On clique-perfect and K-perfect graphs

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Abstract

A graph G is clique-perfect if the cardinality of a maximum clique-independent set of H is equal to the cardinality of a minimum clique-transversal of H, for every induced subgraph H of G. When equality holds for every clique subgraph of G, the graph is c–clique-perfect. A graph G is K-perfect when its clique graph $K(G)$ is perfect. In this work, relations are described among the classes of perfect, K-perfect, clique-perfect and c–clique-perfect graphs. Besides, partial characterizations of K-perfect graphs using polyhedral theory and clique subgraphs are formulated.

Keywords: clique graphs, clique-Helly graphs, clique-perfect graphs, good graphs, K-perfect graphs, perfect graphs.

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1 Introduction

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $N(v)$ the set of neighbours of $v \in V(G)$. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. For $v, w \in V(G)$, say that v is *dominated* by w when $N[v] \subseteq N[w]$. In particular, when $N[v] = N[w]$, then v and w are *twins*.

A *complete set* of G is a subset of vertices pairwise adjacent. A *clique* is a complete set not properly contained in any other. We may also use the term clique to refer to the corresponding complete subgraph.

Let $C(v)$ be the set of cliques containing the vertex v. Denote $m(v) = |C(v)|$.

A *clique cover* of a graph G is a subset of cliques covering all the vertices of G. The *clique-covering number* $\theta(G)$ is the cardinality of a minimum clique cover of G. An *independent set* in a graph G is a subset of pairwise non-adjacent vertices of G. The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set of G.

The *chromatic number* of a graph G is the smallest number of colours that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and is denoted by $\chi(G)$. An obvious lower bound is the maximum cardinality of the cliques of G, the *clique number* of G, denoted by $\omega(G)$.

A graph G is *perfect* when $\theta(H) = \alpha(H)$ for every induced subgraph H of G (or equivalently, when $\chi(H) = \omega(H)$ for every induced subgraph H). If the first equality holds ($\theta(G) = \alpha(G)$) for the graph G, G is *α-good*. If the second one happens ($\chi(G) = \omega(G)$), G is *χ-good*. If both results are verified, we say that G is *good*.

Perfect graphs are very interesting from an algorithmic point of view, see [16]. While determining the clique-covering number, the independence number, the chromatic number and the clique number of a graph are NP-complete problems, they are solvable in polynomial time for perfect graphs [17]. Besides, it has been proved recently that perfect graphs can be recognized in polynomial time [11].

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_c(G)$ and $\alpha_c(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively.

It is easy to see that $\tau_c(G) \geq \alpha_c(G)$ for any graph G. A graph G is *clique-perfect* if $\tau_c(H) = \alpha_c(H)$ for every induced subgraph H of G. If this equality holds for the graph G, we say that G is *clique-good*.

Clique-perfect graphs have been implicitly studied in many articles, as [1, 4, 7, 9, 14, 18, 19]. The terminology "clique-perfect" has been introduced in [18].

Let $H = (V, \mathcal{E})$ be a hypergraph. A sequence $v_1, E_1, \ldots, v_k, E_k$ of distinct vertices $v_1, \ldots, v_k$ and distinct hyperedges $E_1, \ldots, E_k$ of H is a *special cycle* of length k if $k \geq 3$, $v_i, v_{i+1 (\text{mod } k)} \in E_i$ and $E_i \cap \{v_1, \ldots, v_k\} = \{v_i, v_{i+1(\text{mod } k)}\}$, for each i, $1 \leq i \leq k$.

A hypergraph H is *balanced* if it contains no special cycles of odd length [4]. The *clique hypergraph* of a graph G has the same vertex set as G and all cliques of G as hyperedges. Say that a graph G is *balanced* if its clique hypergraph is balanced [13].

A subset $V' \subseteq V$ is a *transversal* of a hypergraph H if every hyperedge contains a vertex from V'. The cardinality of a minimum transversal of H is the *transversal number* of H. A subset $\mathcal{E}' \subseteq \mathcal{E}$ is a *matching* of an hypergraph H if and only if for all distinct $E_i, E_j \in \mathcal{E}'$, $E_i \cap E_j = \emptyset$. The cardinality of a maximum matching of H is the *matching number* of H.

