Homomorphisms and composition operators on algebras of analytic functions of bounded type

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Abstract *Let U and V be convex and balanced open subsets of the Banach spaces X and Y respectively. In this paper we study the following question: Given two Fréchet algebras of holomorphic functions of bounded type on U and V respectively that are algebra-isomorphic, can we deduce that X and Y (or X^* and Y^*) are isomorphic? We prove that if X^* or Y^* has the approximation property and $H_{wu}(U)$ and $H_{wu}(V)$ are topologically algebra-isomorphic, then X^* and Y^* are isomorphic (the converse being true when U and V are the whole space). We get analogous results for $H_b(U)$ and $H_b(V)$, giving conditions under which an algebra-isomorphism between $H_b(X)$ and $H_b(Y)$ is equivalent to an isomorphism between X^* and Y^*. We also obtain characterizations of different algebra-homomorphisms as composition operators, study the structure of the spectrum of the algebras under consideration and show the existence of homomorphisms on $H_b(X)$ with pathological behaviors.*

Introduction

J.C. Díaz and S. Dineen in [12, p. 95] raised the following question: *If X and Y are complex Banach spaces and $X^* \cong Y^*$ (i.e., X^* and Y^* are topologically isomorphic) does this imply that the spaces of continuous m-homogenous polynomials $P(^mX)$ and $P(^mY)$ are also topologically isomorphic for all m?* They gave a partial positive answer. Several authors have recently obtained more partial positive answers to this question. S. Lassalle and I. Zalduendo in [21, Thorem 4] proved that the question has a positive answer if X and Y are symmetrically regular Banach spaces and $X^* \cong Y^*$. In [7, Theorem 1] an analogous result is proved when X is a regular Banach space and $X^* \cong Y^*$ (see below for the required definitions); moreover, it is also proved ([7, Corollary 2]) that, under the same hypothesis on X and Y, the spaces of holomorphic functions of bounded type $H_b(X)$ and $H_b(Y)$ are topologically isomorphic as Fréchet algebras. This kind of results have been recently extended for other Fréchet algebras of vector-valued holomorphic functions in [9]. In this article we study a kind of converse problem. Given two open sets $U \subset X$ and $V \subset Y$ and $\mathcal{F}(U)$ and $\mathcal{F}(V)$ two Fréchet algebras of holomorphic functions of bounded type on U and V respectively, the question that we face is the following: *If $\mathcal{F}(U)$ and $\mathcal{F}(V)$ are topologically isomorphic algebras, can we conclude that X and Y (or X^* and Y^*) are topologically isomorphic?* We are also interested in knowing if it is possible to establish a biholomorphic function between the open sets U and V, to be able to characterize when these topological algebra-isomorphisms are composition operators.

Cartan in the forties proved that given two complete Reinhardt domains U and V in $\mathbb{C}^n$ (i.e., two balanced and n-circled open sets U and V in $\mathbb{C}^n$) the spaces of holomorphic functions $H(U)$ and $H(V)$ are topologically algebra-isomorphic if and only if there exists $f : U \longrightarrow V$ a bijective biholomorphic function. But in 1960 Aizenberg and Mityagin proved in [6] that given any two bounded complete Reinhardt domains U and V, then $H(U)$ and $H(V)$ are topologically isomorphic. It is well known that the euclidean unit ball and the unit polydisc in $\mathbb{C}^n$ are two bounded complete Reinhardt domains that are not biholomorphically equivalent. This is the main reason to restrict ourselves to consider only topological algebra-isomorphisms.

In section one we solve completely our questions in the case that we consider the Fréchet algebras $H_{wu}(U)$ and $H_{wu}(V)$ for U and V convex and balanced open subsets of X and Y respectively, when either X^* or Y^* has the approximation property. As a particular case, we show that $H_{wu}(X)$ and $H_{wu}(Y)$ are topologically algebra-isomorphic if and only if X^* and Y^* are isomorphic Banach spaces. We derive our results from a more general solution to our problems: the case of $H_{w^*u}(U)$ and $H_{w^*u}(V)$ for U and V convex and balanced boundedly-regular open subsets of X^* and Y^*, when either X or Y has the approximation property. We obtain characterizations of the homomorphisms as composition operators.

In the second section we study the case of $H_b(U)$, obtaining positive answers, for example, when every polynomial on the bidual of one of the spaces involved is approximable. In this case, if $H_b(U)$ and $H_b(V)$ are topological algebra-isomorphic then X^* and Y^* are isomorphic Banach spaces, the converse being true when the domains are the whole spaces. We also show that without the hypothesis of approximability the situation is very complex, even if we deal with entire functions. Every homomorphism on an algebra induces a mapping on the spectrum of the algebra. We find that for a wide class of Banach spaces X there are homomorphisms on $H_b(X)$ whose induced mappings have pathological (and unexpected) behaviours. In both sections we study the structure of the spectra of the algebras under consideration.

Throughout the paper X and Y will be complex Banach spaces. For the definitions an basic properties of polynomials and holomorphic functions we refer to [15].

If A is a subset of a Banach space X, $\Gamma(A)$ will denote the smallest convex and balanced set in X that contains A. Let $U \subset X$ be open. We say that $B \subset U$ is a *U-bounded* set if it is bounded and $\text{dist}(B, X \setminus U) > 0$. We say that $\mathcal{B} = (B_n)_{n=1}^{\infty}$ is *a fundamental sequence of U-bounded sets* if it satisfies the two following conditions: (i) B_n is U-bounded for all n and given B a U-bounded set there exists n such that $B \subset B_n$. (ii) There exits a sequence of positive numbers $(r_n)_{n=1}^{\infty}$ such that $B_n + r_n B_X \subset B_{n+1}$ for all n. The fundamental family of U-bounded sets that we use throughout the paper is $(U_n)_{n=1}^{\infty}$ where $U_n = \{x \in U : \|x\| \leq n \text{ and } \text{dist}(x, X \setminus U) \geq \frac{1}{n}\}$, $n \in \mathbb{N}$ (see Remark 2 below). We will denote by $H_b(U)$ the space of holomorphic functions $f : U \to \mathbb{C}$ that are bounded on U-bounded sets, i.e., $\|f\|_B := \sup\{|f(x)| : x \in B\} < \infty$ for all U-bounded set B. $H_b(U)$ is a Fréchet algebra when endowed with the topology of the uniform convergence on U-bounded subsets of U. The sequence of seminorms $(\|f\|_{U_n})_{n=1}^{\infty}$ gives the Fréchet structure of $H_b(U)$. Given $\mathcal{F}$ a Fréchet algebra, its spectrum, that we denote by $M(\mathcal{F})$, is the set of all non null continuous linear and multiplicative mapping $\varphi : \mathcal{F} \longrightarrow \mathbb{C}$.

1 Homomorphisms in $H_{w^*u}(U)$ and $H_{wu}(U)$

Let $U \subset X^*$ be open. We will denote by $H_{w^*u}(U)$ the space of holomorphic functions $f : U \to \mathbb{C}$ that are uniformly $w(X^*, X)$-continuous on U-bounded sets. As $H_{w^*u}(U)$ is a closed subalgebra of $H_b(U)$, it is again a Fréchet algebra endowed with the topology of the the uniform convergence on U-bounded subsets of U. Let $M_{w^*u}(U)$ be the spectrum of $H_{w^*u}(U)$. For $x^* \in U$ we have $\delta_{x^*} \in M_{w^*u}(U)$, where $\delta_{x^*}(f) := f(x^*)$ for $f \in H_{w^*u}(U)$. Since X is contained in $H_{w^*u}(U)$, we can define a projection $\pi : M_{w^*}(U) \to X^*$ as $\pi(\varphi) = \varphi|_X$.

Proposition 1 *Let U be an open subset of X^*. We have*

$$\bigcup_n \overline{U_n}^{w^*} \subset \pi(M_{w^*u}(U)) \subset \bigcup_n \overline{\Gamma(U_n)}^{w^*}.$$

In particular, if U is a convex and balanced open set of X^*, then $\pi(M_{w^*u}(U)) = \bigcup_n \overline{U_n}^{w^*}$.

PROOF. Every $f \in H_{w^*u}(U)$ is uniformly weak-star continuous in each U_n, hence it extends uniquely to a weak-star continuous function $\tilde{f} : \overline{U_n}^{w^*} \longrightarrow \mathbb{C}$ with $\|\tilde{f}\|_{U_n} = \|f\|_{\overline{U_n}^{w^*}}$. Thus, for each $x^* \in \overline{U_n}^{w^*}$, the mapping $\delta_{x^*} : H_{w^*u}(U) \longrightarrow \mathbb{C}$ given by $\delta_{x^*}(f) = \tilde{f}(x^*)$ ($f \in H_{w^*u}(U)$) is a continuous homomorphism.

Therefore, $\bigcup_n \{\delta_{x^*} : x^* \in \overline{U_n}^{w^*}\} \subset M_{w^*u}(U)$ and, since $\pi(\delta_{x^*}) = x^*$, we have $\bigcup_n \overline{U_n}^{w^*} \subset \pi(M_{w^*u}(U))$. For the second inclusion, let $\varphi \in M_{w^*u}(U)$ and $x^* = \pi(\varphi)$. Since φ is continuous, there exists $n \in \mathbb{N}$ such that $|\varphi(f)| \leq \|f\|_{U_n}$ for all $f \in H_{w^*u}(U)$. In particular,

$$|x^*(x)| = |\delta_{x^*}(x)| \leq \sup_{y^* \in U_n} |y^*(x)|.$$

This means that $x^* \in \overline{\Gamma(U_n)}^{w^*}$. □

Remark 2 With the above notation, we set $\tilde{U} := \bigcup_n \overline{U}_n^{w^*}$, and $\delta : \tilde{U} \longrightarrow M_{w^*u}(U)$, $\delta(x^*) := \delta_{x^*}$. We have that $\delta(\tilde{U}) \subset M_{w^*u}(U)$. Note that $\tilde{U} := \cup\{\overline{B}^{w^*} : B \text{ is } U-\text{bounded}\}$. Also, $\tilde{U}$ is an open subset of X^* contained in $\mathring{\overline{U}}^{w^*}$, the $\|.\|^*$-interior of $\overline{U}^{w^*}$. Indeed, if $x^* \in \overline{U_n}^{w^*}$, there exists a net $(x^*_\alpha)_\alpha \subset U_n$ such that $x^*_\alpha \xrightarrow{w^*} x^*$. For $y^* \in \frac{1}{2n(n+1)} B_{X^*}$, we have that $(x^*_\alpha + y^*)_\alpha$ is contained in U_{n+1} and w^*-converges to $x^* + y^*$. Therefore, $x^* + \frac{1}{2n(n+1)} B_{X^*} \subset \overline{U_{n+1}}^{w^*}$ for all $x^* \in \overline{U_n}^{w^*}$ and we obtain

$$\overline{U_n}^{w^*} + \frac{1}{2n(n+1)} B_{X^*} \subset \overline{U_{n+1}}^{w^*} \subset \overline{U}^{w^*} \tag{1.1}$$

for all $n \in \mathbb{N}$.

