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Partial characterizations of coordinated graphs: line graphs and complements of forests

Flavia Bonomo[1,4,⋆], **Guillermo Durán**[2,3,4,⋆⋆], **Francisco Soulignac**[1,∗], **Gabriel Sueiro**[1,∗]

[1] Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina.
e-mail: {fbonomo,fsoulign,gsueiro}@dc.uba.ar.

[2] Departamento de Ingeniería Industrial, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Chile.
e-mail: gduran@dii.uchile.cl.

[3] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina.

[4] CONICET, Argentina.

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Abstract A graph G is *coordinated* if the minimum number of colors that can be assigned to the cliques of H in such a way that no two cliques with non-empty intersection receive the same color is equal to the maximum number of cliques of H with a common vertex, for every induced subgraph H of G. Coordinated graphs are a subclass of perfect graphs. The list of minimal forbidden induced subgraphs for the class of coordinated graphs is not known. In this paper, we present a partial result in this direction, that is, we characterize coordinated graphs by minimal forbidden induced subgraphs when the graph is either a line graph, or the complement of a forest.

Key words complements of forests – coordinated graphs – line graphs – perfect graphs

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Correspondence to: Flavia Bonomo. Depto. de Computación, FCEN, UBA. Ciudad Universitaria, pab. I (1428), Buenos Aires, Argentina.

1 Introduction

Let G be a simple finite undirected graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$ the complement of G. A graph with only one vertex is known as a *trivial* graph. Given two graphs G and G' we say that G *contains* G' if G' is isomorphic to an induced subgraph of G. When we need to refer to the non-induced subgraph containment relation, we will say so explicitly.

A class of graphs $\mathcal{C}$ is *hereditary* if for every $G \in \mathcal{C}$, every induced subgraph of G also belongs to $\mathcal{C}$.

The *neighborhood* of a vertex v is the set $N(v)$ consisting of all the vertices which are adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A vertex v of G is *universal* if $N[v] = V(G)$. Two vertices v and w are *true twins* in G if $N[v] = N[w]$, and *false twins* if they are true twins in $\overline{G}$.

A *complete set* or just a *complete* of a graph is a subset of vertices pairwise adjacent. Denote by K_j the complete with j vertices. A *clique* is a complete set not properly contained in any other. We may also use the term clique to refer to the corresponding complete subgraph. Given a graph G and a vertex v in $V(G)$, we denote by $m(v)$ the number of cliques to which v belongs.

A *stable set* in a graph G is a subset of pairwise non-adjacent vertices of G. A graph is *bipartite* if its vertex set can be partitioned into two stable sets.

Let X and Y be two sets of vertices of G. We say that X is *complete to* Y if every vertex in X is adjacent to every vertex in Y, and that X is *anticomplete to* Y if no vertex of X is adjacent to a vertex of Y.

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being adjacent if and only if the corresponding sets intersect.

The *line graph* $L(G)$ of G is the intersection graph of the edges of G. A graph H is a *line graph* if there exists a graph G such that $L(G) = H$.

A *star* is a graph with one vertex adjacent to every other vertex of the graph and no other adjacencies. The universal vertex is called the *center* of the star.

A *diamond* is the graph isomorphic to $K_4 \setminus \{e\}$, where e is an edge of K_4.

A *tree* is a connected graph with no cycles. A *forest* is a graph with no cycles.

A *hole* is a chordless cycle of length at least 4. An *antihole* is the complement of a hole. A hole or antihole is said to be *odd* if it consists of an odd number of vertices. A hole of length n is denoted by C_n. Denote by P_k the induced path of k vertices.

The *chromatic number* of a graph G is the smallest number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and it is denoted by $\chi(G)$. An obvious lower

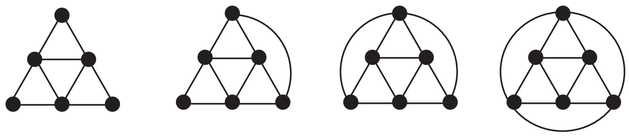

Fig. 1 Forbidden induced subgraphs for the family of HCH graphs

bound is the maximum cardinality of the cliques of G, the *clique number* of G, denoted by $\omega(G)$.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. It has been proved recently that a graph G is perfect if and only if no induced subgraph of G is an odd hole or an odd antihole [4], and that perfect graphs can be recognized in polynomial time [3]. Complete graphs, bipartite graphs, line graphs of bipartite graphs and their complements are perfect [6].

A family of sets S is said to satisfy the *Helly property* if every subfamily of it, consisting of pairwise intersecting sets, has a common element.

A graph G is *clique-Helly* (CH) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* (HCH) if H is clique-Helly for every induced subgraph H of G. A graph G is HCH if and only if G does not contain any of the graphs in Figure 1 as an induced subgraph [12]. These graphs are called 0-pyramid, 1-pyramid, 2-pyramid and 3-pyramid, respectively. A *trinity* is the complement of a 0-pyramid, and it is the only graph whose line graph is a 0-pyramid, as well.

The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. A graph G is *K-perfect* if $K(G)$ is perfect.

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_C(G)$ and $\alpha_C(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. A graph G is *clique-perfect* if $\tau_C(H) = \alpha_C(H)$ for every induced subgraph H of G. Clique-perfect graphs have been implicitly studied in several works but the terminology "clique-perfect" has been introduced in [7].

A *K-coloring* of a graph G is an assignment of colors to the cliques of G in such a way that no two cliques with non-empty intersection receive the same color (equivalently, a K-coloring of G is a coloring of $K(G)$). A *Helly K-complete* of a graph G is a collection of cliques of G with common intersection. The *K-chromatic number* and *Helly K-clique number* of G, denoted by $F(G)$ and $M(G)$, are the sizes of a minimum K-coloring and a maximum Helly K-complete of G, respectively. It is easy to see from the definition that $F(G) = \chi(K(G))$ and that $M(G) = \max_{v \in V(G)} m(v)$. Also, $F(G) \geq M(G)$ for any graph G. A graph G is *coordinated* if $F(H) = M(H)$ for every induced subgraph H of G. Coordinated graphs were defined and studied in [2], where it was proved that they are a subclass of perfect graphs. Bipartite graphs are clique-perfect and coordinated graphs [8,9].

The coordinated graph recognition problem is NP-hard and it is NP-complete even restricted to some subclasses of graphs with $M = 3$ [14]. On the other hand, finding the complete list of minimal forbidden induced subgraphs for coordinated graphs turns out to be a difficult task [13]. In this paper, we present partial results in this direction, that is, we characterize coordinated graphs by minimal forbidden induced subgraphs when the graph is either a line graph, or the complement of a forest. As corollaries of these characterizations and structural properties proved in this work, we can deduce linear time algorithms to recognize coordinated graphs when the graph belongs to these classes.

2 Partial characterizations

2.1 Line graphs

In this section we give a characterization by forbidden induced subgraphs for coordinated graphs within the class of line graphs, that is, we describe all the minimal non-coordinated graphs which are line graphs. Besides, we characterize the structure of those graphs whose line graph is coordinated. This last characterization leads to a linear time recognition algorithm for coordinated graphs when the input graph is a line graph.

This study relies on the known structural properties of line graphs [5, 10, 15] and clique-perfect line graphs [1].

First we need some preliminary results. The family of clique-Helly graphs are of particular interest in the study of clique-perfect and coordinated graphs. The following results show some properties of clique-Helly graphs and their relationships with the operator K and parameters M and F.

Lemma 1 [1] *Let $\mathcal{G}$ be an hereditary class of K-perfect clique-Helly graphs. Then every graph in $\mathcal{G}$ is clique-perfect.*

Using the following lemma, we can prove a similar result for coordinated graphs.

Lemma 2 [2] *Let G be a graph. Then:*

(i) $F(G) = \chi(K(G))$.
(ii) $M(G) \leq \omega(K(G))$.
(iii) *If G is clique-Helly then* $M(G) = \omega(K(G))$.

Lemma 3 *Let $\mathcal{G}$ be an hereditary class of K-perfect clique-Helly graphs. Then every graph in $\mathcal{G}$ is coordinated.*

The following theorem was proved in [1] in order to characterize clique-perfect line graphs.

Theorem 1 [1] *If H is a line graph which contains neither an odd hole nor an induced 0-pyramid, then $K(H)$ is perfect.*

Joining this result with Lemma 3, a characterization (by forbidden induced subgraphs) for coordinated hereditary clique-Helly line graphs is easily obtained. In this section we extend this result avoiding the hypothesis of hereditary clique-Helly. That is, we will prove that given a line graph H, H contains neither an odd hole nor an induced 0-pyramid if and only if H is coordinated.