A hypergraph H has the *König property* if the transversal number of H is equal to the matching number of H (see, for example, [3]).

Clique-perfection of a graph G means that for every induced subgraph G' of G the clique hypergraph of G' has the König property. Berge and Las Vergnas showed that balanced hypergraphs fulfill the König property [4]. These results imply that balanced graphs are clique-perfect, using the terminology of [18].

Strongly chordal graphs [9], dually chordal graphs [7], comparability graphs [1], odd sun-free chordal graphs [19] and short-chorded graphs not containing either a 4-wheel or a 3-fan as induced subgraphs [14] are examples of other clique-perfect graph classes.

Clearly, perfect graphs are not necessarily clique-perfect. On the other hand, clique-perfect graphs are not necessarily perfect. The graph $\overline{C_{6j+3}}$, the complement of a chordless cycle of length $6j + 3$, is clique-perfect but not perfect for any $j \geq 1$ [21].

Perfect matrices are defined in the context of perfect graphs. A matrix $A \in R^{k \times n}$ is *perfect* if the polyhedron $P(A) = \{x \in R^n / Ax \leq \mathbf{1}, x \geq 0\}$ has only integer extreme points.

Let $M_1, \ldots, M_k$ and $v_1, \ldots, v_n$ be the cliques and vertices of a graph G, respectively. A *clique matrix* $A_G \in R^{k \times n}$ of G is a 0-1 matrix whose entry a_{ij} is 1 if $v_j \in M_i$, and 0, otherwise.

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.

Let $A \in R^{r \times n}$ be a 0-1 matrix having no zero columns. The *derived graph* of A is the intersection graph of its columns, that is, a graph of n vertices $v_1, \ldots, v_n$ where v_i is adjacent to v_j if there exists a row l in A such that $a_{li} = a_{lj} = 1$.

The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. Denote by $K^2(G)$ the clique graph of $K(G)$. A graph G is *self-clique* if $K(G)$ is isomorphic to G. A graph G is *K-perfect* if $K(G)$ is perfect.

A family of subsets S satisfies the *Helly property* when every subfamily of it consisting of pairwise intersecting subsets has a common element. A graph is *clique-Helly* (CH) when its cliques satisfy the Helly property. A graph G is *hereditary clique-Helly* (HCH) when H is clique-Helly for every induced subgraph H of G.

A graph is *chordal* when every cycle of length greater than 3 has a chord.

Let G be a graph, M the set of cliques of G and $M' \subseteq M$. Denote by $G_{M'}$ the subgraph of G formed exactly by the vertices and edges corresponding to the cliques of M'. When every clique of $G_{M'}$ is also a clique of G, we say that $G_{M'}$ is a *clique subgraph* of G. A graph G is *c–clique-perfect* if $\tau_c(H) = \alpha_c(H)$ for every clique subgraph H of G.

The paper is organized as follows. In section 2, some basic results on perfect, clique-perfect and good graphs are shown. In section 3, partial characterizations of K-perfect graphs using polyhedral theory and clique subgraphs are formulated.

2 Basic results

2.1 Perfect graphs

Berge defined perfect graphs and stated two famous conjectures [2]. The first one said that a graph is perfect if and only if its complement is perfect. This conjecture was proved by Lóvasz in 1972 [20] and is known as the Perfect Graph Theorem.

Theorem 2.1 (Perfect Graph Theorem) *Given a graph G, the following statements are equivalent:*

(P_1) $\omega(H) = \chi(H)$ *for every induced subgraph H of G.*

(P_2) $\alpha(H) = \theta(H)$ *for every induced subgraph H of G.*

(P_3) $|V(H)| \leq \omega(H)\alpha(H)$ *for every induced subgraph H of G.*

The second one, known as the Strong Perfect Graph Conjecture, was open more than forty years and was recently proved [10].

Theorem 2.2 (Strong Perfect Graph Theorem) *A graph G is perfect if and only if it contains neither induced odd cycle of length at least five nor its complement.*

On the other hand, Chvátal [12] formulated the following theorem which relates perfect matrices with perfect graphs.

Theorem 2.3 *Let A be a* 0-1 *matrix with neither zero columns nor dominated rows. The matrix A is perfect if and only if A is the clique matrix of a perfect graph.*

2.2 Good graphs

In this subsection we prove some results on good graphs and variations of them.

The following concept is related to property P_3 of Theorem 2.1. A graph G is L-good if $|V(G)| \leq \omega(G)\alpha(G)$.

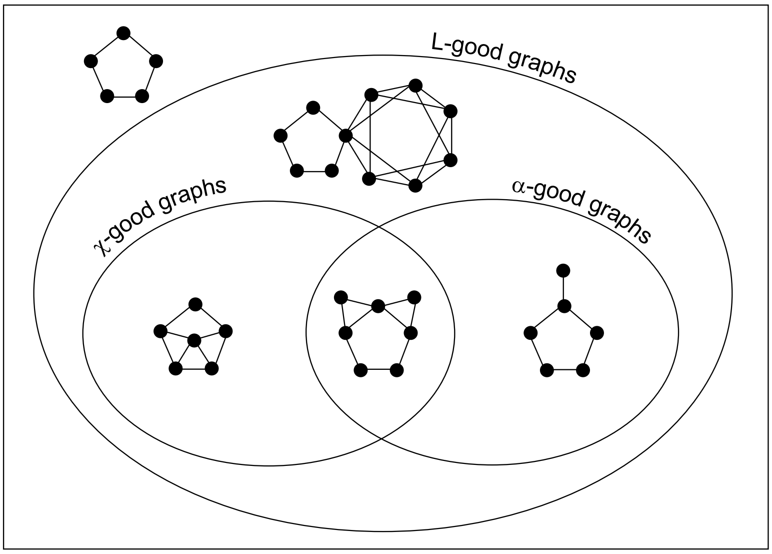

Figure 1: Intersection between classes of good graphs.

Proposition 2.1 *Let G be a graph. Then if G is either χ-good or α-good, then G is L-good.*

Proof: Suppose G is χ-good. As there are at most $\alpha(G)$ vertices in each one of the $\chi(G)$ independent sets in the partition of the vertices induced by an optimal colouring of the graph, it follows that $|V(G)| \leq \alpha(G)\chi(G)$. If $\chi(G) = \omega(G)$, G results L-good.

Now suppose the graph G is α-good. Observe that each one of the $\theta(G)$ cliques in an optimal clique covering covers at most $\omega(G)$ vertices of G. Then, $|V(G)| \leq \omega(G)\theta(G)$. If $\alpha(G) = \theta(G)$, the result follows directly. □

Figure 1 shows that graph classes α-good, χ-good and L-good are not equivalent.

Now, a characterization of χ-good graphs is formulated.

Theorem 2.4 *Let G be a graph, then G is χ-good if and only if there exists a sequence of induced subgraphs $H_0 = G \supset H_1 \supset \cdots \supset H_{\omega(G)-1}$ which verifies for $i = 0, \ldots, \omega(G) - 2$:*

(i) $V(H_{i+1}) = V(H_i) - S_i$, *where* S_i *is an independent set of* H_i,

(ii) $\omega(H_{i+1}) = \omega(H_i) - 1$.

Proof:

$\Rightarrow$) Suppose G is a χ-good graph. Let $F_1, \ldots, F_{\omega(G)}$ be a partition of the vertices of G into $\omega(G)$ independent sets. Consider the following sequence of graphs: $H_0 = G$, H_1 defined as the induced subgraph which contains every vertex of G except for the ones belonging to F_1. In general, define H_i as the subgraph of G induced by the vertices of H_{i-1} except for the vertices belonging to F_i. We need to check that this sequence verifies the previous conditions. Clearly, $H_i \supset H_{i+1}$. It remains to prove that $\omega(H_{i+1}) = \omega(H_i) - 1$. Observe that, since the family $F_1, \ldots, F_{\omega(G)}$ is a partition of the vertices of G into independent sets, it follows that $V(H_{\omega(G)-1}) = F_{\omega(G)}$. As $F_{\omega(G)}$ is an independent set, $\omega(H_{\omega(G)-1}) = 1$. On the other hand, from the fact that the set of vertices $V(H_i - H_{i+1})$ is an independent set, it follows that the difference between $\omega(H_i)$ and $\omega(H_{i+1})$ is either 0 or 1. Then, in each step, the value of ω can decrease by at most 1. If we observe that in the sequence $\omega(H_0), \ldots, \omega(H_{\omega(G)-1})$ there are $\omega(G) - 1$ steps to go from the number $\omega(G)$ to 1, we conclude that $\omega(H_i) - \omega(H_{i+1})$ must be 1.

