# Binomial D-modules

Alicia Dickenstein

Departamento de Matemática - FCEyN

Universidad de Buenos Aires

Joint work with Ezra Miller (UMN) and Laura Matusevich (TAMU)

◇ Motivations and examples

◇ Definition of binomial D-modules

◇ More examples

◇ Questions

◇ Main tools

◇ (Flavour of the) Answers

## ◇ Hypergeometric functions in one variable

Euler (1748), Gauss (1812), Kummer (1836), Riemann (1857), . . . , . . .

Given $\alpha, \beta, \gamma \in \mathbb{C}$, $\gamma \notin \mathbb{Z}_{\leq 0}$ and $(\alpha)_n = \alpha \cdot (\alpha+1) \ldots (\alpha+n-1)$,

- Gauss hypergeometric function

$$F(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!}, \ |x| < 1.$$

For example,

$$F(\alpha, \beta, \beta.x) = (1-x)^{-\alpha} \qquad -xF(1,1,2;x) = \log(1-x).$$

...

$$F(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{x^n}{n!}, \; |x| < 1.$$

- Kummer and Riemann's point of view: $F(\alpha, \beta, \gamma; x)$ satisfies

$$x(1-x)y'' + (\gamma - (\alpha + \beta + 1)x)y' - \alpha\beta y = 0,$$

or, denoting $\Theta := x\frac{d}{dx}$,

$$[\Theta(\Theta + \gamma - 1) - x(\Theta + \alpha)(\Theta + \beta)](y) = 0.$$

Up to normalization, this is a general linear differential equation with 3 regular singular points.

$$A_n := \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}, \quad F(\alpha,\beta,\gamma;x) = \sum_{n\geq 0} A_n x^n.$$

- The coefficients $A_n$ satisfy the following linear recurrence

$$(\gamma+n)(1+n)A_{n+1} - (\alpha+n)(\beta+n)A_n = 0 \qquad (0.2)$$

$$\frac{A_{n+1}}{A_n} = R(n).$$

- (0.2) is equivalent to the fact that $F(\alpha,\beta,\gamma;x)$ satisfies the differential equation

$$[\Theta(\Theta+\gamma-1) - x(\Theta+\alpha)(\Theta+\beta)](y) = 0.$$

## ◇ Hypergeometric functions in several variables

System of hypergeometric PDE's for Horn's function $G_3$

$$\begin{aligned}(x(2\theta_x - \theta_y + a')(2\theta_x - \theta_y + a' + 1) - (-\theta_x + 2\theta_y + a)\theta_x)f = 0 \, , \\ (y(-\theta_x + 2\theta_y + a)(-\theta_x + 2\theta_y + a + 1) - (2\theta_x - \theta_y + a')\theta_y)f = 0 \, .\end{aligned}$$

Its holonomic rank is $4$ ($a, a'$ generic parameters)

- Erdélyi (Acta Mathematica, 1950) noted that, in a neighborhood of a given point, three linearly independent solutions of this system can be obtained through contour integral methods. He also finds a fourth linearly independent solution: the Puiseux monomial $x^{-(a+2a')/3}y^{-(2a+a')/3}$. He remarks that the existence of this elementary solution is puzzling, and offers no explanation for its occurrence.

## ◇ Hypergeometric functions in several variables (suite)

System of hypergeometric PDE's for Appell's function $F_1$

$$\big(x(\theta_x + \theta_y + a)(\theta_x + b) - \theta_x(\theta_x + \theta_y + c - 1)\big) f = 0 \ ,$$
$$\big(y(\theta_x + \theta_y + a)(\theta_y + b') - \theta_y(\theta_x + \theta_y + c - 1)\big) f = 0$$

- For generic values of the parameters $a$, $b$, $b'$ and $c$, the holonomic rank of this system, that is, the dimension of its space of local complex holomorphic solutions around a nonsingular point, is $3 < 2.2 = 4$.

