A CHARACTERIZATION OF COMPOSITION OPERATORS ON ALGEBRAS OF ANALYTIC FUNCTIONS

DANIEL CARANDO

ABSTRACT. We give a characterization of composition operators between algebras of analytic functions on a Banach space. We show (under fairly general conditions) that they are precisely the multiplicative operators that are transposes of operators of between the preduals of the algebras. The special cases of $H_\infty(U)$ and $H_b(U)$ are considered. In this cases, composition operators are those which are pointwise to pointwise continuous and (or) τ_0 to τ_0 continuous (where τ_0 is the compact-open topology). We obtain Banach-Stone type theorems for these algebras.

INTRODUCTION

In this note we give a characterization of composition operators on algebras of analytic functions on a Banach space. Consider $\mathcal{F}(U)$ an algebra of scalar-valued analytic functions on an open subset U of a Banach space E. An operator $A : \mathcal{F}(U) \to \mathcal{F}(V)$ is a composition operator if there exists a mapping $g : V \to U$ (of the vector-valued class $\mathcal{F}$, to be appropriately defined) such that $Af = f \circ g$ for each $f \in \mathcal{F}(U)$. Several authors have dealt with composition operators in different spaces or algebras of analytic functions on infinite dimensional spaces (Aron, Galindo and Lindström [3], Bonet, Domański, Lindström and Taskinen [4], García, Maestre and Sevilla [10], etc.). Many of these works relate properties of the operator A (such as compacity, complete continuity, etc.) with properties of the function g ("size" of the range, different kinds of continuity, etc.) Consequently, the representation of an operator A as a composition operator proves useful to study different characteristics of A. Motivated by this, we want to find out conditions for the existence of such a representation. In other words, we want a characterization of the operators $A : \mathcal{F}(U) \to \mathcal{F}(V)$ which are composition operators. For this, the existence of preduals of the algebra under consideration and the description of the multiplicative elements of this preduals are the main tools. We show under fairly general conditions that a multiplicative operator is a composition operators if and only if it is the transpose of a continuos operator between the preduals of the algebras.

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We also focus on two particular cases: the algebras of bounded holomorphic functions $H_\infty(U)$ and of holomorphic functions of bounded type $H_b(U)$. In both cases, we give more equivalent conditions for a mapping to be a composition operator. This allows us to prove a Banach-Stone type result for these algebras.

In the first section we describe a general construction linearizing functions. This construction is taken from [7] and we briefly sketch it for completeness and to fix notation. The second section deals with the general results. We consider $\mathcal{F}(U)$ an algebra of analytic functions on U where polynomials are weak-star dense (U a $\mathcal{F}$-regular open subset of a Banach space with the bounded approximation property). We show that a multiplicative operator $A : \mathcal{F}(U) \to \mathcal{F}(V)$ is a composition operator if and only if $A = T'$, where T is a continuous linear operator between the preduals of $\mathcal{F}(V)$ and $\mathcal{F}(U)$. In the third section we restrict ourselves to the algebras of bounded holomorphic functions and holomorphic functions of bounded type. We show that in these cases, to be a composition operator is equivalent to be pointwise to pointwise continuous and also to be τ_0 to τ_0 continuous. Finally, we obtain, under certain conditions, that if the algebras $(H_\infty(U), \tau_0)$ and $(H_\infty(V), \tau_0)$ are isomorphic, the Banach spaces containing U and V must be linearly isomorphic. An analogous result holds for the algebra $H_b(U)$.

1. Preliminaries

In [7], a general construction linearizing functions with values in locally convex spaces is given. We briefly describe this construction.

First, we take a space $\mathcal{F}(U)$ of functions $f : U \longrightarrow \mathbb{C}$ which are continuous. A space $\mathcal{F}_*(U)$ and a map $e : U \longrightarrow \mathcal{F}_*(U)$ are constructed such that for any $f \in \mathcal{F}(U)$, there is $L_f \in \mathcal{F}_*(U)'$ such that $L_f \circ e = f$. A sketch of this construction follows.

Consider the vector space $\mathbb{C}^{(U)}$ of finitely supported families of U-indexed complex numbers. A typical element will be denoted by $s = \sum_{x\in U} a_x e_x$, with $e_x(y) = \delta_{xy}$. Note that the sum is finite. For any given $f \in \mathcal{F}(U)$, we define the seminorm

$$p_f(s) = \left| \sum_{x\in U} a_x f(x) \right| .$$

Now

$$\mathcal{N} = \{s \in \mathbb{C}^{(U)} : \sum_{x\in U} a_x f(x) = 0 \text{ for all } f \in \mathcal{F}(U)\}$$

is a subspace of $\mathbb{C}^{(U)}$, and we define X as the quotient

$$X = \mathbb{C}^{(U)}/\mathcal{N}.$$

We will continue to denote the class of e_x by e_x, and the class of s by $s = \sum_{x \in U} a_x e_x$. We endow X (just for a moment) with the topology τ generated by the seminorms p_f. It is clear that a function $f \in \mathcal{F}(U)$ factors through e in the following way

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathbb{C} \\ e \downarrow & \nearrow_{L_f} & \\ X & & \end{array}$$

where $L_f(s) = \sum_x a_x f(x)$ if $s = \sum_x a_x e_x$. The map $\mathcal{F}(U) \longrightarrow X'$ defined by $f \mapsto L_f$ is then an algebraic isomorphism.

Next, we consider on X the strongest locally convex topology compatible with τ for which the map $e : U \longrightarrow X$ is continuous, and denote X with this topology by (X, α). We take $\mathcal{F}_*(U)$ to be the completion of (X, α)

$$\mathcal{F}_*(U) = \widehat{(X, \alpha)}.$$

We still denote by e the continuous map $U \longrightarrow X \longrightarrow \mathcal{F}_*(U)$. In fact, the topology α on $\mathcal{F}(U)'_\alpha$ is the topology of uniform convergence on the equicontinuous τ_p-compact disks of $\mathcal{F}(U)$ (τ_p denotes the topology of pointwise convergence). Here, equicontinuous means equicontinuous as functions on U.

$\mathcal{F}(U)$ identifies with $\mathcal{F}_*(U)'$. Moreover, in many situations $\mathcal{F}(U)$ is the strong dual of $\mathcal{F}_*(U)$. This happens, for example, with $H_b(U)$ or $H_\infty(U)$. Although there may be more than one topology of interest on $\mathcal{F}(U)$, we will consider on $\mathcal{F}(U)$ the topology which makes it the strong dual of $\mathcal{F}_*(U)$.

Given a class $\mathcal{F}(U)$ of continuous scalar-valued functions, and a locally convex space F, we will say that $g : U \longrightarrow F$ is weakly in $\mathcal{F}$ if g is continuous and for every $\gamma \in F'$, $\gamma \circ g \in \mathcal{F}(U)$. We denote by $\omega\mathcal{F}(U, F)$ the space of all such functions. For bounded holomorphic functions or holomorphic functions of bounded type, this coincides with the vector-valued notion of these classes of functions. $\mathcal{F}_*(U)$ also factors functions on $\omega\mathcal{F}(U, F)$:

Theorem 1. [7, Thm 3] *Each function in $\omega\mathcal{F}(U, F)$ factors linearly through e:*

$$\begin{array}{ccc} U & \rightarrow & F \\ e \downarrow & \nearrow_{L} & \\ \mathcal{F}_*(U) & & \end{array}$$

identifying $L(\mathcal{F}_*(U), F)$ with $\omega\mathcal{F}(U, F)$ algebraically.

