Extendible Polynomials on Banach Spaces

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Abstract

We are concerned with the following question: when can a polynomial $P : E \to X$ (E and X are Banach spaces) be extended to a Banach space containing E? We prove that the polynomials that are extendible to any larger space are presicely those which can be extended to $C(B_{E'})$, if X is complemented in its bidual, and $l_\infty(B_{E'})$ in general. We also show that the extendibility is a property that is preserved by Aron-Berner extensions and composition with linear operators. We construct a predual of the space of extendible polynomials for the case that X is a dual space.

Introduction

Throughout, F and X will be Banach spaces over the real or complex field and E an isometric subspace of F. This note is mainly concerned with the following natural question: when can a continuous k-homogeneous polynomial $P : E \to X$ be extended to a polynomial $\tilde{P} : F \to X$? It is not always possible to extend linear operators if E and X are infinite dimensional (the identity operator on c_0 cannot be extended to l_∞ since c_0 is not complemented in l_∞). For the scalar-valued case (or X finite dimensional), the Hahn-Banach extension theorem gives a positive answer for linear functions, but this result cannot be generalized for polynomials of degree $k \geq 2$. For example, l_2 is contained in $C[0, 1]$ but the polynomial $P(x) = \sum_k x_k^2$ on l_2 cannot be extended to $C[0, 1]$, since this last space has the Dunford-Pettis property and consequently any polynomial on $C[0, 1]$ is weakly sequentially continuous [14]. In [2] it is shown that integral scalar-valued polynomials are extendible to any larger space. Many results are known when there is a linear extension morphism for linear functionals $E' \to F'$ (see [1, 3, 6, 8, 16]).

In the first section we recall some facts about the Aron-Berner extension of a polynomial from a Banach space to its bidual and generalize some known results for the scalar-valued case to the vector-valued case. We show that the Aron-Berner extension of a weakly compact polynomial has its range in the same space as the polynomial. Unlike the linear case, the converse is not true. In the second section we study the space of polynomials $P : E \to X$ which can be extended to a fixed space F containing E, while in the third one we look at those polynomials which can be extended to any larger space. These last polynomials (*extendible* polynomials) turn out to be the polynomials that can be extended to some particular spaces ($C(B_{E'})$ if X is complemented in its bidual and $l_\infty(B_{E'})$ in general). Thus, we can apply the results of section 2 and this enables us to define a natural norm in the space of extendible polynomials and to find some properties such as the stability of the class of extendible polynomials under Aron-Berner extensions and compositions with linear operators. We also find a predual of the space of extendible polynomials in the case that X is a dual space.

Identifying homogeneous polynomials with linear functionals is a useful tool when studying extensions of polynomials, since it sometimes enables us to use the Hahn-Banach extension theorem. It is known [15] that given a polynomial $P \in \mathcal{P}({}^kE, X)$ there is a unique linear operator $T_P : \otimes_s^k E \to X$ such that $P(x) = T_P(x \otimes \cdots \otimes x)$. Moreover, if we endow the tensor product $\otimes_s^k E$ with the projective norm π, the correspondence between $\mathcal{P}({}^kE, X)$ and $\mathcal{L}(\otimes_{s,\pi}^k E, X)$ is an isometric isomorphism. In particular, $\mathcal{P}({}^kE) \simeq \left(\otimes_{s,\pi}^k E\right)'$.

In the case that X is a dual space, say $X = Y'$, we can also define the linear functional P^* on $\left(\otimes_s^k E\right) \otimes Y$ given by $P^*(s \otimes y) = T_P(s)(y)$, for $s \in \otimes_s^k E$ and $y \in Y$. This correspondence gives an isometric isomorphism between $\left(\left(\otimes_{s,\pi}^k E\right) \otimes_\pi Y\right)'$ and $\mathcal{P}({}^kE, X)$. Changing the $\pi-$norm by other norms gives rise to different spaces of polynomials. We are interested in those norms for which the extendibility of a polynomial is related to the continuity of the associated operator.

We refer to [5] and [12] for notation and results regarding polynomials. The author wishes to thank his advisor Nacho Zalduendo, without whose help this work would not have been possible; Andrew Tonge, for some suggestions given to Nacho and Suárez Granero, for the idea of using the Sobczyk theorem in the example after Theorem 3.1.

1 Extensions to the bidual

In [1], Aron and Berner found a way of extending any continuous homogeneous polynomial from E to its bidual (see also [16]). There are several ways of defining this extension. One of them is the following, which we will show for 2-homogeneous polynomials but is easily generalized. Let $P : E \to X$ be a 2-homogeneous polynomial and consider its associated symmetric bilinear function

$$\Phi : E \times E \longrightarrow X.$$

Fix $x \in E$, $z' \in X'$; then $z'(\Phi(x, \cdot))$ is an element of E'. This gives a mapping $E \times X' \to E'$. If we do this again, we will get $X' \times E'' \to E'$ and if we insist we finally obtain the (not necessarily symmetric) bilinear function:

$$\bar{\Phi} : E'' \times E'' \longrightarrow X''.$$

The polynomial $AB(P)(x'') = \bar{\Phi}(x'', x'')$ (from E'' to X'') is called the Aron-Berner extension of P. Observe that if P is a linear operator, the process described above gives the bitranspose of P. Moreover, if T is a linear operator, P a polynomial and we apply this process to the polynomial $T \circ P$ we obtain that $AB(T \circ P) = T'' \circ AB(P)$. In particular, if $\gamma \in X'$, $AB(\gamma \circ P)(z) = AB(P)(z)(\gamma)$, for any $z \in E''$. Now the following characterization of the Aron-Berner extension is an immediate consequence of the result proved in [16] for scalar-valued polynomials. Remember that the *differential* of a polynomial $P \in \mathcal{P}(^kE, X)$ is the $(k-1)-$homogeneous polynomial $DP : E \to \mathcal{L}(E; F)$ given by

$$DP(x) = k \overset{\vee}{P} (k - 1\underbrace{x, \dots, x}, \cdot),$$

where $\overset{\vee}{P}$ is the symmetric k-linear function associated to P.

Proposition 1.1 *If $Q \in \mathcal{P}^k(E'', X'')$ is such that $Q|_E = P$, then $Q = AB(P)$ if and only if*
a) For every $x \in E$, $DQ(x) : E'' \to X''$ is $w - w* -$continuous.*
b) For every $z \in E''$ and $(x_\alpha) \subset E$ such that $x_\alpha \xrightarrow{w*} z$, $DQ(z)(x_\alpha) \xrightarrow{w*} DQ(z)(z)$ in X''.

As a consequence of proposition 1.1 we have that the Aron-Berner extension is a linear morphism from $\mathcal{P}(^kE, X)$ to $\mathcal{P}(^kE'', X'')$, since conditions $a)$ and $b)$ are preserved by sums and scalar multiplications. In [3], Davie and Gamelin proved that in the scalar-valued case the Aron-Berner extension is actually an isometry. This allows us to identify the symmetric tensor product $\otimes^k_{s,\pi} E''$ with a subspace of $\mathcal{P}(^kE)' = \left(\otimes^k_{s,\pi} E\right)''$:

$$\begin{array}{ccc} \otimes^k_{s,\pi} E'' & \longrightarrow & \mathcal{P}(^kE)' \\ z \otimes \cdots \otimes z & \mapsto & e_z \end{array} \tag{1}$$

where $e_z(Q) = AB(Q)(z)$ for $Q \in \mathcal{P}(^kE)$. The Davie and Gamelin's result implies that $\|e_z\| = \|z\|$ for all $z \in E''$.

The following lemma gives an expression for the Aron-Berner extension of a vector-valued polynomial which is sometimes easier to handle than the one given above.

Lemma 1.2 *a) If $\Delta : E \to \otimes^k_s E$ is the polynomial $\Delta(x) = x \otimes \cdots \otimes x$ then $AB(\Delta) : E'' \to \mathcal{P}(^kE)'$ is given by $AB(\Delta)(z) = e_z$.*
b) Let $P : E \to X$ be a $k-$homogeneous polynomial and T_P its associated linear operator. Then $AB(P)(z) = T''_P(e_z)$.

Proof: a) Let $\Delta_0 : E'' \to \mathcal{P}(^kE)'$ be given by $\Delta_0(z) = e_z$. Clearly, $\Delta_0|_E = \Delta$, so we only need to show that Δ_0 satisfies the conditions of proposition 1.1. First note that if $z_1, \dots, z_k \in E''$ and $Q \in \mathcal{P}(^kE)$, then $\overset{\vee}{\Delta}_0 (z_1, \dots, z_k)(Q) = \overset{\vee}{AB(Q)} (z_1, \dots, z_k)$ ($\overset{\vee}{P}$ denotes the symmetric $k-$linear mapping associated to P). Therefore, if $z, w \in E''$,

$$\begin{aligned} (D\Delta_0(z)(w))(Q) &= k \overset{\vee}{\Delta}_0 (z, \dots, z, w)(Q) \\ &= k \overset{\vee}{AB(Q)} (z, \dots, z, w) \\ &= D(AB(Q))(z)(w). \end{aligned}$$

If we put $z = x \in E$, the last expression is $w*-$continuous in w (since $AB(Q)$ satisfies the conditions of the proposition) and consequently $D\Delta_0(z)$ is $w*-w*-$continuous. Similarly we see that Δ_0 satisfies condition $b)$ of proposition 1.1. Hence, $\Delta_0 = AB(\Delta)$.
b) Since $P = T_P \circ \Delta$, $AB(P)(z) = T_P''(AB(\Delta)(z)) = T_P''(e_z)$. ∎

With the help of the lemma, we generalize the Davie-Gamelin result [3] to the vector-valued case:

Proposition 1.3 $AB : \mathcal{P}(^kE, X) \to \mathcal{P}(^kE'', X'')$ *is an isometry.*

Proof: For $z \in E''$ we have

$$\begin{aligned} \|AB(P)(z)\| &= \|T_P''(e_z)\| \leq \|T_P''\| \, \|e_z\| \\ &= \|T_P\| \, \|z\|^k = \|P\| \|z\|^k. \end{aligned}$$

This implies that $\|AB(P)\| \leq \|P\|$ and since $AB(P)(x) = P(x)$ for every $x \in E$, the equality holds. ∎

Unfortunately, if X is not reflexive, the Aron-Berner extension of a polynomial is not an extension in the meaning we give to this word: an extension of $P : E \to X$ to E'' should be a polynomial $\tilde{P} : E'' \to X$ extending P. This sometimes fails to exist: we have already mentioned that the identity operator on c_0 cannot be extended to $c_0'' = l_\infty$. Note that in this case, the Aron-Berner extension is the identity operator on l_∞. If X is complemented in its bidual, there always exists an extension of P to E''. We recall that a Banach space X is called a $\mathcal{C}_l$-space if X is complemented in its bidual with a linear projection $p : X'' \to X$ with $\|p\| \leq l$ (see [1]). In this case, $AB_p(P) = p \circ AB(P)$ is an extension of P to its bidual and $\|AB_p(P)\| \leq l \, \|P\|$.

The Gantmacher theorem (see [9]) states that an operator $T : E \to X$ is weakly compact if and only if $T''(z)$ belongs to X for every $z \in E''$. This means that T is weakly compact if and only if T'' is an extension of T. We say that a polynomial $P : E \to X$ is weakly compact if $P(B_E)$ is relatively weakly compact. For these polynomials we have:

Proposition 1.4 *If* $P : E \to X$ *is a weakly compact* k*-homogeneous polynomial then* $AB(P)(z)$ *belongs to* X *for every* $z \in E''$.

Proof: Since the unit ball of $\otimes_s^k E$ is the closed absolutely convex hull of $\otimes_s^k B_E$, the closure of its image by T_P is the closed absolutely convex hull of $P(B_E)$, which is weakly compact. Therefore, T_P is a weakly compact linear operator and by the Gantmacher theorem the range of T_P'' is contained in X. Using the identification (1) and Lemma 1.2, the associated linear operator of $AB(P)$ is the restriction of T_P'' to the subspace $\otimes_s^k E''$ of $\mathcal{P}(^kE)'$. Consequently, the range of $AB(P)$ is contained in X. ∎

The converse is not true. Indeed, let $P : l_2 \to l_1$ be the polynomial given by $P(x) = (x_n^2)_n$. $P(B_{l_2})$ is the unit ball of l_1 (in the complex case) and therefore is not weakly compact. However, $AB(P)(z) = P(z) \in l_1$ for every $z \in l_2'' = l_2$. Note that any operator from a reflexive space is weakly compact, while this is not true for polynomials, as the example shows.

2 Extending polynomials to a fixed space.

Let E be a closed subspace of a Banach space F. The inclusion $i : E \hookrightarrow F$ induces a one-to-one mapping between the k-fold symmetric tensor products:

$$\otimes_s^k i : \otimes_s^k E \longrightarrow \otimes_s^k F$$

The projective norm π on $\otimes_s^k F$ induces via this mapping a norm on $\otimes_s^k E$, which will be denoted η_F. Then, for $s \in \otimes_s^k E$, we have:

$$\|s\|_{\eta_F} = \left\|\otimes_s^k i(s)\right\|_{\pi,F}$$

A scalar-valued polynomial $P \in \mathcal{P}(^k E)$ can be extended to a continuous polynomial on F if and only if its associated linear functional T_P on $\otimes_s^k E$, is η_F-continuous. This is not true for vector-valued polynomials: for $k = 1$ the η_F-norm is just the norm on E, every continuous operator is η_F-continuous but they are not always extendible. We will call $\mathcal{P}_{e_F}(^k E, X)$ the space of all $P \in \mathcal{P}(^k E, X)$ that can be extended to a polynomial $\tilde{P} \in \mathcal{P}(^k F, X)$ (for scalar-valued polynomials we will write $\mathcal{P}_{e_F}(^k E)$). In this space we can define the norm

$$\|P\|_{e_F} = \inf\left\{\left\|\tilde{P}\right\| : \tilde{P} : F \to X \text{ extends } P\right\} \tag{2}$$

Clearly, $\|P\| \leq \|P\|_{e_F}$. Moreover, if

$$\rho : \mathcal{P}(^k F, X) \longrightarrow \mathcal{P}_{e_F}(^k E, X)$$

is the restriction map, $\|P\|_{e_F} = \inf\left\{\left\|\tilde{P}\right\| : \rho(\tilde{P}) = P\right\}$. It follows that $\left(\mathcal{P}_{e_F}(^k E, X), \|\ \|_{e_F}\right)$ can be seen as the quotient space $\mathcal{P}(^k F, X)/\ker\rho$. We also have the following:

Proposition 2.1 *Let $E \subseteq F$ and X be Banach spaces.*
a) $\left(\mathcal{P}_{e_F}(^k E, X), \|\ \|_{e_F}\right)$ is a Banach space.
b) In the scalar-valued case, $\left(\otimes_{s,\eta_F}^k E\right)' = \left(\mathcal{P}_{e_F}(^k E), \|\ \|_{e_F}\right)$ isometrically.

Proof: a) $\left(\mathcal{P}_{e_F}(^k E, X), \|\ \|_{e_F}\right)$ is a quotient space of a Banach space by a closed subspace.
b) Let T be an η_F-continuous functional on $\otimes_s^k E$ and P_T its associated polynomial. Since $\otimes_{s,\eta_F}^k E$ is an isometric subspace of $\otimes_{s,\pi}^k F$, T extends by the Hahn-Banach theorem to a

linear functional $\tilde{T}$ on $\otimes_{s,\pi}^k F$ with $\|\tilde{T}\|_\pi = \|T\|_{\eta_F}$. The associated polynomial $P_{\tilde{T}}$ extends P to F with $\|P_{\tilde{T}}\| = \|T\|_{\eta_F}$ and therefore $\|P_T\|_{e_F} \leq \|T\|_{\eta_F}$.

On the other hand, if $\tilde{P}$ is an extension of P to F, $T_{\tilde{P}}$ is an extension of T_P to $\otimes_s^k F$ and $\|T_P\|_{\eta_F} \leq \|T_{\tilde{P}}\| = \|\tilde{P}\|$. Taking infimum over all possible extensions of P we get the other inequality. ∎

From the proof of b) it follows that for scalar-valued polynomials, the infimum in (2) is actually a minimum. The same is true for polynomials taking values in a dual space, as we will see below.

Since $\|P\| \leq \|P\|_{e_F}$, if every polynomial $P \in \mathcal{P}({}^kE, X)$ extends to F then $\| \ \|$ and $\| \ \|_{e_F}$ are equivalent norms on $\mathcal{P}({}^kE, X) = \mathcal{P}_{e_F}({}^kE, X)$, since this space is complete with both norms. Therefore, if every k-homogeneous polynomial from E to X is extendible to F, there exists a constant c such that for every $P \in \mathcal{P}({}^kE, X)$, there is an extension $\tilde{P} \in \mathcal{P}({}^kF, X)$ with $\left\|\tilde{P}\right\| \leq c\,\|P\|$. It is not always possible to set $c = 1$, even in scalar-valued polynomials on finite dimensional spaces: in [12], P. Mazet showed the existence of a finite dimensional space F and an hyperplane $H \subset F$ for which $c \geq 2$ in the real case and $c \geq \frac{7}{3}$ in the complex case.

We have already seen that for vector-valued polynomials, η_F-continuity does not assure extendibility to F. In fact, η_F-continuity is related to a weaker notion of extendibility:

Definition 2.2 *A polynomial $P : E \to X$ is said to be weakly extendible to F if for every linear functional γ on X, the scalar-valued polynomial $\gamma \circ P$ extends to F.*

If P extends to a polynomial $\tilde{P}$ on F, then for every $\gamma \in X'$, $\gamma \circ \tilde{P}$ is an extension of $\gamma \circ P$ to F. Thus, a polynomial that extends to F is weakly extendible to F. The converse is not true, since by the Hahn-Banach theorem every linear operator is weakly extendible to any space. The next proposition shows the connection between weak extendibility and η_F-continuity.

Proposition 2.3 *A polynomial $P \in \mathcal{P}({}^kE, X)$ is weakly extendible to F if and only if its associated operator $T_P : \otimes_s^k E \to X$ is η_F-continuous.*

Proof: If $T_P : \otimes_s^k E \to X$ is η_F-continuous, then for every $\gamma \in X'$, $\gamma \circ T_P$ is an η_F-continuous linear functional on $\otimes_s^k E$. It follows from Proposition 2.1 *b)* that $\gamma \circ P$, being its associated polynomial, extends to F.

Now suppose that P is weakly extendible to F. This means that for every $\gamma \in X'$ the polynomial $\gamma \circ P$ extends to F. By the proposition, the linear functional $T_{\gamma \circ P}$ on $\otimes_s^k E$ is η_F-continuous. Since $T_{\gamma \circ P} = \gamma \circ T_P$ it follows that the image by T_P of the unit ball of

$\otimes^k_{s,\eta_F} E$ is a weakly bounded (and therefore bounded) subset of X. This means that T_P is η_F-continuous. ■

We end this section with the construction of a predual of $\mathcal{P}_{e_F}(^kE, X)$ in the case that X is a dual space: $X = Y'$.

Let $P \in \mathcal{P}(^kE, Y')$ and P^* be its associated linear functional (as defined in the introduction). For any pair of tensor norms α and β, we will denote by $\|P^*\|_{\alpha,\beta}$ the norm of P^* as a linear functional on $\left(\otimes^k_{s,\alpha} E\right) \otimes_\beta Y$. We want to find α and β such that the $\alpha, \beta-$norm of a functional coincides with the e_F-norm of the associated polynomial.

We have that $\otimes^k_{s,\eta_F} E$ is an isometric subspace of $\otimes^k_{s,\pi} F$. However, the (non-symmetric) tensor product $\left(\otimes^k_{s,\eta_F} E\right) \otimes_\pi Y$ is not necessarily a subspace of $\left(\otimes^k_{s,\pi} F\right) \otimes_\pi Y$ (not even isomorphically). In fact, it is a subspace if and only if every continuous operator from $\otimes^k_{s,\eta_F} E$ to Y' extends to a continuous operator from $\otimes^k_{s,\pi} F$ to Y'. Using the correspondence given in the previous proposition, this is equivalent to the following fact: every polynomial from E to Y' which is weakly extendible to F, extends to F. In this case we have:

$$\left((\otimes^k_{s,\eta_F} E) \otimes_\pi Y\right)' = \mathcal{L}\left(\otimes^k_{s,\eta_F} E, Y'\right) = \mathcal{P}_{e_F}(^kE, Y')$$

isomorphically.

For the general case, the one-to-one mapping:

$$\left(\otimes^k_s i\right) \otimes I_Y : \left(\otimes^k_{s,\eta_F} E\right) \otimes Y \longrightarrow \left(\otimes^k_{s,\pi} F\right) \otimes_\pi Y$$

where I_Y is the identity operator on Y, induces a norm on $\left(\otimes^k_{s,\eta_F} E\right) \otimes Y$ which will be denoted λ_F. Now we have:

Proposition 2.4 *a) If every polynomial from E to Y' which is weakly extendible to F, extends to F, then $\left((\otimes^k_{s,\eta_F} E) \otimes_\pi Y\right)' = \mathcal{P}_{e_F}(^kE, Y')$ isomorphically.*
b)$\left((\otimes^k_{s,\eta_F} E) \otimes_{\lambda_F} Y\right)' = \mathcal{P}_{e_F}(^kE, Y')$ isometrically.

Proof: a) It was proved above.
b) Let $P : E \to X$ be a polynomial whose associated linear functional P^* belongs to $\left((\otimes^k_{s,\eta_F} E) \otimes_{\lambda_F} Y\right)'$. By Hahn-Banach, it extends to a functional on $\left(\otimes^k_{s,\pi} F\right) \otimes_\pi Y$ with the same norm. This gives an extension of P to F with norm $\|P^*\|_{\eta_F,\lambda_F}$ and so we have that $\|P\|_{e_F} \leq \|P^*\|_{\eta_F,\lambda_F}$. On the other hand, any extension of P to F gives an extension of P^* to $\left(\otimes^k_{s,\pi} F\right) \otimes_\pi Y$. Then, $\|P^*\|_{\eta_F,\lambda_F} \leq \|Q\|$ for any extension Q of P. