Compactified Jacobians of curves with spine decompositions

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- $P = \coprod_\chi P^\chi$, where $P^\chi := \{[L] \in P \mid \chi(L) = \chi\}$.
- Riemann–Roch: $\chi(L) = \deg(L) + 1 - g$.

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- Furthermore, there is a well-defined Abel map,

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sending N to $\mathcal{O}_C(N)$.
- The Abel map is an embedding if $C \not\cong \mathbb{P}^1_k$.

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- For instance, if C is an irreducible nodal cubic, then P^χ is isomorphic to $\mathbb{G}_m$, the multiplicative group of k.
- If C is reducible, then P^χ is not even of finite type over k.

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- $P^\chi \subseteq J^\chi$ open.

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- A *polarization* is a n-tuple $\mathfrak{a} = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{Q}_+$ such that $\sum a_i = 1$.
- For instance, if C is Gorenstein, ω_C is ample and $g > 1$, we have the canonical polarization $\mathfrak{w} := (w_1, \ldots, w_n)$, where

$$w_i := \frac{\deg(\omega_C|_{C_i})}{2g-2}.$$

Semistability 8

- A torsion-free, rank-1 sheaf I on C is $\mathfrak{a}$-*semistable* ($\mathfrak{a}$-*stable*) if

$$\chi(I_Y) \geq (>)\, a_Y \chi(I)$$

for every proper subcurve $Y \subset C$, where $I_Y := I|_Y/(\text{torsion})$ and $a_Y = \sum_{C_i \subseteq Y} a_i$.

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- Seshadri (1982): There is a projective scheme $U(\mathfrak{a}, \chi)$ corepresenting $J^{ss}(\mathfrak{a}, \chi)$.
- $U(\mathfrak{a}, \chi) = \{\text{S-equivalence classes of } \mathfrak{a}\text{-semistable sheaves}\}$, a coarse moduli space.

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- E. (2001): $\widetilde{J}$ is a scheme, universally closed over k.

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- Let $Q \in X$ in the nonsingular locus. We say that a torsion-free, rank-1 sheaf I on C is $\mathfrak{a}$-quasistable w.r.t. Q if

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- $J^{Q}(\mathfrak{a},\chi) := \{[I] \in J^{ss}(\mathfrak{a},\chi) \,|\, I \text{ is } \mathfrak{a}\text{-quasistable w.r.t. } Q\}$ is open in $\widetilde{J}^{ss}(\mathfrak{a},\chi)$ and proper over k.

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- For ϵ small, $J^Q(\mathfrak{a},\chi) = J^s(\mathfrak{b},\chi) = \widetilde{J}^{ss}(\mathfrak{b},\chi)$.

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- Φ is surjective.
- Denoting by $U^s(\mathfrak{a}, \chi) \subseteq U(\mathfrak{a}, \chi)$ the open subscheme parametrizing $\mathfrak{a}$-stable sheaves, we have

$$\Phi^{-1}(U^s(\mathfrak{a}, \chi)) = J^s(\mathfrak{a}, \chi)$$

and $\Phi^s \colon J^s(\mathfrak{a}, \chi) \to U^s(\mathfrak{a}, \chi)$ is an isomorphism.

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- **Theorem.** (Caporaso, Coelho, –) Assume C is Gorenstein, ω_C is ample, and $C \not\cong \mathbb{P}^1_k$. If C has no separating nodes then

$$A\colon C \to J^Q(\mathfrak{w}, 2-g),$$

taking N to $\mathcal{I}^*_{N/C}$, is well-defined and an embedding.

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- When do "universal" sheaves over $U(\mathfrak{a}, \chi)$ exist?

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- Álvaréz–King (2007): These "divisors" are enough to understand $U(\mathfrak{a}, \chi)$, at least in characteristic zero.

Main result 16

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- **Theorem.** Assume that every subcurve $Y \subseteq C$ for which $a_Y\chi \in \mathbb{Z}$ is a spine or contains Q. Then the S-map $\Phi\colon J^Q(\mathfrak{a},\chi) \to U(\mathfrak{a},\chi)$ is a bijective closed embedding. Moreover, if C is locally planar, then Φ is an isomorphism.

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- It induces a decomposition $X = Z_1 \cup \cdots \cup Z_q$ in subcurves with finite pairwise intersection and a filtration

$$0 = I_0 \subset I_1 \subset \cdots \subset I_q = I$$

such that I_j/I_{j-1} is a torsion-free, rank-1 sheaf on Z_j with $\chi(I_j/I_{j-1}) = a_{Z_j}\chi$ for each $j = 1, \ldots, q$.

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- $\mathfrak{a}|_{Z_j}$ is the collection of a_i/a_{Z_j} for all $C_i \subseteq Z_j$.
- (Two sheaves are S-equivalent if their associated graded sheaves are isomorphic.)

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- Since all the Z_j are spines, there is an isomorphism

$$\begin{aligned}\Lambda\colon \widetilde{J}^{\chi}_C \longrightarrow & \coprod_{m_1+\cdots+m_q=\chi} \widetilde{J}^{m_1}_{Z_1}\times\cdots\times\widetilde{J}^{m_q}_{Z_q}\\ [K]\mapsto & (\dots,[K|_{Z_j}\otimes\mathcal{O}_{Z_j}\Big(-\sum_{\ell>j}Z_\ell\cap Z_j\Big)],\dots).\end{aligned}$$

Proof, part II

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$$[K] \mapsto (\dots,[K|_{Z_j}\otimes\mathcal{O}_{Z_j}\Big(-\sum_{\ell>j} Z_\ell\cap Z_j\Big)],\dots).$$

- In our case $[I]$ is taken to $(\dots,[I_j/I_{j-1}],\dots)$, so inside

$$\prod_j J^s(\mathfrak{a}|_{Z_j},\chi_j).$$

Proof, part III

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Proof, part III

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- That "theta divisor" is associated to a locally free sheaf on Z_j. We lift it carefully to a locally free sheaf E on C in such a way that $\Theta_{E|_{Z_j}}$ is also defined for $j \neq i$ and $[I_j/I_{j-1}] \notin \Theta_{E_{Z_j}}$.

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- Then it is a matter of showing that

$$(\Lambda^{-1}_{\chi_1,\dots,\chi_q})^*\Theta_E = \sum_{i=1}^{q} J_{Z_1}^{\chi_1} \times \cdots \times \Theta_{E|_{Z_j}} \times \cdots \times J_{Z_q}^{\chi_q}.$$

Abel map, part II 20

- There is a hidden lemma in the proof above: that if an $\mathfrak{a}$-quasistable sheaf I induces a decomposition $X = Z_1 \cup \cdots \cup Z_q$ where the Z_j are spines, then $d\Phi_{[I]}$ is injective.

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- **Proof.** It is enough to show that all sheaves in the image of A induce a decomposition of C in spines.