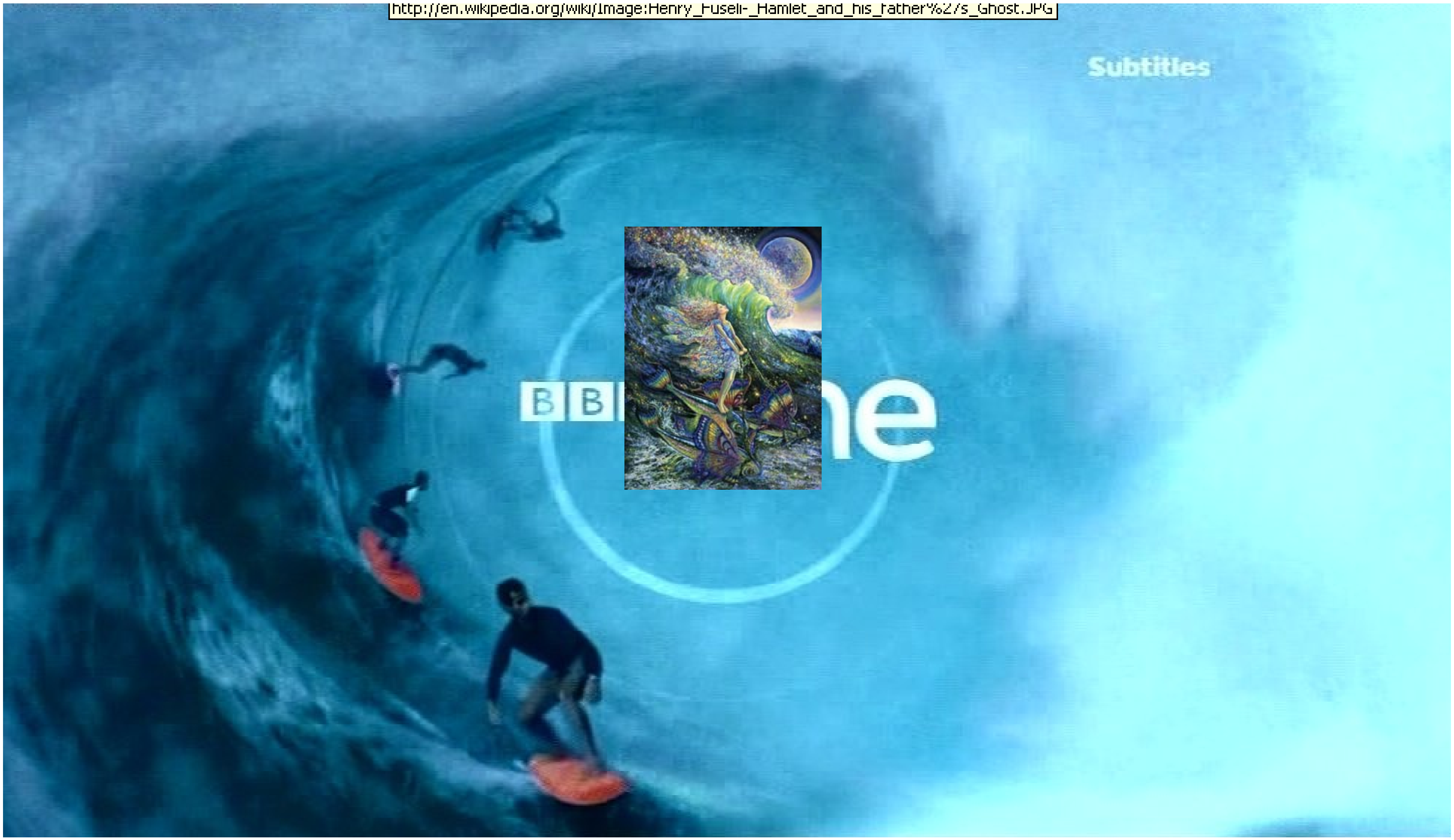
$$< \, , \, >_{L,h,0}\colon \frac{\mathbf{B}}{Ann_{\mathbf{B}}(h)} \times \frac{\mathbf{B}}{Ann_{\mathbf{B}}(h)} \xrightarrow{\cdot} \frac{\mathbf{B}}{Ann_{\mathbf{B}}(h)} \xrightarrow{L} \mathbb{R}$$

$$L : \frac{\mathbf{B}}{Ann_{\mathbf{B}}(h)} \longrightarrow \mathbb{R} \ , \ \ L(\frac{J_{\mathbf{B}}}{h}) > 0$$

Let $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be as before, and let X_t be a family of real analytic vector fields with algebraically isolated singularities satisfying

$$d(f - t)X_t = h_t(f - t)$$

i.e. X_t is tangent to V_t.


http://en.wikipedia.org/wiki/Image:Henry_Fuseli-_Hamlet_and_his_Father%27s_Ghost.JPG
Subtitles

We can do Taylor series expansion of this family of bilinear forms, to obtain a formula for the index on the left and on the right of the form

$$\sum_{m\geq 1} \tau_m \quad , \quad \sum_{m\geq 1} (-1)^m \tau_m$$

Using a Residue Map, the bilinear forms in the 'Taylor series expansion' can be 'transported' to the ring A (local version of Poincaré-Hopf Theorem) and we obtain:

Theorem ([GGM)]: If 0 is an algebraically isolated critical point of V_0:
If n is odd, then

$$Ind_{V_{+,0}}(X) = Ind_{V_{-,0}}(X) = \sigma_{\mathbf{B},h,0} - \sigma_{\mathbf{A},h,0}.$$

If n is even:

$$Ind_{V_{\pm,0}}(X) = \sigma_{\mathbf{B},h,0} + K_{\pm}$$

$$K_+ = \sum_{m\geq 1} \sigma_{\mathbf{A},f,m} \quad , \quad K_- = \sum_{m\geq 1} (-1)^m \sigma_{\mathbf{A},f,m}.$$

Conjecture: There is a relationship between the filtration of the cohomology groups $H^k(V_t^{\mathbb{R}}, \mathbb{Q})$ and $\sigma_{\mathbf{A},f,m}$ (using the Riemann-Hodge bilinear relations).