2. General results

Let U be an open subset of the a locally convex space E. If the class $\mathcal{F}(U)$ contains E', the mapping e defined in the previous section is one to one [7, Prop. 1]. Moreover, if E is a Banach space, e is in fact bicontinuous:

Lemma 2. *If E is a Banach space and $\mathcal{F}(U)$ contains E', then U is homeomorphic to $e(U)$.*

Proof. Since e is always continuous, we only need show that the topology α on $e(U)$ is stronger than that induced by the norm. This is a consequence of the fact that the unit ball of E' is one of the equicontinuous τ_p-compact disks in $\mathcal{F}(U)$ which define the topology α. □

If $E' \subset \mathcal{F}(U)$, we can consider the projection $\pi : \mathcal{F}(U)' \to E''$ given by $\pi(\phi) = \phi\mid_{E'}$. When dealing with some algebras of analytic functions, evaluation on elements of the bidual E'' is defined, so the range of π contains elements in E''. However, the elements of $\mathcal{F}_*(U)$ are always mapped into E, i.e., $\pi(\mathcal{F}_*(U))$ is contained in E:

Proposition 3. *Let E be a Banach space and suppose $E' \subset \mathcal{F}(U)$. If $\phi \in \mathcal{F}_*(U)$, then $\pi(\phi)$ belongs to E.*

Proof. Let $z = \pi(\phi)$ and take $(s_t)_t \subset X$ such that $s_t \to \phi$ in the topology α. Each s_t can be written as

$$s_t = \sum a_i^t e_{x_i^t}.$$

Let $x_t = \sum a_i^t x_i^t$. For $\gamma \in B_{E'}$ we have that

$$\gamma(x_t) = \sum a_i^t \gamma(x_i^t) = L_\gamma(s_t),$$

which converges to $L\gamma(\phi) = z(\gamma)$ uniformly for $\gamma \in B_{E'}$. This means that x_t converges to z in norm and then $z \in E$. □

We now assume that $\mathcal{F}(E)$ is an algebra of analytic functions and denote by $\mathcal{F}_*^M(E)$ the set of elements in $\mathcal{F}_*(E))$ which are multiplicative. Clearly, $e(E)$ is contained in $\mathcal{F}_*(E)$. We see that these two sets often coincide.

Proposition 4. *Let E be a Banach space with the bounded approximation property and $\mathcal{F}(E)$ be an algebra of analytic functions where polynomials are $\sigma(\mathcal{F}(E), \mathcal{F}_*(E))$-dense. Then, $\mathcal{F}_*^M(E) = e(E)$.*

Proof. Let $\Phi \in \mathcal{F}_*^M(E)$ and $z = \Phi|_{E'}$. Proposition 3 implies that $z \in E$. Let us see that $\Phi(P) = P(z)$ for every $P \in P(^kE)$, for any $k \in \mathbb{N}$.

We fix P. By the bounded approximation property, for every compact subset $K \subset E$ and every $\varepsilon > 0$ there exists a finite type polynomial $P_{K,\varepsilon}$ such that $\|P - P_{K,\varepsilon}\|_K < \varepsilon$ and the collection $(P_{K,\varepsilon})_{K,\varepsilon}$ is bounded. Note that, since $P_{K,\varepsilon}$ is a finite type polynomial and Φ is linear and multiplicative,

$$\Phi(P_{K,\varepsilon}) = P_{K,\varepsilon}(z). \tag{1}$$

Let $(s_t)_t \subset X$ such that $s_t \stackrel{\alpha}{\to} \Phi$. If we denote $B = \Gamma\overline{\{P_{K,\varepsilon}\}\bigcup\{P\}} \subset P({}^kE)$, then $s_t \to \Phi$ uniformly on B (B is bounded and therefore equicontinuous and τ_p-compact). Given $\varepsilon > 0$, there exists t_0 such that

$$|L_Q(s_{t_0}) - \Phi(Q)| < \varepsilon \quad \forall Q \in B. \tag{2}$$

Since s_{t_0} can be written as $s_{t_0} = \sum a_i x_i$ (finite sum) there exists a finite type polynomial $P_0 \in B$ such that:

$$|L_{P_0}(s_{t_0}) - L_P(s_{t_0})| < \varepsilon \tag{3}$$

$$|L_{P_0}(z) - L_P(z)| < \varepsilon \text{ (equivalently, } |P_0(z) - P(z)| < \varepsilon) \tag{4}$$

So we have:

$$\begin{aligned} |\Phi(P) - P(z)| &\leq |\Phi(P) - L_P(s_{i_0})| + |L_P(s_{i_0}) - L_{P_0}(s_{i_0})| \\ &\quad + |L_{P_0}(s_{i_0}) - \Phi(P_0)| + |\Phi(P_0) - P_0(z)| + |P_0(z) - P(z)| \\ &< \varepsilon + \varepsilon + \varepsilon + 0 + \varepsilon \end{aligned}$$

by (2), (3), (2), (1) and (4).