The idea of the proof is to analyze the non-Helly subfamilies of the cliques of $H = L(G)$ that induce cliques in $K(H)$. We divide the non-Helly subfamilies into two types and study them in Lemma 7 and Lemma 8.

Towards the end of the section, we rewrite this characterization in terms of the inverse image of a line graph (Theorem 5). This new characterization leads to linear time recognition algorithms for recognizing a coordinated line graph and for determining $M(H)$ when $H = L(G)$ is coordinated (Theorem 6).

Let G be a graph and $E \subseteq E(G)$ a set of edges. Let W be the set of vertices that are incident to some edge of E. We define $G[E]$ as the graph with vertex set W and edge set E. Let $\mathcal{E}$ be a family of sets of edges. We define $G[\mathcal{E}] = G[\bigcup_{E \in \mathcal{E}} E]$. Note that given a set of edges E of a graph G, $L(G[E])$ is the subgraph of $L(G)$ induced by the set of vertices corresponding to E in $L(G)$.

Let T be a triangle, $T = \{v_1, v_2, v_3\}$. We denote by E_T the set of edges $\{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$. If $\mathcal{T}$ is a family of triangles, we define $E_{\mathcal{T}} = \bigcup_{T \in \mathcal{T}} E_T$.

Two edges of a graph are said to be *adjacent* if they are incident to a common vertex. Note that if G is a graph then a clique of $L(G)$ corresponds to a maximal set of pairwise adjacent edges of G.

The following lemmas are easy results about completes and cliques of line graphs.

Lemma 4 *Let G be a graph and $E \subseteq E(G)$. Then $L(G[E])$ is a complete if and only if $G[E]$ is a star or a triangle.*

Lemma 5 *Let G be a graph and let Q_1, Q_2 be two different cliques of $L(G)$. Then $|Q_1 \cap Q_2| = 2$ if and only if $G[Q'_1]$ is a triangle and $G[Q'_2]$ is a maximal star with center in some vertex of $G[Q'_1]$ (or viceversa), where Q'_1 and Q'_2 are the set of edges of G corresponding to the cliques Q_1 and Q_2 of $L(G)$, respectively.*

Lemma 6 *Let G be a graph and let Q_1, Q_2 be two different cliques of $L(G)$. Then $|Q_1 \cap Q_2| < 3$.*

Let G be a graph and let $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ a family of cliques of $L(G)$ pairwise intersecting. We say that $\mathcal{Q}$ is maximal if there is no other clique of $L(G)$ that intersects all cliques of $\mathcal{Q}$. In other words, $\mathcal{Q}$ is maximal if the vertices of $K(L(G))$ corresponding to the cliques of $\mathcal{Q}$ induce a clique in $K(L(G))$. Besides, we say that $\mathcal{Q}$ is type 2 if there exist i and j, $1 \leq i < j \leq k$, such that $|Q_i \cap Q_j| = 2$; and $\mathcal{Q}$ is type 1 if for every i, j with

$1 \leq i < j \leq k$, $|Q_i \cap Q_j| = 1$. Note that, by Lemma 6, every set of pairwise intersecting cliques of a line graph is either type 1 or type 2.

The following two lemmas analyze the type 1 and type 2 families of cliques that do not verify the Helly property.

Lemma 7 *Let G be a graph such that $L(G)$ contains no odd hole; $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ a maximal family of cliques of $L(G)$; $Q'_1, \ldots, Q'_k$ the sets of edges of G corresponding to the cliques $Q_1, \ldots, Q_k$ of $L(G)$, respectively; and $\mathcal{Q}' = \{Q'_1, \ldots, Q'_k\}$. Then $\mathcal{Q}$ is type 2 and does not verify the Helly property if and only if $k = 4$ and one of the following statements holds:*

(i) *For some i, $G[Q'_i]$ is a triangle and for each l, $1 \leq l \leq 4$ with $l \neq i$, $G[Q'_l]$ is a maximal star with center in a vertex of $G[Q'_i]$,*
(ii) *$G[\mathcal{Q}']$ contains a K_4 where for some i, $G[Q'_i]$ is a star with center in a vertex $v \in K_4$ and for every l, $1 \leq l \leq 4$ with $l \neq i$, $G[Q'_l]$ is a triangle of K_4 that contains v.*

Proof $\Leftarrow$) Trivial.

$\Rightarrow$) Since $\mathcal{Q}$ is type 2, then are two distinct cliques of $\mathcal{Q}$, suppose Q_1 and Q_2, such that $|Q_1 \cap Q_2| = 2$. By Lemma 5, without loss of generality we can suppose that $G[Q'_1]$ is a triangle $T = \{v_1, v_2, v_3\}$ and $G[Q'_2]$ is a maximal star with center in some vertex of T, suppose v_1.

Since $\mathcal{Q}$ does not verify the Helly property, there is a clique of $\mathcal{Q}$, suppose Q_3, such that $(v_1, v_3) \notin Q'_3$.

Suppose $(v_1, v_2) \notin Q'_3$. Since the cliques of $\mathcal{Q}$ are pairwise intersecting, Q'_3 has a common edge with Q'_1 and another with Q'_2. Then, since $(v_1, v_3) \notin Q'_3$ and $(v_1, v_2) \notin Q'_3$, it follows that $(v_2, v_3) \in Q'_3$. But then, since $(v_2, v_3) \notin Q'_2$, Q'_3 must have an edge e that is incident with v_1 (because all edges of Q'_2 are incident with v_1) and with v_2 or v_3 (because all edges of Q'_3 are adjacent). Then, $(v_1, v_2) \in Q'_3$ or $(v_1, v_3) \in Q'_3$, which is a contradiction. Therefore, $(v_1, v_2) \in Q'_3$.

Using a similar argument, there is another clique of $\mathcal{Q}$, suppose Q_4, such that $(v_1, v_2) \notin Q'_4$ and $(v_1, v_3) \in Q'_4$.

Now, let us prove that $\mathcal{Q} = \{Q_1, \ldots, Q_4\}$. By Lemma 6, $|Q_1 \cap Q_3| \leq 2$ and $|Q_1 \cap Q_4| \leq 2$. Therefore, there are four cases.

Case 1: $|Q_1 \cap Q_3| = 1$ and $|Q_1 \cap Q_4| = 1$. Then, by Lemma 5, $G[Q'_3]$ and $G[Q'_4]$ are not the maximal stars with center v_2 and v_3 respectively. Therefore, by Lemma 4 $G[Q'_3]$ is a triangle $T_2 = \{v_1, v_2, w\}$ and $G[Q'_4]$ is a triangle $T_3 = \{v_1, v_3, z\}$. If $w \neq z$, then $C = \{v_1, w, v_2, v_3, z\}$ is an odd cycle in G and the edges of C induce an odd hole in $L(G)$, which is a contradiction. Therefore $w = z$ and we name $v_4 = w = z$. Note that the family $\{Q'_1, Q'_2, Q'_3, Q'_4\}$ satisfies the sentence (ii) of the lemma with $i = 2$.

Let us see now that $k \leq 4$. Suppose there is a clique $Q_5 \in \mathcal{Q}$ with $Q_5 \neq Q_i$ for every i, $1 \leq i \leq 4$. By Lemma 4, $G[Q'_5]$ is a triangle or a maximal star. If $G[Q'_5]$ is the maximal star with center v_i $(2 \leq i \leq 4)$, suppose $i = 2$, then $Q_5 \cap Q_4 = \emptyset$, which is a contradiction. Therefore, $G[Q'_5]$ is a triangle.

Since $Q_5 \cap Q_2 \neq \emptyset$, then $v_1 \in V(G[Q'_5])$. Since $Q_5 \cap Q_1 \neq \emptyset$, then we can suppose without loss of generality that $v_2 \in V(G[Q'_5])$. Since $Q_5 \cap Q_4 \neq \emptyset$ and $v_3 \notin V(G[Q'_5])$, then $v_4 \in V(G[Q'_5])$. It follows that $V(G[Q'_5]) = \{v, v_2, v_4\}$, that is, $Q_5 = Q_3$, which is a contradiction. Therefore, $k = 4$.