## ◇ Hypergeometric functions in one variable, revisited GKZ style

GKZ = Gel'fand, Kapranov and Zelevinsky (89)

Consider the configuration in $\mathbb{R}^3$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

$$\ker_{\mathbb{Z}}(A) = \langle (1,1,-1,-1) \rangle \quad (1,1,-1,-1) = (1,1,0,0) - (0,0,1,1)$$

- The following system of equations in four variables $x_1, x_2, x_3, x_4$ is a nice encoding for Gauss equation in one variable:

$$\begin{cases} (\partial_1\partial_2 - \partial_3\partial_4)(\varphi) = 0 \\ (x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4)(\varphi) = \beta_1\varphi \\ (x_2\partial_2 + x_3\partial_3)(\varphi) = \beta_2\varphi \\ (x_2\partial_2 + x_4\partial_4)(\varphi) = \beta_3\varphi \end{cases}$$

$$\begin{cases} (\partial_1\partial_2 - \partial_3\partial_4)\,(\varphi) = 0 \\ (x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4)\,(\varphi) = \beta_1\varphi \\ (x_2\partial_2 + x_3\partial_3)\,(\varphi) = \beta_2\varphi \\ (x_2\partial_2 + x_4\partial_4)\,(\varphi) = \beta_3\varphi \end{cases} \tag{0.3}$$

- Given any $(\beta_1, \beta_2, \beta_3)$ and $\mathbf{v} \in \mathbb{C}^n$ such that $A \cdot \mathbf{v} = (\beta_1, \beta_2, \beta_3)$ and $v_1 = 0$, any solution $\varphi$ of (0.4) can be written as

$$\varphi(x) = x^v\, f\left(\frac{x_1x_2}{x_3x_4}\right),$$

where $f(z)$ satisfies Gauss equation with

$$\alpha = v_2\,,\ \beta = v_3\,,\ \gamma = v_4 + 1.$$

$$\begin{cases} (\partial_1\partial_2 - \partial_3\partial_4)(\varphi) &= 0 \\ (x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4)(\varphi) &= \beta_1\varphi \\ (x_2\partial_2 + x_3\partial_3)(\varphi) &= \beta_2\varphi \\ (x_2\partial_2 + x_4\partial_4)(\varphi) &= \beta_3\varphi \end{cases} \tag{0.4}$$

- Given any $(\beta_1, \beta_2, \beta_3)$ and $\mathbf{v} \in \mathbb{C}^n$ such that $A \cdot \mathbf{v} = (\beta_1, \beta_2, \beta_3)$ and $v_1 = 0$, any solution $\varphi$ of (0.4) can be written as

$$\varphi(x) = x^v\, f\left(\frac{x_1 x_2}{x_3 x_4}\right),$$

where $f(z)$ satisfies Gauss equation with

$$\alpha = v_2\,,\ \beta = v_3\,,\ \gamma = v_4 + 1.$$

The binomial operator $(\partial_1\partial_2 - \partial_3\partial_4)$ "represents" the hypergeometric recursion on the coefficients of the series.

◇ Hypergeometric functions in several variables, roots of generic univariate polynomials

Birkeland, Mayr, Mellin, Sturmfels, . . . ,
Cattani – D'Andrea – D.('99), Passare – Tsikh ('04), D. – Sadykov ('07)

Given coprime integers $0 < k_1 < \ldots < k_m < n$, set

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & k_1 & \ldots & k_m & n \end{pmatrix},$$

and $\beta = (0, -1)$.

- The local roots $\rho(x)$ of the generic sparse polynomial $(f(x, \rho(x)) = 0)$

$$f(x; t) := x_0 + x_{k_1} t^{k_1} + \cdots + x_{k_m} t^{k_m} + x_n t^n ,$$

viewed as functions of the coefficients, are algebraic solutions to the associated $A$-hypergeometric system.

For example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix},$$

$$\beta = (0, -1), \quad f(x;t) = x_0 + x_1 t + x_3 t^3 + x_4 t^4, \quad \theta_i := x_i \partial_i,$$

- The corresponding $A$-hypergeometric system is given by:

$$\begin{aligned} \left(\partial_0^2 \partial_3 - \partial_1^3\right)(\varphi) &= 0 \\ \left(\partial_0^3 \partial_4 - \partial_1^4\right)(\varphi) &= 0 \\ \left(\partial_0 \partial_4 - \partial_1 \partial_3\right)(\varphi) &= 0 \\ \left(\partial_1 \partial_4^2 - \partial_3^3\right)(\varphi) &= 0 \\ (\theta_0 + \theta_1 + \theta_3 + \theta_4)(\varphi) &= 0 \\ (\theta_1 + 3\theta_3 + 4\theta_4 + 1)(\varphi) &= 0. \end{aligned}$$

Also residues, periods, generating functions of intersection numbers in moduli spaces are **hypergeometric**.

