On Hilbert schemes of scrolls of genus $g$

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General problem

a. Given $S \subset \mathbb{P}^r$ smooth surface of degree $d$, sectional genus $g$, etc., study properties of the Hilbert scheme $\text{Hilb}(d, g, r)$ parametrizing such $S$'s?

b. Existence of "particular" curves on $S$ can influence the behaviour of the component(s) of $\text{Hilb}(d, g, r)$ to which $S$ belong?

c. Given $S$ sufficiently general in a component of $\text{Hilb}(d, g, r)$, what kind of "limits" $S$ admits (embedded degenerations)?

d. Conversely, given a configuration $X = \bigcup_i V_i \subset \mathbb{P}^r$, is it smoothable to an element of $\text{Hilb}(d, g, r)$?

Namely, $\exists \ X \to \Delta$ s.t. $X_0 = X$ and $X_t = S$, for $t \neq 0$ and $[S]$ general in a component of $\text{Hilb}(d, g, r)$?

e. If $X \to \Delta$ in d actually exists and if we know the combinatorial data of the configuration $X_0 = X$:

(i) what kind of properties can we deduce for $S = X_t$, $t \neq 0$?

(ii) what kind of properties can we deduce for $\text{Hilb}(d, g, r)$ from the fact that $[X = X_0]$ is a Hilbert point?

(iii) applications to other parameter spaces of some other "related" geometric objects?
Motivations and inspirations

Classical papers:

[C. Segre]

[Severi]

[Zappa]


More recent papers:

[Ghione]

[Oxbury]

[Fuentes-Garcia, Pedreira]

Main subject of this talk

Given $d > 0$ and $g \geq 0$ integers, we will give some answers to the previous questions in the case of **scrolls** of degree $d$, genus $g$, "sufficiently" general.

Our approach


Notation and general assumptions

From now on:

(1) $C$ smooth, irreducible projective curve of genus $g \geq 0$,

$F \xrightarrow{\rho} C$ geometrically ruled surface, i.e.

$$F = \mathbb{P}(\mathcal{F}),$$

$\mathcal{F}$ rank-two vector bundle (equiv. loc. free sheaf) on $C$

Assume further:

- $\deg(\mathcal{F}) := \deg(\det(\mathcal{F})) = d$;
- $h^0(C, \mathcal{F}) = r + 1$, with $r \geq 3$;
- $|\mathcal{O}_F(1)|$ is b.p.f. and the induced morphism $\Phi : F \rightarrow \mathbb{P}^r$ is birational to its image.

Then

$$\Phi(F) := S \subset \mathbb{P}^r$$

is a scroll of degree $d$ (sectional) genus $g$ (determined by $(\mathcal{F}, C)$).

Remark: $S$ smooth $\Leftrightarrow \mathcal{F}$ v.a.; otherwise $F$ is its minimal desingularization.

For any $x \in C$, $f_x := \rho^{-1}(x) \cong \mathbb{P}^1$ and $l_x := \Phi(f_x)$ is a line of the ruling of $S$. 
(2) For $A \in \text{Pic}(C)$, any $B_1 \in |O_F(1) \otimes \rho^*(A)| \neq \emptyset$ is a unisecant curve of $F$.

An irreducible unisecant $B$ is called a section of $F$.

\[ \uparrow \quad 1:1 \text{ correspondence} \]

\[ 0 \to N \to \mathcal{F} \to L = L_B \to 0. \]

If $B = B_L \subset F$ section, $L \in \text{Pic}(C)$, let $\Gamma := \Phi(B) \subset S$.

$\Phi|_B$ birational $\Rightarrow$ $\Gamma$ section (or directrix) of $S$.

$\Phi|_B$ $n:1 \Rightarrow \Gamma$ $n$-directrix of $S$.

(3) Riemann-Roch

\[ r + 1 := h^0(\mathcal{O}_F(1)) = h^0(\mathcal{F}) = d - 2g + 2 + h^1 \]

\[ h^1 := h^1(\mathcal{O}_F(1)) = h^1(\mathcal{F}) = \text{speciality of the scroll}. \]

$S$ special scroll if $h^1 > 0$, non-special otherwise.

Since $r \geq 3 \Rightarrow d \geq 2g + 2 - h^1$.

From now on

\[ d \geq 2g + 2 \]

(necessary bound for linearly normal, non-special scrolls).
**Bounds on speciality:** Riemann-Roch thm. for \( F \) on \( C \):

\[
0 \leq h^1 \leq g
\]

\( h^1 = g \) cones [Segre - Ghione],

\( h^1 = 0 \) non-special scrolls.

Any intermediate value \( 1 \leq h^1 \leq g - 1 \) can be realized.

**Example.** Let \( g \geq 3 \), \( d \geq 4g - 1 \), \( 1 \leq h^1 \leq g - 1 \).

\(|L|\) b.p.f. with \( h^1(L) = h^1 \).

\( N \) general l.b. of degree \( d - \text{deg}(L) \).

\( \text{deg}(L) \leq 2g - 2 \) and \( d \geq 4g - 1 \) \( \Rightarrow \) \( \text{deg}(N) \geq 2g + 1 \) i.e. \(|N|\) very ample.

Let \( F = L \oplus N \); then \( \mathcal{O}_F(1) \) b.p.f. and \( h^1(\mathcal{O}_F(1)) = h^1 \).

