Partial characterizations by forbidden induced subgraphs
Chordal graphs

A graph is chordal when every cycle of length at least four has a chord. Chordal graphs have polynomial time recognition (Rose, Tarjan and Lueker, 1976).

Theorem (Lehel and Tuza, 1986)

Let $G$ be a chordal graph. Then the following are equivalent:

1. $G$ does not contain odd suns.
2. $G$ is balanced.
3. $G$ is clique-perfect.

The recognition of clique-perfect chordal graphs can be reduced to the recognition of balanced graphs, which is solvable in polynomial time.
Chordal graphs

Chordal graphs $\subseteq$ perfect graphs

Hereditary $K$-perfect

Balanced $=$ Clique-perfect

Flavia Bonomo

On clique-perfect graphs
Operations preserving clique-perfectness

- **Twin vertices:** Two vertices \( v \) and \( w \) are twins in \( G \) if \( N[v] = N[w] \), or, equivalently, if they belong to exactly the same cliques of \( G \).

**Example:**

- If \( v \) and \( w \) are twins in \( G \), then \( G \) is clique-perfect if and only if \( G - v \) is. Moreover, \( \alpha_c(G) = \alpha_c(G - v) \) and \( \tau_c(G) = \tau_c(G - v) \).
Operations preserving clique-perfectness

- **Disjoint union:** Let $G = (V, E)$ and $G' = (V', E')$ with $V \cap V' = \emptyset$. Then the disjoint union $G \cup G'$ is the graph with vertex set $V \cup V'$ and edge set $E \cup E'$.

**Example:**

- Under these conditions, $G \cup G'$ is clique-perfect if and only if $G$ and $G'$ are. Moreover, $\alpha_c(G \cup G') = \alpha_c(G) + \alpha_c(G')$ and $\tau_c(G \cup G') = \tau_c(G) + \tau_c(G')$.
- Every graph is the disjoint union of its connected components.
Operations preserving clique-perfectness

- **Join**: Let $G = (V, E)$ and $G' = (V', E')$ with $V \cap V' = \emptyset$. Then the join $G \vee G'$ is the graph with vertex set $V \cup V'$ and edge set $E \cup E' \cup V \times V'$, that is, $\overline{G \vee G'} = \overline{G} \cup \overline{G'}$.

**Example:**

- Under these conditions, $G \vee G'$ is clique-perfect if and only if $G$ and $G'$ are. Moreover, $\alpha_c(G \vee G') = \min\{\alpha_c(G), \alpha_c(G')\}$ and $\tau_c(G \vee G') = \min\{\tau_c(G), \tau_c(G')\}$.

- $G$ is anticonnected if $\overline{G}$ is connected. Otherwise, it is the join of its anticomponents (the subgraphs of $G$ induced by the vertices of the connected components of $\overline{G}$).
Cographs

A cograph is a $P_4$-free graph, that is, a graph with no four vertices inducing $P_4$.

Theorem (Corneil, Lerchs and Stewart Burlingham, 1981)

Let $G$ be a cograph. Then $G$ is either trivial (it has only one vertex) or the disjoint union or the join of smaller cographs.

So, by induction and based on the properties stated previously about disjoint union and join, it can be proved that all the cographs are clique-perfect.
**$P_4$-sparse graphs**

A spider is a graph whose vertex set can be partitioned into three sets $S$, $C$ and $R$, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; $s_i$ is adjacent to $c_j$ if and only if $i = j$ (a thin spider), or $s_i$ is adjacent to $c_j$ if and only if $i \neq j$ (a thick spider); $R$ can be empty but if not, then $R$ is complete to $C$ and anticomplete to $S$.

![Graphs with partitions S, C, R](image)

Note that if $G$ is a thin spider, then $\alpha_c(G) = \tau_c(G) = k$ ($C$ is a clique-transversal an the legs are a clique-independent set). Note also that a thick spider (that is not thin, so $k \geq 3$) contains an induced 3-sun.
**P₄-sparse graphs**

**Def:** Every set of five vertices contains at most one induced $P₄$.

**Theorem (Hoàng, 1985)**

Let $G$ be a $P₄$-sparse graph. Then $G$ satisfies one of this statements:

- $G$ is trivial
- $G$ is the disjoint union of smaller $P₄$-sparse graphs
- $G$ is the join of smaller $P₄$-sparse graphs
- $G$ is a spider $(S, C, R)$, and $R$ induces a $P₄$-sparse graph.

**Theorem**

The only minimally clique-imperfect $P₄$-sparse graph is the 3-sun. So, if $G$ is $P₄$-sparse, it is clique-perfect if and only if it does not contain 3-sun as an induced subgraph.
$P_4$-sparse graphs

$P_4$-sparse graphs $\subseteq$ perfect graphs
Paw-free graphs

A paw-free graph is a graph with no four vertices inducing a paw.

Looking at the complement of a paw, it can be easily proved that if a paw-free graph is not anticonnected, then its anticomponents are stable sets.

It is also not difficult to prove that if a paw-free graph is connected and anticonnected, it contains no triangles. So, if it contains no odd-holes, then it is bipartite.

Then, the only minimally clique-imperfect paw-free graphs are the odd holes.
Paw-free graphs

We have then the following characterization, since odd antiholes of length at least 7 have induced paws.

**Theorem**

Let $G$ be a paw-free graph. Then the following are equivalent:

1. $G$ does not contain odd holes.
2. $G$ is perfect.
3. $G$ is clique-perfect.
Paw-free graphs

For the K-perfectness, every bipartite graph is K-perfect, because if $G$ is bipartite, then $K(G) = L(G)$ and line graphs of bipartite graphs are perfect.

A non-anticonnected graph having an anticomponent of size less than three is also K-perfect:

- if $G$ has an anticomponent of size one $\{v\}$, then $v$ is a universal vertex, so $K(G)$ is complete.
- if $G$ has an anticomponent of size two $\{v, w\}$, then every clique of $G$ contains either $v$ or $w$. The cliques containing $v$ form a complete in $K(G)$, and so the cliques containing $w$. Therefore, $K(G)$ is the complement of a bipartite graph, thus perfect.
Paw-free graphs

It can be seen that the join of three stable sets of size three is not \( K \)-perfect. So, we obtain this characterization.

**Theorem**

Let \( G \) be a paw-free graph. Then \( G \) is hereditary \( K \)-perfect iff each connected component \( H \) satisfies one of these conditions:

1. \( H \) is bipartite.
2. \( H \) is not anticonnected and at most two anticomponents of \( H \) have more than two vertices.

**Theorem**

The minimally \( K \)-imperfect paw-free graphs are odd holes and \( 3\overline{K_3} \).
Paw-free graphs

- Paw-free graphs
  - Clique-perfect = perfect
  - Hereditary K-perfect
Diamond-free graphs

Let $G$ be a diamond-free graph. Then the following are equivalent:

1. $G$ contains no odd generalized sun.
2. $G$ is clique-perfect.
3. $G$ is hereditary $K$-perfect.

Diamond-free odd generalized suns are odd generalized suns without non-proper edges. In this case, the characterization is not formulated by minimal subgraphs yet.
Sketch of proof

**Theorem**

Let $G$ be a diamond-free graph. Then the following are equivalent:

1. $G$ contains no odd generalized sun.
2. $G$ is clique-perfect.
3. $G$ is hereditary K-perfect.

We prove $1 \Leftrightarrow 3$, and then the equivalence with 2 holds one way, because odd generalized suns are not clique perfect, and on the other way, by using that diamond-free graphs are hereditary clique-Helly.

Since $K(diamond-free) = diamond-free$, $K(G)$ cannot contain odd antiholes of length at least 7. Suppose $K(G)$ contains an odd hole. Consider the cliques $M_1, \ldots, M_{2k+1}$ of $G$ inducing that odd hole. Taking $v_i$ in $M_i \cap M_{i+1}$ we have an odd cycle in $G$. It is easy to prove that if there is a non-proper edge in that cycle, then there is a diamond. So, that cycle induces an odd generalized sun with no improper edges.
Theorem

Let $G$ be a $\{\text{gem}, W_4, \text{bull}\}$-free graph. Then the following are equivalent:

1. $G$ contains no odd holes.
2. $G$ is perfect.
3. $G$ is clique-perfect.
4. $G$ is hereditary $K$-perfect.

In this case the proof is also based on the $K$-perfectness, but the arguments are more involved.
Line graphs

Let $H$ be a graph. Its line graph $L(H)$ is the intersection graph of the edges of $H$. A graph $G$ is a line graph if there exists a graph $H$ such that $G = L(H)$. Line graphs have polynomial time recognition (Lehot, 1974).

**Theorem**

Let $G$ be a line graph. Then the following are equivalent:

1. $G$ contains no odd holes.
2. $G$ is perfect.
3. $G$ is hereditary K-perfect.

The proof is based on the structure of graphs whose line graph is perfect (several results by Trotter, de Werra and Maffray), and uses some of the operations preserving perfection or K-perfection.
For clique-perfection, we have this characterization.

**Theorem**

Let $G$ be a line graph. Then the following are equivalent:

1. no induced subgraph of $G$ is an odd hole, or a 3-sun.
2. $G$ is clique-perfect.