◇ Hypergeometric functions in several variables, revisited GKZ style

System of hypergeometric PDE's for Horn's function $G_3$

$$\Big(x(2\theta_x - \theta_y + a')(2\theta_x - \theta_y + a' + 1) - (-\theta_x + 2\theta_y + a)\big(1\theta_x + 0\theta_y\big)\Big) f = 0 \,,$$
$$\Big(y(-\theta_x + 2\theta_y + a)(-\theta_x + 2\theta_y + a + 1) - (2\theta_x - \theta_y + a')\theta_y\Big) f = 0 \,.$$

Explanation (D.– Matusevich – Sadykov ('05):

Look at the binomials

$$q_1 = \partial_1^2 \partial_4^0 - \partial_2^1 \partial_3^1, \; q_2 = \partial_1 \partial_4 - \partial_2^2$$

in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$.

$$\left(x(2\theta_x - \theta_y + a')(2\theta_x - \theta_y + a' + 1) - (-\theta_x + 2\theta_y + a)\big(1\theta_x + 0\theta_y\big)\right) f = 0 \,,$$
$$\big(y(-\theta_x + 2\theta_y + a)(-\theta_x + 2\theta_y + a + 1) - (2\theta_x - \theta_y + a')\theta_y\big) f = 0 \,.$$

System of hypergeometric PDE's for Horn's function $G_3$

Explanation (D.– Matusevich – Sadykov ('05):

Look at the binomials

$$q_1 = \partial_1^2 \partial_4^0 - \partial_2^1 \partial_3^1, \; q_2 = \partial_1^1 \partial_4^0 - \partial_2^2$$

in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$.
Its zero set has the component "at infinity" $\{\partial_1 = \partial_2 = 0\}$ , with multiplicity equal to the intersection multiplicity $\mu_0$ at the origin of the system of $2$ binomials in $2$ variables

$$p_1 = \partial_1^a - \partial_2^b, \; p_2 = \partial_1^c - \partial_2^d, \quad a = 2, b = 1, c = 1, d = 2$$

$$\mu_0 = \min\{|a \cdot d|, |b \cdot d|\} = 1,$$

which equals the dimension of the space of solutions to the Horn system which have finite support.

◇ Hypergeometric functions in several variables, revisited GKZ style

System of hypergeometric PDE's for Appell's function $F_1$

$$\Big(x(1\theta_x + 1\theta_y + a)(1\theta_x + 0\theta_y + b) - (1\theta_x + 0\theta_y)(1\theta_x + 0\theta_y + c - 1)\Big) f = 0\,,$$
$$\Big(y(\theta_x + \theta_y + a)(\theta_y + b') - \theta_y(\theta_x + \theta_y + c - 1)\Big) f = 0$$

Explanation (D.– Matusevich – Sadykov ('05):

Look at the binomials

$$q_1 = \partial_1^1 \partial_3^1 - \partial_2^1 \partial_4^1,\ q_2 = \partial_1\partial_5 - \partial_2\partial_6$$

in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$.

System of hypergeometric PDE's for Appell's function $F_1$

$$\big(x(1\theta_x + 1\theta_y + a)(1\theta_x + 0\theta_y + b) - (1\theta_x + 0\theta_y)(1\theta_x + 0\theta_y + c - 1)\big)f = 0 \,,$$
$$\big(y(\theta_x + \theta_y + a)(\theta_y + b') - \theta_y(\theta_x + \theta_y + c - 1)\big)f = 0$$

Explanation (D.– Matusevich – Sadykov ('05):

Look at the binomials

$$q_1 = \partial_1^1 \partial_3^1 - \partial_2^1 \partial_4^1, \; q_2 = \partial_1 \partial_5 - \partial_2 \partial_6$$

in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$.
Its zero set has the component "at infinity" $\{\partial_1 = \partial_2 = 0\}$ , with multiplicity equal to the intersection multiplicity $\mu_0$ at the origin of the system of $2$ binomials in $2$ variables

$$p_1 = \partial_1^a - c_1 \partial_2^b, \; p_2 = \partial_1^c - c_2 \partial_2^d, \quad a = 1, b = 1, c = 1, d = 1,$$
$$\mu_0 = \min\{|a \cdot c|, |b \cdot d|\} = 1,$$

but since $(1, -1), (1, -1)$ are linearly dependent, it does NOT give any solution to the Horn system for generic values of the parameters. Thus, there are only $3 = 4 - 1$ linearly independent local solutions.