$\Leftarrow$) Suppose there exists a sequence $H_0 = G \supset H_1 \supset \cdots \supset H_{\omega(G)-1}$ verifying the hypothesis of the theorem. We are going to construct a partition of the vertices of G into $\omega(G)$ independent sets. Define F_1 as the set of vertices belonging to H_0 and not to H_1. In general, for $i \leq \omega(G) - 1$ define F_i as the set of vertices which belong to H_{i-1} and not to H_i. Observe that from the fact that $\omega(H_i) - \omega(H_{i+1}) = 1$, it follows that $\omega(H_{\omega(G)-1}) = 1$. This means that the vertices of $H_{\omega(G)-1}$ are an independent set. Hence, by defining $F_{\omega(G)}$ as the set of vertices of $H_{\omega(G)-1}$, it follows that the family $F_1, \ldots, F_{\omega(G)}$ is a partition of the vertices G into $\omega(G)$ independent sets. □

It is a trivial result that a graph G is χ-good if and only if $\overline{G}$ is α-good. So, we may formulate an analogous characterization for α-good graphs.

Theorem 2.5 *Let* G *be a graph, then* G *is* α*-good if and only if there exists a sequence of induced subgraphs* $L_0 = G \supset L_1 \supset \cdots \supset L_{\alpha(G)-1}$ *which verifies for* $j = 0, \ldots, \alpha(G) - 2$*:*

(i) $V(L_{j+1}) = V(L_j) - C_j$, *where* C_j *is a complete set of* L_j,

(ii) $\alpha(L_{j+1}) = \alpha(L_j) - 1$.

Clearly, graphs H_i of Theorem 2.4 are χ-good and graphs L_j of Theorem 2.5 are α-good.

Clique-good graphs and graphs whose clique graph is α-good are related by Theorem 2.6. To prove this theorem, we will need the following lemma:

Lemma 2.1 [15] *Let G be a clique-Helly graph and let $K(G)$ be its clique graph. Then, each clique L of $K(G)$ has an associated vertex v_L in G such that the vertices of L in $K(G)$ are exactly those corresponding to the cliques of $C(v_L)$ in G.*

Theorem 2.6 *Let G be a graph. Then:*

(i) $\alpha_c(G) = \alpha(K(G))$.

(ii) $\tau_c(G) \geq \theta(K(G))$.

(iii) *If G is clique-Helly then $\tau_c(G) = \theta(K(G))$.*

Proof:

(i) It follows from the fact that independent cliques of G correspond to non adjacent vertices in $K(G)$, and conversely.

(ii) Let $v_1, \ldots, v_{\tau_c(G)}$ be a clique-transversal set of G. For each i, analyse the $m(v_i)$ vertices in $K(G)$ corresponding to the cliques in G that contain the vertex v_i. They form a complete set of $K(G)$. This complete set must be included in some clique L_i of $K(G)$. Observe that these cliques L_i ($i = 1, \ldots, \tau_c(G)$) are not all necessarily different. Let us see that these at most $\tau_c(G)$ cliques are a clique cover of $K(G)$. Let w be a vertex of $K(G)$. Then w corresponds to some clique M_w of G. As the set $v_1, \ldots, v_{\tau_c(G)}$ intersects all the cliques of G, there is some vertex v_j that belongs to M_w. This means that the corresponding vertex of M_w in $K(G)$ belongs to the clique L_j, i.e, $w \in L_j$. Then, the size of the minimum clique cover of $K(G)$ is at most the size of this clique cover which is at most $\tau_c(G)$.