Case 2: $|Q_1 \cap Q_3| = 2$ and $|Q_1 \cap Q_4| = 1$. By Lemma 5, $G[Q'_4]$ is a triangle $T_3 = \{v_1, v_3, z\}$ and $G[Q'_3]$ is the maximal star with center in v_2. But then $Q_4 \cap Q_3 = \emptyset$, which is a contradiction.

Case 3: $|Q_1 \cap Q_3| = 1$ and $|Q_1 \cap Q_4| = 2$. This case is analogous to Case 2.

Case 4: $|Q_1 \cap Q_3| = 2$ and $|Q_1 \cap Q_4| = 2$. Then, by Lemma 5, $G[Q'_3]$ is the maximal star with center v_2 and $G[Q'_4]$ is the maximal star with center v_3. Note that the family $\{Q'_1, Q'_2, Q'_3, Q'_4\}$ satisfies the sentence (i) of the lemma with $i = 1$.

Suppose that $k > 4$. Then, there is a clique $Q_5 \in \mathcal{Q}$ with $Q_5 \neq Q_i$ for every i, $1 \leq i \leq 4$. By Lemma 4, $G[Q'_5]$ is a maximal star or a triangle. If $G[Q'_5]$ is a triangle, then, since for every i, $2 \leq i \leq 4$, $G[Q'_i]$ is the maximal star with center v_{i-1} and $Q_5 \cap Q_i \neq \emptyset$. It follows that $G[Q'_5]$ is the triangle $\{v_1, v_2, v_3\}$, which contradicts that $Q_5 \neq Q_1$. Therefore, $G[Q'_5]$ is a maximal star. Since $Q_1 \cap Q_5 \neq \emptyset$, then $G[Q'_5]$ is the maximal star with center in some vertex of T. But then $Q_5 = Q_i$ for some i, $2 \leq i \leq 4$, which is a contradiction. □

Lemma 8 *Let G be a graph such that $L(G)$ contains no odd hole; $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ a maximal family of cliques of $L(G)$; $Q'_1, \ldots, Q'_k$ the sets of edges of G corresponding to the cliques $Q_1, \ldots, Q_k$ of $L(G)$, respectively; and $\mathcal{Q}' = \{Q'_1, \ldots, Q'_k\}$. Then $\mathcal{Q}$ is type 1 and does not verify the Helly property if and only if $k = 4$ and one of the following statements holds:*

(i) *For every i, $1 \leq i \leq 4$, $G[Q'_i]$ is the maximal star with center v_i and the set $\{v_1, v_2, v_3, v_4\}$ induces K_4.*
(ii) *$G[\mathcal{Q}'] = K_4$ and for every i, $1 \leq i \leq 4$, $G[Q'_i]$ is a triangle of $G[\mathcal{Q}']$.*

Proof ⇐) Trivial.

⇒) Since $\mathcal{Q}$ is type 1, then for every pair of different cliques $Q_a, Q_b \in \mathcal{Q}$, $|Q_a \cap Q_b| = 1$.

Let $Q_1 \cap Q_2 = \{e_1\}$. Since $\mathcal{Q}$ does not verify the Helly property, there is a clique $Q_3 \in \mathcal{Q}$ such that $e_1 \notin Q_3$. Let $Q_1 \cap Q_3 = \{e_2\}$ and $Q_2 \cap Q_3 = \{e_3\}$.

Clearly, the set of vertices $\{e_1, e_2, e_3\}$ induces a complete in $L(G)$. Therefore there is a clique Q of $L(G)$ such that $\{e_1, e_2, e_3\} \subseteq Q$ (and let Q' be the corresponding set of edges of G).

Since for every i, $1 \leq i \leq 3$, $|Q \cap Q_i| = 2$, then by Lemma 5 either $G[Q']$ is a triangle or every $G[Q'_i]$ is a triangle of G. In both cases, since $\mathcal{Q}$ is type 1, then $Q \notin \mathcal{Q}$.

Case 1: $G[Q']$ is a triangle $T = \{v_1, v_2, v_3\}$ and every $G[Q'_i]$ is the maximal star with center v_i $(1 \leq i \leq 3)$.

Since $\mathcal{Q}$ is maximal, there must be a clique $Q_4 \in \mathcal{Q}$ such that $Q_4 \cap Q = \emptyset$, otherwise Q could be added to $\mathcal{Q}$. Since $|Q_4 \cap Q_1| = 1$, then by Lemma 5,

$G[Q'_4]$ is not a triangle and by Lemma 4, $G[Q'_4]$ is a maximal star with center $w \in V(G)$. Clearly, since for every i, $1 \leq i \leq 3$, $Q_4 \neq Q_i$, then $w \neq v_i$. On the other hand, Q'_4 has an edge from each Q'_i (with $1 \leq i \leq 3$) and since $G[Q'_i]$ and $G[Q'_4]$ are stars, it follows that (v_i, w) is an edge of G. Note that in this case the family $\{Q'_1, Q'_2, Q'_3, Q'_4\}$ satisfies the statement (i) of the lemma.

Now, suppose that $k > 4$, that is, there is a clique Q_5 of $L(G)$ such that Q'_5 has an edge of each Q'_i for every i, $1 \leq i \leq 4$. Since $|Q_5 \cap Q_1| = 1$, by Lemma 5, $G[Q'_5]$ is not a triangle and by Lemma 4, $G[Q'_5]$ is a maximal star with center $z \in V(G)$. Clearly, since $Q_1, \ldots, Q_5$ are all different cliques then $z \neq v_1, v_2, v_3, w$.

Since the cliques of $\mathcal{Q}$ are pairwise intersecting, then Q_5 intersects Q_1, Q_2, Q_3, Q_4. Then, since every $G[Q'_i]$ $(1 \leq i \leq 5)$ is a maximal star, it follows that all the edges between their centers are edges of G. Therefore w, z, v_2, v_3, v_1 induce a complete in G that contains as a subgraph a cycle C of length 5. Then, the edges of C induce an odd hole in $L(G)$, which is a contradiction. We conclude that $k = 4$.

Case 2: $G[Q']$ is a maximal star with center v. Then, by Lemma 5, for every i, $1 \leq i \leq 3$, $G[Q'_i]$ is a triangle in G that contains v and two edges of Q'. Since the sets of edges of these triangles are pairwise intersecting, it follows that $G[(Q'_1 \cup Q'_2 \cup Q'_3)]$ is a complete of four vertices $\{v, v_1, v_2, v_3\}$, where we can assume that $(v, v_i), (v, v_{i+1}) \in Q'_i$ for each i, $1 \leq i \leq 3$ (the index sums must be understood modulo 3). That is, $V(G[Q'_i]) = \{v_i, v_{i+1}, v\}$.

Since $\mathcal{Q}$ is maximal, there must be a clique $Q_4 \in \mathcal{Q}$ such that $Q_4 \cap Q = \emptyset$, otherwise Q could be added to $\mathcal{Q}$. Since $|Q_4 \cap Q_1| = 1$, then by Lemma 4 and Lemma 5, $G[Q'_4]$ is a triangle. On the other hand, since $Q \cap Q_4 = \emptyset$, then no edge incident with v belongs to Q'_4. It follows that $V(G[Q'_4]) = \{v_1, v_2, v_3\}$. Note that the family $\{Q'_1, Q'_2, Q'_3, Q'_4\}$ satisfies the statement (ii) of the lemma.

Suppose now that $k > 4$, that is, there is a clique $Q_5 \in \mathcal{Q}$, with $Q_5 \neq Q_i$, for every i, $1 \leq i \leq 4$. Since $Q_5 \cap Q_4 \neq \emptyset$ and $Q_5 \neq Q_4$, without loss of generality we can assume that $(v_1, v_2) \in Q'_5$ and $(v_2, v_3) \notin Q'_5$. Since $Q_5 \cap Q_2 \neq \emptyset$ then $(v, v_2) \in Q'_5$. The only other edge that can belong to Q'_5 so that $Q_5 \cap Q_3 \neq \emptyset$ is (v, v_1). This is a contradiction because Q_5 and Q_1 are different cliques of $L(G)$. $\square$

Now we can prove the main result of this section.

Theorem 2 *Let H be a line graph, then H is coordinated if and only if H contains neither an odd hole nor a 0-pyramid as induced subgraph.*

Proof $\Rightarrow$) Coordinated graphs contain no odd hole because they are perfect. It is easy to see that the 0-pyramid is not coordinated.

$\Leftarrow$) Since the property of being a line graph with neither an odd hole nor a 0-pyramid is hereditary, then it suffices to prove that H verifies $F(H) = M(H)$. By Theorem 1, H is K-perfect and so $\omega(K(H)) = \chi(K(H))$. By

Lemma 2, $F(H) = \chi(K(H))$, therefore it is enough to show that $\omega(K(H)) = M(H)$.

By Lemma 2, $\omega(K(H)) \geq M(H)$. Suppose that $\omega(K(H)) > M(H)$. Let $\mathcal{Q} = \{Q_1, \ldots, Q_{\omega(K(H))}\}$ a pairwise intersecting set of cliques of H whose corresponding vertices induce a maximum clique of $K(H)$. Since $\omega(K(H)) > M(H)$, then $\mathcal{Q}$ does not verify the Helly property. By Lemma 6, $\mathcal{Q}$ is type 1 or type 2. By Lemma 7 and Lemma 8, $\omega(K(H)) = 4$. Therefore, $M(H) < 4$.

Let G be a graph such that $L(G) = H$; $Q'_1, \ldots, Q'_4$ the sets of edges of G corresponding to the cliques $Q_1, \ldots, Q_4$ of H, respectively; and $\mathcal{Q}' = \{Q'_1, \ldots, Q'_4\}$. Suppose that some subgraph (not necessarily induced) J of $G[\mathcal{Q}']$ is isomorphic to a diamond with vertices v_1, v_2, v_3, v_4, where v_3 is not adjacent to v_4. Let R'_i be the set of edges of the maximal star of J with center v_i for $i \in \{1, 2\}$. Let R'_i be the set of edges of the triangle of J containing v_i, for $i \in \{3, 4\}$. Let $R_1, \ldots, R_4$ be cliques of $L(J)$ corresponding to $R'_1, \ldots, R'_4$, respectively. Let e be the vertex of H corresponding to the edge $e' = (v_1, v_2)$ of G. Since $e' \in \bigcap_{i=1}^{4} R'_i$, then $m_H(e) \geq m_{L(J)}(e) \geq 4$, which is a contradiction. Therefore, no subgraph of $G[\mathcal{Q}']$ is isomorphic to a diamond. Then, by Lemma 8, $\mathcal{Q}$ is not type 1 and we conclude that $\mathcal{Q}$ is type 2.

By Lemma 7 and since the case (ii) of that lemma cannot hold by the arguments above, it follows that for some i, $1 \leq i \leq 4$, $G[Q'_i]$ is a triangle $T = \{v_1, v_2, v_3\}$ and for every l, $1 \leq l \leq 4$ with $l \neq i$, $G[Q'_l]$ is a maximal star with center in some vertex of T. We can assume that $i = 1$.

Since no subgraph of $G[\mathcal{Q}']$ is isomorphic to a diamond, then for every i, j, $2 \leq i < j \leq 4$ and for every pair of edges $e \in Q'_i \setminus E_T$ and $e' \in Q'_j \setminus E_T$ it holds that e and e' are not adjacent. For every i, $2 \leq i \leq 4$, since Q_i is maximal, there is an edge $e_i \in Q'_i \setminus E_T$. If follows that G contains a trinity, so H contains an induced 0-pyramid, which is a contradiction. Therefore, $\omega(K(H)) = M(H)$. □

The following result is a corollary of Theorem 2 and the fact that perfect graphs can be recognized in polynomial time (it is easy to see that odd antiholes are not line graphs).

Theorem 3 *The problem of determining if a line graph is coordinated can be solved in polynomial time.*

In the rest of this section we characterize coordinated line graphs in terms of graph G instead of $H = L(G)$. This characterization leads to linear time algorithms for the coordinated line graph recognition problem and the problem of determining $M(H)$ for a coordinated line graph $H = L(G)$. Define a graph G as *line-coordinated* if $L(G)$ is coordinated.

Theorem 4 [11,15] *Let G be a graph. Then the following statements are equivalent:*

1. $H = L(G)$ *is perfect.*
2. *No subgraph of G is an odd cycle of length at least* 5.

3. *Every subgraph G' of G satisfies at least one of the following conditions:*
   (i) *G' is bipartite.*
   (ii) *$G' = K_4$.*
   (iii) *$V(G') = \{a, b\} \cup X$ where $X = N(a) \cup N(b)$ is a non-empty stable set.*
   (iv) *G' has a cut-vertex.*

Given a graph G and $S \subseteq V(G)$, we denote by $E_G(S)$ the set of edges of G between vertices of S, that is, $E_G(S) = E(G) \cap (S \times S)$. We say that $S \subseteq V(G)$ is an *edge separator* of G if every path in G between two vertices of S uses only edges in $E_G(S)$.

Let $T = \{v_1, v_2, v_3\}$ be a triangle of a graph G. We say that v_1, v_2, v_3 is a *well ordered permutation* of T, or just *well ordered*, if $d(v_1) \leq d(v_2) \leq d(v_3)$ and $d(v_1) < d(v_2)$, or $N[v_3]$ is either equal to both $N[v_1]$ and $N[v_2]$ or to none of $N[v_1]$ and $N[v_2]$. We will denote by $\langle v_1, v_2, v_3 \rangle$ a triangle with v_1, v_2, v_3 being well ordered.

Lemma 9 *Every triangle admits a well ordered permutation.*

Proof Let $T = \{v_1, v_2, v_3\}$ be a triangle of a graph G, $d(v_1) \leq d(v_2) \leq d(v_3)$. Suppose that $d(v_1) = d(v_2)$. If $N[v_3] = N[v_1]$ but $N[v_3] \neq N[v_2]$, then v_1, v_3, v_2 is a well order for T. Otherwise, if $N[v_3] = N[v_2]$ but $N[v_3] \neq N[v_1]$, then v_2, v_3, v_1 is a well order for T. □

Now, we can give a characterization of line-coordinated graphs.

Theorem 5 *Let G be a graph and $\mathcal{T}$ be the set of triangles of G. Then the following statements are equivalent:*

1. *G is line-coordinated.*
2. *$G \setminus E_{\mathcal{T}}$ is bipartite and every triangle $\langle v_1, v_2, v_3 \rangle \in \mathcal{T}$ satisfies one of the following statements:*
   (i) *$d(v_1) = 2$ and $N[v_2] \cap N[v_3]$ is an edge separator of G.*
   (ii) *$d(v_1) = 3$, v_1 and v_2 are true twins and $N[v_1]$ is an edge separator of G.*

Proof 1. ⇒ 2.) Since coordinated graphs are perfect, if G is line-coordinated then it is line-perfect. Therefore, by Theorem 2 and Theorem 4, no subgraph of G is an odd cycle of length at least 5, so $G \setminus E_{\mathcal{T}}$ is bipartite.

Let $T = \langle v_1, v_2, v_3 \rangle$ be a triangle of G. Suppose that v_1 and v_2 have different neighbors w_1 and w_2 respectively, where $w_1, w_2 \notin T$. If v_3 has a neighbor $w_3 \notin T$ where $w_3 \neq w_1$ and $w_3 \neq w_2$, then $T \cup \{w_1, w_2, w_3\}$ is a trinity, which is a contradiction by Theorem 2. Consequently, since v_1, v_2, v_3 is well ordered then v_3 is adjacent to both w_1, w_2; thus v_1, w_1, v_3, w_2, v_2 is an odd cycle, which also contradicts Theorem 2. Therefore, v_1 and v_2 do not have different neighbors out of T and so we need to consider only two cases:
Case A: $d(v_1) = 2$.
Case B: $d(v_1) = 3$ and v_1 and v_2 are true twins.

Case A: Let $S = N[v_2] \cap N[v_3]$. Suppose S is not an edge separator, that is, there exists a path P between two vertices of S having one edge in $E(G) \setminus E_G(S)$. Denote by $E(P)$ the edges of P. Clearly $G' = G[E_G(S) \cup E(P)]$ is neither bipartite nor isomorphic to K_4, and it does not have any cut-vertex and does not satisfy the condition (iii) of Theorem 4. Therefore, by Theorem 4, some subgraph of G is an odd cycle of length at least 5 which again contradicts Theorem 2. We conclude then that S is an edge separator and the condition (i) is satisfied.

Case B: Let $S = N[v_1]$ and u be the neighbor of v_1 which is neither v_2 nor v_3. Suppose that S is not an edge separator, that is, there is a path P between two vertices of S having one edge in $E(G) \setminus E_G(S)$. Denote by $E(P)$ the edges of P. Clearly $G' = G[E_G(S) \cup E(P)]$ is neither bipartite nor isomorphic to K_4 and it does not have any cut-vertex. Since v_1 and v_2 are true twins, then $\{v_1, v_2\}$ is anticomplete to $G \setminus S$. So, P contains a subpath between u and v_3. Therefore, G' does not satisfy the condition (iii) of Theorem 4 and it follows that some subgraph of G is an odd cycle of length at least 5, which is a contradiction by Theorem 2. We conclude that S is an edge separator and the condition (ii) is satisfied.

2. $\Rightarrow$ 1.) Suppose that $L(G)$ is not coordinated, then, by Theorem 2 and Theorem 4, some subgraph (not necessarily induced) of G is isomorphic to an odd cycle of length at least 5 or a trinity.

Case A: Some subgraph C of G is isomorphic to an odd cycle of length at least 5. Since $G \setminus E_{\mathcal{T}}$ is bipartite then some edge e of C belongs to a triangle $T = \langle v_1, v_2, v_3 \rangle$. If $d(v_1) = 2$, let $S = N[v_2] \cap N[v_3]$; if $d(v_1) = 3$, let $S = N[v_1]$. By hypothesis, in both cases, S is an edge separator and clearly $e \in S$. Therefore, every path between the endpoints of e contains only edges of $E_G(S)$, which implies that $V(C) \subseteq S$. Thus, if $d(v_1) = 3$ then C has at most four vertices (recall that $S = N[v_1]$), a contradiction. Hence $d(v_1) = 2$; v_2 and v_3 belong to C; and $S = N[v_2] \cap N[v_3]$. Since $V(C) \subseteq S$ and $|V(C)| \geq 5$, then $d(v_3) \geq d(v_2) > 3$ and there exists a vertex w in C (different from v_1) which is adjacent to v_2 and v_3, and $d(w) \geq 3$. Let T_1 be a new triangle formed by vertices w, v_2 and v_3. Let us see that T_1 does not verify conditions (i) or (ii), a contradiction. If $d(w) \geq 4$, there is no vertex in T_1 with degree ≤ 3, and if $d(w) = 3$, then w is not adjacent to v_1 and hence w is true twin neither of v_2 nor of v_3.

Case B: Some subgraph of G is isomorphic to a trinity. So G contains a triangle $\langle v_1, v_2, v_3 \rangle$ and three different vertices u_1, u_2, u_3 with v_i adjacent to u_i, for $i \in \{1, 2, 3\}$. Therefore, $d(v_1) \geq 4$ or v_1 and v_2 are not true twins, which is a contradiction. $\square$

We now briefly discuss the recognition algorithm and its complexity, but the missing details are left to the reader. First of all, we need to find the set of well ordered triangles $\mathcal{T}$ of G. In a BFS tree with a root of maximum degree, every triangle with a two degree vertex contains a leaf. Similarly, every triangle with two true twins of degree three contains two brothers. So, $\mathcal{T}$ can be found in $O(|V(G)| + |E(G)|)$ time.

Now let $T = \langle v_1, v_2, v_3 \rangle$ be a well ordered triangle. If $d(v_1) = 2$, define $u_T = v_2$ and $v_T = v_3$. If $d(v_1) = 3$ and v_1, v_2 are true twins, then define u_T, v_T as the two vertices of $N(v_1) \cap N(v_2)$. Finally, define $C(G)$ as the graph having both vertices u_T, v_T for each triangle of $\mathcal{T}$ and one vertex c_A for each connected component A of $G \setminus E(\mathcal{T})$, where u_T, v_T are adjacent and u_T (v_T) is adjacent to c_A if and only if u_T (v_T) belongs to A.

Clearly, the graph $C(G)$ is obtained from G in $O(|V(G)|+|E(G)|)$ time. By Theorem 5, testing if G is line-coordinated is equivalent to test if $G \setminus E(\mathcal{T})$ is bipartite and $C(G)$ is a forest. Therefore, determining whether G is line-coordinated takes $O(|V(G)| + |E(G)|)$ time. Finally, given a graph H, in $O(V(H))$ time we can find a graph G such that $L(G) = H$ or say that H is not a line graph [10]. Since $O(|V(G)| + |E(G)|) = O(|V(H)|)$, determining whether H is a coordinated line graph is solvable in $O(|V(H)|)$ time.

Finally, given a coordinated line graph, we can compute the parameters M and F in linear time.

Theorem 6 *Given a line graph H, the problem of determining $M(H)$ can be solved in $O(|V(H)|)$ time when H is coordinated.*

Proof A graph G such that $L(G) = H$ can be found in $O(|V(H)|)$ time [10]. We will prove that given G we can calculate $M(H)$ in $O(|E(G)|)$ time. Let $e' = (v, w)$ an edge of G and let $t(e')$ be the number of triangles to which e' belongs. By Lemma 4 any clique of H containing the vertex e (corresponding to the edge e' of G) is generated by a triangle of G containing the edge e' or a star of G with center v or w. Therefore, $m_H(e)$ can be calculated in the following way:

- If $t(e') = 0$ then $m_H(e) = 1$ when $d_G(v) = 1$ or $d_G(w) = 1$; $m_H(e) = 2$ otherwise.
- If $t(e') = 1$ then $m_H(e) = 1$ when $d_G(v) = 2$ and $d_G(w) = 2$; $m_H(e) = 3$ when $d_G(v) > 2$ and $d_G(w) > 2$; $m_H(e) = 2$ otherwise.
- If $t(e') \geq 2$ then $m_H(e) = t(e') + 2$.

Since a line coordinated graph G has a linear number of triangles which can be found in linear time, then $M(H)$ can be calculated in $O(V(H))$ time. □

Corollary 1 *Given a line graph H, the problem of determining $F(H)$ can be solved in $O(|V(H)|)$ time when H is coordinated.*

2.2 Complements of forests

In this section we give a characterization by forbidden induced subgraphs for coordinated graphs within the class of complements of forests.

Let $2P_4$ be the graph obtained from the union of two disjoint induced paths of four vertices. We define R to be the graph obtained by adding to $2P_4$ an edge joining the second vertex of each path. Both $\overline{2P_4}$ and $\overline{R}$ are not coordinated, since $M(\overline{2P_4}) = M(\overline{R}) = 6$ and $F(\overline{2P_4}) = F(\overline{R}) = 7$. We will show that if G is the complement of a forest, then G is coordinated if and only if G contains neither $\overline{2P_4}$ nor $\overline{R}$ as induced subgraph.

The idea of the proof is to show that if G is the complement of a forest containing neither $\overline{2P_4}$ nor $\overline{R}$ as induced subgraph then G is K-perfect and $\omega(K(G)) = M(G)$. The K-perfection is consequence of the fact that $K(G)$ is the complement of a bipartite graph (Lemma 11). To show that $\omega(K(G)) = M(G)$ we prove that $K(G)$ admits a bipartition such that one of the sets of the partition (V_2) is a maximum clique of $K(G)$ and the family of cliques corresponding to the vertices of V_2 verifies the Helly property (Lemma 16).

Lemma 10 *Let G be a non-trivial graph and $v \in V(G)$ with $d(v) = |V(G)| - 2$. Then $K(G)$ is the complement of a bipartite graph with bipartition V_1, V_2 and each one of V_1' and V_2' (the families of cliques in G corresponding to the sets of vertices V_1 and V_2 in $K(G)$) verifies the Helly property.*

Proof Let w be the only vertex not adjacent to v. Let V_1' be the family of cliques of G containing v and V_2' the family of cliques of G containing w. Clearly V_1, V_2 (their corresponding sets of vertices in $K(G)$) define a bipartition of $\overline{K(G)}$ and both V_1' and V_2' verify the Helly property. □

We denote by $S(G)$ the graph $K(\overline{G})$. In other words, $S(G)$ is the intersection graph of the maximal stable sets of G.

The following lemma is a direct consequence of Lemma 10.

Lemma 11 *Let F be a forest. Then $S(F)$ is either trivial or the complement of a bipartite graph.*

We say that a bipartition V_1, V_2 of $V(G)$ is *cumulative* (on V_2) if V_2 is complete and there exist orderings $v_1, \dots, v_{|V_1|}$ of V_1 and $w_1, \dots, w_{|V_2|}$ of V_2 such that, for each $i = 1, \dots, |V_1|$, w_i is not adjacent to v_i but w_i is adjacent to v_j for every j with $i < j \leq |V_1|$.