## ◇ What is a binomial D-module?

Data:

- An integer matrix $\boldsymbol{A} \in \mathbb{Z}^{d\times n}$ such that the cone generated by the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ of $\boldsymbol{A}$ contains no lines, all of the $\boldsymbol{a}_i$ are nonzero, and $\mathbb{Z}\boldsymbol{A} = \mathbb{Z}^d$
- A *binomial ideal* is an ideal generated by *binomials* $\boldsymbol{\partial}^u - \boldsymbol{\lambda}\boldsymbol{\partial}^v$, where $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^n$ are column vectors and $\boldsymbol{\lambda} \in \mathbb{C}$.

## ◇ What is a binomial D-module?

Data:

- An integer matrix $\boldsymbol{A} \in \mathbb{Z}^{d \times n}$ such that the cone generated by the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ of $\boldsymbol{A}$ contains no lines , all of the $\boldsymbol{a}_i$ are nonzero, and $\mathbb{Z}\boldsymbol{A} = \mathbb{Z}^d$.
- A *binomial ideal* is an ideal generated by *binomials* $\boldsymbol{\partial}^{\boldsymbol{u}} - \boldsymbol{\lambda}\boldsymbol{\partial}^{\boldsymbol{v}}$, where $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^n$ are column vectors and $\boldsymbol{\lambda} \in \mathbb{C}$.

- $\boldsymbol{A}$ induces a $\mathbb{Z}^d$-grading of the polynomial ring $\mathbb{C}[\boldsymbol{\partial}_1, \ldots, \boldsymbol{\partial}_n] = \mathbb{C}[\boldsymbol{\partial}]$, $\mathbf{deg}(\boldsymbol{\partial}_i) = -\boldsymbol{a}_i$.
- An ideal of $\mathbb{C}[\boldsymbol{\partial}]$ is *$\boldsymbol{A}$-graded* if it is generated by elements that are homogeneous for the $\boldsymbol{A}$-grading.

## ◇ What is a binomial D-module?

Data:

- An integer matrix $A \in \mathbb{Z}^{d\times n}$ such that the cone generated by the columns $a_1, \dots, a_n$ of $A$ contains no lines, all of the $a_i$ are nonzero, and $\mathbb{Z}A = \mathbb{Z}^d$
- A *binomial ideal* is an ideal generated by *binomials* $\partial^u - \lambda\partial^v$, where $u, v \in \mathbb{Z}^n$ are column vectors and $\lambda \in \mathbb{C}$.

A binomial ideal is $A$-graded precisely when it is generated by binomials $\partial^u - \lambda\partial^v$ each of which satisfies either $Au = Av$ or $\lambda = 0$

## ◇ What is a binomial D-module? (suite)

- The Weyl algebra $D = D_n$ of linear partial differential operators (in $n$ variables) written with the variables $x$ and $\partial$, is also $A$-graded by additionally setting $\mathbf{deg}(x_i) = a_i$.

## ◇ What is a binomial D-module? (suite)

- For each $i \in \{1, \ldots, d\}$, the $i$-th *Euler operator* is

$$E_i = a_{i1}\theta_1 + \cdots + a_{in}\theta_n.$$

- Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$.
- These operators are $A$-homogeneous of degree $0$.

## ◇ What is a binomial D-module? (suite)

- For an $A$-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_A(I, \beta)$ the left ideal $I + \langle E - \beta \rangle$ in the Weyl algebra $D$.
- Finally, the *binomial D-module* associated to $I$ is $D/H_A(I, \beta)$.

## ◇ What is a binomial D-module? (suite)

- For an $A$-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_A(I,\beta)$ the left ideal $I + \langle E - \beta \rangle$ in the Weyl algebra $D$.
- Finally, the *binomial D-module* associated to $I$ is $D/H_A(I,\beta)$.

- When $I$ equals the toric ideal $I_A$ we have an $A$-hypergeometric system.
- When $I$ is a lattice basis ideal, we have a Horn system (in binomial version).

## ◇ What is a binomial D-module? (suite)

- A *binomial D-module* is a quotient by a left $D$-ideal generated by an $A$-homogeneous binomial ideal $I$ with constant coefficients plus Euler operators associated to the row span of $A$.