Consequently, $\Phi(P) = P(z)$ for every $P \in P({}^kE)$, for any $k \in \mathbb{N}$ and, by density, $\Phi = e_z$. □

Let U be an open subset of the Banach space E and $\mathcal{F}(U)$ be an algebra of analytic functions in which polynomials are $\sigma(\mathcal{F}(U), \mathcal{F}_*(U))$-dense. We define

$$U_{\mathcal{F}} := \pi(\mathcal{F}_*^M(U))$$

Given Φ in $\mathcal{F}_*^M(U)$ we set $z = \pi(\Phi)$. As in the previous proposition, we can see that $\Phi(P) = P(z)$ for each polynomial P on E. As a consequence, if $\pi(\Phi_1) = \pi(\Phi_2)$, we obtain $\Phi_1(P) = \Phi_2(P)$ for all P and, by density, $\Phi_1 = \Phi_2$. Therefore, we have that π is a bijection between $U_{\mathcal{F}}$ and $\mathcal{F}_*^M(U)$.

If $f \in \mathcal{F}(U)$, we can define $f_{\mathcal{F}} : U_{\mathcal{F}} \to \mathbb{C}$ by $f_{\mathcal{F}}(z) = \Phi(f)$, where $\pi(\Phi) = z$. The remarks in the previous paragraph show that $f_{\mathcal{F}}$ is well defined. Note that the mapping $f \mapsto f_{\mathcal{F}}$ is linear and multiplicative. Moreover, if we take $z \in U_{\mathcal{F}}$ and define $e^{\mathcal{F}}(z)$ as $e^{\mathcal{F}}(z)(f) = f_{\mathcal{F}}(z)$, we have a mapping $e^{\mathcal{F}} : U_{\mathcal{F}} \to \mathcal{F}_*^M(U)$ which is the inverse of π. Also, $e^{\mathcal{F}}$ is an extension of the mapping e defined in the previous section. With this notation we have:

Lemma 5. *Let E be a Banach space with the bounded approximation property. Let $\mathcal{F}(U)$ be an algebra of analytic functions where polynomials are $\sigma(\mathcal{F}(U), \mathcal{F}_*(U))$-dense. Then $\mathcal{F}_*^M(U) = e^{\mathcal{F}}(U_{\mathcal{F}})$.*

We will say that an open subset U of E is *$\mathcal{F}$-regular* if $U_{\mathcal{F}} = U$. Clearly, if U is $\mathcal{F}$-regular, e coincides with $e^{\mathcal{F}}$. In what follows, V will be an open subset of a Banach space G. When necessary, we will use the notation e_U for the mapping e associated to an open set U.

In this setting, a mapping $A : \mathcal{F}(U) \to \mathcal{F}(V)$ will be called a *composition operator* if there exists $g \in w\mathcal{F}(V,U)$ such that $Af = f \circ g$ for all $f \in \mathcal{F}(U)$. We have the following characterization of composition operators:

Theorem 6. *Let E be a Banach space with the bounded approximation property, $\mathcal{F}(U)$ be an algebra of analytic functions on U where polynomials are $\sigma(\mathcal{F}(U), \mathcal{F}_*(U))$-dense. Suppose that U is $\mathcal{F}$-regular and let $A : \mathcal{F}(U) \to \mathcal{F}(V)$ be an homomorphism. A is a composition operator if and only if $A = T'$, where $T : \mathcal{F}_*(V) \to \mathcal{F}_*(U)$ is a continuous operator.*