(iii) All we need to prove is that if G is clique-Helly, then $\tau_c(G) \leq \theta(K(G))$. Let $L_1, \ldots, L_{\theta(K(G))}$ be a clique cover of $K(G)$. Let $v_{L_1}, \ldots, v_{L_{\theta(K(G))}}$ be the associated vertices of those $\theta(K(G))$ cliques given by Lemma 2.1. We want to prove that they form a clique-transversal set of G. Let M be a clique of G and w_M its corresponding vertex in $K(G)$.

Then there is an index j such that w_M belongs to the clique L_j in $K(G)$. It follows that the associated vertex v_{L_j} belongs to M in G. □

A trivial corollary of this theorem is the following.

Corollary 2.1 *Let G be a clique-good graph. Then the clique graph $K(G)$ is α-good. Besides, if G is a clique-Helly graph, the converse also holds.*

2.3 Clique-perfect graphs

We describe classes of graphs which are not clique-perfect.

An *r-sun*, $r \geq 3$, is a chordal graph G of $2r$ vertices whose vertex set can be partitioned into two sets: $W = \{w_1, \ldots, w_r\}$ and $U = \{u_1, \ldots, u_r\}$, such that W is an independent set and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1 \pmod{r}$.

Let G be a graph and C be a cycle of G not necessarily induced. An edge of C is *non proper* if it forms a triangle with some vertex of C.

An *r-generalized sun*, $r \geq 3$, is a graph G whose vertex set can be partitioned into two sets: a cycle C of r vertices, with non proper edges $\{e_j\}_{j \in J}$ (J could be an empty set) and an independent set $U = \{u_j\}_{j \in J}$, such that for each $j \in J$, u_j is adjacent only to the endpoints of e_j.

An r-sun or an r-generalized sun is said to be *odd* if r is odd. Figure 2 shows some examples of odd generalized suns.

A hole is a chordless cycle of length at least 4. An antihole is the complement of a hole. A hole is said to be odd if the length of the chordless cycle is odd, and even, otherwise.

Clearly, odd suns and odd holes are odd generalized suns.

Antiholes of length at least 5, except for $\overline{C_{3k}}$, are not clique-perfect [21]. The same holds for odd generalized suns, as follows.

Theorem 2.7 *Odd generalized suns are not clique-perfect.*

Proof: Let G be a $(2k + 1)$-generalized sun. Then its vertex set can be partitioned into a $(2k + 1)$-cycle with a set E of non proper edges and an

Figure 2: Examples of odd generalized suns.

independent set $\{u_e\}_{e \in E}$, as it is said in the definition. Each edge e of the cycle either is proper (and then is a clique of G) or it forms a triangular clique with the vertex u_e. Let T be a minimum clique-transversal set of G and let e be an edge of the cycle. If e is not covered by T, then e must be non proper and the vertex u_e must belong to T. Replacing u_e by an endpoint of e, we obtain a minimum clique-transversal set which also covers the edge e. Repeating this procedure, we can obtain a minimum clique-transversal set that covers all the edges of the cycle, so $\tau_c(G) \geq k+1$. On the other hand, every clique of G contains at least two vertices of the cycle, so $\alpha_c(G) \leq k$. Therefore G is not clique-good and, in consequence, is not clique-perfect too. □

It remains as an open problem to characterize clique-perfect graphs. In addition, is there such a characterization by forbidden subgraphs? Finally, another open question is whether clique-perfect graphs can be recognized in polynomial time.

3 K-perfect graphs

3.1 A partial characterization by polyhedral theory

In this section, we relate the concept of K-perfection for a clique-Helly graph G with properties of the polyhedron $\{x \in R^k / A_G^t x \leq \mathbf{1}, x \geq 0\}$, where A_G is a clique matrix of G.