- Binomial differential operators annihilating a (multivariate Puiseux) series are equivalent to (special) linear recurrences satisfied by its coefficients.
- Euler operators prescribe $A$-homogeneity (infinitesimally).

## ◇ A (non holonomic) example

Consider

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that

$$H(B, \beta) = \langle \partial_1\partial_3 - \partial_2, \partial_1\partial_4 - \partial_2 \rangle + \langle x_1\partial_1 - x_2\partial_2 - \beta_1, x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - \beta_2 \rangle.$$

If $\beta_1 = 0$, then any (local holomorphic) bivariate function $f(x_3, x_4)$ annihilated by the operator $x_3\partial_3 + x_4\partial_4 - \beta_2$ is a solution of $H(B, \beta)$.

The space of such functions is infinite-dimensional; in fact, it has uncountable dimension, as it contains all monomials

$$x_3^{w_3} x_4^{w_4}$$

with $w_3, w_4 \in \mathbb{C}$ and $w_3 + w_4 = \beta_2$.

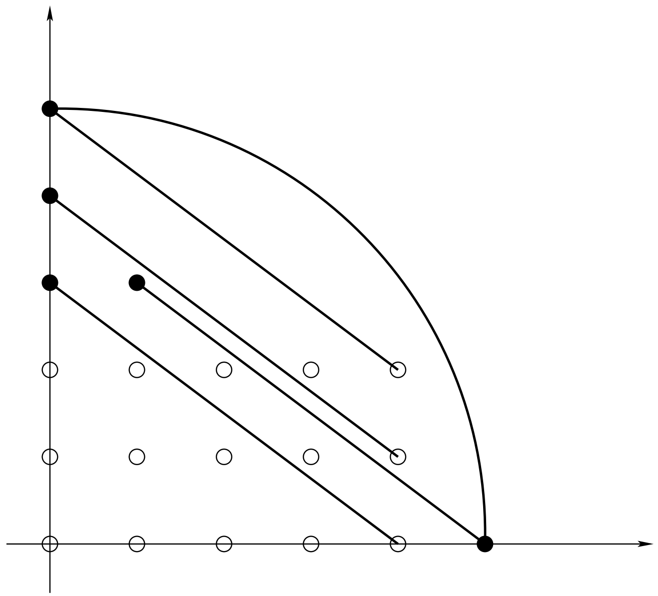
◇ Finding polynomial solutions of binomial ideals

$$M = \begin{pmatrix} 4 & 5 \\ -3 & -5 \end{pmatrix}$$

The system $H(I(M), 0)$ is defined by the operators

$$\frac{\partial^4}{\partial x_1^4} - \frac{\partial^3}{\partial x_2^3}, \qquad \frac{\partial^5}{\partial x_1^5} - \frac{\partial^5}{\partial x_2^5}.$$

It has $\mathbf{15}$ linearly independent polynomial solutions, with the following minimal supports:



## ◇ Description of solutions to CI Horn binomial D-modules

Consider the matrices:

$$B = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \\ 4 & 5 & 0 \\ -3 & -5 & 0 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 10 & 0 & 7 & 6 \end{bmatrix}$$

$$I = I(B) = x_2 x_5^3 - x_4^4, x_1 x_5^5 - x_3 x_4^5, x_1^2 - x_2 x_3.$$

We concentrate on the decomposition:

$$M = \begin{bmatrix} 4 & 5 \\ -3 & -5 \end{bmatrix}; \quad N = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \hat{B} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

Note that $\gcd\{2, -1, -1\} = 1$, so there is only one associated prime coming from this decomposition:

$$I_{\left[\begin{smallmatrix} 1 & 1 & 1 \\ 5 & 10 & 0 \end{smallmatrix}\right]} + \langle \partial_4, \partial_5 \rangle$$