Lemma 12 *Let G be a graph with a bipartition V_1, V_2 of $V(G)$ cumulative on V_2. Then V_2 is a maximum clique of G.*

Proof Let $v_1, \dots, v_{|V_1|}$ and $w_1, \dots, w_{|V_2|}$ be the orderings of V_1 and V_2 in the cumulative bipartition. Let K be a maximum clique of G, with $|V_1 \cap K|$ minimum.

Suppose that $|V_1 \cap K| > 0$. Let $v_{i_1}, \dots, v_{i_n}$ be the vertices of $K \cap V_1$, with $i_1 < \dots < i_n$. Since $i_1 < i_j$, for every j, $1 < j \leq n$, then w_{i_1} is adjacent to every vertex of $K \cap V_1$, except v_{i_1}. Besides, since V_2 is complete, w_{i_1} is adjacent to every vertex of $K \cap V_2$. It follows that w_{i_1} is adjacent to every vertex of $K \setminus \{v_{i_1}\}$. Then, $K' = (K \setminus \{v_{i_1}\}) \cup \{w_{i_1}\}$ is another maximum clique, $|V_1 \cap K'| < |V_1 \cap K|$, contradicting the fact that $|V_1 \cap K|$ is minimum. □

Given a tree N and a vertex v of N, a *subtree* is any connected component of $N \setminus \{v\}$. Note that a subtree is also a tree.

We say that a tree N is a *c-tree* if there is a distinguished vertex v in N such that:

- Every vertex of $N \setminus \{v\}$ has degree at most 2 in N.
- The diameter of N is at most 7.
- The distance from every vertex of N to v is less than 5.
- N has no false twins.

Lemma 13 *Let N be a tree. Then N is a c-tree if and only if N has no false twins and contains neither $2P_4$ nor R as induced subgraph.*

Proof $\Rightarrow$) Suppose that N contains an induced $2P_4$. Let v be the distinguished vertex of N from the c-tree definition. If v belongs to one of the two P_4 then, since there is no vertex of degree greater than 2 (except v), the distance from one of the ends of the other P_4 to v is at least 5. If v does not belong to any of the two P_4 then, since the interior vertices of the paths have degree 2, for each one of the paths the distance from v to one of the ends is at least 4, hence the diameter of N is at least 8. In any case, N is not a c-tree.

If N contains R as induced subgraph, N has at least two vertices of degree at least 3, contradicting that N is a c-tree.

Finally, since N is a c-tree, N has no false twins.

$\Leftarrow$) Case 1: every vertex of N has degree at most 2, so N is isomorphic to a path $p_1, \ldots, p_k$ for some k. If $k > 8$ then N contains an induced $2P_4$. If not, N is a c-tree with $v = p_{\lceil k/2 \rceil}$.

Case 2: there is a vertex in $V(N)$ of degree at least 3. Let v be that vertex. Suppose there is a vertex d_1 ($d_1 \neq v$) which has degree at least 3. Let t be the neighbor of v in the same subtree of $N \setminus \{v\}$ than d_1 (perhaps $t = d_1$), and let d_2 and d_3 be two different neighbors of d_1 ($d_2 \neq v$, $d_3 \neq v$). Since d_2 and d_3 are not false twins, one of them (say d_2) has a new neighbor d_4. Since N has no false twins, then at most one neighbor of v is a leaf. Then, let s_1 be a non-leaf neighbor of v ($s_1 \neq t$), and call s_2 the neighbor of s_1 different to v. Since $d(v) \geq 3$, v has a neighbor s_3 different from t and s_1. It follows that s_3, v, s_1, s_2 and d_4, d_2, d_1, d_3 induce two disjoint P_4, with the only possible edge between them being (v, d_1) (which occurs when d_1 is equal to t). But then, N contains an induced $2P_4$ if d_1 is different to t, or N contains an induced R if d_1 is equal to t; a contradiction. Therefore, every vertex of N has degree at most 2 (except v).

It remains to see that there is no vertex at distance 5 from v and that the diameter of N is at most 7.

If there is a vertex at distance 5 from v, the subtree of $N \setminus \{v\}$ containing that vertex contains an induced P_4 anticomplete to v. In a similar way as done above, choose s_1, s_2 and s_3 such that s_3, v, s_1, s_2 induce a P_4 anticomplete to the other P_4, contradicting that N contains no induced $2P_4$. If follows that N has no vertex at distance 5 from v.

Now suppose that there are two different vertices at distance 4 from v, then those two vertices are in different subtrees of $N \setminus \{v\}$ (recall that N has no false twins). Each one of these subtrees contains P_4, and there are no edges between both P_4, contradicting that N does not contain an induced $2P_4$. So, there is at most one vertex at distance 4 from v, which implies that the diameter of N is at most 7. □

We say that a forest F is a *c-forest* if every connected component is a c-tree and at most one connected component has more than 2 vertices.

Lemma 14 *Let F be a forest. Then, F is a c-forest if and only if F contains neither $2P_4$ nor R as induced subgraph and none of its connected components has false twins.*

Proof ⇒) Trivial, by Lemma 13 and the fact that at most one connected component of F has more than 2 vertices.

⇐) The proof is by induction on the number of trees of F. The base case (only one tree) is given by Lemma 13.

Let F be a forest with $k+1$ trees ($k \geq 1$) and let N be a tree with $|V(N)|$ minimum. By inductive hypothesis, $F \setminus N$ is a c-forest.

If $|V(N)| \leq 2$, then F is a c-forest by definition.
If $|V(N)| = 3$, then N (which is a connected component of F) has false twins, a contradiction.
If $|N| \geq 4$, since N has no false twins, then N contains a P_4. But, since N is a tree of F with $|V(N)|$ minimum, there is another tree of F which contains another P_4, and there are no edges between both P_4, again a contradiction.
□

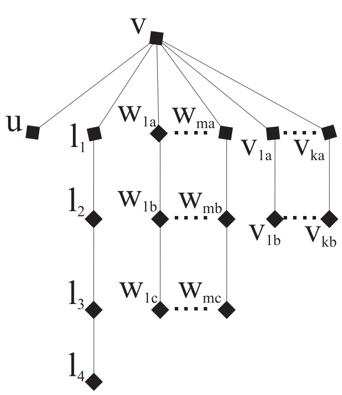

Fig. 2 General structure of a c-tree.

Lemma 15 *Let N be a non trivial c-tree. Then there is a bipartition V_1, V_2 of $V(S(N))$, cumulative on V_2, such that V_2' (the family of maximal stable sets in N corresponding to the set of vertices V_2 in $S(N)$) verifies the Helly property.*

Proof Let v be the distinguished vertex of N. Since N is a c-tree, there is no vertex (except v) with degree at least 3. Therefore every subtree of $N \setminus \{v\}$ is a path. There is at most one subtree L containing a vertex at distance 4 from v ($V(L) = \{l_1, l_2, l_3, l_4\}$). There is at most one subtree u being a single vertex. The remaining subtrees are induced paths of 2 or 3 vertices. Let $H_1, \ldots, H_k$ be the subtrees of $N \setminus \{v\}$ isomorphic to P_2, and denote $V(H_i) = \{v_{ia}, v_{ib}\}$, where v_{ia} is adjacent to v and to v_{ib}. Let $W_1, \ldots, W_m$ be the subtrees of $N \setminus \{v\}$ isomorphic to P_3, and denote $V(W_i) = \{w_{ia}, w_{ib}, w_{ic}\}$ where w_{ia} is adjacent to v and to w_{ib}, and w_{ib} is adjacent to w_{ic} (see Figure 2).

Divide the maximal stable sets of N into two classes: those that contain v (type I), and those that do not contain v (type II). Let I_j be the set $\{1, \ldots, j\}$ for every non-negative integer j (note that I_0 is the empty set). Depending on the existence of L or u in N, we have four cases.

Case 1: L and u do not exist.

In this case, the type I and type II sets are:

- Type I: $\{v\} \cup \{v_{ib}\}_{i \in I_k} \cup \{w_{ib}\}_{i \in C} \cup \{w_{ic}\}_{i \in (I_m \setminus C)}$ with $C \subseteq I_m$.
- Type II: $\{v_{ia}\}_{i \in A} \cup \{v_{ib}\}_{i \in (I_k \setminus A)} \cup \{w_{ia}\}_{i \in (I_m \setminus B)} \cup \{w_{ib}\}_{i \in B} \cup \{w_{ic}\}_{i \in (I_m \setminus B)}$ with $A \subseteq I_k$, $B \subseteq I_m$, excluding the case in which A and $I_m \setminus B$ are simultaneously empty.