In order to clarify the properties of the extension $\tilde{f}$ of any $f \in H_{w^*u}(U)$, we are going to introduce a new class of Fréchet algebras. Let U be an open subset of X^* and $\mathcal{B} = (B_n)_{n=1}^{\infty}$ a countable family of weak-star closed U-bounded sets satisfying $\cup_{n=1}^{\infty} B_n = U$ and such that for each n there is $\varepsilon_n > 0$ with $B_n + \varepsilon_n B_{X^*} \subset B_{n+1}$. We define the Frechét algebra $H_{\mathcal{B}w^*u}(U) := \{f \in H(U) : f|_{B_n}$ is weak-star uniformly continuous on B_n for all $n = 1, \ldots\}$, endowed with the family of seminorms $(\|.\|_{B_n})_{n=1}^{\infty}$. If $\mathcal{B}$ is a fundamental sequence of U-bounded sets then $H_{\mathcal{B}w^*u}(U) = H_{w^*u}(U)$ algebraically and topologically.

Proposition 3 *(i) Let U be a balanced open subset of X^* and $\mathcal{B} = (\overline{U_n}^{w^*})_{n=1}^{\infty}$. Every $f \in H_{w^*u}(U)$ extends uniquely to an $\tilde{f} \in H_{\mathcal{B}w^*u}(\tilde{U})$ and the mapping $i : H_{w^*u}(U) \longrightarrow H_{\mathcal{B}w^*u}(\tilde{U})$, $i(f) := \tilde{f}$ is an topological algebra-isomorphism.*

(ii) If U is a convex balanced open subset of X^* and X has the approximation property then $M_{w^*}(U) = \delta(\tilde{U})$.

PROOF. (i) Let U be a balanced open set. If $f \in H_{w^*u}(U)$ and $\sum_{m=1}^{\infty} P_m$ is the power series expansion of f at 0, then it converges to f in $H_{w^*u}(U)$ and, by [3, Lemma 2.1], $P_n \in H_{w^*u}(X)$ for all n. Since there exists $r_n > 1$ such that $r_n U_n \subset U_{n+1}$ for every n, we have

$$r_n^m \|P_m\|_{\overline{U_n}^{w^*}} = r_n^m \|P_m\|_{U_n} = \|P_m\|_{r_n U_n} \leq \|P_m\|_{U_{n+1}} \leq \|f\|_{U_{n+1}}, \tag{1.2}$$

for all n. Thus $\sum_{m=0}^{\infty} P_m$ converges absolutely and uniformly on $\overline{U_n}^{w^*}$ for all n and $\tilde{f} = \sum_{m=0}^{\infty} P_m \in H_{\mathcal{B}w^*u}(\tilde{U})$.

(ii) By Proposition 1, $\pi(M_{w^*u}(U)) = \tilde{U}$ and by part (i) (and its proof), $\tilde{f} \in H_{\mathcal{B}w^*u}(\tilde{U})$ and the polynomials in $H_{w^*u}(X^*)$ are dense in $H_{\mathcal{B}w^*u}(\tilde{U})$. Every polynomial of that class can be approximated uniformly on the bounded sets by weak-star finite type polynomials [5, Thm. 2]. This means that the algebra spanned by X is dense in $H_{\mathcal{B}w^*u}(\tilde{U})$ and therefore, $\varphi(f) = (i^{-1})^*(\varphi)(\tilde{f}) = \tilde{f}(\pi(\varphi)) = \delta_{\pi(\varphi)}(f)$ for all $f \in H_{w^*}(U)$ and all $\varphi \in M_{w^*u}(U)$. □

The above proposition is a generalization of [5, theorem 3], where the authors show that if X is a Banach space with the approximation property, then $M_{w^*}(X) = \{\delta_{x^*} : x^* \in X^*\}$. Next example shows that the result cannot be extended to an arbitrary open set, even in the finite dimensional case.

Example 4 Let $U = \{z \in \mathbb{C}^2 : \frac{1}{2} < \|x\|_\infty < 1\}$. We have $U = \tilde{U}$. But it is known [20, Remark and Example, p. 91] that every holomorphic function $f \in H(U)$ extends uniquely to a function $\tilde{f} \in H(W)$, where $W = \{z \in \mathbb{C}^2 : \|x\|_\infty < 1\}$ and moreover, the mapping $i : H(U) \longrightarrow H(\tilde{U})$ defined as $i(f) := \tilde{f}$ is an algebra topological isomorphism. Therefore, $M(H(U)) = M_{w^*u}(U) = M_{w^*u}(W)$. Since W is a convex and balanced open set, by above proposition we have $M(H(U)) = \{\delta_x : x \in \mathbb{C}^2 : \|x\|_\infty < 1\}$.

In the previous example, next remark and the examples following it we exhibit some characteristics of the relationship between U, $\tilde{U}$ and $M_{w^*u}(U)$. This characteristics relay not only in the *holomorphic* nature of the involved functions (as in the previous proposition) but also in topological and geometrical aspects of the open set U.

Remark 5 (i) *If U is the polar of a bounded open set in X, then $\tilde{U} = U$.* Indeed, U is the dual unit ball of an equivalent norm in X. In this case, we have that the w^*-closure of each U_n coincides with its norm closure and then $\tilde{U} = U$.

(ii) On the other hand it is natural to ask about the relationship between $\tilde{U}$ and $\overset{\circ}{\overline{U}^{w^*}}$, the $\|.\|^*$-interior of $\overline{U}^{w^*}$. *If U is a bounded, convex and balanced open set in X^*, then* $\tilde{U} = \overset{\circ}{\overline{U}^{w^*}}$. This is a consequence of the fact that we can take $U_n = \frac{n}{n+1}U$ $n \in \mathbb{N}$ and hence $\tilde{U} = \cup_{n=1}^{\infty} \frac{n}{n+1}\overline{U}^{w^*} = \overset{\circ}{\overline{U}^{w^*}}$. Moreover, the sequence $(\overline{U_n}^{w^*})_{n=1}^{\infty} = (\frac{n}{n+1}\overline{U}^{w^*})_{n=1}^{\infty}$ is a fundamental sequence of $\tilde{U}$-bounded sets. Another trivial case in which the equality in $\tilde{U} = \overset{\circ}{\overline{U}^{w^*}}$ holds is whenever $U = X^*$. But we do not know if $\tilde{U}$ coincides with $\overset{\circ}{\overline{U}^{w^*}}$ in general. This is equivalent to say that any $x^* \in \overset{\circ}{\overline{U}^{w^*}}$ is the w^*-limit of a U-bounded net in U. In [8] we have constructed, in any infinite dimensional X^*, a balanced open set U and an element $x^* \in \overline{U}^{w^*}$ which is not the w^*-limit of any bounded net in U. Also we give an example of an absolutely convex open set in ℓ_1 with the same property. In both examples, we follow some ideas from [10, Lemma 1].

Example 6 shows that the equality $\tilde{U} = U$ is not true in general, even if U is bounded and absolutely convex.

Example 6 Let $U = \{x \in \ell_1 : p(x) := \sum_{k=1}^{\infty} |x_k| + 2|\sum_{k=1}^{\infty} x_k| < \frac{5}{2}\}$. Since $p(e_1 - e_m + x) \leq 2 + 3\|x\|$ for all $x \in \ell_1$, we have that $e_1 - e_m + \frac{1}{6}B_{\ell_1} \subset U$ for all $m \in \mathbb{N}$. Hence, $\cup_{m=1}^{\infty} e_1 - e_m + \frac{1}{7}B_{\ell_1}$ is a U-bounded set. But $p(e_1 + x) \geq 3(1 - \|x\|)$ for all $x \in \ell_1$, and $e_1 - e_m + x$ converges to $e_1 + x$ in the weak-star topology. Thus, $e_1 + \frac{1}{7}B_{\ell_1} \subset \tilde{U} \setminus U$ and we conclude that $U \subsetneq \tilde{U}$.

Another natural question raised by Proposition 3 is the following: given U an open set in X^* and (U_n) a fundamental sequence of U-bounded sets, is $(\overline{U_n}^{w^*})$ a fundamental sequence of $\tilde{U}$-bounded sets? Whenever the answer is positive we call U a *boundedly-regular open set.* The whole dual space X^* and every convex, balanced and bounded open set are examples of boundedly-regular open sets (see Remark 5(ii)). By Proposition 3, if U a balanced boundedly-regular open set then the mapping $\Phi : H_{w^*u}(\tilde{U}) \longrightarrow H_{w^*u}(U)$ defined as $\Phi(f) := f|_U$ for all $f \in H_{w^*u}(\tilde{U})$ is a topological algebra-isomorphism. But, in general, neither all balanced open sets are boundedly-regular nor Φ is a topological algebra-isomorphism, as the next example shows.

Example 7 This is an example of a balanced open set $U \subset X^*$ which is not boundedly-regular. Moreover, we will see that the spaces $H_{\mathcal{B}w^*u}(\tilde{U})$ and $H_{w^*u}(\tilde{U})$ do not coincide for $\mathcal{B} = \{\overline{U_n}^{w^*} : n \in \mathbb{N}\}$, where $\{U_n : n \in \mathbb{N}\}$ is a fundamental sequence of U-bounded sets in U. We consider $X^* = \ell_1$.

For each $k > 4$ and $x \in \ell_1$ we set

$$p_k(x) = k \sum_{i \neq k} |x_{2i+1}| + |\sum_i x_{2i} + kx_{2k+1}|.$$

Let $V_k = \{x \in \ell_1 : p_k(x) < 2\}$. Now we define U as:

$$U = \bigcup_{k>4} V_k + \frac{1}{4} B_{\ell_1}.$$

In order to see that U is not boundedly-regular, we construct a $\tilde{U}$-bounded set D such that there is no bounded set $B \subset U$ with $D \subset \overline{B}^{w^*}$. We fix $k > 4$. If $m \in \mathbb{N}$, $p_k(e_{2k+1} - ke_{2m}) = 0$ and therefore, $e_{2k+1} - ke_{2m}$ belongs to V_k. Moreover, if $\|x\| < \frac{1}{8}$, $\text{dist}(e_{2k+1} - ke_{2m} + x, \ell_1 \setminus U) > \frac{1}{8}$ and we have that $\{e_{2k+1} - ke_{2m} + x : m \in \mathbb{N}\}$ is U-bounded. Since

$$w^* - \lim_{m \to \infty} (e_{2k+1} - ke_{2m} + x) = e_{2k+1} + x,$$

we have that $e_{2k+1} + x \in \tilde{U}$ whenever $\|x\| < \frac{1}{8}$. In other words, $\text{dist}(e_{2k+1}, \ell_1 \setminus \tilde{U}) \geq \frac{1}{8}$ and consequently the set $D := \{e_{2n+1} : n > 4\}$ is $\tilde{U}$-bounded.

Suppose there exists a bounded $B \subset U$ such that $D \subset \overline{B}^{w^*}$. For each $n > 4$, $e_{2n+1} \in \overline{B}^{w^*}$ and there exists $y^n \in B$ such that $|y^n_{2n+1}| > 1 - \frac{1}{4}$. We will see that $(y^n)_n$ is not bounded.

Let $k > 4$ such that $y^n \in V_k + B(0, \frac{1}{4})$ and let $x^n \in V_k$ with $\|y^n - x^n\| < \frac{1}{4}$. Note that $|x^n_{2n+1}| > \frac{1}{2}$.

First, we see that n and k coincide: if $n \neq k$,

$$p_k(x^n) = k \sum_{i \neq k} |x^n_{2i+1}| + |\sum_i x^n_{2i} + kx^n_{2k+1}| \geq k|x^n_{2n+1}| > \frac{k}{2} > 2,$$

which means that x^n is not in V_k, a contradiction. Therefore, $k = n$.

Now, we estimate $\|x^n\|$. Since $x^n \in V_n$,

$$2 > |p_n(x^n)| \geq |\sum_i x^n_{2i} + n x^n_{2n+1}| \geq \left| |\sum_i x^n_{2i}| - n|x^n_{2n+1}| \right|$$

Therefore,

$$|\sum_i x^n_{2i}| > n|x^n_{2n+1}| - 2 > \frac{n}{2} - 2. \tag{1.3}$$

Since $|\sum_i x^n_{2i}| \leq \|x^n\|$, we obtain that $\|x^n\| > \frac{n}{2} - 2$.

Finally, we observe that $\|y^n\| > \frac{n}{2} - 2 - \frac{1}{4}$ and $y^n \in B$ for all n to conclude that B cannot be bounded.

Now we define $g_h(x) := (\frac{5}{4} x_{2h+1})^h$, $x \in \ell_1$ and $h = 1, 2, \ldots$. Clearly the set $\{g_h : h \geq 2\}$ is contained in $H_{w^*u}(\ell_1)$. As

$$g_h(e_{2h+1}) = (\frac{5}{4})^h$$

for all $h \geq 2$, $\{g_h : h \geq 2\}$ is not bounded on B (which is $\tilde{U}$-bounded). Hence $\{g_h : h \geq 2\}$ is not a bounded subset of $H_{w^*u}(\tilde{U})$.

On the other hand, we are going to check that $\{g_h : h \geq 2\}$ is a bounded subset of $H_{w^*u}(U)$ and, consequently, of $H_{\mathcal{B}w^*u}(\tilde{U})$. Let C be U-bounded and $M > 0$ such that $\|x\| < M - \frac{1}{4}$ for all $x \in C$. We take $h \in \mathbb{N}$ such that $h > (M+2)2$. If $x \in C$, we write $x = y + z$, with $\|z\| < 1/4$ and $y \in V_k$ for some $k \geq 5$. We have two possibilities:
(i) If $k \neq h$, we have that $k|y_{2h+1}| < 2$ and then $|y_{2h+1}| < \frac{2}{5}$. Thus

$$|g_h(x)| = |(\frac{5}{4}(y_{2h+1} + z_{2h+1}))^h < (\frac{5}{4}(\frac{2}{5} + \frac{1}{4}))^h = (\frac{13}{16})^h < 1$$

(ii) If $k = h$, we claim that $|y_{2h+1}| \leq \frac{1}{2}$. Indeed if $|y_{2h+1}| > \frac{1}{2}$, equation (1.3) applied to y gives $\|y\| > \frac{h}{2} - 2$. But $\|y\| < M$ and hence $h < (M+2)2$, a contradiction. Thus

$$|g_h(x)| = |(\frac{5}{4}(y_{2h+1} + z_{2h+1}))^h < (\frac{5}{4}(\frac{1}{2} + \frac{1}{4}))^h = (\frac{15}{16})^h < 1,$$

and we obtain

$$\sup\{|g_h(x)| : x \in C, h > (M+2)2\} \leq 1.$$

Since $\sup\{|g_h(x)| : x \in C, 2 \leq h \leq (M+2)2\} < \infty$, then $\{g_h : h \geq 2\}$ is a bounded subset of $H_{w^*u}(U)$. As a consequence the spaces $H_{\mathcal{B}w^*u}(\tilde{U})$ and $H_{w^*u}(\tilde{U})$ are topologically different. We also have that $H_{\mathcal{B}w^*u}(\tilde{U})$ is in fact a proper subset of $H_{w^*u}(\tilde{U})$. This can be deduced from the open mapping theorem or, directly, by noting that

$$g(x) = \sum_h (\frac{8}{9})^h g_h(x)$$

belongs to $H_{\mathcal{B}w^*u}(\tilde{U})$ but not to $H_{w^*u}(\tilde{U})$.

The following result (see also [5, prop. 5] and its proof) states rather general conditions under which all (continuous) homomorphisms from $H_{w^*u}(U)$ to $H_{w^*u}(V)$ are composition operators.

Proposition 8 *Let U be a convex and balanced open subset of X^*, suppose X has the approximation property and let $V \subset Y^*$ be a balanced open set. If $A : H_{w^*u}(U) \to H_{w^*u}(V)$ is a continuous multiplicative operator, then there exists $g : \tilde{V} \longrightarrow \tilde{U}$ holomorphic such that the restriction of g to any V-bounded subset of V is weak-star to weak-star uniformly continuous and $\widetilde{Af}(y^*) = \tilde{f} \circ g(y^*)$ for all $f \in H_{w^*u}(U)$ and $y^* \in \tilde{V}$.*

PROOF. Let $y^* \in \tilde{V} \subset M_{w^*u}(V)$. Since A is multiplicative, $\delta_{y^*} \circ A$ is an element of $M_{w^*u}(U) = \delta(\tilde{U})$. Therefore, we can define $g(y^*)$ as the element in $\tilde{U}$ such that $\delta_{y^*} \circ A = \delta_{g(y^*)}$. By definition, $\widetilde{Af}(y^*) = \tilde{f} \circ g(y^*)$ for all $f \in H_{w^*u}(U)$ and $y^* \in \tilde{V}$. Since we can identify X with a subset of $H_{w^*u}(U)$, we have

$$(x \circ g)(y^*) = \widetilde{Ax}(y^*), \tag{1.4}$$

for all $y^* \in \tilde{V}$ and $x \in X$. We take $(V_n)_{n=1}^{\infty}$ a fundamental system of V-bounded sets and $\mathcal{B} = (\overline{V_n}^{w^*})_{n=1}^{\infty}$. Since V is balanced, by applying Proposition 3, we get $x \circ g \in H_{\mathcal{B}w^*u}(\tilde{V})$ for all $x \in X$. By a classical result of Dunford [16] and Grothendieck [19] on weak-star holomorphic mappings, g is holomorphic on $\tilde{V}$. From (1.4) it is straightforward to check that the restriction of g to any V-bounded set is weak-star to weak-star uniformly continuous. □

The next theorem answer our question for spaces H_{w^*u}.

Theorem 9 *Let X and Y be Banach spaces, one of them having the approximation property. Let $U \subset X^*$ and $V \subset Y^*$ be convex and balanced open sets. If $H_{w^*u}(U)$ and $H_{w^*u}(V)$ are topologically algebra-isomorphic, then X and Y are isomorphic Banach spaces.*

PROOF. Suppose X has the approximation property and let $A : H_{w^*u}(U) \to H_{w^*u}(V)$ be the algebra-isomorphism. We consider $g : \tilde{V} \longrightarrow \tilde{U}$ obtained in Proposition 8.

Consider the mappings $\theta_A : M_{w^*u}(V) \to M_{w^*u}(U)$ and $\theta_{A^{-1}} : M_{w^*u}(U) \to M_{w^*u}(V)$ given by $\theta_A(\varphi) = \varphi \circ A$ and $\theta_{A^{-1}}(\psi) = \psi \circ A^{-1}$. Since they are the restrictions of A^* and $(A^{-1})^*$, the transposes of A and A^{-1}, to the corresponding sets of homomorphisms, we have $\theta_{A^{-1}}(\theta_A(\varphi)) = \varphi$ for all $\varphi \in M_{w^*u}(V)$. By Proposition 3, we can define $h : \tilde{U} \to \tilde{V}$ by $h = \pi \circ \theta_{A^{-1}} \circ \delta$. Since we can identify Y as a subset of $H_{w^*u}(V)$, we have

$$(y \circ h)(x^*) = h(x^*)(y) = \pi(\theta_{A^{-1}}(\delta_{x^*}))(y) = \pi(\delta_{x^*} \circ A^{-1})(y) = A^{-1}(y)(x^*), \tag{1.5}$$

for all $y \in Y$ and $x^* \in \tilde{U}$. Hence $y \circ h \in H_{\mathcal{B}w^*u}(\tilde{U})$ for all $y \in Y$, where $\mathcal{B} = (\overline{U_n}^{w^*})_{n=1}^{\infty}$. Analogously to (1.4), we obtain that h is holomorphic on $\tilde{U}$ and that $h|_U$ is weak-star to weak-star uniformly continuous on U-bounded sets. For $y^* \in \tilde{V}$, we have that $(h \circ g)(y^*) = \theta_{A^{-1}}(\theta_A(\delta_{y^*})) \;|_Y = \delta_{y^*} \circ A \circ A^{-1} \;|_Y = y^*$. Since $h \circ g = id_{\tilde{V}}$, differentiating at 0 we have that $dh(g(0)) \circ dg(0) = id_{Y^*}$ and Y^* can be identified with a complemented subspace of X^*. We know that for $x \in X$, $x \circ g \in H_{w^*u}(V)$ and then $x \circ dg(0) = d(x \circ g)(0)$ is weak-star continuous on bounded sets [3, Lemma 2.2]. This means that $dg(0)$ is weak-star to weak-star continuous on bounded sets.

As $\tilde{U} = \cup_{n=1}^{\infty} \overline{U_n}^{w^*}$, there exists $n \in \mathbb{N}$ such that $g(0) \in \overline{U_n}^{w^*}$. By (1.1) the open set $g(0) + \frac{1}{2n(n+1)} B_{X^*}$ is included in $\overline{U_{n+1}}^{w^*}$. Thus, $h \in H_{w^*u}(g(0) + \frac{1}{2n(n+1)} B_{X^*})$. Just as above, we get that $dh(g(0))$ is weak-star to weak-star continuous on bounded sets.

Consequently, there are operators $\alpha : X \to Y$ and $\beta : Y \to X$ such that $dg(0) = \alpha^*$ and $dh(g(0)) = \beta^*$. Since $\alpha \circ \beta = id_Y$, Y is a complemented subspace of X, it inherits the approximation property and so $M_{w^*u}(V) = \delta(\tilde{V})$. Now we can proceed as above to show that $g \circ h = id_{\tilde{U}}$, which ends the proof. □

Corollary 10 *Let X and Y be Banach spaces, one of them having the approximation property. $H_{w^*}(X^*)$ and $H_{w^*}(Y^*)$ are topologically algebra-isomorphic if and only if X and Y are isomorphic Banach spaces.*

To simplify the notation we will write $g \in H_{w^*u}(\tilde{V}, \tilde{U})$ if g is holomorphic, g is weak-star to weak-star uniformly continuous on $\tilde{U}$-bounded sets and such that g maps $\tilde{U}$-bounded sets into $\tilde{V}$ bonded sets.

If $U \subset X^*$ is open, $V \subset Y^*$ is a balanced boundedly-regular open set and $A : H_{w^*u}(U) \longrightarrow H_{w^*u}(V)$ is a continuous multiplicative operator, then the mapping $\tilde{A} : H_{w^*u}(\tilde{U}) \longrightarrow H_{w^*u}(\tilde{V})$ defined as $\tilde{A}(f) := \widetilde{A(f|_U)}$, for all $f \in H_{w^*u}(\tilde{U})$ is also an homomorphism. If in addition U is balanced and boundedly-regular then A is an algebra-isomorphism if and only if $\tilde{A}$ is an algebra-isomorphism.

Theorem 11 *Let X and Y be Banach spaces, one of them having the approximation property. Let $U \subset X^*$ and $V \subset Y^*$ be convex, balanced, boundedly-regular open sets. A mapping $A : H_{w^*u}(U) \longrightarrow H_{w^*u}(V)$ is a continuous homomorphism if and only if there exists a function $g \in H_{w^*u}(\tilde{V}, \tilde{U})$ such that the operator $\tilde{A} : H_{w^*u}(\tilde{U}) \longrightarrow H_{w^*u}(\tilde{V})$ is the composition operator generated by g (i.e., $\tilde{A}f = f \circ g$ for all $f \in H_{w^*u}(\tilde{U})$).*

PROOF. Let $A : H_{w^*u}(U) \longrightarrow H_{w^*u}(V)$ be a continuous homomorphism. The hypotheses on U and V imply that $H_{w^*u}(U)$ and $H_{w^*u}(\tilde{U})$ are topologically algebra-isomorphic, as well as $H_{w^*u}(V)$ and $H_{w^*u}(\tilde{V})$, and that $\tilde{A} : H_{w^*u}(\tilde{U}) \to H_{w^*u}(\tilde{V})$ is a continuous homomorphism too. It follows that $\tilde{\tilde{U}} = \tilde{U}$ and $\tilde{\tilde{V}} = \tilde{V}$. Applying Proposition 8 we obtain a holomorphic function $g : \tilde{V} \longrightarrow \tilde{U}$ such that $\tilde{A}f = f \circ g$ for all $f \in H_{w^*u}(\tilde{U})$) and such that the restriction of g to $\tilde{V}$-bounded sets is weak-star to weak-star uniformly continuous. We need to prove that given a $\tilde{V}$-bounded set B, then $g(B)$ is $\tilde{U}$-bounded set. We can assume B to be weak-star closed and therefore, $g(B)$ is weak-star compact and thus is bounded. If $\mathrm{dist}(g(B), X^* \setminus \tilde{U}) = 0$, since $(\overline{U_n}^{w^*})_{n=1}^\infty$ is a fundamental sequence of $\tilde{U}$-bounded sets, we can find a sequence $(y_n^*) \subset B$ such that $g(y_n^*) \notin \overline{U_{n+1}}^{w^*}$. By the Hahn-Banach theorem applied to the weak-star topology, there exists a sequence $(x_n) \subset X$ such that, $x_n(g(y_n^*)) > 1$ and $|x_n(x^*)| \leq 1$ for all $x^* \in \overline{U_{n+1}}^{w^*}$ and all $n \in \mathbb{N}$. But, given n, there exists a $R_n > 1$ such that $R_n \overline{U_n}^{w^*} \subset \overline{U_{n+1}}^{w^*}$. Hence

$$|R_n x_n(x^*)| \leq 1 \quad \forall x^* \in \overline{U_n}^{w^*}.$$

Let (α_n) a sequence of positive numbers such that $\lim_n R_n^{\alpha_n} = +\infty$. We consider $h_n := (R_n x_n)^{\alpha_n} \in H_{w^*u}(\tilde{U})$. Given $m \in \mathbb{N}$ we have $\|h_n\|_{\overline{U_m}^{w^*}} \leq 1$ for all $n \geq m$. Hence

$$\sup_{n \in \mathbb{N}} \|h_n\|_{\overline{U_m}^{w^*}} \leq \max\{1, \|h_1\|_{\overline{U_m}^{w^*}}, \ldots, \|h_{m-1}\|_{\overline{U_m}^{w^*}}\} < \infty,$$

for all $m \in \mathbb{N}$, and the family $(h_n)_{n=1}^\infty$ is bounded in $H_{w^*u}(\tilde{U})$. Thus $(\tilde{A}h_n)_{n=1}^\infty$ is bounded in $H_{w^*u}(\tilde{V})$. In particular,

$$\sup\{|\tilde{A}h_n(y^*)| : n \in \mathbb{N}, y^* \in B\} < \infty.$$

But $Ah_n(y_n^*) = h_n \circ g(y_n^*) > R_n^{\alpha_n}$ for all $n \in \mathbb{N}$. A contradiction.

For the converse it is enough to observe that if $B : H_{w^*u}(\tilde{U}) \longrightarrow H_{w^*u}(\tilde{V})$ is a continuous linear operator then $A : H_{w^*u}(U) \longrightarrow H_{w^*u}(V)$ defined by $A(f) := B(\tilde{f})|_V$, $f \in H_{w^*u}(U)$ is again an homomorphism and $\tilde{A} = B$. □

Corollary 12 *Let X and Y be Banach spaces, one of them having the approximation property. Let $U \subset X^*$ and $V \subset Y^*$ be convex, balanced boundedly-regular open sets. There exists $A : H_{w^*u}(U) \longrightarrow H_{w^*u}(V)$ a topological algebra-isomorphism if and only if there exists a biholomorphic function $g \in H_{w^*u}(\tilde{V}, \tilde{U})$ with $g^{-1} \in H_{w^*u}(\tilde{U}, \tilde{V})$, such that the operator $\tilde{A} : H_{w^*u}(\tilde{U}) \longrightarrow H_{w^*u}(\tilde{V})$ is the composition operator generated by g (i.e $\tilde{A}f = f \circ g$ for all $f \in H_{w^*u}(\tilde{U})$). In that case, X and Y are isomorphic Banach spaces.*

PROOF. The result follows from Theorem 11 and the proof Theorem 9. □

Remark 13 We now consider U a balanced open subset of X and define

$$\hat{U} := \bigcup_n \overline{U}_n^{w^*} = \bigcup_n \overline{U}_n^{w(X^{**}, X^*)} \subset X^{**}.$$

By uniform continuity, given $f \in H_{wu}(U)$ there exists a unique $\hat{f} : \hat{U} \longrightarrow \mathbb{C}$ such that $\hat{f}|_U = f$, it is weak-star uniformly continuous when restricted to $\overline{U}_n^{w^*}$ and

$\|f\|_{\overline{U_n}^{w^*}} = \|f\|_{U_n}$ for all $n \in \mathbb{N}$. Hence, given $z \in \hat{U}$, the mapping $\delta_z(f) := \hat{f}(z)$ for all $f \in H_{wu}(U)$ is a continuous homomorphism and we can define $\delta : \hat{U} \longrightarrow M_{wu}(U)$ as $\delta(z) := \delta_z$. The Aron-Berner extension implies that $\hat{f} \in H(\hat{U})$. We consider $\mathring{\overline{U}}^{w^*}$, the $\|.\|^{**}$-interior of $\overline{U}^{w^*}$. In [17, Remark 1.4 and the proof of Theorem 1.5], it is shown that if U is a convex and balanced open set, for each $\mathring{\overline{U}}^{w^*}$- bounded set $D \subset \mathring{\overline{U}}^{w^*}$there exists a U-bounded set $C \subset U$ such that $D \subset \overline{C}^{w^*}$. Hence $\hat{U} = \mathring{\overline{U}}^{w^*}$, $(\overline{U_n}^{w^*})_{n=1}^{\infty}$ is a fundamental sequence of $\hat{U}$-bounded sets and $\hat{U}$ is a convex and balanced boundedly-regular open set in the dual space $(X^*)^*$. Moreover, $\tilde{\hat{U}} = \bigcup_n \overline{\overline{U_n}^{w^*}}^{w^*} = \mathring{\overline{U}}^{w^*} = \hat{U}$. Consequently, if U is a convex and balanced open set, $H_{wu}(U)$ and $H_{w^*u}(\mathring{\overline{U}}^{w^*})$ are topologically algebra-isomorphic and the above results can be translated to these algebras of holomorphic functions on convex and balanced open sets:

Corollary 14 *Let U be an open convex and balanced open subset of X, then $\pi(M_{wu}(U)) = \mathring{\overline{U}}^{w^*}$. Moreover, if X^* has the approximation property, $M_{wu}(U) = \delta(\mathring{\overline{U}}^{w^*})$.*

Theorem 15 *Let U be a convex and balanced open subset of X, suppose X^* has the approximation property and let $V \subset Y$ be open, convex and balanced. A mapping $A : H_{wu}(U) \to H_{wu}(V)$ is a continuous multiplicative operator, if and only if there exists $g \in H_{w^*u}(\mathring{\overline{V}}^{w^*}, \mathring{\overline{U}}^{w^*})$ such that $Af = \hat{f} \circ g|_U$ for all $f \in H_{wu}(U)$.*

If $A : H_{wu}(U) \longrightarrow H_{wu}(V)$ is a continuous multiplicative linear operator we define $\hat{A} : H_{w^*u}(\mathring{\overline{U}}^{w^*}) \longrightarrow H_{w^*u}(\mathring{\overline{V}}^{w^*})$ as $\hat{A}(f) = \widehat{A(f|_U)}$ for all $f \in H_{w^*u}(\mathring{\overline{U}}^{w^*})$. By the above remarks, $\hat{A}$ is a continuous multiplicative operator too, and A is a topological algebra-isomorphism if and only if $\hat{A}$ is a topological algebra-isomorphism.