## ◇ Description of solutions to CI Horn binomial D-modules (suite)

- The quatrinomial $\boldsymbol{p = 5x_4^4x_5^2 + 2x_4^5 + 2x_5^5 + 40x_4x_5^3}$ is a solution of the constant coefficient system $\boldsymbol{I(M)}$.
- Let $\boldsymbol{f}$ be a solution of the $\left[\begin{smallmatrix} 1 & 1 & 1 \\ 5 & 10 & 0 \end{smallmatrix}\right]$-hypergeometric system that is homogeneous of appropriate degree.
- Then the following function is a solution of $\boldsymbol{H(B,\beta)}$:

$$\boldsymbol{5x_4^4x_5^2\partial_2^2\partial_3 f + 2x_4^5\partial_1\partial_2 f + 2x_5^5\partial_2\partial_3 f + 40x_4x_5^3\partial_1 f}$$

## ◇ Questions

- For which parameters does the space of local holomorphic solutions around a nonsingular point of a binomial Horn system have finite dimension as a complex vector space?
- What is a combinatorial formula for the minimum such dimension, over all possible choices of parameters?
- Which parameters are generic, in the sense that the minimum dimension is attained?
- How do (the supports of) series solutions centered at the origin of a binomial Horn system look, combinatorially?
- When is $D/H_A(I, \beta)$ a holonomic $D$-module?
- When is $D/H_A(I, \beta)$ a *regular* holonomic $D$-module?

◇ Holonomic rank

$$P = \sum k_{\alpha,\beta} x^\alpha \partial^\beta, \quad P \neq 0$$

$$\sigma(P) = in_{(0,e)}(P) = \sum_{|\beta| = \mathbf{ord}(P)} k_{\alpha,\beta} x^\alpha \xi^\beta \in \mathbb{C}[x, \xi]$$

$J \subset D_n = D_n(\mathbb{C})$ left ideal, $\quad in_{(0,e)}(J) := \langle \sigma(P), P \in J, P \neq 0 \}$.

The holonomic rank rank($J$) of $J$ is the dimension over $\mathbb{C}$ of the space of its local holomorphic solutions around a (generic = non singular) point. It equals

$$\mathbf{rank}(J) = \dim_{\mathbb{C}(x)} \left( \mathbb{C}(x)[\xi] / \mathbb{C}(x)[\xi] . in_{(0,e)}(J) \right).$$

**Basic block:** $A$-hypergeometric binomial $D$-modules are always holonomic with non zero holonomic rank [GKZ'89], [Adolphson'94].

## ◇ Our main tools

- Precise description of combinatorial commutative algebra of binomial ideals in semigroup rings (based on: *Binomial ideals*, Eisenbud – Sturmfels ('94))

## ◇ Our main tools

- Precise description of combinatorial commutative algebra of binomial ideals in semigroup rings (based on: *Binomial ideals*, Eisenbud – Sturmfels ('94))

- Functorial translation of those results into $\boldsymbol{D}$-module theory by means of Cayley-Koszul complexes (based on: *Homological methods for hypergeometric families*, Matusevich – Miller – Walther ('05))

## ◇ Our main tools

- Precise description of combinatorial commutative algebra of binomial ideals in semigroup rings (based on: *Binomial ideals*, Eisenbud – Sturmfels ('94))

- Functorial translation of those results into $\boldsymbol{D}$-module theory by means of Cayley-Koszul complexes (based on: GKZ('89,'90), Adolphson('94), Matusevich – Miller – Walther ('05))

- And, of course, some direct generalizations of the beautiful theory of $\boldsymbol{A}$-hypergeometric systems developed by Gel'fand, Kapranov, and Zelevinsky!

## ◇ Primary components of binomial ideals

- Eisenbud-Sturmfels: An irredundant primary decomposition of an arbitrary binomial ideal $I \subseteq \mathbb{C}[\partial]$ is given by

$$I = \bigcap_{I_{\rho,J} \in \mathrm{Ass}(I)} \mathrm{Hull}(I + I_\rho + \langle \partial_j : j \notin J \rangle^e)$$

for any large integer $e$, where $\mathbf{Hull}$ means to discard the primary components for embedded (i.e., nonminimal associated) primes.

- We make this Hull operation explicit, by characterizing the monomials in each primary component of $I$.

Basic blocks:

- "Prime binomial ideals $=$ toric ideals "
- Zero dimensional ideals $I(M)$, where $M$ is a $m \times m$ mixed square matrix.

For example: $I = \langle x^2 - y, x^3 - y^2 \rangle, \quad 0 \in V(I)$

- Which are all the monomials that are present in $I_0$?
- Our answer: $I_0 = I + \langle x^3, xy, y^2 \rangle$, and these are exactly all the monomials in $I_0$.
- For instance, a SINGULAR computation gives a standard basis $\{y - x^2, x^3\}$ (for local lex order with $y > x$).