Proof. If $A : \mathcal{F}(U) \to \mathcal{F}(V)$ is a composition operator, say $A(f) = f \circ g$, we can consider the mapping $e_U \circ g : V \to \mathcal{F}_*(U)$. For each $L = L_f \in (\mathcal{F}_*(U))'$, we have

$$L_f \circ e_E \circ g = f \circ g = A(f) \in \mathcal{F}(V).$$

Therefore, $e_E \circ g$ belongs to $w\mathcal{F}(V, \mathcal{F}_*(U)))$ and there exists a continuous operator $T = T_{e_E \circ g} : \mathcal{F}_*(V) \to \mathcal{F}_*(U)$ such that $e_U \circ g = T \circ e_V$. It is easy to check that $A = T'$

Conversely, suppose $A = T'$. Since A is a multiplicative operator, $T(e(V))$ is contained in $\mathcal{F}_*^M(U) = e(U)$ (the last equality follows from the previous lemma and the fact that U is $\mathcal{F}$-regular). Therefore, we can define $g : V \to U$ by $g(y) = x$ if $T(e_y) = e_x$. Now, $Af(y) = T'(L_f)(e_y) = L_f(T(e_y)) = L_f(e_{g(y)}) = f(g(y))$ and then $Af = f \circ g$.

Let us see that g belongs to $w\mathcal{F}(V,U)$. By lemma 2, U and V are homeomorphic to $e(U)$ and $e(V)$ and, since T is continuous, g is continuous. Moreover, if $\gamma \in E'$, $\gamma \circ g = A\gamma \in \mathcal{F}(V)$. So, $g \in w\mathcal{F}(V,U)$. □

Suppose U is not $\mathcal{F}$-regular but there exists $g \in w\mathcal{F}(V,E)$ whose image is contained in $U_{\mathcal{F}}$ such that $Af(x) = f_\tau \circ g(x)$ for all $x \in V$. Then we can find $T : \mathcal{F}_*(V) \to \mathcal{F}_*(V)$ such that $A = T'$ just as before. Conversely, if $A = T'$, the previous construction leads us to a mapping $g : V \to U_{\mathcal{F}}$ such that $Af(x) = f_\tau \circ g(x)$. In fact, g belongs to $w\mathcal{F}(V,E)$. To see this, we must show that g is continuous. But $g = \pi_U \circ T \circ e_V$. Since $B_{E'}$ is an equicontinuous τ_p-compact subset of $\mathcal{F}(U)$, π_U is continuous from $(\mathcal{F}_*^M(U), \alpha)$ to $(U_{\mathcal{F}}, \|\cdot\|)$ and therefore so is g. Also, $\gamma \circ g = A(\gamma) \in \mathcal{F}(U)$ for all $\gamma \in E'$. We have shown the following:

Corollary 7. *Let E be a Banach space with the bounded approximation property, $\mathcal{F}(U)$ be an algebra of analytic functions on U where polynomials are $\sigma(\mathcal{F}(U), \mathcal{F}_*(U))$-dense. Let $A : \mathcal{F}(U) \to \mathcal{F}(V)$ be an homomorphism. The following are equivalent:*

a) There exists a function $g \in w\mathcal{F}(V,E)$ whose image is contained in $U_{\mathcal{F}}$ such that $Af(x) = f_{\mathcal{F}} \circ g(x)$ for all $x \in V$

b) $A = T'$, for some continuous operator $T : \mathcal{F}_*(V) \to \mathcal{F}_*(U)$.

In the previous results, the hypothesis were used only in one direction. With the same proof, we see that a composition operator is the transpose of an operator between the preduals in a much more general situation:

Proposition 8. *Let E and G be locally convex spaces and $U \subset E$, $V \subset G$ open subsets. If $A : \mathcal{F}(U) \to \mathcal{F}(V)$ is a composition operator then there exists a continuous operator $T : \mathcal{F}_*(V) \to \mathcal{F}_*(U)$ such that $A = T'$.*