**Remark** For large values of \( h^1 \), \( \mathcal{O}_F(1) \) in general not v.a.

- \( h^1 = g - 1 \) \( \Rightarrow \) \( |L| = g_2^1 \) \( \Rightarrow \) \( S \) has a linear 2-directrix

- \( h^1 = g - 2 \) \( \Rightarrow \) \( |L| = g_3^1 \) or \( |L| = 2g_2^1 \) or \( |L| = g_4^2 \). \( S \) is smooth only if \( |L| = g_4^2 \) with \( g = 3 \).

**Remark** For any section \( \Gamma \) of \( S \),

\[
h^1(\mathcal{O}_\Gamma(1)) := \text{speciality of } \Gamma \leq h^1.
\]
Hilbert schemes of l.n. non-special scrolls

Rational case:

Proposition 1 (Classical). Let $d \geq 2$ and $r = d + 1$.

The Hilbert scheme $\mathcal{H}_{d,0}$ parametrizing rational normal scrolls of degree $d$ in $\mathbb{P}^r$ is irreducible, generically smooth.

The general point of $\mathcal{H}_{d,0}$ represents a smooth, balanced scroll.

$\dim(\mathcal{H}_{d,0}) = (r + 1)^2 - 7$.

Proof: $S \subset \mathbb{P}^r$ any smooth, rational normal scroll. Consider

$$0 \to T_S \to T_{\mathbb{P}^r}|_S \to N_{S/\mathbb{P}^r} \to 0.$$ 

Euler sequence restricted to $S + S$ is a scroll

$$\Downarrow$$

$$h^1(T_{\mathbb{P}^r}|_S) = h^1(N_{S/\mathbb{P}^r}) = 0$$

so

$$h^0(N_{S/\mathbb{P}^r}) = h^0(T_{\mathbb{P}^r}|_S) - \chi(T_S) = (r + 1)^2 - 1 - 6.$$ 

$h^1(N_{S/\mathbb{P}^r}) = 0 \Rightarrow [S]$ smooth point of the Hilbert scheme of such scrolls. Therefore

$$h^0(N_{S/\mathbb{P}^r}) = \dim_s(\mathcal{H}_{d,0}) = \dim(T_s(\mathcal{H}_{d,0})).$$

Finally, one uses the well-known fact: $S_{a,b}$ degenerates to $S_{h,f}$ $\iff a + b = h + f$ and $|a - b| < |h - f|$ ♣

Remark In particular, for $[S] \in \mathcal{H}_{d,0}$ general, there are $\infty^6$ projectivities of $\mathbb{P}^r$ fixing $S$. 
Irregular case, i.e. $g \geq 1$:

**Theorem 1** [C, C, –, M] Let $r = d - 2g + 1$, where

- $d \geq 5$, if $g = 1$
- $d \geq 2g + 4$, if $g \geq 2$.

Then, there exists a unique irreducible component $\mathcal{H}_{d,g}$ of Hilbert scheme of scrolls of degree $d$ and genus $g$ in $\mathbb{P}^r$, whose general point $[S] \in \mathcal{H}_{d,g}$ is a smooth, linearly normal scroll $S \subset \mathbb{P}^r$ (equiv. $h^1(\mathcal{O}_S(1)) = 0$).

Moreover,

(i) $\mathcal{H}_{d,g}$ contains suitable reducible surfaces $[T]$ as smooth points of $\mathcal{H}_{d,g}$;

(ii) $\mathcal{H}_{d,g}$ generically smooth,

$$\dim(\mathcal{H}_{d,g}) = h^0(S, \mathcal{N}_{S/\mathbb{P}^r}) = (r + 1)^2 + 7(g - 1)$$

and

$$h^1(S, \mathcal{N}_{S/\mathbb{P}^r}) = h^2(S, \mathcal{N}_{S/\mathbb{P}^r}) = 0.$$ 

(iii) $\mathcal{H}_{d,g}$ dominates $\mathcal{M}_g$, i.e. $S$ is with general moduli.

**Remark.** No gaps w.r.t. the initial condition $d \geq 2g + 2$. In other words, the bounds on $d$ in Theorem 1 are sharp.

Indeed, no smooth, linearly normal scrolls in $\mathbb{P}^r$ if either $d = 2g + 2$ and $g \geq 1$ or $d = 2g + 3$ and $g \geq 2$. 
Proof: Induction on $g$ + degeneration techniques.

Step 1: $g = 0$, ok from Proposition 1.

Step 2: Let $g \geq 1$. Construct suitable reducible (precisely Zappatic) surfaces in $\mathcal{H}_{d,g}$.

Let $[\tilde{S}] \in \mathcal{H}_{d-2,g-1}$ general $\Rightarrow \tilde{S} \subset \mathbb{P}^r$, $r = (d-2) - 2(g-1) + 1 = d - 2g + 1$.

Let $l_1$ and $l_2$ general lines of the ruling of $\tilde{S}$.

$\langle l_1, l_2 \rangle = \Lambda \cong \mathbb{P}^3$. Let $Q \subset \Lambda$ general quadric through $l_1$ and $l_2$.

Fact $Q$ is smooth and $\tilde{S} \cap Q = l_1 \cup l_2$ transverse.

Let $T := \tilde{S} \cup Q$

reducible surface with g.n.c. $\Rightarrow$

$\text{Sing}(T) = l_1 \cup l_2 : = R$.

