Binomial D-modules

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Motivations and examples

Definition of binomial D-modules

More examples

Questions

Main tools

(Flavour of the) Answers
Hypergeometric functions in one variable

Euler (1748), Gauss (1812), Kummer (1836), Riemann (1857), …, …

Given $\alpha, \beta, \gamma \in \mathbb{C}$, $\gamma \notin \mathbb{Z}_{\leq 0}$ and $(\alpha)_n = \alpha \cdot (\alpha + 1) \ldots (\alpha + n - 1)$,

• Gauss hypergeometric function

$$F(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{x^n}{n!}, \ |x| < 1.$$  

For example,

$$F(\alpha, \beta, \beta, x) = (1 - x)^{-\alpha} - xF(1, 1, 2; x) = \log(1 - x).$$
\[ F(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} \frac{(\alpha)_n(\beta)_n x^n}{(\gamma)_n n!}, \quad |x| < 1. \]

- Kummer and Riemann’s point of view: \( F(\alpha, \beta, \gamma; x) \) satisfies

\[
x(1 - x)y'' + (\gamma - (\alpha + \beta + 1)x)y' - \alpha\beta y = 0,
\]

or, denoting \( \Theta := x \frac{d}{dx} \),

\[
[\Theta(\Theta + \gamma - 1) - x(\Theta + \alpha)(\Theta + \beta)](y) = 0.
\]

Up to normalization, this is a general linear differential equation with 3 regular singular points.
\[ A_n := \frac{(\alpha)_n(\beta)_n}{(\gamma)_nn!}, \quad F(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} A_n x^n. \]

- The coefficients \( A_n \) satisfy the following linear recurrence

\[
(\gamma + n)(1 + n)A_{n+1} - (\alpha + n)(\beta + n)A_n = 0 \tag{0.2}
\]

\[
\frac{A_{n+1}}{A_n} = R(n).
\]

- (0.2) is equivalent to the fact that \( F(\alpha, \beta, \gamma; x) \) satisfies the differential equation

\[
[\Theta(\Theta + \gamma - 1) - x(\Theta + \alpha)(\Theta + \beta)](y) = 0.
\]
Hypergeometric functions in several variables

System of hypergeometric PDE’s for Horn’s function $G_3$

\[
\begin{align*}
(x(2\theta_x - \theta_y + a')(2\theta_x - \theta_y + a' + 1) &- (-\theta_x + 2\theta_y + a)\theta_x)f = 0, \\
y(-\theta_x + 2\theta_y + a)(-\theta_x + 2\theta_y + a + 1) &- (2\theta_x - \theta_y + a')\theta_y)f = 0.
\end{align*}
\]

Its holonomic rank is 4 ($a$, $a'$ generic parameters)

- Erdélyi (Acta Mathematica, 1950) noted that, in a neighborhood of a given point, three linearly independent solutions of this system can be obtained through contour integral methods. He also finds a fourth linearly independent solution: the Puiseux monomial $x^{-(a+2a')/3}y^{-(2a+a')/3}$. He remarks that the existence of this elementary solution is puzzling, and offers no explanation for its occurrence.
Hypergeometric functions in several variables (suite)

System of hypergeometric PDE’s for Appell’s function $F_1$

\[
\begin{align*}
(x(\theta_x + \theta_y + a)(\theta_x + b) - \theta_x(\theta_x + \theta_y + c - 1))f &= 0, \\
(y(\theta_x + \theta_y + a)(\theta_y + b') - \theta_y(\theta_x + \theta_y + c - 1))f &= 0
\end{align*}
\]

• For generic values of the parameters $a, b, b'$ and $c$, the holonomic rank of this system, that is, the dimension of its space of local complex holomorphic solutions around a nonsingular point, is $3 < 2.2 = 4$. 
Hypergeometric functions in one variable, revisited \textit{GKZ} style

\textit{GKZ} = Gel’fand, Kapranov and Zelevinsky (89)

Consider the configuration in $\mathbb{R}^3$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$  

$\ker_\mathbb{Z}(A) = \langle (1, 1, -1, -1) \rangle$ \quad $(1, 1, -1, -1) = (1, 1, 0, 0) - (0, 0, 1, 1)$