3. Operators on $H_\infty(U)$ and $H_b(U)$

$H_\infty(U)$ denotes the space of bounded holomorphic functions on U. This is a Banach space when equipped with the supremum norm. J. Mujica [12] has constructed a Banach predual $G_\infty(U)$ of $H_\infty(U)$. A characterization of this predual is

$$G_\infty(U) = \{\phi \in H_\infty(U)' : \phi \text{ is } \tau_0\text{-continuous on the unit ball of } H_\infty(U)\}.$$

The construction given in the first section coincides with $G_\infty(U)$ [7]. We obtain the following characterization of composition operators on $H_\infty(U)$:

Theorem 9. *Let $A : H_\infty(U) \to H_\infty(V)$ be a multiplicative operator. If E has the bounded approximation property and U is bounded and absolutely convex, then the following conditions are equivalent:*

a) A is a composition operator.

b) $A = T'$, for some continuous operator $T : G_\infty(V) \to G_\infty(U)$

c) A is $\tau_p - \tau_p$-continuous.

d) A is $\tau_0 - \tau_0$-continuous.

Proof. $a) \Rightarrow c)$ and $a) \Rightarrow d)$ are straightforward. We suppose $c)$ holds and let $\phi \in G_\infty(V)$. By $c)$ and the characterization of $G_\infty(V)$, $\phi \circ A$ is τ_p-continuous on the unit ball of $H_\infty(U)$. The topologies τ_p and τ_0 coincide on this unit ball [8, Lemma 3.25]. We then have that $\phi \circ A$ is τ_0-continuous on the unit ball of $H_\infty(U)$ and consequently $\phi \circ A$ belongs to $G_\infty(U)$. This shows that $A' : H_\infty(V)' \to H_\infty(U)'$ maps $G_\infty(V)$ in $G_\infty(U)$. Therefore, A is the transpose of the restriction of A' to $G_\infty(V)$ and we have $b)$. In an analogous way we have that $d) \Rightarrow b)$. Finally, to see that $b) \Rightarrow a)$ let us check that we may apply theorem 6. Since U is absolutely convex, it is H_∞-regular. Indeed, if $z \in U_{H_\infty} \setminus U$, we choose $\gamma \in E'$ such that $\gamma(z) = 1$ and $\sup_{x \in U} |\gamma(x)| < 1$. The sequence $(\gamma^n)_n$ is in the unit ball of $H_\infty(U)$ and τ_0-converges to 0. If Φ is such that $\pi(\Phi) = z$, we have on the one hand that $\Phi(\gamma^n) \to 0$. On the other hand, $\Phi(\gamma^n) = \Phi(\gamma)^n = \gamma(z)^n = 1$, which is a contradiction. Moreover, polynomials are $(H_\infty(U), G_\infty(U))$-dense: the Taylor series expansion of a function $f \in H_\infty(U)$ τ_0-converges to f and is bounded. Since elements in $G_\infty(U)$ are τ_0-continuous on bounded subsets of $H_\infty(U)$, density follows. Now $a)$ is a consequence of theorem 6. $\square$

Now we turn our attention to holomorphic functions of bounded type. We say that $B \subset U$ is a U-*bounded* set if it is bounded and $\mathrm{dist}(B, E \setminus U) > 0$. We will denote by $H_b(U)$ the space of holomorphic functions $f : U \to \mathbb{C}$ that are bounded on U-bounded sets, i.e., $\|f\|_B := \sup\{|f(x)| : x \in B\} < \infty$ for all U-bounded set B. $H_b(U)$ is a Fréchet algebra when endowed with the topology of the uniform convergence on U-bounded subsets of U. A fundamental sequence of U-bounded sets is given by $U_n = \{x \in U : \|x\| < n \text{ and } d(x, U^c) > \frac{1}{n}\}$. The topology on $H_b(U)$ is given by uniform convergence on each U_n.

When the construction described in the first section is applied to $H_b(U)$, we obtain the space $\mathcal{P}_b(U)$, constructed by P. Galindo, D. García, and M. Maestre in [9]. A characterization of this space was given in [13]: for each sequence of positive numbers $(\alpha_n)_n$, B^α denotes:

$$B^\alpha = \{f \in H_b(U) : \sup_{U_n} |f| \leq \alpha_n\}.$$

Now, $\mathcal{P}_b(U)$ is the set of all $\phi \in H_b(U)'$ which are τ_0-continuous on B^α for all α. In [9, Proposition 3], the authors give a description of the multiplicative elements of $\mathcal{P}_b(E)$ when E has the approximation property.