Step 3: Some invariants of $T$

Using e.g.


\[ g(T) = 0, \quad \chi(\mathcal{O}_T) = 1 - g, \quad p_\omega(T) := h^0(\omega_T) = 0. \]

Step 4: Some cohomological property of $T$.

(a) From

\[ 0 \to \mathcal{O}_T(1) \to \mathcal{O}_{\tilde{S}}(1) \oplus \mathcal{O}_Q(1) \to \mathcal{O}_R(1) \to 0 \]

one has

\[ h^1(\mathcal{O}_T(1)) = 0 \]
(b) \( \mathcal{N}_T \) and \( \mathcal{I}_T \) be the normal and "tangent" sheaf of \( T \) in \( \mathbb{P}^r \). Then
\[
(\ast) \quad 0 \to \mathcal{I}_T \to \mathcal{I}_{\mathbb{P}^r}|_T \xrightarrow{\tau} \mathcal{N}_T \to T^1 := \text{Coker}(\tau) \to 0.
\]

[Friedman] \( T \) with g.n.c. \( \Rightarrow T^1 \cong \mathcal{N}_{R/\tilde{S}} \otimes \mathcal{N}_{R/Q} \cong 0_R \).

(c) From \( T = \tilde{S} \cup Q \) and \( h^1(\mathcal{N}_S) = 0 \) by induction
\[
\Downarrow
\]
\[
h^1(\mathcal{N}_T) = h^2(\mathcal{N}_T) = 0 \quad h^0(\mathcal{N}_T) = \chi(\mathcal{N}_T) = (r + 1)^2 + 7(g - 1)
\]
and
\[
H^0(\mathcal{N}_T) \xrightarrow{\alpha} H^0(T^1).
\]

Step 5: \( h^1(\mathcal{N}_T) = 0 \Rightarrow [T] \) smooth point of the Hilbert scheme of surfaces of degree \( d \) and genus \( g \) in \( \mathbb{P}^r \).
\[
\Downarrow
\]
[\( T \)] belongs to a unique component \( \mathcal{H}_{d,g} \) of this Hilbert scheme, of dimension \( \chi(\mathcal{N}_T) = h^0(\mathcal{N}_T) \).

\( \alpha \) surjective \( \Rightarrow \) a general tangent vector to \( \mathcal{H}_{d,g} \) at \( [T] \) represents an infinitesimal embedded deformation of \( T \) which smooths \( \Gamma = \text{Sing}(T) \).
\[
\Downarrow
\]
The general point of \( \mathcal{H}_{d,g} \) represents a smooth, irreducible surface \( S \) degenerating to \( T \).
\[
\Downarrow
\]
From Step 3 and e.g.

\[
\Downarrow
\]
\( S \) is necessarily ruled.

From Step 4 - (a) and semi-continuity, \( h^1(\mathcal{O}_S(1)) = 0 \), i.e. \( S \subset \mathbb{P}^r \) is linearly normal.
Step 6: Using


S degenerates to $T \Rightarrow K^2_S = 8(1 - g)$

then $S$ is necessarily geometrically ruled.

Adjunction theory implies $S$ is a scroll: otherwise, $0 < d \leq 4(g - 1) + K^2_S$, contradicting $K^2_S = 8(1 - g)$.

Step 7: From

$$0 \to \mathcal{I}_S \to \mathcal{I}_{\mathbb{P}^r|S} \to \mathcal{N}_{S/\mathbb{P}^r} \to 0,$$

we get

$$H^0(\mathcal{N}_{S/\mathbb{P}^r}) \to H^1(\mathcal{I}_S),$$

(Euler sequence gives $h^1(\mathcal{I}_{\mathbb{P}^r|S}) = 0$).

From $S \xrightarrow{\rho} C$ one has $H^1(\mathcal{I}_S) \to H^1(\mathcal{I}_C) \Rightarrow$

$$H^0(\mathcal{N}_{S/\mathbb{P}^r}) \to H^1(\mathcal{I}_C),$$

i.e. $\mathcal{H}_{d,g}$ dominates $\mathcal{M}_g$.

Step 8: Uniqueness of the component $\mathcal{H}_{d,g}$.

It follows from the previous steps and the **Classical result**:

*For any smooth scroll $S$ of degree $d$, there exists $\varphi : Y := C \times \mathbb{P}^1 \dashrightarrow S$ birational which is the composition of $d$ elementary transformations at $d$ distinct points $\Theta := \{y_1, \ldots, y_d\} \subset Y$ lying on $d$ distinct $\mathbb{P}^1$-fibres of $Y$. ♣*
Remarks. (1) Some previous results partially related: [Arrondo, Pedreira, Sols; 1988] projections of scrolls into $\mathbb{P}^3$ and then study curves in $\mathbb{G}(1, 3)$.

(2) With a similar approach, one can reprove Proposition 1 on $\mathcal{H}_{d,0}$.

(3) From the proof of Theorem 1

\[ \downarrow \]

$[S] \in \mathcal{H}_{d,g}$ general degenerates to a reducible surface like $T$, which is a Zappatic surface with g.n.c. singularities

(4) Pushing further degeneration techniques and using once again the results in:


we prove that $\mathcal{H}_{d,g}$ contains reducible surfaces which are union of planes, with only double lines and further singular points (the so called $R_3$- and $S_4$-points, cf. [C, C, --, M] "On the $K^2$ of degenerations of surfaces and the multiple point formula", *Annals of Mathematics*, 165 (2007), no. 2, 335-365.)