The proof is by induction, using as a base case when the graph is hereditary clique-Helly, because there we can use the K-perfection. Otherwise, we look how the other pyramids can appear and decompose the graph reducing the problem to smaller cases.
Claw-free hereditary clique-Helly graphs

Theorem (Chudnovsky and Seymour 2005)

Let $G$ be a claw-free graph. Then either $G \in S_0 \cup \cdots \cup S_6$, or $G$ admits twins, or a non-dominating W-join, or a coherent W-join, or a 0-join, or a 1-join, or a generalized 2-join, or a hex-join, or $G$ is antiprismatic.
Claw-free hereditary clique-Helly graphs

The characterization obtained for HCH claw-free graphs is the following:

**Theorem**

Let $G$ be a hereditary clique-Helly claw-free graph. Then the following are equivalent:

1. no induced subgraph of $G$ is an odd hole, or $\overline{C_7}$.
2. $G$ is clique-perfect.
3. $G$ is perfect.
4. $G$ is hereditary $K$-perfect.
Sketch of proof

Theorem

Let $G$ be a hereditary clique-Helly claw-free graph. Then the following are equivalent:

1. no induced subgraph of $G$ is an odd hole, or $\overline{C_7}$.
2. $G$ is clique-perfect.
3. $G$ is perfect.
4. $G$ is hereditary $K$-perfect.

The main part of the proof is $1 \iff 4$. The proof is by induction, based on the claw-free graphs decomposition theorem of Chudnovsky and Seymour. We prove it for the basic classes, and then we do induction using that if $G$ in non-basic then it admits a decomposition. For some of the decompositions (1-join, 2-join) the idea is that these decompositions lead to some decompositions of the clique graph preserving perfection. Some other cases are more complicated, and require “brute force” to find the way of using induction.
Helly circular-arc graphs

Recall that a graph $G$ is HCA if there exists a family of arcs of a circle verifying the Helly property and such that $G$ is the intersection graph of this family.

**Theorem**

Let $G$ be a Helly circular-arc graph. Then the following are equivalent:

1. $G$ does not contain any of the graphs in the figure, where the dotted lines replace an induced path of length at least one.
2. $G$ is clique-perfect.
Sketch of proof

Theorem

Let $G$ be a Helly circular-arc graph. Then the following are equivalent:

1. $G$ does not contain any of the graphs in the figure, where the dotted lines replace an induced path of length at least one.

2. $G$ is clique-perfect.

To prove $1 \Rightarrow 2$, we show that Helly circular-arc graphs which do not contain the graphs of the figure as induced subgraphs are $K$-perfect. This is the hardest part of the proof, and the idea is to “bring back” to $G$ the odd holes and odd antiholes of $K(G)$. The remaining part is based in the fact that Helly circular-arc graphs that are not HCH have $\alpha_c = \tau_c$ or they are clique-complete without a universal vertex, and then we use a characterization of clique-complete graphs by Szwarcfiter, Lucchesi and P. de Mello, 1998.
Recognition algorithm

**Input:** A HCA graph \( G \); **Output:** TRUE if \( G \) is clique-perfect and FALSE if \( G \) is not.

1. Check if \( G \) contains a 3-sun. Case yes, return FALSE.
2. Check for odd holes and \( C_7 \): check if \( G \) is perfect. Case not, return FALSE.
3. Check for vikings and 2-vikings:
   - For every 7-tuple...
   - If return FALSE
   - Else... is \( G' \) perfect?

4. Check for \( S^1_k \) and \( S^2_k \):
   - For every 8-tuple...
   - If return FALSE
   - Else... is \( G' \) perfect?

5. If no forbidden subgraph is found, return TRUE.
Partial characterizations by forbidden induced subgraphs

Summary

<table>
<thead>
<tr>
<th>Class</th>
<th>Forbidden induced subgraphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chordal</td>
<td>odd suns</td>
</tr>
<tr>
<td>$P_4$-sparse</td>
<td>3-sun</td>
</tr>
<tr>
<td>Paw-free</td>
<td>odd holes</td>
</tr>
<tr>
<td>Diamond-free</td>
<td>odd generalized suns</td>
</tr>
<tr>
<td>${\text{gem, } W_4, \text{bull}}$-free</td>
<td>odd holes</td>
</tr>
<tr>
<td>Line graphs</td>
<td>odd holes, 3-sun</td>
</tr>
<tr>
<td>HCH claw-free</td>
<td>odd holes, $\overline{C_7}$</td>
</tr>
<tr>
<td>HCA</td>
<td>3-sun, odd holes, $\overline{C_7}$,</td>
</tr>
<tr>
<td></td>
<td>vikings, 2-vikings, $S_k^1$, $S_k^2$</td>
</tr>
</tbody>
</table>

![Graphs of forbidden induced subgraphs](image)