Corollary 16 *Let X and Y be Banach spaces, such that X^* or Y^* has the approximation property. Let $U \subset X$ and $V \subset Y$ be convex and balanced open sets. Then $A : H_{wu}(U) \longrightarrow H_{wu}(V)$ is a topological algebra-isomorphism if and only if there is a biholomorphic function $g \in H_{w^*u}(\mathring{\overline{V}}^{w^*}, \mathring{\overline{U}}^{w^*})$ whose inverse is in $H_{w^*u}(\mathring{\overline{U}}^{w^*}, \mathring{\overline{V}}^{w^*})$ such that $\hat{A}(f) = f \circ g$ for all $f \in H_{w^*u}(\mathring{\overline{U}}^{w^*})$.*

In this case, we have that X^* and Y^* must be isomorphic Banach spaces.

As a consequence of a result of Lassalle and Zalduendo [21, Proposition 6] (see also [9, Proposition 3.4]), if X^* and Y^* are isomorphic, $H_{wu}(X)$ and $H_{wu}(Y)$ are isomorphic algebras. Therefore we have:

Corollary 17 *Let X and Y be Banach spaces, one of their duals having the approximation property. Then $H_{wu}(X)$ and $H_{wu}(Y)$ are topologically isomorphic algebras if and only if X^* and Y^* are isomorphic Banach spaces.*

2 Morphisms on $H_b(U)$

In [1], Aron and Berner showed that any $f \in H_b(X)$ can be extended in a natural way to a function $\bar{f} \in H_b(X^{**})$ so that the map $AB : H_b(X) \longrightarrow H_b(X^{**})$, $AB(f) = \bar{f}$, is a topological algebra-isomorphism into the image. We state the formula of the extension for n-homogeneous continuous polynomials. Let $P : X \to \mathbb{C}$ be a n-homogeneous continuous polynomial and $L : X \times \cdots \times X \to \mathbb{C}$ its associated $n-$linear symmetric mapping. Then the Aron-Berner extension of P is given by $\bar{P}(z) = \bar{L}(z, \cdots, z)$ for all $z \in X^{**}$, where $\bar{L} : X^{**} \times \cdots \times X^{**} \to \mathbb{C}$ is defined as

$$\bar{L}(z_1, ..., z_n) = \lim_{x_{\alpha_1} \xrightarrow{w^*} z_1} \cdots \lim_{x_{\alpha_n} \xrightarrow{w^*} z_n} L(x_{\alpha_1}, ..., x_{\alpha_n})$$

(w^* denotes the weak-star topology $w(X^{**}, X^*)$). These results were extended to functions in $H_b(B_X)$ by Davie and Gamelin [11, Theorem 3] and slightly improved in [17, Theor. 1.3 and 1.5] where it is shown that given U a convex and balanced open subset of X there exists a multiplicative extension operator $AB : H_b(U) \longrightarrow H_b(\overset{\circ}{\overline{U}^{w^*}})$ such that $\|\bar{f}\|_{\overline{B}^{w^*}} = \|f\|_B$ for all $f \in H_b(U)$ and all U-bounded set B.

Let $M_b(U)$ denote the spectrum of $H_b(U)$ and $\pi : M_b(U) \to X^{**}$ the mapping given by $\pi(\varphi) = \varphi\mid_{X^*}$. As in Proposition 1, we obtain the following "bounds" for $\pi(M_b(U))$.

Proposition 18 *Let U be an open subset of X. Then* $\bigcup_n \overline{U_n}^{w^*} \subset \pi(M_b(U)) \subset \bigcup_n \overline{\Gamma(U_n)}^{w^*}$.

PROOF. If $x^{**} \in \overline{U_n}^{w^*}$, there exists a net $(x_\alpha)_{\alpha \in I}$ in U_n w^*-converging to x^{**}. Let Δ be a cofinal ultrafilter on I and define $\varphi(f) = \lim_\Delta f(x_\alpha)$. Note that φ is well defined and continuous since $|\varphi(f)| \leq \sup_\alpha |f(x_\alpha)| \leq \|f\|_{U_n}$ for $f \in H_b(U)$.

For $x^* \in X^*$, $\varphi(x^*) = \lim_\Delta x^*(x_\alpha) = x^{**}(x^*)$ and then $\pi(\varphi) = x^{**}$. Therefore, we have that $\bigcup_n \overline{U_n}^{w^*} \subset \pi(M_b(U))$. The second inclusion follows exactly as in Proposition 1. □

In Remark 2 we saw that $\delta(\tilde{U})$ is contained in $M_{w^*u}(U)$. In Section 1 we set $\hat{U} := \bigcup_n \overline{U}_n^{w^*}$. For $x_0 \in X$, $C(U \cup \{x_0\})$ is the algebra of all norm-continuous functions on $U \cup \{x_0\}$ endowed with the pointwise topology. Let $x_0 \in X$ and assume that there exists a multiplicative and continuous extension operator $B : H_b(U) \longrightarrow C(U \cup \{x_0\})$, with $B(g)(x_0) = g(x_0)$ for all $g \in H_b(X)$. In this case we denote by δ_{x_0} the continuous homomorphism $\delta_{x_0}(f) := B(f)(x_0)$, where $f \in H_b(U)$. The previous proposition asserts that $\hat{U}$ is contained in $\pi(M_b(U))$. However, the following example shows that for a reflexive X, $\delta(\hat{U})$ is not necessarily contained in $M_b(U)$ and that in general, $\pi(M_b(U))$ is strictly larger than $M_b(U) \bigcap \delta(X)$.

Example 19 We consider the following open set:

$$U = \{x \in \ell_2 : \mathbf{Re}\,(\sum_k x_k^2) > \frac{1}{2}\}.$$

First, we see that $\hat{U} = \ell_2$: given $x \in \ell_2$, we can find $\lambda > 0$ large enough for the set $\{x + \lambda e_m : m \in \mathbb{N}\}$ to be U-bounded. Therefore, it is contained in U_n for some n and $x \in \overline{U_n}^{w^*} = \overline{U_n}^{w}$. By Proposition 18, $\pi(M_b(U)) = \ell_2$.

On the other hand, take $y \notin U$. Since $\mathbf{Re}\,(\sum_k y_k^2) \leq \frac{1}{2}$, the function $f(x) = \frac{1}{1-e^{\sum_k y_k^2 - x_k^2}}$ is in $H_b(U)$ and the family $\{f_n(x) := \sum_{m=0}^n e^{m\sum_{k=0}^\infty y_k^2 - x_k^2}\}_{n=1}^\infty$ is bounded in $H_b(U)$. If $\delta_y \in M_b(U)$, then $(\delta_y(f_n))_{n=1}^\infty$ would be a bounded sequence. But as $f_n \in H_b(\ell_2)$, we would have $\delta_y(f_n) = f_n(y) = n$ for all n, a contradiction.

If U is convex and balanced, let $\bar{\delta} : \overset{\circ}{\overline{U}^{w^*}} \subset X^{**} \to M_b(U)$ be given by $\bar{\delta}(x^{**}) = \bar{\delta}_{x^{**}}$, with $\bar{\delta}_{x^{**}}(f) = \bar{f}(x^{**})$, for $f \in H_b(U)$ and $x^{**} \in \overset{\circ}{\overline{U}^{w^*}}$. Since $\|\bar{f}\|_{\overline{U_n}^{w^*}} = \|f\|_{U_n}$ for all $n \in \mathbb{N}$ we have $\bar{\delta}(\overset{\circ}{\overline{U}^{w^*}}) \subset M_b(U)$. Therefore, we have the following lemma.

Lemma 20 *If U is a convex and balanced open set in X, then $\pi(M_b(U)) = \overset{\circ}{\overline{U}^{w^*}}$ and $\bar{\delta}(\overset{\circ}{\overline{U}^{w^*}}) \subset M_b(U)$.*

If U and V are open sets in the Banach spaces X and Y respectively and $A : H_b(U) \to H_b(V)$ is a continuous multiplicative operator, we can define the mapping $\theta_A : M_b(V) \to M_b(U)$ by

$$\theta_A(\varphi)(f) = \varphi(A(f)).$$

Note that θ_A is just is the transpose A^* restricted to $M_b(V)$. If in addition U and V are convex and balanced we define $\bar{A} : H_b(\overset{\circ}{\overline{U}^{w^*}}) \to H_b(\overset{\circ}{\overline{V}^{w^*}})$ by $\bar{A}(f) := \overline{A(f|_U)}$. By Remark 13, $\bar{A}$ is well defined and it is a continuous multiplicative operator too. Nevertheless, a big difference with the situation of Section 1 is that if X is not reflexive, $\bar{A}$ is never a topological algebra-isomorphism: if we take $\upsilon \neq 0$ in $X^{(3)}$, the topological dual of X^{**}, such that $\upsilon|_X = 0$, then $\bar{A}(\upsilon) = 0$.