Clearly, a type I set is determined by the set C and a type II set is determined by the sets A and B.

Let X be a type I set, and let $C \subseteq I_m$ the set that determines X. Let Y be a type II set, and let $A \subseteq I_k$ and $B \subseteq I_m$ the sets that determine Y. If $A \neq I_k$, then X and Y intersect in some v_{ib}. If $C \cap B \neq \emptyset$, then X and Y intersect in some w_{ib}. If $(I_m \setminus C) \cap (I_m \setminus B) \neq \emptyset$, then X and Y intersect in some w_{ic}. Therefore, X and Y do not intersect if and only if $A = I_k$ and $B = I_m \setminus C$. It follows that every type I set intersects all but at most one type II sets and viceversa.

Let Y_1, Y_2 be two type II sets, and let $A_1, A_2 \subseteq I_k$ and $B_1, B_2 \subseteq I_m$ the sets that determine Y_1 and Y_2, respectively. If $A_1 \cap A_2 \neq \emptyset$, then Y_1 and Y_2 intersect in some v_{ia}. If $B_1 \cap B_2 \neq \emptyset$, then Y_1 and Y_2 intersect in some w_{ib}. If $(I_k \setminus A_1) \cap (I_k \setminus A_2) \neq \emptyset$, then Y_1 and Y_2 intersect in some v_{ib}. If $(I_m \setminus B_1) \cap (I_m \setminus B_2) \neq \emptyset$, then Y_1 and Y_2 intersect in some w_{ia}. Therefore, Y_1 and Y_2 do not intersect if and only if $A_1 = I_k \setminus A_2$ and $B_1 = I_m \setminus B_2$. It follows that every type II set intersects all but at most one type II sets.

We will prove now that there is a bipartition V_1, V_2 of $V(S(N))$, cumulative on V_2, and such that V_2' (the family of maximal stable sets in N corresponding to the set of vertices V_2 in $S(N)$) verifies the Helly property. To this end we will consider two cases: $k \geq 1$ and $k = 0$.

Case 1A: If $k \geq 1$ (v_{1a} and v_{1b} exist), let $V_1' = \{$maximal stable sets of N containing $v_{1a}\}$ and $V_2' = \{$maximal stable sets of N containing $v_{1b}\}$. Since v_{1b} is a leaf, by Lemma 10, V_1, V_2 (the corresponding sets of vertices in $S(N)$) define a bipartition of $V(S(N))$, and V_2' verifies the Helly property. Let us see that for each maximal stable set $X \in V_1'$, there is a maximal stable set $Y \in V_2'$ such that Y do not intersect X but intersects every other element of V_2'.

Note that, since every type I set is in V_2', the elements of V_1' are type II sets.

Let $X \in V_1'$, X is type II. Let $A \subseteq I_k$ and $B \subseteq I_m$ the sets which determine X. If $A = I_k$, let $C = I_m \setminus B$, and let Y be the type I set determined by C. Clearly, X and Y do not intersect. Then, since Y intersects all but at most one type II sets, Y intersects every element of V_1', except X. If $A \neq I_k$, let $A' = I_k \setminus A$ and $B' = I_m \setminus B$. Let Y be the type II set determined by A' and B'. Again, X and Y do not intersect. Then, since every type II set intersects all but at most one type II sets, Y intersects every element of V_1', except X.

Case 1B: If $k = 0$, let $V_1' = \{\text{type II sets}\}$ and $V_2' = \{\text{type I sets}\}$. As $k = 0$, $m \geq 1$ (because N is non trivial) and so, V_1' is not empty. Since all type I sets have v as a common vertex, then V_2' verifies the Helly property and V_2 (the corresponding set of vertices) induces a complete in $S(N)$.

Let $X \in V_1'$ be a type II set, and let B be the set that determines X (in this case, since $I_k = \emptyset$, the set A is not needed). Let $C = I_m \setminus B$, and let Y be the type I set determined by C. Clearly, X and Y do not intersect. Since Y is a type I set, Y intersects all but at most one type II sets. Therefore, Y intersects every set of V_1', except X.

It follows that, in both cases, any ordering of the vertices of V_1 defines an ordering of the vertices of V_2 which leads to a cumulative partition.

Case 2: L does not exist, but u exists.

The type I sets are the same that in Case 1. The type II sets are obtained by adding u to every type II set of Case 1, without excluding in this case the set for which $A = \emptyset$ and $I_m \setminus B = \emptyset$.

Let $V_1' = \{\text{type I sets}\}$ and $V_2' = \{\text{type II sets}\}$. Since u is a leaf, by Lemma 10, V_1, V_2 (the corresponding sets of vertices) define a bipartition of $V(S(N))$, V_2' verifies the Helly property and V_2 induces a complete in $S(N)$.

Let X be a the type I set determined by a set C; and let Y be a type II set determined by sets A and B. Like in the Case 1, X intersects Y unless $A = I_k$ and $B = I_m \setminus C$. Note that, unlike the Case 1, in this case for each set X of type I there is always a set Y of type II such that X does not intersect Y. Then we conclude that for every type I set there is only one set of type II which it does not intersect; and for every type II set there is at most one type I set which it does not intersect.

It follows that any ordering of the vertices of V_1 defines an ordering of the vertices of V_2, which leads to a cumulative partition.

Case 3: L exists, but u does not exist.

For $R \subseteq I_4$, define $L_R = \cup_{i \in R}\{l_i\}$. In this case, the type I and type II sets are:

- Type I: $\{v\} \cup \{v_{ib}\}_{i \in I_k} \cup \{w_{ib}\}_{i \in C} \cup \{w_{ic}\}_{i \in (I_m \setminus C)} \cup L_D$ with $C \subseteq I_m, D \in \{\{2,4\},\{3\}\}$

– Type II: $\{v_{ia}\}_{i\in A}\cup\{v_{ib}\}_{i\in(I_k\setminus A)}\cup\{w_{ia}\}_{i\in(I_m\setminus B)}\cup\{w_{ib}\}_{i\in B}\cup\{w_{ic}\}_{i\in(I_m\setminus B)}\cup L_E$, with $A\subseteq I_k$, $B\subseteq I_m, E\in\{\{1,3\},\{2,4\},\{1,4\}\}$, excluding the case where $E=\{2,4\}$, $A=\emptyset$ and $I_m\setminus B=\emptyset$, which does not determine a maximal stable set.

The type I sets are determined by sets C and D; and type II sets, by sets A, B and E.

Let $V_1'=\{$maximal stable sets containing $l_3\}$ and $V_2'=\{$maximal stable sets containing $l_4\}$. Since l_4 is a leaf, by Lemma 10, V_1, V_2 (the corresponding sets of vertices) define a bipartition of $V(S(N))$, V_2' verifies the Helly property and V_2 induces a complete in $S(N)$.

Let Z be the type II set determined by $A=I_k$, $B=I_m$ and $E=\{1,3\}$. Since $l_3\in Z$, then $Z\in V_1'$.

We order the vertices of V_1 in the following way. First, we list all the vertices corresponding to the type I sets of V_1', and then we list all the vertices corresponding to the type II sets of V_1', with the only condition that z (the vertex corresponding to Z) is the last one.

Let $X\in V_1'$ be a type I set. Let C and D be the sets that determine X. Since $l_3\in X$, then $D=\{3\}$. Let $A=I_k$, $B=I_m\setminus C$ and $E=\{1,4\}$, and let $Y\in V_2'$ the type II set determined by A, B and E. Clearly, Y does not intersect X. Let $X'\in V_1'$, $X'\neq X$. If X' is of type I, let C' and D' be the sets that determine X'. Again, $D'=\{3\}$. Then, since $X\neq X'$, it follows that $C'\neq C$, and therefore Y intersects X' in some w_{ib}. If X' is of type II, let A, B and E be the sets that determine X'. Since $l_3\in X'$, then $E=\{1,3\}$, therefore Y intersects X' in l_1. We conclude that Y intersects every element of V_1', except X.

Let $X\in V_1'$ be a type II set, $X\neq Z$. Let A, B and E be the sets that determine X. Since $l_3\in X$, then $E=\{1,3\}$. Let $A'=I_k\setminus A$, $B'=I_m\setminus B$ and $E'=\{2,4\}$, and let $Y\in V_2'$ be the type II set determined by A', B' and E'. Clearly, Y does not intersect X (note that if $X=Z$, then Y would not be a maximal stable set). Let $X'\in V_1'$ of type II, $X'\neq X$. Let A'', B'' and E'' be the sets that determine X'. Again, $E''=\{1,3\}$. Since $X'\neq X$, then $A''\neq A$ or $B''\neq B$. Therefore, Y intersects X' in some v_{ia} or in some w_{ib} (note that if X' is of type I, then Y does not necessarily intersect X'; this explains the ordering given to the vertices of V_1).