These are examples of planar Zappatic surfaces (studied in the above 2 papers.)
Precisely, we prove:

**Proposition 2** \([C, C, -, M] \mathcal{H}_{d,g}\) contains planar Zappatic surfaces \([X_{d,g}]\) such that

- if \(g = 0\), \(X_{d,0}\) has \(d - 1\) double lines and \(d - 2\) \(R_3\)-points
- if \(g \geq 1\), \(X_{d,g}\) has \(d + 2g + 2\) double lines, \(d - 2g + 2\) \(R_3\)-points and \(2g - 2\) \(S_4\)-points.

In other words, for \(g \geq 0\), the general \([S] \in \mathcal{H}_{d,g}\) degenerates to planar Zappatic surfaces \(X_{d,g}\) as above.

This in particular answers (improving the expected bound) to Zappa’s original questions (1940-50):

**Zappa’s questions:** If \(S \subset \mathbb{P}^r\) is a scroll of genus \(g\), degree \(d \geq 3g + 2\) and "sufficiently general"

- can \(S\) degenerate to a union of planes?
- If yes, is it possible to have in the limit surface only double lines and \(E_3\)-, \(R_3\)- and \(S_4\)-points?

**Remark.** Zappa (1940-50) realized that \textbf{g.n.c.} singularities are not sufficient to study \underline{embedded degenerations} of surfaces, even with general moduli.
Connections with vector bundles

For any \( g \geq 1 \), for any smooth \( C \) of genus \( g \) and for any \( d \)

\[ U_C(d) \]

the moduli space of degree \( d \), rank-two **semistable** vector bundles on \( C \).

**Theorem 2** \([C, C, −, M]\) Let \( g \geq 1 \) and num. assumptions as in **Thm1**.

Let \([S] \in \mathcal{H}_{d,g}\) be general. Then:

(i) \( S \) is determined by \((\mathcal{F}, C)\), where \([C] \in \mathcal{M}_g\) general and \([\mathcal{F}] \in U_C(d)\) general.

(ii) In particular, if \( g \geq 2 \), \( \mathcal{F} \) is stable and, if \( G_S := \text{sgr. of projectivities of } \mathbb{P}^r \text{ fixing } S \), then \( G_S = \{Id\} \).

**Sketch Proof.**

**Step 1**: Small Lemma: \( d \geq 2g \) and \([\mathcal{F}] \in U_C(d)\) general \( \Rightarrow h^1(C, \mathcal{F}) = 0 \).

**Step 2**: From Step 1, \([\mathcal{F}] \in U_C(d)\) general \( \Rightarrow h^0(\mathcal{F}) = d - 2g + 2 = r + 1 \).

We can costruct a morphism

\[
\Psi : \mathcal{U} \xrightarrow{(C, \mathcal{F}, \sigma)} \text{Hilb}(d, g, r) \xrightarrow{\sigma(S)} \sigma(S)
\]

where

\([C] \in \mathcal{M}_g, \ [\mathcal{F}] \in U_C(d), \ \sigma \in \text{PGL}(r + 1) \) and \( S = \Phi(\mathcal{F}) \).

\( S \) is smooth for \( \mathcal{F} \) general.

**Step 3**: \( \mathcal{U} \) irreducible and, from **Theorem 1**, \( \Psi(\mathcal{U}) \subseteq \mathcal{H}_{d,g} \).

Semistability is an open condition \( \Rightarrow \Psi \) dominant on \( \mathcal{H}_{d,g} \).

**Step 4**: any possible element \( Id \neq \tau \in G_S \) induces a non-trivial automorphism of \( F \), contradicting \( \mathcal{F} \) stable (so simple) and \( C \) with general moduli. ★
Remarks.

(1) For $g \geq 2$, $\mathcal{U}$ depends on the following parameters:

- $3g - 3$ for $\mathcal{C}$;
- $\dim(U_C(d)) = 4g - 3$ for $\mathcal{F}$;
- $(r + 1)^2 - 1 = \dim(PGL(r + 1, \mathbb{C}))$, i.e.
  $$\dim(\mathcal{U}) = \dim(H_{d,g}) = 7g - 7 + (r + 1)^2.$$ 

This gives a parametric representation of $H_{d,g}$.

From Theorem 2 (ii), more precisely $\Psi$ is birational.

(2) For $g = 1$, statement (ii) and behaviour of $\Psi$ are more involved (depends on the parity of $d$).

(3) **Attention**: we are not saying that all smooth scrolls parametrized by $H_{d,g}$ comes from semistable bundles.

**Example.** $\mathcal{E}$ any unstable bundle on $\mathcal{C}$ of degree $e$. Let $A$ be ample of degree $a$ and consider

$$\mathcal{F} := \mathcal{E} \otimes A^k, \quad k \gg 0.$$ 

So $\mathcal{F}$ v.a., $h^1(\mathcal{F}) = 0$, thus $(\mathcal{F}, \mathcal{C})$ determines a scroll $[S_\mathcal{F}] \in H_{e+2ka,g}$ coming from $\mathcal{F}$ unstable.

Such scrolls fill up closed sub-loci of $H_{e+2ka,g}$.