Theorem 3.1 *Let G be a graph, A_G a clique matrix of G and $A_{K(G)}$ a clique matrix of $K(G)$ with the vertices in the same order as their corresponding cliques in A_G. Then, the following statements are equivalent:*

(i) *G is clique-Helly.*

(ii) *The matrix A^t_G without the dominated rows is a clique matrix of $K(G)$.*

(iii) *The polyhedron $\{x \in R^k / A^t_G x \leq \mathbf{1}, x \geq 0\}$ is the same as the polyhedron $\{x \in R^k / A_{K(G)} x \leq \mathbf{1}, x \geq 0\}$.*

Proof:

(i) $\Rightarrow$ (ii) Let G be a clique-Helly graph and $V(G) = \{v_1, v_2, \ldots, v_n\}$ the vertices of G. Note that $C(v_1), \ldots, C(v_n)$ are identified by the columns of A_G, so by Lemma 2.1, every clique of $K(G)$ is identified by a column of A_G and every column of A_G represents a complete set of $K(G)$. Then the submatrix of A^t_G obtained by removing the dominated rows is a clique matrix of $K(G)$.

(ii) $\Rightarrow$ (iii) As the variables are nonnegative, the dominated rows of the matrix A^t_G can be removed without loosing restrictions.

(iii) $\Rightarrow$ (i) Suppose that G is not clique-Helly. Let $M_1, M_2, \ldots, M_r$, $r \geq 2$, be a pairwise intersecting family of cliques in G without common intersection. Without loss of generality, we can assume that those cliques correspond to the first r rows of A_G. Then, for each vertex v_j there exists a clique M_{i_j} not containing it. For every column j of the clique matrix of G, there is some $i_j \leq r$ such that $a_{i_j j} = 0$. Let $x = (x_i)$ be the vector: $x_i = \frac{1}{r-1}$ for $1 \leq i \leq r$, and $x_i = 0$ for $r + 1 \leq i \leq k$ and compute $(A^t_G x)_j = \sum_{i=1}^{r} a_{ij} x_i$ $+ \sum_{i=r+1}^{k} a_{ij} 0$. As for each j there is at least one $i_j \leq r$ such that $a_{i_j j} = 0$, then $(A^t_G x)_j \leq \frac{r-1}{r-1} = 1$ Then, the vector x belongs to the polyhedron $\{x \in R^k / A^t_G x \leq \mathbf{1}, x \geq 0\}$. Now, let $A_{K(G)} = \{b_{ij}\}$ be the clique matrix of $K(G)$. As $M_1, M_2, \ldots, M_r$ are a pairwise intersecting family of cliques of G, there must be a clique in $K(G)$ containing their corresponding vertices in the clique graph. Therefore, there is a row i in $A_{K(G)}$ such that $b_{ij} = 1$ for $j \leq r$. Then $(A_{K(G)} x)_i = \frac{r}{r-1} > 1$ and so x does not belong to the polyhedron $\{x \in R^k / A_{K(G)} x \leq \mathbf{1}, x \geq 0\}$, which is a contradiction. □

Now, we are able to characterize clique-Helly K-perfect graphs.

Corollary 3.1 *Let G be a graph and let A_G be a clique matrix of G. Then G is clique-Helly and K-perfect if and only if A^t_G is a perfect matrix.*

Proof:

⇒) Let $A_{K(G)}$ be a clique matrix of $K(G)$ with the vertices in the same order than their corresponding cliques in A_G. According to Theorem 2.3, G is K-perfect if and only if $A_{K(G)}$ is perfect. By Theorem 3.1, if G is a clique-Helly graph, the polyhedra $P(A_G^t)$ and $P(A_{K(G)})$ are the same, so A_G^t is a perfect matrix.

⇐) Let A be the matrix obtained removing the dominated rows of A_G^t. Then A is a 0-1 matrix with neither zero columns nor dominated rows. Since $P(A_G^t)$ and $P(A)$ are the same, A is a perfect matrix. So by Theorem 2.3, A is the clique matrix of a perfect graph. The derived graph of A is $K(G)$, because it is the same as the derived graph of A_G^t. Therefore, $K(G)$ is perfect and A is a clique matrix of $K(G)$. Finally, by Theorem 3.1, it follows that G is a clique-Helly graph. □

3.2 A partial characterization by clique subgraphs

In this section we give a characterization of hereditary clique-Helly K-perfect graphs in terms of clique subgraphs.