Again to simplify the notation we write $g \in H_b(\overset{\circ}{\overline{V}^{w^*}}, \overset{\circ}{\overline{U}^{w^*}})$ if g is holomorphic and g maps $\overset{\circ}{\overline{V}^{w^*}}$-bounded sets into $\overset{\circ}{\overline{U}^{w^*}}$-bounded sets. We recall that a n-homogeneous continuous polynomial on a Banach space X is called approximable if it is in the norm-closure of the polynomials generated by $\{x_1^* \dots x_n^* : x_j^* \in X^*,\ j = 1, \dots n\}$ [15, Def. 2.1]. The next three results are our first positive answer to our question for the case of algebras of holomorphic functions of bounded type.

Theorem 21 *Let $U \subset X$ and $V \subset Y$ be convex and balanced open sets and suppose* *that every polynomial on X is approximable. A mapping $A : H_b(U) \to H_b(V)$ is a continuous multiplicative operator if and only if there exists $g \in H_b(\overset{\circ}{\overline{V}}^{w^*}, \overset{\circ}{\overline{U}}^{w^*})$ such that $\bar{A}$ is the composition operator $\overline{A}f = \overline{f} \circ g$.*

PROOF. Since every polynomial on X is approximable, we have $H_b(U) = H_{wu}(U)$ algebraically and topologically and $M_b(U) = \bar{\delta}(\overset{\circ}{\overline{U}}^{w^*})$. This last fact can be deduced as in the proof of Proposition 3 (see also [3, Thm. 3.3]). If $y^{**} \in \overset{\circ}{\overline{V}}^{w^*}$, we define $g : \overset{\circ}{\overline{V}}^{w^*} \longrightarrow \overset{\circ}{\overline{U}}^{w^*}$ by $g(y^{**}) = x^{**}$ where x^{**} satisfies $\bar{\delta}_{y^{**}} \circ A = \bar{\delta}_{x^{**}}$. Now an argument analogous to the proof of Proposition 8 shows that $g \in H_b(\overset{\circ}{\overline{V}}^{w^*}, X^{**})$.

We know that $(\overline{U_n}^{w^*})_{n=1}^{\infty}$ is a fundamental sequence of $\overset{\circ}{\overline{U}}^{w^*}$-bounded sets. If there were a $\overset{\circ}{\overline{V}}^{w^*}$-bounded set B such that $g(B)$ is not $\overset{\circ}{\overline{U}}^{w^*}$-bounded, we could find $(y_n^{**}) \subset B$, $(x_n^*) \subset X^*$ such that $x_n^*(g(y_n^{**})) > 1$ and $|x_n^*(x^{**})| \leq 1$ for all $x^{**} \in \overline{U_{n+1}}^{w^*}$ an all $n \in \mathbb{N}$. Now an argument like the one in Theorem 12 leads to a contradiction. $\square$

Corollary 22 *Let X and Y be Banach spaces. Let $U \subset X$ and $V \subset Y$ be convex and balanced open sets and suppose that every polynomial on X^{**} is approximable. There exists a topological algebra-isomorphism $A : H_b(U) \longrightarrow H_b(V)$ if and only there exists a biholomorphic function $g \in H_{w^*u}(\overset{\circ}{\overline{V}}^{w^*}, \overset{\circ}{\overline{U}}^{w^*})$ whose inverse is in $H_{w^*u}(\overset{\circ}{\overline{U}}^{w^*}, \overset{\circ}{\overline{V}}^{w^*})$ such that $\overline{Af} = \overline{f} \circ g$ for all $f \in H_b(U)$. In this case X^* and Y^* must be isomorphic Banach spaces.*

PROOF. We first prove that every polynomial on Y^{**} is approximable by making suitable modifications of the proofs of Theorems 9 and 12. We consider the mapping $g \in H_b(\overset{\circ}{\overline{V}}^{w^*}, \overset{\circ}{\overline{U}}^{w^*})$ obtained in Theorem 21. We define $h : \overset{\circ}{\overline{U}}^{w^*} \to \overset{\circ}{\overline{V}}^{w^*}$ by $h = \pi \circ \theta_{A^{-1}} \circ \bar{\delta}$. Differentiating, we obtain that Y^{**} can be identified with a complemented subspace of X^{**}. Then every polynomial on Y^{**} (and on Y) is approximable. Hence $H_b(V) = H_{wu}(V)$ and, since every polynomial on X is approximable, $H_b(U) = H_{wu}(U)$. Now the conclusion follows from Corollary 16. There, the approximation property of X^* or Y^* is used only to ensure that under the rest of the conditions, $M_{wu}(U) = \overset{\circ}{\overline{U}}^{w^*}$ and $M_{wu}(V) = \overset{\circ}{\overline{V}}^{w^*}$. This is also true under our present hypotheses. $\square$

If every polynomial on X^{**} is approximable and X^* and Y^* are isomorphic Banach spaces, we have that X^{**} and Y^{**} are isomorphic and then every polynomial on Y^{**} is also approximable. It follows that $H_b(X)$ and $H_b(Y)$ are algebra-isomorphic (see also [21]). Therefore, in an analogous way to Corollary 17 we obtain the following:

Corollary 23 *If every polynomial on X^{**} is approximable, $H_b(X)$ and $H_b(Y)$ are algebra-isomorphic if and only if X^* and Y^* are isomorphic Banach spaces.*

Example 24 The original Tsirelson space T^* satisfies the condition of Corollaries 22 and 23. Since T^* is reflexive, if $U \subset T^*$ and $V \subset Y$ are convex and balanced open sets and $H_b(U)$ and $H_b(V)$ are topologically algebra-isomorphic, then T^* and Y are isomorphic. Moreover, $H_b(T^*)$ and $H_b(Y)$ are isomorphic if and only if T^* and Y are isomorphic.

Example 25 The Tsirelson-James space T_J^* is a quasi-reflexive space on which every polynomial is approximable. By [13, Lemma 19], all polynomials on its bidual are also approximable. Therefore, for $U \subset T_J^*$ and $V \subset Y$ as before, if $H_b(U)$ and $H_b(V)$ are topologically algebra-isomorphic then T_J^{**} and Y^* are isomorphic. Moreover, $H_b(T_J^*)$ and $H_b(Y)$ are isomorphic if and only if T_J^{**} and Y^* are isomorphic.

Now we face the far more difficult situation in which we do not assume that every continuous polynomial on X^{**} is approximable and we are going to restrict ourselves to the case of entire functions of bounded type.

A complex Banach space X is said to be *(symmetrically) regular* if every continuous (symmetric) linear mapping $T : X \to X^*$ is weakly compact. Recall that T is *symmetric* if $< Tx, y >=< x, Ty >$ for all $x, y \in X$. From now on, every Banach space is assumed to be symmetrically regular. If $\varphi \in M_b(X)$ we define *the sheet of* φ as the set $S(\varphi) := \{\varphi \circ \tau_z : z \in X^{**}\}$, where $\tau_z(f) = \bar{f}(\cdot + z)$. We consider on $M_b(X)$ the Riemann analytic manifold structure on X^{**} given in [4, Corollary 2.2]. With this structure, $\pi : M_b(X) \to X^{**}$ is the local homeomorphism over X^{**}, so that each sheet is an analytic copy of X^{**} and $M_b(X)$ is the disjoint union of those sheets (the sheets being the connected components of $M_b(X)$). Our starting point is Theorem 30, hence it is for us relevant to know if θ_A produces a continuous mapping when the spectra are endowed with this analytic structure. The answer is, in general, negative as we show in Theorems 32 and 35.

It can be seen that for fixed $\varphi \in M_b(X)$ and $f \in H_b(X)$, the mapping $x^{**} \longmapsto \varphi \circ \tau_{x^{**}}(f)$, $x^{**} \in X^{**}$ is an analytic function of bounded type [15, Proof of Prop. 6.30]. In particular, for $x^* \in X^* \subset X^{(3)}$, the mapping $x^{**} \mapsto x^*(\pi \circ \theta_A(\varphi \circ \tau_{x^{**}})) = \pi \circ \theta_A(\varphi \circ \tau_{x^{**}})(x^*) = \varphi \circ \tau_{x^{**}}(A(x^*))$ is analytic. As this happens for each $x^* \in X^*$, $x^{**} \mapsto \pi \circ \theta_A(\varphi \circ \tau_{x^{**}})$ is analytic [18] (see also [15, Prop. 3.7, Ex. 3.8g]). Since π is the local homeomorphism which gives the analytic structure on each sheet, if θ_A maps the sheet $S(\varphi)$ into a single sheet in $M_b(X)$ then θ_A is continuous (in fact it is analytic) on $S(\varphi)$. Note that in this case, $S(\varphi)$ is mapped into the sheet $S(\theta_A(\varphi))$. On the other hand, continuous functions map connected sets into connected sets, so the next lemma holds.

Lemma 26 *θ_A is continuous on $S(\varphi)$ if and only if $\theta_A(S(\varphi)) \subset S(\theta_A(\varphi))$.*

In particular, θ_A is continuous on $\bar{\delta}(Y^{**})$ if and only if $\theta_A(\bar{\delta}(Y^{**}))$ lies on a single sheet of $M_b(X)$. The following lemma gives an equivalent condition.

Lemma 27 *Let X and Y be symmetrically regular Banach spaces, $A : H_b(X) \to H_b(Y)$ be a homomorphism and $\varphi_0 \in M_b(X)$. Then $\theta_A(\bar{\delta}(Y^{**})) \subset S(\varphi_0)$ if and only if there exists $g \in H_b(Y^{**}, X^{**})$ such that $\overline{A(f)}(y^{**}) = \varphi_0 \circ \tau_{g(y^{**})}(f)$ for all $f \in H_b(U)$ and all $y^{**} \in Y^{**}$.*

PROOF. Assume that $\theta_A(\bar{\delta}(Y^{**})) \subset S(\varphi_0)$. Given $y^{**} \in Y^{**}$ there exists a unique $x^{**} \in X^{**}$ such that $\theta_A(\bar{\delta}_{y^{**}}) = \varphi_0 \circ \tau_{x^{**}}$. Define $g : Y^{**} \to X^{**}$ by $g(y^{**}) = x^{**}$. We have that $\overline{A(f)}(y^{**}) = \bar{\delta}_{y^{**}}(A(f)) = \theta_A(\bar{\delta}_{y^{**}})(f) = \varphi_0 \circ \tau_{g(y^{**})}(f)$. To see that $g \in H_b(Y^{**}, X^{**})$ note that if $x^* \in X^*$, then $\overline{A(x^*)}(y^{**}) = \varphi_0 \circ \tau_{g(y^{**})}(x^*) = \varphi_0(x^*) + g(y^{**})(x^*)$. Therefore, the mapping $y^{**} \mapsto x^* \circ g(y^{**})$ belongs to $H_b(Y^{**})$ for any $x^* \in X^*$, which means that $g \in H_b(Y^{**}, X^{**})$ [18] (see also [15, Prop. 3.7 or Ex. 3.8g]).

Conversely, if $y^{**} \in Y^{**}$, $\theta_A(\bar{\delta}_{y^{**}})(f) = \bar{\delta}_{y^{**}}(A(f)) = \overline{A(f)}(y^{**}) = \varphi_0 \circ \tau_{g(y^{**})}(f)$, which belongs to $S(\varphi_0)$. $\square$

We will say that $A : H_b(X) \to H_b(Y)$ is an AB-composition homomorphism if there exists $g \in H_b(Y^{**}, X^{**})$ such that $\overline{A(f)}(y^{**}) = \overline{f}(g(y^{**}))$ for all $f \in H_b(U)$ and all $y^{**} \in Y^{**}$.

Corollary 28 *Let X and Y be symmetrically regular Banach spaces and let $A : H_b(X) \to H_b(Y)$ be a continuous multiplicative operator.*

(1) A is an AB -composition homomorphism if and only if $\theta_A(\bar{\delta}(Y^{**})) \subset \bar{\delta}(X^{**})$.

(2) A is a composition homomorphism if and only if $\theta_A(\delta(Y)) \subset \delta(X)$.

PROOF. Just take $\varphi_0 = \delta(0)$ in the previous lemma. $\square$

If $g : Y \to X$ is a biholomorphic mapping of bounded type and $z \in Y^{**} \setminus Y$, then $A(f)(x) = \bar{f}\bar{g}(x+z)$ is a topological algebra-isomorphism between $H_b(X)$ and $H_b(Y)$ which is not an AB-composition homomorphism.

Corollary 29 *Let X be a symmetrically regular Banach space. If X^* has the approximation property, the following are equivalent:*

a) Every polynomial on X weakly continuous on bounded sets.

b) Every homomorphism $A : H_b(X) \to H_b(Y)$ is an AB -composition one, for any symmetrically regular Banach space Y.

c) Every homomorphism $A : H_b(X) \to H_b(X)$ is an AB -composition one.

PROOF. *a)* $\Rightarrow$ *b)* If every polynomial on X is approximable, then $M_b(X) = \bar{\delta}(X^{**})$ [3, Thm. 3.3] and *b)* follows from Corollary 28.

b) $\Rightarrow$ *c)* This is clear.

c) $\Rightarrow$ *a)* If there is a polynomial which is not weakly continuous on bounded sets, then by [3, Thm. 3.3] we have that $M_b(X) \neq \bar{\delta}(X^{**})$. If $\varphi_0 \in M_b(X) \setminus \bar{\delta}(X^{**})$, the homomorphism $A : H_b(X) \to H_b(X)$ given by $A(f)(x) = \varphi_0 \circ \tau_x(f)$ is not an AB-composition operator by Corollary 28. $\square$

Theorem 30 *Let X and Y be symmetrically regular Banach spaces and $A : H_b(X) \to H_b(Y)$ be an isomorphism. Suppose that there exist non-empty open subsets $\mathcal{V} \subset M_b(Y)$ and $\mathcal{U} \subset M_b(X)$ such that $\theta_A : \mathcal{V} \to \mathcal{U}$ is an homeomorphism. Then X^{**} and Y^{**} are isomorphic.*

PROOF. We fix $\varphi_0 \in \mathcal{V}$ and define $\mathcal{V}_0 = \mathcal{V} \cap S(\varphi_0)$. We have that $\mathcal{U}_0 := \theta_A(\mathcal{V}_0)$ is an open subset of $M_b(X)$ contained in $S(\theta_A(\varphi_0))$ (this follows from the bicontinuity of θ_A and the fact that the sheets are the connected components of the spectrum). Proceeding as in Lemma 27, we can find open subsets $V_0 \subset Y^{**}$ and $U_0 \subset X^{**}$ and holomorphic functions $g \in H_b(V_0, U_0)$ and $h \in H_b(U_0, V_0)$ such that

$$\theta_A(\varphi_0 \circ \tau_{y^{**}}) = \theta_A(\varphi_0) \circ \tau_{g(y^{**})} \quad \text{for all } y^{**} \in V_0$$

and

$$\theta_{A^{-1}}(\theta_A(\varphi_0) \circ \tau_{x^{**}}) = \varphi_0 \circ \tau_{h(x^{**})} \quad \text{for all } x^{**} \in U_0.$$

We have that g and h are inverse to each other and in particular, for any $y^{**} \in V_0$ the differential $dg(y^{**})$ is an isomorphism between Y^{**} and X^{**}. $\square$

Let us consider $X = c_0(\ell_2^n)$ and $Y = c_0(\ell_2^n) \oplus \ell_2$, the preduals of the examples of Stegall [23]. It is known that X^{**} and Y^{**} are isomorphic, but X^* and Y^* are not. The space X has approximation property, the Dunford Pettis property (since X^* is Schur) and does not contain ℓ_1. This means that the finite type polynomials are dense in $H_b(X)$. Corollary 23 says that $H_b(Y)$ cannot be isomorphic to $H_b(X)$. This could also be deduced from the fact that $H_b(X)$ is separable and, since Y contains a complemented copy of ℓ_2, $H_b(Y)$ is not separable. The density of the finite type polynomials also show that X is symmetrically regular (and consequently, $Y = X \oplus \ell_2$ is also symmetrically regular). Note that the isomorphism between the biduals forces the sheets on $M_b(X)$ and $M_b(Y)$ to be homeomorphic. However, $M_b(X)$ and $M_b(Y)$ are not homeomorphic, since $M_b(X) = \bar{\delta}(X^{**})$ while $M_b(Y)$ consists of infinitely many sheets.

In light of Lemma 27, θ_A is continuous on $\bar{\delta}(Y^{**})$ if and only if there exist $\varphi_0 \in M_b(X)$ and $g \in H_b(Y^{**}, X^{**})$ such that $\overline{A(f)}(y^{**}) = \varphi_0 \circ \tau_{g(y^{**})}(f)$ for all $f \in H_b(X)$

and $y^{**} \in Y^{**}$. In particular, if $g \in H_b(Y,X)$ and we consider the composition homomorphism $A_g : H_b(X) \to H_b(Y)$ given by $A_g(f)(x) = f \circ g(x)$, the mapping $\theta_{A_g} : M_b(Y) \to M_b(X)$ is continuous on $\bar{\delta}(Y^{**})$. However, θ_{A_g} is not necessarily continuous on $M_b(Y)$ even if g is a continuous homogeneous polynomial. First we state the following lemma, the proof of which is straightforward.

Lemma 31 *Let X and Y be symmetrically regular Banach spaces. If $g \in H_b(Y,X)$ then:*

a) $\theta_{A_g}(\varphi \circ \tau_{y^{**}})(f) = \varphi\left(y \mapsto \bar{f} \circ \bar{g}(y + y^{**})\right)$

b) $\theta_{A_g}(\varphi) \circ \tau_{x^{**}}(f) = \varphi\left(y \mapsto \bar{f}\left(\bar{g}(y) + x^{**}\right)\right)$

for all $\varphi \in M_b(Y)$, $y^{**} \in Y^{**}$, $x^{**} \in X^{**}$ and $f \in H_b(X)$.

We refer to [22, page 48] for the definition of a finite dimensional Schauder decomposition in a Banach space X.

Theorem 32 *Let X be a symmetrically regular Banach space with an unconditional finite dimensional Schauder decomposition and suppose that there exists a continuous N-homogeneous polynomial which is not weakly sequentially continuous. Then there exists a $(N+1)$-homogeneous polynomial $P : X \to X$ such that θ_{A_P} is not continuous.*

PROOF. Let Q be a N-homogeneous polynomial that is not weakly sequentially continuous. Following [14, Prop. 1.6] we can choose $\varepsilon > 0$ and a weakly null semi-normalized block sequence $(u_j)_{j=1}^{\infty}$ such that $|Q(u_j)| > \varepsilon$ for all j. Let $\sigma_j : X \to X$ be the projection on the support of u_j. By [14, Prop. 1.3], there exists a constant $C > 0$ such that

$$\sum_{j=1}^{\infty} \left| \frac{Q(\sigma_j(x))}{Q(u_j)} \right| \leq C\|x\|^N. \tag{2.6}$$

Choose $x^* \in X^*$ be such that $x^*(u_1) = 1$ and $x^*(u_j) = 0$ for $j > 1$. We define the $(N+1)$-homogeneous polynomial $P : X \to X$ by

$$P(x) = x^*(x) \sum_{j=1}^{\infty} \frac{Q(\sigma_{2j}(x))}{Q(u_{2j})}\, u_j,$$

which is well defined and continuous by inequality (2.6). Let $A_P : H_b(X) \to H_b(X)$ be the composition operator $A_P(f) = f \circ P$.

We now define $\varphi \in M_b(X)$ by $\varphi(f) = \lim_\Gamma f(u_n)$, where Γ is any ultrafilter on $\mathbb{N}$ containing $\{\{2n, 2(n+1), 2(n+2), \ldots\} : n \in \mathbb{N}\}$. Let us see that θ_{A_P} is not continuous on $S(\varphi)$. If it were, $\theta_{A_P}(S(\varphi))$ should be contained in $S(\theta_{A_P}(\varphi))$. In particular, there should exist $x^{**} \in X^{**}$ such that

$$\theta_{A_P}(\varphi \circ \tau_{u_1}) = \theta_{A_P}(\varphi) \circ \tau_{x^{**}}. \tag{2.7}$$

Note that, for $f \in H_b(X)$, $\theta_{A_P}(\varphi) \circ \tau_{x^{**}}(f) = \varphi\left(y \mapsto \bar{f}(P(y) + x^{**})\right)$. Since $P(u_{2n}) = 0$, $\bar{f}(P(u_{2n}) + x^{**}) = \bar{f}(x^{**})$ for all $n \in \mathbb{N}$, we have

$$\theta_{A_P}(\varphi) \circ \tau_{x^{**}}(f) = \varphi(\bar{f}(x^{**})1_X) = \bar{f}(x^{**}) \tag{2.8}$$

for all $f \in H_b(X)$, where 1_X is the constant one function on X.

Let's now compute $\theta_{A_P}(\varphi \circ \tau_{u_1})$. We have that

$$\theta_{A_P}(\varphi \circ \tau_{u_1})(f) = \varphi\left(x \mapsto f \circ P(x + u_1)\right).$$

Since $P(u_{2n} + u_1) = u_n$, we have that $\varphi\left(x \mapsto f \circ P(x + u_1)\right) = \lim_\Gamma f(u_n)$ and then

$$\theta_{A_P}(\varphi \circ \tau_{u_1})(f) = \lim_\Gamma f(u_n) \tag{2.9}$$

for all $f \in H_b(X)$.

In particular, if $x^* \in X^*$, we obtain from equations (2.7), (2.8) and (2.9) that $x^{**}(x^*) = \overline{x}^*(x^{**}) = \lim_\Gamma x^*(u_n) = 0$. Since this happens for any $x^* \in X^*$, we obtain that $x^{**} = 0$. But if we consider $F(x) = \sum_{j=1}^\infty \frac{Q(\sigma_j(x))}{Q(u_j)}$, $\lim_\Gamma F(u_n) = 1$. This should be equal to $F(0) = 0$. From this contradiction we conclude that θ_{A_P} is not continuous on $S(\varphi)$. $\square$

If $A : H_b(X) \to H_b(Y)$ is not an $AB-$composition homomorphism, θ_A may not even be continuous on $\bar{\delta}(Y^{**})$.

Corollary 33 *Let X be a symmetrically regular Banach space with an unconditional finite dimensional Schauder decomposition and suppose that there exists a continuous N-homogeneous polynomial which is not weakly sequentially continuous. Then there exists a continuous homomorphism $A : H_b(X) \to H_b(X)$ such that θ_A is not continuous in $\bar{\delta}(X^{**})$.*

PROOF. Let P and φ be defined as in the previous theorem. If we define $A : H_b(X) \to H_b(X)$ by $A(f)(x) = \varphi \circ \tau_x(f)$, then θ_A is not continuous over $\delta(X)$. To see this, note that $\theta_A(\delta_x)(f) = A(f)(x) = \varphi \circ \tau_x(f \circ P) = \theta_{A_P}(\varphi \circ \tau_x)(f)$. Therefore, $\theta_A(\delta_0) = \theta_{A_P}(\varphi)$ and $\theta_A(\delta_{u_1}) = \theta_{A_P}(\varphi \circ \tau_{u_1})$, hence $\theta_A(\delta(X))$ is not contained in a single sheet of $M_b(X)$. Consequently θ_A is not continuous on $\bar{\delta}(X^{**})$. $\square$

Note that the homomorphism A given in the previous corollary can be written as $Af(x) = \lim_\Gamma f \circ P(x + u_n)$. If we consider $X = \ell_p$, $1 \le p < \infty$, the polynomial P in Theorem 32 can be chosen to have a much simpler expression. Proceeding as in the proof of Theorem 32 and taking in the last step $F(x) = \sum_{j=1}^\infty x_j^N$, with $N \ge p$, we obtain:

Example 34 Let $P : \ell_p \to \ell_p$ be given by

$$P(x) = x_1 \sum_{j=1}^{\infty} x_{2j} e_j.$$

Then θ_{A_P} is not continuous on $M_b(\ell_p)$. Moreover, if $A : H_b(\ell_p) \to H_b(\ell_p)$ is defined by $A(f)(x) = \lim_\Gamma f \circ P(x + e_n)$, then θ_A is not continuous over $\delta(\ell_p)$.

If the Banach space X has a weakly null symmetric basis $\{e_n\}_n$ (see [22, Definition 3.a,.1]) and the N-homogeneous polynomial Q satisfies $\lim Q(e_n) \neq 0$, we can even obtain a composition *isomorphism* $A : H_b(X) \to H_b(X)$ such that θ_A is not continuous.

Theorem 35 *Let X be a symmetrically regular Banach space with a weakly null symmetric basis $\{e_n\}_n$ and suppose there exists a homogeneous polynomial Q such that $\lim_n Q(e_n) \neq 0$. Then there exists a biholomorphic polynomial $g : X \to X$ such that the composition algebra-isomorphism $A_g : H_b(X) \to H_b(X)$ given by $A_g f = f \circ g$ induces a non-continuous θ_{A_g}.*

PROOF. Take $\varepsilon > 0$ and a subsequence $(e_{n_k})_k$ such that $|Q(e_{n_k})| > \varepsilon$ for all k. We may suppose that $n_1 > 1$. If $x \in X$, $x = \sum_n x_n e_n$, we define:

$$P(x) = x_1 \sum_{k=1}^{\infty} x_{n_{2k}} e_{n_{2k-1}}.$$

and the projection

$$\Pi(x) = x - \sum_{k=1}^{\infty} x_{n_{2k-1}} e_{n_{2k-1}}.$$

Now we set $g : X \to X$ by $g(x) = x + P(x)$, $x \in X$. Since $P(\Pi(x)) = P(x)$ and $\Pi(P(x)) = 0$ for all $x \in X$, it is easy to check that $g^{-1}(y) = y - P(y)$ for all $y \in X$, which shows that g is biholomorphic.

Let $\varphi \in M_b(X)$ be given by $\varphi(f) = \lim_\Gamma f(e_{n_{2k}})$, where Γ is any ultrafilter on $\mathbb{N}$ containing $\{\{k, (k+1), (k+2), \ldots\} : k \in \mathbb{N}\}$. Let us compute $\theta_{A_g}(\varphi \circ \tau_{e_1})(f)$, for $f \in H_b(X)$. First, $\tau_{e_1}(f \circ g)(e_{n_{2k}}) = f \circ g(e_{n_{2k}} + e_1) = f(e_{n_{2k}} + e_1 + e_{n_{2k-1}})$. Therefore,

$$\theta_{A_g}(\varphi \circ \tau_{e_1})(f) = \varphi(\tau_{e_1}(f \circ g)) = \lim_\Gamma f(e_{n_{2k}} + e_1 + e_{n_{2k-1}}) \tag{2.10}$$

On the other hand, let $x^{**} \in X^{**}$. Since $g(e_{n_{2k}}) = e_{n_{2k}} + P(e_{n_{2k}}) = e_{n_{2k}}$, it follows that $(\tau_{x^{**}}(f) \circ g)(e_{n_{2k}}) = \tau_{x^{**}}(f)(e_{n_{2k}}) = \overline{f}(e_{n_{2k}} + x^{**})$. So we have

$$\theta_{A_g}(\varphi) \circ \tau_{x^{**}}(f) = \lim_{\Gamma}(\tau_{x^{**}}(f) \circ g)(e_{n_{2k}}) = \lim_{\Gamma} \overline{f}(e_{n_{2k}} + x^{**}). \tag{2.11}$$

Suppose there exists $x^{**} \in X^{**}$ such that $\theta_{A_g}(\varphi \circ \tau_{e_1}) = \theta_{A_g}(\varphi) \circ \tau_{x^{**}}$. The right-hand sides of equations (2.10) and (2.11) must coincide, in particular, for any $f \in X^*$. The basis $\{e_n\}_n$ is weakly null and then $x^{**} = e_1$.

Now we set $f_0(x) = Q(\sum_{k=1}^{\infty} x_{n_{2k-1}} e_{n_{2k-1}})$. We have $\overline{f}_0(e_{n_{2k}} + x^{**}) = f_0(e_{n_{2k}} + e_1) = Q(0) = 0$. By equation (2.11), $\theta_{A_g}(\varphi) \circ \tau_{x^{**}}(f_0) = 0$. However, $|f_0(e_{n_{2k}} + e_1 + e_{n_{2k-1}})| = |Q(e_{n_{2k-1}})| > \varepsilon$ which, by equation (2.10), means that $\theta_{A_g}(\varphi \circ \tau_{e_1})(f_0) \neq 0$, a contradiction.

Hence $\theta_A(\varphi \circ \tau_{e_1})$ does not belong to $S(\theta_A(\varphi))$ and therefore θ_A is not continuous on $S(\varphi)$. □

Examples fulfilling the hypotheses of this theorem are the spaces ℓ_p for $1 < p < \infty$ and $Q(x) = \sum_{n=1}^{\infty} x_n^r$ with $r \in \mathbb{N}, r \geq p$.

Maybe it is not clear at first sight, but above theorem is partially based on Henon mappings $h : \mathbb{C}^2 \to \mathbb{C}^2$, $h(z, u) := (f(z) - cu, z)$ where $f(z)$ is an entire function and c a nonzero complex constant. We want to thank Lawrence Harris for pointing out their existence to us.

In the second part of this section we have studied the continuity of mappings on $M_b(X)$ which were induced by multiplicative linear operators on $H_b(X)$. We now show an example of a natural mapping $\theta : M_b(X) \to M_b(X)$ which is continuous on each sheet but is not associated to any homomorphism $A : H_b(X) \to H_b(X)$. Recall that given φ, $\psi \in M_b(X)$ we can define their convolution $\varphi * \psi$ as

$$\varphi * \psi(f) = \psi(x \mapsto \varphi \circ \tau_x(f)).$$

In [4, Example 3.4 and Remark 3.5], the authors present examples of pairs of elements φ_0, ψ_0 in $M_b(X)$ with $\varphi_0 * \psi_0 \neq \psi_0 * \varphi_0$. Define $\theta : M_b(X) \to M_b(X)$ as $\theta(\psi) = \psi * \varphi_0$. Since $\psi \circ \tau_{x^{**}} = \delta_{x^{**}} * \psi$, we have that $\theta(\psi \circ \tau_{x^{**}}) = (\delta_{x^{**}} * \psi) * \varphi_0 = \delta_{x^{**}} * (\psi * \varphi_0) = \theta(\psi) \circ \tau_{x^{**}}$. This means that θ is continuous (in fact it is analytic) on each sheet. Suppose now that $\theta = \theta_A$ for some homomorphism $A : H_b(X) \to H_b(X)$. In this case, $A(f)(x) = \theta(\delta_x)(f) = \delta_x * \varphi_0(f) = \varphi_0 \circ \tau_x(f)$ and consequently $\theta(\psi_0)(f) = \theta_A(\psi_0)(f) = \psi_0(Af) = \psi_0(x \mapsto \varphi_0 \circ \tau_x(f)) = \varphi_0 * \psi_0(f)$. This is a contradiction, since by definition $\theta(\psi_0) = \psi_0 * \varphi_0 \neq \varphi_0 * \psi_0$.

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