(4) Some previous results partially related: [Pedreira; 1988].
Applications and consequences

Using degeneration techniques as in **Theorem 1** and construction as in **Theorem 2**, we get also the following results:

(1) $S$ is a **general ruled surface** [Ghione, 1981], i.e.

\[
\text{Div}_S^{1,m}
\]

has the expected dimension $e$; it is smooth; it is irreducible when $e > 0$.

\[\downarrow\]

effective existence results of general ruled surfaces (specifying bounds on $d$), whereas Ghione’s existence results were only asymptotic.

(2) Sections of **minimal degree** on $S$: compute their degree and the dimension of the family.

(3) Enumerative results on $\text{Div}_S^{1,m}$: cardinality (when $e = 0$), index, genus computation, monodromy, etc…..

(4) $M_n(\mathcal{F})$: Scheme parametrizing sub-line bundles of degree $n$ of $\mathcal{F}$.

Then:

\[
M_n(\mathcal{F}) \cong \text{Div}_S^{1,m}
\]

where $m + n = d$.

\[\downarrow\]

• alternative proofs (via proj. geom.) of some results on sub-line bundles of **maximal degree** $\bar{n}$, e.g. [Maruyama], [Lange-Narashiman], [Oxbury].

• affirmative answer to a **Conjecture** of [Oxbury] (2005) in the rank 2 case: connectedness of $M_{\bar{n}}(\mathcal{F})$, on any $C$, when $\dim > 0$.

(7) Study $W_n(\mathcal{F}) := \text{Im}(M_n(\mathcal{F})) \subset \text{Pic}^n(C)$ and some questions on the **Brill-Noether theory** of the line-bundles in $W_n(\mathcal{F})$.

**Consequences**: we can compute $\dim(|\emptyset_S(\Gamma)|)$, for $[\Gamma] \in \text{Div}_S^{1,m}$ general and for any admissible $m$.

(8) We show that any irreducible component of $W_n^1(\mathcal{F})$ has the expected dimension.
First possible questions

(1) Is $\mathcal{H}_{d,g}$ the only component of the Hilbert scheme whose general point parametrizes a smooth scroll of degree $d$ and genus $g$ in $\mathbb{P}^r$?

(2) If another such component $\mathcal{Z}$ actually exists,

- $\dim(\mathcal{Z}) = ?$

- $\mathcal{Z}$ generically smooth ?

- general point of $\mathcal{Z}$ ?

- rank-two vector bundle determining the general point of $\mathcal{Z}$ ?

- image of $\mathcal{Z}$ via the natural map $\mathcal{Z} \to \mathcal{M}_g$?

All these questions naturally lead to the study of special scrolls
Hilbert schemes of l.n. special scrolls

From now on

\[ S \subset \mathbb{P}^r \]

smooth scroll of genus \( g \), degree \( d \) and speciality \( h^1 \), with

\[ 0 < h^1 < g \quad \text{and} \quad r = d - 2g + 1 + h^1. \]

Main Point: existence of a special section on \( S \).

**Proposition 3** [C. Segre, 1889. Revisited C, C, −, M, 2008] Let \( g \geq 3 \) and \( d \geq 4g - 2 \). Let \( S \subset \mathbb{P}^r \) be a smooth, linearly normal, special scroll of degree \( d \), genus \( g \) and speciality \( h^1 \). Then:

(i) \( S \) contains a unique, special section \( \Gamma \subset \mathbb{P}^h \) of degree \( m \) such that \( m - h = g - h^1 \). Furthermore:

- \( \Gamma \) is linearly normally embedded in \( \mathbb{P}^h \), i.e. \( H^0(\mathcal{F}) \rightarrow H^0(L_\Gamma) \).
- \( h^1(\Gamma, \mathcal{O}_\Gamma(H)) = h^1 \)
- \( \Gamma \) curve (different from a line) of minimal degree
- \( \Gamma \) unique section with non-positive self-intersection (\( \Gamma^2 < 0 \)).

(ii) If moreover \( S \) has general moduli, then

- either \( g \geq 4h^1, \ h \geq 3 \), or
- \( g = 3, \ h^1 = 1, \ h = 2 \).

**Remark** (1) Existence results of special sections, with no assumptions on \( d \) and \( g \), are given (via completely different techniques) by [Fuentes-Garcia, Pedreira, 2005-2006].

On the other hand, differently from Segre’s approach, no information about its uniqueness and its speciality.

(2) Conditions in (ii) follow from Brill-Noether theory and smoothness. Namely

\[ \rho(g, h, m) := g - (h + 1)h^1 \geq 0. \]
Corollary 1 Suppose $S$ determined by a pair $(F, C)$. Let
\begin{align*}
(*) 
0 & \to N \to F \to L \to 0
\end{align*}
where $L$ corresponds to the special section $\Gamma$.

Then $F$ is unstable. If, moreover, $d \geq 6g - 5$ then $F = L \oplus N$.

Proof $\Gamma^2 = 2m - d < 0 \implies \deg(N) = d - m > \mu(F) = \frac{d}{2} \implies F$ is unstable.

From $(*)$, $[F] \in \text{Ext}^1(L, N) \cong H^1(C, N \otimes L^\vee)$.

$L$ special and $d \geq 6g - 5 \implies \deg(N \otimes L^\vee) = d - 2m \geq 2g - 1 \implies N \otimes L^\vee$ is non-special $\implies (\ast)$ splits.

Lemma 1 Assume $\text{Aut}(C) = \{Id\}$ (in particular, when $C$ has general moduli).