Theorem 3.2 *Let G be a clique-Helly K-perfect graph. Then G is c–clique-perfect.*

Proof: Let H be a clique subgraph of G. Since the cliques of H are cliques of G, and G is a clique-Helly graph, H is CH too. As $K(H)$ is an induced subgraph of $K(G)$, $K(H)$ is perfect. According to Theorem 2.6, it follows that $\alpha_c(H) = \tau_c(H)$. □

This theorem allows us to conclude that dually chordal graphs and balanced graphs are c–clique-perfect since they are clique-Helly and K-perfect [6, 8, 22].

Let H be a graph. The *generator* G of H is the induced subgraph of H obtained identifying twin vertices and then removing dominated vertices. If a graph G is the generator of H, we say that H is generated by G.

Lemma 3.1 [15] *Let G be a clique-Helly graph. Then $K^2(G)$ is the generator of G.*

Corollary 3.2 *Let G be a perfect clique-Helly graph. Then $K(G)$ is c–clique-perfect.*

Proof: By Lemma 3.1 $K^2(G)$ is an induced subgraph of G. So, if G is perfect, then $K^2(G)$ is perfect too. By Theorem 3.2, $K(G)$ is c–clique-perfect. □

Corollary 3.3 *If G is K-perfect and clique-Helly, then the generator of G is c–clique-perfect.*

Proof: Combining Corollary 3.2 with Lemma 3.1 the proof is straightforward. □

Lemma 3.2 *Let G be a clique-Helly graph and let H be a clique-Helly graph generated by G. Then $K(H)$ is generated by $K(G)$. In particular, if G is self-clique, then $K(H)$ contains G as an induced subgraph. Conversely, if G is a self-clique clique-Helly graph, then every clique-Helly graph H such that $K(H) = G$ is generated by G.*

Proof:

$\Rightarrow$) Let H be a clique-Helly graph generated by G. Then, by Lemma 3.1, $K^2(H) = G$. So $K^2(K(H)) = K^3(H) = K(G)$. Since $K(H)$ is clique-Helly, by Lemma 3.1 it is generated by $K(G)$.

$\Leftarrow$) Let H be a clique-Helly graph such that $K(H) = G$. Then $K^2(H) = K(K(H)) = K(G) = G$. So by Lemma 3.1, G is the generator of H. □

Now, we are able to characterize hereditary clique-Helly K-perfect graphs by clique subgraphs.

Theorem 3.3 *Let G be an hereditary clique-Helly graph. Then the following statements are equivalent:*

(i) *The matrix A_G^t is perfect.*

(ii) *G is K-perfect.*

(iii) *G is c–clique-perfect.*

(iv) *G does not contain as a clique subgraph a graph generated either by an odd cycle of length at least five or by $\overline{C_7}$.*

Proof: Corollary 3.1 implies that items (i) and (ii) are equivalent. By Theorem 3.2, item (ii) implies item (iii).

(iii) $\Rightarrow$ (ii) Let U be an induced subgraph of $K(G)$. Since G is hereditary clique-Helly, there exists a clique subgraph H of G such that $K(H) = U$ [5]. As G is c–clique-perfect $\alpha_c(H) = \tau_c(H)$ and since H is clique-Helly, by Theorem 2.6, $\alpha(U) = \theta(U)$.

Let us see that items (ii) and (iv) are equivalent:

(ii) $\Rightarrow$ (iv) Let us suppose that H is a clique subgraph of G generated by an odd cycle of length at least five or by $\overline{C_7}$. Then H is a clique-Helly graph generated by a self-clique clique-Helly graph. By Lemma 3.2, $K(H)$ contains the generator of H as induced subgraph, and so $K(G)$. Therefore, $K(G)$ is not a perfect graph.

(iv) $\Rightarrow$ (ii) If $K(G)$ is not a perfect graph, by the Strong Perfect Graph Theorem (Theorem 2.2), $K(G)$ must contain an odd cycle of length at least five or its complement as induced subgraph. Since the complement of an odd cycle of length greater than 7 is not HCH, $K(G)$ must contain an induced subgraph U isomorphic to an odd cycle of length at least five or $\overline{C_7}$. Then U is self-clique and clique-Helly. Since G is HCH, there exists a clique subgraph H of G such that $K(H) = U$ [5]. Since H is clique-Helly, by Lemma 3.2 it is generated by U. □

It remains an open question whether the last theorem holds for (general) clique-Helly graphs.

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