## ◇ Euler-Koszul homology

- Euler-Koszul homology allows us to functorially translate the commutative algebra of $A$-graded primary decomposition directly into the $D$-module setting and to pull apart the "contributions" of each of the primary components of binomial ideals in a binomial $D$-module.

## ◇ Euler-Koszul homology

- For $z$ $A$-homogeneous in an $A$-graded left $D$-module, define
$$(E_i - \beta_i) \circ z = (E_i - \beta_i - \deg_i(z))z,$$
- Fix $\beta \in \mathbb{C}^d$ and an $A$-graded ideal $I$. The *Euler-Koszul complex*
$$\mathcal{K}_.(E - \beta; \mathbb{C}[\partial]/I)$$
is the Koszul complex of left $D$-modules defined by the sequence $E - \beta$ of commuting endomorphisms on the left $D$-module $D \otimes_{\mathbb{C}[\partial]} V$, ($V = \mathbb{C}[\partial]/I$), concentrated in homological degrees $d$ to $0$.
- The $i$-th *Euler-Koszul homology* is $\mathcal{H}_i(E - \beta; V) = H_i(\mathcal{K}_.(E - \beta; V))$.
- The *binomial Horn $D$-module* with parameter $\beta$ is $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I)$.

---

- Each graded piece of the Euler-Koszul complex coincides with a commutative Koszul complex over the ring $\mathbb{C}[\theta_1, \ldots, \theta_n]$.

## ◇ Euler-Koszul homology

- The $i$-th Euler-Koszul homology of the quotient $V = \mathbb{C}[\partial]/I_\rho$ corresponding to a *toral* primary component $I_\rho$ of $I$ is nonzero for some $i \geq 1$ if and only if $-\beta$ lies in the Zariski closure of the degrees $\alpha \in \mathbb{Z}^d$ such that the $\alpha$-graded piece of the local cohomology module $H^i_{\mathfrak{m}}(V)_\alpha$ is non zero for some $i < d$.
- This Zariski closure is a finite union of linear subspaces.

## ◇ General answers

- We explicitly classify all primary components of $I$ (in particular, all monomials that are present), their multiplicities, their behaviour with respect to the grading *(toral and Andean components)*, and their holonomic rank.
- We explicitly define two finite subspace arrangements associated to the Andean components *(Andean arrangement)* and to the pairwise intersections of two components *(primary arrangement)* as the Zariski closure of parameters for which the corresponding piece in certain local cohomology modules is non zero, which account for non generic behaviour of the complex parameter $\beta$.
- The basic building blocks are associated $A$-hypergeometric systems (for several different $A$).

- The dimension is finite exactly for $-\beta$ not in the Andean arrangement.
- The generic (minimum) rank is $\sum \mu(L, J)\cdot \mathrm{vol}(A_J)$, the sum being over all toral associated sublattices with $\mathbb{C}A_J = \mathbb{C}^d$, where $\mathrm{vol}(A_J)$ is the volume of the convex hull of $A_J$ and the origin, normalized so a lattice simplex in $\mathbb{Z}A_J$ has volume $1$.
- The minimum rank is attained precisely when $-\beta$ lies outside of an explicit affine subspace arrangement determined by certain local cohomology modules, containing the Andean arrangement.
- When the Horn system is regular holonomic and $\beta$ is general, there are $\mu(L, J) \cdot \mathrm{vol}(A_J)$ linearly independent solutions supported on (translates of) the $L$-bounded classes, with hypergeometric recursions determining the coefficients.
- Only $g \cdot \mathrm{vol}(A)$ many Gamma series solutions have full support, where $g = |\ker(A)/\mathbb{Z}B|$ is the index of $\mathbb{Z}B$ in its saturation $\ker(A)$.
- Holonomicity is equivalent to finite dimension of the (local holomorphic) solutions spaces.
- Holonomicity is equivalent to regular holonomicity when $I$ is standard $\mathbb{Z}$-graded—i.e., the row-span of $A$ contains the vector $[1 \cdots 1]$. Conversely, if there exists a parameter $\beta$ for which $D/H_A(I, \beta)$ is regular holonomic, then $I$ is standard $\mathbb{Z}$-graded.

. . .

**Thanks for your attention!**