If $G_S \subset \text{PGL}(r + 1, \mathbb{C})$ sub-group of projectivities of $\mathbb{P}^r$ fixing $S$, then $G_S \cong \text{Aut}(S)$ and
\begin{align*}
\dim(G_S) = \left\{ \begin{array}{ll}
h^0(N \otimes L^\vee) & \text{if } F \text{ indecomposable} \\
h^0(N \otimes L^\vee) + 1 & \text{if } F \text{ decomposable}
\end{array} \right.
\end{align*}

Proof We want to show the obvious inclusion $G_S \hookrightarrow \text{Aut}(S)$ is an isomorphism.

Let $\sigma \in \text{Aut}(S)$. By Theorem 3, $\sigma(\Gamma) = \Gamma$ and since $\text{Aut}(C) = \{Id\}$, $\sigma$ fixes $\Gamma$ pointwise.

Now $H \sim \Gamma + \rho^*(N) \implies \sigma^*(H) = \sigma^*(\Gamma) + \sigma^*(\rho^*(N)) = \Gamma + \rho^*(N) \sim H \implies \sigma$ is induced by a projective trasformation.

The rest of the claim directly follows from [Maruyama, 1970].

Remark Different behaviour from $[S] \in \mathcal{H}_{d,g}$. Indeed, from Theorem 2 we know that for $S$ general non-special, $F$ is stable and $G_S = \{Id\}$. 

Components with general moduli

Let

\[ \text{Hilb}(d, g, h^1) \]

the open subset of the Hilbert scheme parametrizing smooth scrolls \( S \subset \mathbb{P}^r \) of genus \( g \), degree \( d \) and speciality \( h^1 \).

**Theorem 3 [C, C, –, M]** Let \( g \geq 3 \) and \( d \geq 4g - 2 \). Let \( m \) be any integer such that either

- \( m = 4 \), if \( g = 3 \), \( h^1 = 1 \), or
- \( g + 3 - h^1 \leq m \leq \overline{m} := \left\lfloor \frac{g}{h^1} - 1 \right\rfloor + g - h^1 \), otherwise.

(i) If \( h^1 = 1 \), \( \text{Hilb}(d, g, 1) \) consists of a unique component \( \mathcal{H}_{d,g,1}^{2g-2} \) whose general point parametrizes a smooth, linearly normal scroll \( S \subset \mathbb{P}^r \) with a canonical curve as the unique special section.

Furthermore,

1. \( \dim(\mathcal{H}_{d,g,1}^{2g-2}) = 7(g - 1) + r(r + 1) \),
2. \( \mathcal{H}_{d,g,1}^{2g-2} \) is generically smooth and dominates \( \mathcal{M}_g \).

Moreover, scrolls with \( h^1 = 1 \), containing a special section of degree \( m < 2g - 2 \) fill up an irreducible subscheme of \( \mathcal{H}_{d,g,1}^{2g-2} \) which also dominates \( \mathcal{M}_g \) and whose codimension is \( 2g - 2 - m \).

(ii) If \( h^1 \geq 2 \) then, for any \( g \geq 4h^1 \), \( d \geq 4g - 2 \) and for any \( m \) as above, \( \text{Hilb}(d, g, h^1) \) contains a unique component \( \mathcal{H}_{d,g,h^1}^{m} \) whose general point parametrizes a smooth, linearly normal scroll \( S \subset \mathbb{P}^r \), having general moduli whose special section \( \Gamma \) has degree \( m \) and speciality \( h^1 \).

Furthermore,

1. \( \dim(\mathcal{H}_{d,g,h^1}^{m}) = 7(g - 1) + (r + 1)(r + 1 - h^1) + (d - m - g + 1)h^1 - (d - 2m + g - 1) \),
2. \( \mathcal{H}_{d,g,h^1}^{m} \) is generically smooth.
Remarks (1) Bounds on $m$: Since $d \geq 4g - 2$, from Proposition 3, $S$ contains a unique, special section $\Gamma$ of speciality $h^1$.

$\Gamma$ is the image of $C$ via $|L| = g^h_m$.

In order to have $C$ with general moduli, $\rho(g, h, m) = g - (h + 1)h^1 \geq 0$

\[ m \leq \overline{m} = \lfloor \frac{g}{h^1} - 1 \rfloor + g - h^1. \]

On the other hand, $S$ smooth $\Rightarrow h = m - g + h^1 \geq 2$.

If $h^1 \geq 2$ and the scroll has general moduli, then $h \geq 3$.

If $h^1 = 1$ and $h = 2$, then $m = 4$ and $g = 3$.

(2) Reducibility: $\text{Hilb}(d, g, h^1)$ is reducible as soon as $h^1 \geq 2$.

If moreover $h^1 \geq 3$, it is also not equidimensional.

The component of maximal dimension is

$\mathcal{H}^{g+3-h^1}_{d,g,h^1}$

whereas the component of minimal dimension is

$\mathcal{H}^{\overline{m}}_{d,g,h^1}$

with $\overline{m}$ as above.

(3) By Corollary 1, all smooth scrolls in $\mathcal{H}^{m}_{d,g,h^1}$, for any $h^1 \geq 1$, correspond to unstable bundles.
To prove Theorem 3, no degeneration techniques.