To complete this case, it remains to show that there is a set $Y\in V_2'$ such that Y does not intersect Z. Such Y is the type I set determined by $C=\emptyset$ and $D=\{2,4\}$.

It follows that the ordering given to the vertices of V_1 defines an ordering of the vertices of V_2, which leads to a cumulative partition.

Case 4: L and u exist.

The type I sets are the same as in Case 3. The type II sets are obtained by adding u to every type II set of Case 3, without excluding in this case the type II set for which $A=\emptyset$, $B=\emptyset$ and $E=\{2,4\}$.

Let $V_1'=\{$type I sets$\}$ and $V_2'=\{$type II sets$\}$. Since u is a leaf, by Lemma 10, V_1, V_2 (the corresponding sets of vertices) define a bipartition of $V(S(N))$, V_2' verifies the Helly property and V_2 induces a complete in $S(N)$.

Let $X \in V_1'$, X is of type I. Let C and D be the sets that determine X.

If $D = \{2,4\}$, let $A = I_k$, $B = I_m \setminus C$ and $E = \{1,3\}$, and let Y be the type II set determined by A, B and E. Clearly, X and Y do not intersect. Let $X' \in V_1'$, $X' \neq X$. Then X' is of type I. Let C' and D' be the sets that determine X'. If $D' = \{3\}$, then Y intersects X' in l_3. If $D' = \{2,4\}$, since $X \neq X'$, then $C \neq C'$ and therefore Y intersects X' in some w_{ib}. It follows that Y intersects every set of V_1', except X.

If $D = \{3\}$, Y is defined in a similar way, but with $E = \{2,4\}$. Again, Y intersects every set of V_1', except X.

It follows that any ordering of the vertices of V_1 defines an ordering of the vertices of V_2, which leads to a cumulative partition. □

Lemma 16 *Let F be a c-forest with no isolated vertices. Then there is a bipartition W_1, W_2 of $V(S(F))$, cumulative on W_2, such that W_2' (the family of maximal stable sets in F corresponding to the set of vertices W_2 in $S(F)$) verifies the Helly property.*

Proof The proof is by induction on the number of trees of F. The base case is the Lemma 15.

Let F be a c-forest with $k+1$ c-trees ($k \geq 1$). Let N be a c-tree of F with $|V(N)|$ minimum. By definition of c-forest, $|V(N)| \leq 2$. But, since F has no isolated vertices, $|V(N)| = 2$. Denote $V(N) = \{a, b\}$.

By inductive hypothesis, there is a bipartition V_1, V_2 of $V(S(F \setminus N))$ cumulative on V_2, with V_2' (the family of maximal stable sets in $F \setminus N$ corresponding to the set of vertices V_2 in $S(F \setminus N)$) verifying the Helly property. Let $X_1, \ldots, X_k$ be the ordering of the vertices of V_1, and $Y_1, \ldots, Y_j$ ($j \geq k$) the ordering of the vertices of V_2, given by the definition of cumulative partition. Denote by $X_1', \ldots, X_k'$ and $Y_1', \ldots, Y_j'$, the corresponding maximal stable sets in $F \setminus N$.

Let Z' be a maximal stable set of $F \setminus N$. Then $Z'^a = Z' \cup \{a\}$ and $Z'^b = Z' \cup \{b\}$ are maximal stable sets of F (Z^a and Z^b are the corresponding vertices of $S(F)$). Note that every maximal stable set of F can be built as a maximal stable set of $F \setminus N$ plus the vertex a or the vertex b.

Let $W_1 = \{X^a\}_{X \in V_1} \cup \{X^b\}_{X \in V_1}$ and $W_2 = \{Y^a\}_{Y \in V_2} \cup \{Y^b\}_{Y \in V_2}$ a bipartition of $S(F)$, and W_1' and W_2' the corresponding families of maximal stable sets in F. Consider the following ordering of W_1: $X_1^a, X_1^b, \ldots, X_k^a, X_k^b$, and the following ordering of W_2: $Y_1^a, Y_1^b, \ldots, Y_k^a, Y_k^b, \ldots, Y_j^a, Y_j^b$.

Since V_2 is complete and V_2' verifies the Helly property, there is a vertex h in $F \setminus N$ such that $h \in \bigcap_{Y' \in V_2'} Y'$. But then, by the definition of W_2, it follows that $h \in \bigcap_{Y' \in W_2'} Y'$. Therefore W_2 is complete and W_2' verifies the Helly property.

Finally, using that V_1, V_2 is a cumulative bipartition of $V(F \setminus N)$, it is easy to check that the orderings of W_1 and W_2 defined above lead this bipartition of $S(F)$ to be cumulative. □

Lemma 17 *Let F be a c-forest. Then $\omega(S(F)) = M(\overline{F})$.*

Proof If F has an isolated vertex, then every maximal stable set of F contains that vertex. Therefore $S(F)$ is a complete, meaning that $\omega(S(F)) = M(\overline{F})$.

If F has no isolated vertices, by Lemma 16 there is a partition V_1, V_2 of $V(S(F))$ cumulative on V_2, with V_2' (the corresponding family of maximal stable sets of F) verifying the Helly property. By Lemma 12, V_2 is a maximum clique of $S(F)$. But, since V_2' verifies the Helly property, then $M(\overline{F}) = \omega(S(F))$. □

Now, we can give a characterization of coordinated complements of forests.

Theorem 7 *Let G be a forest. Then the following statements are equivalent:*

(i) *$\overline{G}$ is coordinated.*
(ii) *G contains neither $2P_4$ nor R as induced subgraph.*
(iii) *The forest G' obtained by identifying the false twins of G is a c-forest.*

Proof (i) ⇒ (ii)) As it was mentioned at the beginning of this subsection, $\overline{2P_4}$ and $\overline{R}$ are not coordinated.

(ii) ⇒ (iii)) Let G' be the result of identifying all the false twins of G. Since G contains neither $2P_4$ nor R as induced subgraph, then G' contains neither $2P_4$ nor R. Then, by Lemma 14, G' is a c-forest.

(iii) ⇒ (i)) Since the class of graphs that are complements of forests is hereditary, it suffices to prove that $F(\overline{G}) = M(\overline{G})$.

By Lemma 11, $S(G)$ is either trivial or the complement of a bipartite graph. In both cases, $S(G)$ is perfect. But, since $S(G) = K(\overline{G})$, then $\overline{G}$ is K-perfect hence $\omega(K(\overline{G})) = \chi(K(\overline{G}))$. By definition $\chi(K(\overline{G})) = F(\overline{G})$. Then it is enough to show that $\omega(K(\overline{G})) = M(\overline{G})$, that is, $\omega(S(G)) = M(\overline{G})$.

Since G' is a c-forest, then by Lemma 17, $\omega(S(G')) = M(\overline{G'})$. Since G' is the result of identifying all the false twins of G, it is easy to see that $S(G')$ is isomorphic to $S(G)$ and $M(\overline{G'}) = M(\overline{G})$. If follows that $\omega(S(G)) = M(\overline{G})$. □

Corollary 2 *Let G be the complement of a forest. Then G is coordinated if and only if G contains neither $\overline{2P_4}$ nor $\overline{R}$ as induced subgraph.*

Corollary 3 *The graphs $\overline{2P_4}$ and $\overline{R}$ are minimally not coordinated.*

Theorem 8 *The problem of determining if the complement of a forest is coordinated can be solved in linear time.*

Proof Let G be the complement of a forest. By Theorem 7 we only need to check whether the forest obtained by identifying the false twins of $\overline{G}$ is a c-forest or not. This can be easily done in linear time. □

3 Summary

These results allow us to formulate partial characterizations of coordinated graphs by minimal forbidden induced subgraphs, as it is shown in Table 1.

Graph classes	**Forbidden induced subgraphs**	**Recognition**	**Ref.**
Line graphs	0-pyramid, odd holes	linear	Thms 2,5
Complements of forests	$\overline{R}$, $\overline{2P_4}$	linear	Thm 7

Table 1 Minimal forbidden induced subgraphs for coordinated graphs in each studied class.

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