- The following system of equations in four variables $x_1, x_2, x_3, x_4$ is a nice encoding for Gauss equation in one variable:

$$\begin{cases} (\partial_1 \partial_2 - \partial_3 \partial_4)(\varphi) = 0 \\ (x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4)(\varphi) = \beta_1 \varphi \\ (x_2 \partial_2 + x_3 \partial_3)(\varphi) = \beta_2 \varphi \\ (x_2 \partial_2 + x_4 \partial_4)(\varphi) = \beta_3 \varphi \end{cases}$$
\[
\begin{cases}
(x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4) (\varphi) = \beta_1 \varphi \\
(x_2 \partial_2 + x_3 \partial_3) (\varphi) = \beta_2 \varphi \\
(x_2 \partial_2 + x_4 \partial_4) (\varphi) = \beta_3 \varphi
\end{cases}
\] (0.3)

Given any \((\beta_1, \beta_2, \beta_3)\) and \(v \in \mathbb{C}^n\) such that \(A \cdot v = (\beta_1, \beta_2, \beta_3)\) and \(v_1 = 0\), any solution \(\varphi\) of (0.4) can be written as
\[
\varphi(x) = x^v \, f \left( \frac{x_1 x_2}{x_3 x_4} \right),
\]
where \(f(z)\) satisfies Gauss equation with
\[
\alpha = v_2, \quad \beta = v_3, \quad \gamma = v_4 + 1.
\]
\[
\begin{aligned}
\left\{ 
\begin{array}{l}
(\partial_1 \partial_2 - \partial_3 \partial_4)(\varphi) = 0 \\
(x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4)(\varphi) = \beta_1 \varphi \\
(x_2 \partial_2 + x_3 \partial_3)(\varphi) = \beta_2 \varphi \\
(x_2 \partial_2 + x_4 \partial_4)(\varphi) = \beta_3 \varphi
\end{array}
\right.
\end{aligned}
\]

(0.4)

- Given any \((\beta_1, \beta_2, \beta_3)\) and \(v \in \mathbb{C}^n\) such that \(A \cdot v = (\beta_1, \beta_2, \beta_3)\) and \(v_1 = 0\), any solution \(\varphi\) of (0.4) can be written as

\[
\varphi(x) = x^v \ f\left(\frac{x_1 x_2}{x_3 x_4}\right),
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where \(f(z)\) satisfies Gauss equation with

\[
\alpha = v_2, \ \beta = v_3, \ \gamma = v_4 + 1.
\]

The binomial operator \((\partial_1 \partial_2 - \partial_3 \partial_4)\) "represents" the hypergeometric recursion on the coefficients of the series.
Hypergeometric functions in several variables, roots of generic univariate polynomials


Given coprime integers $0 < k_1 < \ldots < k_m < n$, set

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & k_1 & \ldots & k_m & n \end{pmatrix} ,$$

and $\beta = (0, -1)$.

- The local roots $\rho(x)$ of the generic sparse polynomial $(f(x, \rho(x)) = 0)$

$$f(x; t) := x_0 + x_{k_1} t^{k_1} + \cdots + x_{k_m} t^{k_m} + x_n t^n ,$$

viewed as functions of the coefficients, are algebraic solutions to the associated $A$-hypergeometric system.
For example:

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}, \]

\[ \beta = (0, -1), \quad f(x; t) = x_0 + x_1 t + x_3 t^3 + x_4 t^4, \quad \theta_i := x_i \partial_i, \]

- The corresponding \( A \)-hypergeometric system is given by:
  
  \[ (\partial^2_0 \partial_3 - \partial^3_1) (\varphi) = 0 \]
  \[ (\partial^3_0 \partial_4 - \partial^4_1) (\varphi) = 0 \]
  \[ (\partial_0 \partial_4 - \partial_1 \partial_3) (\varphi) = 0 \]
  \[ (\partial_1 \partial^2_4 - \partial^3_3) (\varphi) = 0 \]
  \[ (\theta_0 + \theta_1 + \theta_3 + \theta_4) (\varphi) = 0 \]
  \[ (\theta_1 + 3\theta_3 + 4\theta_4 + 1) (\varphi) = 0. \]

Also residues, periods, generating functions of intersection numbers in moduli spaces are hypergeometric.
Hypergeometric functions in several variables, revisited \textit{GKZ} style

System of hypergeometric PDE's for \textit{Horn's function} $G_3$

\[ (x(2\theta_x - \theta_y + a')(2\theta_x - \theta_y + a' + 1) - (-\theta_x + 2\theta_y + a)(1\theta_x + 0\theta_y))f = 0 , \]
\[ (y(-\theta_x + 2\theta_y + a)(-\theta_x + 2\theta_y + a + 1) - (2\theta_x - \theta_y + a')\theta_y)f = 0 . \]

Explanation (D.– Matusevich – Sadykov ('05):

Look at the binomials

\[ q_1 = \partial_1^2 \partial_4^0 - \partial_2^1 \partial_3^1, \quad q_2 = \partial_1 \partial_4 - \partial_2^2 \]

in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$. 
\[(x(2\theta_x-\theta_y+a')(2\theta_x-\theta_y+a'+1)) - (-\theta_x + 2\theta_y + a)(1\theta_x + 0\theta_y)) f = 0 ,
(y(-\theta_x + 2\theta_y + a)(-\theta_x + 2\theta_y + a + 1) - (2\theta_x - \theta_y + a')\theta_y) f = 0 .
\]

System of hypergeometric PDE’s for Horn’s function \( G_3 \)

Explanation (D.– Matusevich – Sadykov ('05):

Look at the binomials
\[ q_1 = \partial_1^2\partial_4^0 - \partial_2^1\partial_3^1, \quad q_2 = \partial_1^1\partial_4^0 - \partial_2^2 \]
in the commutative polynomial ring \( \mathbb{C}[\partial_1, \ldots, \partial_4] \).

Its zero set has the component “at infinity” \( \{\partial_1 = \partial_2 = 0\} \), with multiplicity equal to the intersection multiplicity \( \mu_0 \) at the origin of the system of 2 binomials in 2 variables
\[ p_1 = \partial_1^a - \partial_2^b, \quad p_2 = \partial_1^c - \partial_2^d, \quad a = 2, \ b = 1, \ c = 1, \ d = 2 \]
\[ \mu_0 = \min\{|a \cdot d|, |b \cdot d|\} = 1, \]
which equals the dimension of the space of solutions to the Horn system which have finite support.
Hypergeometric functions in several variables, revisited GKZ style

System of hypergeometric PDE’s for Appell’s function $F_1$

\[(x(1\theta_x + 1\theta_y + a)(1\theta_x + 0\theta_y + b) - (1\theta_x + 0\theta_y)(1\theta_x + 0\theta_y + c - 1))f = 0 ,
(y(\theta_x + \theta_y + a)(\theta_y + b') - \theta_y(\theta_x + \theta_y + c - 1))f = 0\]

Explanation (D.– Matusevich – Sadykov ('05):

Look at the binomials

\[q_1 = \partial_1^1 \partial_3^1 - \partial_2^1 \partial_4^1, q_2 = \partial_1 \partial_5 - \partial_2 \partial_6\]

in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$. 
System of hypergeometric PDE’s for Appell’s function $F_1$

$$(x(1\theta_x + 1\theta_y + a)(1\theta_x + 0\theta_y + b) - (1\theta_x + 0\theta_y)(1\theta_x + 0\theta_y + c - 1))f = 0,$$

$$(y(\theta_x + \theta_y + a)(\theta_y + b') - \theta_y(\theta_x + \theta_y + c - 1))f = 0$$

Explanation (D.– Matusevich – Sadykov (’05):

Look at the binomials

$$q_1 = \partial_1^1 \partial_3^1 - \partial_2^1 \partial_4^1, \quad q_2 = \partial_1 \partial_5 - \partial_2 \partial_6$$

in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$.

Its zero set has the component “at infinity” $\{\partial_1 = \partial_2 = 0\}$, with multiplicity equal to the intersection multiplicity $\mu_0$ at the origin of the system of 2 binomials in 2 variables

$$p_1 = \partial_1^a - c_1 \partial_2^b, \quad p_2 = \partial_1^c - c_2 \partial_2^d, \quad a = 1, b = 1, c = 1, d = 1,$$

$$\mu_0 = \min\{|a \cdot c|, |b \cdot d|\} = 1,$$

but since $(1, -1), (1, -1)$ are linearly dependent, it does NOT give any solution to the Horn system for generic values of the parameters. Thus, there are only $3 = 4 - 1$ linearly independent local solutions.
What is a binomial D-module?

**Data:**

- An integer matrix $A \in \mathbb{Z}^{d \times n}$ such that the cone generated by the columns $a_1, \ldots, a_n$ of $A$ contains no lines, all of the $a_i$ are nonzero, and $\mathbb{Z}A = \mathbb{Z}^d$.

- A **binomial ideal** is an ideal generated by binomials $\partial^u - \lambda \partial^v$, where $u, v \in \mathbb{Z}^n$ are column vectors and $\lambda \in \mathbb{C}$. 
What is a binomial D-module?

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- A binomial ideal is an ideal generated by binomials $\partial^u - \lambda \partial^v$, where $u, v \in \mathbb{Z}^n$ are column vectors and $\lambda \in \mathbb{C}$.

- $A$ induces a $\mathbb{Z}^d$-grading of the polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_n] = \mathbb{C}[\partial]$, $\deg(\partial_i) = -a_i$.
- An ideal of $\mathbb{C}[\partial]$ is $A$-graded if it is generated by elements that are homogeneous for the $A$-grading.
What is a binomial D-module?

Data:

- An integer matrix $A \in \mathbb{Z}^{d \times n}$ such that the cone generated by the columns $a_1, \ldots, a_n$ of $A$ contains no lines, all of the $a_i$ are nonzero, and $\mathbb{Z}A = \mathbb{Z}^d$.
- A binomial ideal is an ideal generated by binomials $\partial^u - \lambda \partial^v$, where $u, v \in \mathbb{Z}^n$ are column vectors and $\lambda \in \mathbb{C}$.

A binomial ideal is $A$-graded precisely when it is generated by binomials $\partial^u - \lambda \partial^v$ each of which satisfies either $Au = Av$ or $\lambda = 0$. 
• The Weyl algebra $D = D_n$ of linear partial differential operators (in $n$ variables) written with the variables $x$ and $\partial$, is also $A$-graded by additionally setting $\deg(x_i) = a_i$. 
• For each $i \in \{1, \ldots, d\}$, the $i$-th Euler operator is

$$E_i = a_{i1} \theta_1 + \cdots + a_{in} \theta_n.$$ 

• Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$.

• These operators are $\mathcal{A}$-homogeneous of degree 0.
For an $A$-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_A(I, \beta)$ the left ideal $I + \langle E - \beta \rangle$ in the Weyl algebra $D$.

Finally, the \textit{binomial D-module} associated to $I$ is $D/H_A(I, \beta)$.
• For an \( A \)-graded binomial ideal \( I \subseteq \mathbb{C}[\partial] \), we denote by \( H_A(I, \beta) \) the left ideal \( I + \langle E - \beta \rangle \) in the Weyl algebra \( D \).
• Finally, the binomial \( D \)-module associated to \( I \) is \( D/H_A(I, \beta) \).

• When \( I \) equals the toric ideal \( I_A \) we have an \( A \)-hypergeometric system.
• When \( I \) is a lattice basis ideal, we have a Horn system (in binomial version).
What is a binomial D-module? (suite)

- A binomial D-module is a quotient by a left $D$-ideal generated by an $A$-homogeneous binomial ideal $I$ with constant coefficients plus Euler operators associated to the row span of $A$.

- Binomial differential operators annihilating a (multivariate Puiseux) series are equivalent to (special) linear recurrences satisfied by its coefficients.
- Euler operators prescribe $A$-homogeneity (infinitesimally).
A (non holonomic) example

Consider

\[
A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

so that

\[
H(B, \beta) = \langle \partial_1 \partial_3 - \partial_2, \partial_1 \partial_4 - \partial_2 \rangle + \langle x_1 \partial_1 - x_2 \partial_2 - \beta_1, x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4 - \beta_2 \rangle.
\]

If \( \beta_1 = 0 \), then any (local holomorphic) bivariate function \( f(x_3, x_4) \) annihilated by the operator \( x_3 \partial_3 + x_4 \partial_4 - \beta_2 \) is a solution of \( H(B, \beta) \).

The space of such functions is infinite-dimensional; in fact, it has uncountable dimension, as it contains all monomials

\[
x_3^{w_3} x_4^{w_4}
\]

with \( w_3, w_4 \in \mathbb{C} \) and \( w_3 + w_4 = \beta_2 \).
Finding polynomial solutions of binomial ideals

\[ M = \begin{pmatrix} 4 & 5 \\ -3 & -5 \end{pmatrix} \]

The system \( H(I(M), 0) \) is defined by the operators

\[ \frac{\partial^4}{\partial x_1^4} - \frac{\partial^3}{\partial x_2^3}, \quad \frac{\partial^5}{\partial x_1^5} - \frac{\partial^5}{\partial x_2^5}. \]

It has 15 linearly independent polynomial solutions, with the following minimal supports:
Description of solutions to CI Horn binomial D-modules

Consider the matrices:

\[
B = \begin{bmatrix}
0 & -1 & 2 \\
-1 & 0 & -1 \\
0 & 1 & -1 \\
4 & 5 & 0 \\
-3 & -5 & 0
\end{bmatrix}; \quad A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
5 & 10 & 0 & 7 & 6
\end{bmatrix}
\]

\[
I = I(B) = x_2x_5^3 - x_4^4, x_1x_5^5 - x_3x_4^5, x_1^2 - x_2x_3.
\]