If $U \subset E$ is open and absolutely convex, with the same proof that showed the H_∞-regularity of U we can see that it is H_b-regular. Also, polynomials are $(H_b(U), \mathcal{P}_b(U))$-dense. In fact, they are dense with the strong dual topology on $H_b(U)$. Indeed, the Taylor series expansion of a function $f \in H_b(U)$ converges uniformly on U-bounded sets. But uniform convergence on U-bounded sets is precisely the strong dual topology on $H_b(U)$. We have:

Theorem 10. *Let E be a Banach space with the bounded approximation property and let $U \subset E$ and $V \subset G$ be open subsets, U absolutely convex. If $A : H_b(U) \to H_b(V)$ is an homomorphism, the following are equivalent:*
a) A is a composition operator
b) $A = T'$, where $T : \mathcal{P}_b(V) \to \mathcal{P}_b(U)$ is a continuous operator.
c) A is τ_p to τ_p-continuous.
d) A is τ_0 to τ_0-continuous.

Proof. Again, $a) \Rightarrow c)$ and $a) \Rightarrow d)$ are straightforward. We suppose $c)$ holds and let $\phi \in \mathcal{P}_b(V)$. By $c)$ and the characterization of $\mathcal{P}_b(V)$, $\phi \circ A$ is τ_p-continuous on each B^α. Since the subsets B_α are locally bounded, the topologies τ_p and τ_0 coincide on each B^α [8, Lemma 3.25]. We have that $\phi \circ A$ is τ_0-continuous on each B^α and consequently $\phi \circ A$ belongs to $\mathcal{P}_b(U)$. This shows that $A' : H(V)' \to H(U)'$ maps $\mathcal{P}_b(V)$ in $\mathcal{P}_b(U)$. Therefore, A is the transpose of the restriction of A' to $\mathcal{P}_b(V)$ and we have $b)$. In an analogous way we have that $d) \Rightarrow b)$. $b) \Rightarrow a)$ follows from theorem 6. $\square$

If both U and V satisfy the condition of theorem 9 and the operator A is an isomorphism, the function g is invertible, with g^{-1} in $H_\infty(U,V)$. Therefore, an isomorphism between the algebras $(H_\infty(U),\tau_0)$ and $(H_\infty(U),\tau_0)$ is equivalent to the analytic equivalence of U and V. The existence of such biholomorphic functions implies that the spaces E and G are isomorphic as Banach spaces, since the differential at any point is an isomorphism. Moreover, if U and V are the unit balls of E and G, the isomorphism is isometric ([11]). The situation for $H_b(U)$ is analogous. So we have the following results (see also [6, 16, 17], where similar results have been obtained for different algebras):

Corollary 11. *Let $U \subset E$ and $V \subset G$ be bounded, open and absolutely convex and suppose that E and G have the bounded approximation property. $(H_\infty(U),\tau_0)$ and $(H_\infty(V),\tau_0)$ are topological and algebraically isomorphic if and only if U and V are holomorphically equivalent. In this case, E and G are isomorphic Banach spaces. If U and V are the unit balls of E and G, the spaces are isometric.*

Corollary 12. *Let $U \subset E$ and $V \subset G$ open and absolutely convex and suppose that E and G have the bounded approximation property. $(H_b(U),\tau_0)$ and $(H_b(V),\tau_0)$ are topological and algebraically isomorphic if and only if U and V are holomorphically equivalent. In this case, E and G are isomorphic Banach spaces. If U and V are the unit balls of E and G, the spaces are isometric.*

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Departamento de Matemática, Universidad de San Andrés, Vito Dumas 284 (B1644BID) Victoria, Buenos Aires, Argentina.

E-mail address: daniel@udesa.edu.ar