Steps of the Proof:

Step 1: For any $m$ and $h$, we construct a morphism

$$
\psi_{h,m} : \mathcal{U}_{h,m} \rightarrow \text{Hilb}(d, g, h^1)
$$

where

$$
(C, L, N, \xi, \sigma) \rightarrow (C, L, N, \xi, \sigma) \rightarrow \sigma(S)
$$

where

$$
[C] \in \mathcal{M}_g, \quad L \in W^h_m(C), \quad N \in \text{Pic}^{d-m}(C), \quad \sigma \in \text{PGL}(r + 1),
$$

whereas

$$
\xi = \begin{cases} 
0 & \text{if } \text{Ext}^1(L, N) = 0 \\
\mathbb{P}(\text{Ext}^1(L, N)) & \text{if } \text{Ext}^1(L, N) \neq 0
\end{cases}
$$

and

$$
\mathcal{F} = \begin{cases} 
L \oplus N & \text{if } \xi = 0 \\
\mathcal{F}_{\xi} & \text{corresponding to } \xi
\end{cases}
$$

and

$$
S = \Phi(F).
$$

$\mathcal{H}^m_{d,g,h^1}$ is defined as the closure of the image of $\psi_{h,m}$.

Step 2: Given $[S] \in \mathcal{H}^m_{d,g,h^1}$ general, $C$, $L$ and $N$ are uniquely determined.

From Step 2, $\dim \psi_{h,m}^{-1}([S]) = \dim(G_S)$.

Step 3: From Lemma 1, we know $\dim(G_S) \Rightarrow$ we know $\dim(\mathcal{H}^m_{d,g,h^1})$. 
Step 4: Compute cohomology of $N_{S/P^r}$.

$C$ is with general moduli and Castelnuovo’s Lemma for surjectivity of multiplication maps of sections of suitable $H^0$’s

\[
\begin{align*}
(i) \quad h^0(S, N_{S/P^r}) &= 7(g - 1) + (r + 1)(r + 1 - h^1) + (d - m - g + 1)h^1 - \left( d - 2m + g - 1 \right); \\
(ii) \quad h^1(S, N_{S/P^r}) &= h^1(d - m - g + 1) - (d - 2m + g - 1); \\
(iii) \quad h^2(S, N_{S/P^r}) &= 0.
\end{align*}
\]

Step 5: By comparing $\dim(\mathcal{H}^m_{d,g,h^1})$ in Step 3 and Step 4 (i)

\[
\dim[\mathcal{S}](\mathcal{H}^m_{d,g,h^1}) = h^0(S, N_{S/P^r})
\]

$\mathcal{H}^m_{d,g,h^1}$ generically smooth.

Step 6: $h^1 = 1$: suppose $L \neq \omega_C$. In particular, $g \geq 4$ and $m < 2g - 2$.

$|\omega_C \otimes L^\vee| = g^0_{2g-2-m}$, i.e.

\[
\omega_C \otimes L^\vee \cong \mathcal{O}_C(p_1 + \cdots + p_{2g-2-m}),
\]

where $p_j$ general points on $C$, $1 \leq j \leq 2g - 2 - m$, hence

\[
L \cong \omega_C(-p_1 - \cdots - p_{2g-2-m}).
\]

**Fact** Any bundle $\mathcal{G}$ on $C$ such that

\[
0 \to N \to \mathcal{G} \to \omega_C(-p_1 - \cdots - p_{2g-2-m}) \to 0
\]

is a degeneration of a vector bundle $\mathcal{F}$ fitting in

\[
0 \to N(p_1 + \cdots + p_{2g-2-m}) \to \mathcal{F} \to \omega_C \to 0.
\]

$\mathcal{H}^m_{d,g,1}$, with $m < 2g - 2$, sits in the closure of $\mathcal{H}^{2g-2}_{d,g,h^1}$. 
Step 7: $h^1 \geq 2$. From the dimension count in Step 3 and Step 4 (i)

\[
\downarrow
\]

$\mathcal{H}^m_{d,g,h^1}$ generically smooth component of $\text{Hilb}(d, g, h^1)$. ♣

Remarks (1) Different construction of $\mathcal{H}_{d,g;1}^{2g-2}$ is given by [Fuentes-Garcia, Pedreira, 2006]: they use internal projections from decomposable scrolls.

(2) With similar approach **Theorem 3** holds also for

\[
\frac{7}{2}g - h^1 + 1 \leq d \leq 4g - 3 \quad \text{and} \quad h^1 \geq 2.
\]

- Existence of a special section $\Gamma$: use [Fuentes-Garcia, Pedreira, 2005-06] instead of [C. Segre];

- $\Gamma$ is the unique special section, $\Gamma^2 < 0$ and it is the curve (different from a line) of minimal degree: use assumptions on $d$ and some projective geometry arguments;

- computation of $h^i(S, \mathcal{N}_{S/P^r})$: same approach as in **Theorem 3**; use surjectivity results of suitable multiplication maps of $H^0$'s of [Green] and [Butler], instead of Castelnuovo's lemma.

- the related vector bundles are still **unstable**, as in the Segre's case $d \geq 4g - 2$. 
Other components of the Hilbert scheme

Other components with **general moduli**.

Let \( s = d - 2g + 1 + k \), with \( 0 \leq k < h^1 \).

Consider the family \( Y_{k, h^1}^m \) whose general element is a general projection to \( \mathbb{P}^s \) of the general scroll in \( \mathcal{H}_{d, g, h^1}^m \).

**Questions** With assumptions as above:

- is \( Y_{k, h^1}^m \) contained in \( \mathcal{H}_{d, g, k}^n \) for some \( n \), if \( k > 0 \)?

- is \( Y_{k, h^1}^m \) contained in \( \mathcal{H}_{d, g} \), if \( k = 0 \)?

**Proposition 4** [CCFM] *In the above setting:*

(i) if \( k > 0 \), \( Y_{k, h^1}^m \) sits in an irreducible component of the Hilbert scheme different from \( \mathcal{H}_{d, g, k}^n \), for any \( n \);

(ii) if \( h^1 > 1 \), \( Y_{0, h^1}^m \) sits in an irreducible component of the Hilbert scheme different from \( \mathcal{H}_{d, g} \), for any \( m \);

(iii) \( Y_{0, 1}^{2g-2} \) is a divisor inside \( \mathcal{H}_{d, g} \), whose general point is a smooth point for \( \mathcal{H}_{d, g} \). ♣

**Remark** In cases (i) and (ii), by construction, we find other components of \( \text{Hilb}(d, g, s) \) which always dominates \( \mathcal{M}_g \).
Components with **special moduli**

\[[C] \in \mathcal{M}_g \Rightarrow \text{gonality of } C\]

\[\gamma := \begin{cases} \frac{g+2}{2} & \text{if } g \text{ even,} \\ \frac{g+3}{2} & \text{se } g \text{ odd,} \end{cases}\]

For any \(2 \leq k \leq \gamma\), stratification of \(\mathcal{M}_g\)

\[\mathcal{M}^1_{g,2} \subset \mathcal{M}^1_{g,3} \subset \ldots \subset \mathcal{M}^1_{g,k} \subset \ldots \subset \mathcal{M}_g,\]

where

\[\mathcal{M}^1_{g,k} := \{[C] \in \mathcal{M}_g | C \text{ has a } g^1_k\}\]

is the \(k\)-gonal locus.

[Arbarello-Cornalba, 1981] \(\mathcal{M}^1_{g,k}\) irreducible of dimension \(2g + 2k - 5\), if \(k < \gamma\). Moreover, \([C] \in \mathcal{M}^1_{g,k}\) general has a unique \(g^1_k\).

**Fact** \([C] \in \mathcal{M}^1_{g,k}\) general is such that \(\text{Aut}(C) = \{Id\}\).

**Example** Let \(g \geq 3\), \(d \geq 6g - 5\) and \([C] \in \mathcal{M}^1_{g,k}\) general, with \(3 \leq k < \gamma\).

\(|A| = g^1_k\) on \(C\) and \(L := \omega_C \otimes A^\vee \Rightarrow m := \deg(L) = 2g - 2 - k\) and \(h^1(L) = 2\).

[Kim,Kim, 2004] if \(3k(k-1) \leq 2g - 1\), then \(L\) very ample

\(N \in \text{Pic}^{d-m}(C) \Rightarrow \deg(N) = d + k - 2g + 2 \geq 4g - 4 + k \Rightarrow N\) very-ample and non-special.

\(\mathcal{F} := N \oplus L;\) not restrictive, since \(d \geq 6g - 5\) (**Corollary 1**)

\(\downarrow\)

\(\mathcal{F}\) very-ample, unstable with \(h^1(\mathcal{F}) = 2 \Rightarrow S = \Phi(F) \subset \mathbb{P}^{d-2g+3}\).

\(\text{Aut}(C) = \{Id\} \Rightarrow \text{Lemma 1 still holds } \Rightarrow \dim(G_S) = \dim(\text{Aut}(S))\) known.

\(\downarrow\)

We get components \(\mathcal{H}_k\) of \(\text{Hilb}(d,g,2)\) such that

- \(\dim(\mathcal{H}_k) > \dim(\mathcal{H}_{d,g})\), so different from \(\mathcal{H}_{d,g}\),
- \(\mathcal{H}_k\) with special moduli. ♣

**Remark** Other examples with higher speciality: take \(|A^\otimes r|\) instead of \(|A|\). Thus, \(L_r := \omega_C \otimes (A^\otimes r)^\vee\) is of speciality \(r + 1\).
Open questions

(1) Classify all the components of the Hilbert scheme of non-special and special scrolls.

(2) What are their images in $M_g$?

(3) Singular loci? (we have yet descriptions of some singular points of both $H_{d,g}$ and $H_{d,g,h}^n$)

(4) Non-special scrolls

(i) Brill-Noether loci $W_n^p(F) \subset \text{Pic}^n(C)$, for $p \geq 1$:
   • existence?
   • dimension?
   • structure?
   • Petri’s type conjectures?

(ii) Torelli’s type results: when $\dim(M_n(F)) = 0$ we have $2^g$ sub-line bundles of maximal degree. Therefore, we have
   $$U_C(d) \rightarrow \text{Sym}^{2g}(\text{Pic}^n(C)).$$
   Can we reconstruct $F$?

(5) Special scrolls

(i) Degenerations?

(ii) Families of unisecants?

(iii) Brill-Noether theory?

(iv) If $d < \frac{7}{2}g - h^1 + 1$, $F$ can be stable even if special?

**Remark** (iv) would give existence results of Brill-Noether loci in $U_C(d)$, for $C$ with general moduli, not covered by the results of [Teixidor I Bigas, 2005].