We concentrate on the decomposition:

\[
M = \begin{bmatrix}
4 & 5 \\
-3 & -5
\end{bmatrix}; \quad N = \begin{bmatrix}
0 & -1 \\
-1 & 0 \\
0 & 1
\end{bmatrix}; \quad \hat{B} = \begin{bmatrix}
2 \\
-1 \\
-1
\end{bmatrix}
\]

Note that \(\gcd\{2, -1, -1\} = 1\), so there is only one associated prime coming from this decomposition:

\[
I[1, 1, 1] + \langle \partial_4, \partial_5 \rangle
\]
• The quatrinomial $p = 5x_4^4x_5^2 + 2x_4^5 + 2x_5^5 + 40x_4x_5^3$ is a solution of the constant coefficient system $I(M)$.

• Let $f$ be a solution of the $\left[\begin{array}{ccc} 1 & 1 & 15 \\ 1 & 0 & 10 \\ 0 & 0 & 0 \end{array}\right]$-hypergeometric system that is homogeneous of appropriate degree.

• Then the following function is a solution of $H(B, \beta)$:

$$5x_4^4x_5^2\partial_2^2\partial_3 f + 2x_4^5\partial_1\partial_2 f + 2x_5^5\partial_2\partial_3 f + 40x_4x_5^3\partial_1 f$$
Questions

- For which parameters does the space of local holomorphic solutions around a nonsingular point of a binomial Horn system have finite dimension as a complex vector space?
- What is a combinatorial formula for the minimum such dimension, over all possible choices of parameters?
- Which parameters are generic, in the sense that the minimum dimension is attained?
- How do (the supports of) series solutions centered at the origin of a binomial Horn system look, combinatorially?
- When is $D/H_A(I, \beta)$ a holonomic $D$-module?
- When is $D/H_A(I, \beta)$ a regular holonomic $D$-module?
Holonomic rank

$P = \sum k_{\alpha,\beta} x^\alpha \partial^\beta, \quad P \neq 0$

$\sigma(P) = \text{in}_{(0,e)}(P) = \sum_{|\beta| = \text{ord}(P)} k_{\alpha,\beta} x^\alpha \xi^\beta \in \mathbb{C}[x, \xi]$

$J \subset D_n = D_n(\mathbb{C})$ left ideal, \( \text{in}_{(0,e)}(J) := \langle \sigma(P), P \in J, P \neq 0 \rangle \).

The holonomic rank \( \text{rank}(J) \) of \( J \) is the dimension over \( \mathbb{C} \) of the space of its local holomorphic solutions around a (generic = non singular) point. It equals

$$\text{rank}(J) = \dim_{\mathbb{C}(x)} \left( \mathbb{C}(x)[\xi]/\mathbb{C}(x)[\xi].\text{in}_{(0,e)}(J) \right).$$

Basic block: \( A \)-hypergeometric binomial \( D \)-modules are always holonomic with non zero holonomic rank [GKZ’89], [Adolphson’94].
Our main tools

- Precise description of combinatorial commutative algebra of binomial ideals in semigroup rings (based on: \textit{Binomial ideals}, Eisenbud – Sturmfels ('94))
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- Functorial translation of those results into $D$-module theory by means of Cayley-Koszul complexes (based on: GKZ('89,'90), Adolphson('94), Mateus-vich – Miller – Walther ('05))

- And, of course, some direct generalizations of the beautiful theory of $A$-hypergeometric systems developed by Gel’fand, Kapranov, and Zelevinsky!
Primary components of binomial ideals

- **Eisenbud-Sturmfels:** An irredundant primary decomposition of an arbitrary binomial ideal $I \subseteq \mathbb{C}[\partial]$ is given by

$$I = \bigcap_{I_{\rho}, J \in \text{Ass}(I)} \text{Hull}(I + I_{\rho} + \langle \partial_j : j \notin J \rangle^e)$$

for any large integer $e$, where $\text{Hull}$ means to discard the primary components for embedded (i.e., nonminimal associated) primes.

- We make this Hull operation explicit, by characterizing the monomials in each primary component of $I$.

Basic blocks:

- “Prime binomial ideals = toric ideals”
- Zero dimensional ideals $I(M)$, where $M$ is a $m \times m$ mixed square matrix.
Primary components of binomial ideals

For example: $I = \langle x^2 - y, x^3 - y^2 \rangle$, $0 \in V(I)$

- Which are all the monomials that are present in $I_0$?

- Our answer: $I_0 = I + \langle x^3, xy, y^2 \rangle$, and these are exactly all the monomials in $I_0$.

- For instance, a SINGULAR computation gives a standard basis $\{y - x^2, x^3\}$ (for local lex order with $y > x$).
Euler-Koszul homology allows us to functorially translate the commutative algebra of $A$-graded primary decomposition directly into the $D$-module setting and to pull apart the “contributions” of each of the primary components of binomial ideals in a binomial $D$-module.
For \( z \) \( A \)-homogeneous in an \( A \)-graded left \( D \)-module, define

\[
(E_i - \beta_i) \circ z = (E_i - \beta_i - \deg_i(z)) z,
\]

Fix \( \beta \in \mathbb{C}^d \) and an \( A \)-graded ideal \( I \). The Euler-Koszul complex

\[
\mathcal{K}.(E - \beta; \mathbb{C}[\partial]/I)
\]

is the Koszul complex of left \( D \)-modules defined by the sequence \( E - \beta \) of commuting endomorphisms on the left \( D \)-module \( D \otimes_{\mathbb{C}[\partial]} V \), (\( V = \mathbb{C}[\partial]/I \)), concentrated in homological degrees \( d \) to 0.

The \( i \)-th Euler-Koszul homology is \( \mathcal{H}_i(E - \beta; V) = H_i(\mathcal{K}.(E - \beta; V)) \).

The binomial Horn \( D \)-module with parameter \( \beta \) is \( \mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I) \).

Each graded piece of the Euler-Koszul complex coincides with a commutative Koszul complex over the ring \( \mathbb{C}[\theta_1, \ldots, \theta_n] \).
• The $i$-th Euler-Koszul homology of the quotient $V = \mathbb{C}[\partial]/I_\rho$ corresponding to a \textit{toral} primary component $I_\rho$ of $I$ is nonzero for some $i \geq 1$ if and only if $-\beta$ lies in the Zariski closure of the degrees $\alpha \in \mathbb{Z}^d$ such that the $\alpha$-graded piece of the local cohomology module $H^i_m(V)_\alpha$ is non zero for some $i < d$.

• This Zariski closure is a \textit{finite union of linear subspaces}.
• We explicitly classify all primary components of $I$ (in particular, all monomials that are present), their multiplicities, their behaviour with respect to the grading (toral and Andean components), and their holonomic rank.

• We explicitly define two finite subspace arrangements associated to the Andean components (Andean arrangement) and to the pairwise intersections of two components (primary arrangement) as the Zariski closure of parameters for which the corresponding piece in certain local cohomology modules is non zero, which account for non generic behaviour of the complex parameter $\beta$.

• The basic building blocks are associated $\mathcal{A}$-hypergeometric systems (for several different $\mathcal{A}$).
• The dimension is finite exactly for $-\beta$ not in the Andean arrangement.

• The generic (minimum) rank is $\sum \mu(L, J) \cdot \text{vol}(A_J)$, the sum being over all toral associated sublattices with $CA_J = \mathbb{C}^d$, where $\text{vol}(A_J)$ is the volume of the convex hull of $A_J$ and the origin, normalized so a lattice simplex in $\mathbb{Z}A_J$ has volume 1.

• The minimum rank is attained precisely when $-\beta$ lies outside of an explicit affine subspace arrangement determined by certain local cohomology modules, containing the Andean arrangement.

• When the Horn system is regular holonomic and $\beta$ is general, there are $\mu(L, J) \cdot \text{vol}(A_J)$ linearly independent solutions supported on (translates of) the $L$-bounded classes, with hypergeometric recursions determining the coefficients.

• Only $g \cdot \text{vol}(A)$ many Gamma series solutions have full support, where $g = |\ker(A)/\mathbb{Z}B|$ is the index of $\mathbb{Z}B$ in its saturation $\ker(A)$.

• Holonomicity is equivalent to finite dimension of the (local holomorphic) solutions spaces.

• Holonomicity is equivalent to regular holonomicity when $I$ is standard $\mathbb{Z}$-graded—i.e., the row-span of $A$ contains the vector $[1 \cdots 1]$. Conversely, if there exists a parameter $\beta$ for which $D/H_A(I, \beta)$ is regular holonomic, then $I$ is standard $\mathbb{Z}$-graded.
Thanks for your attention!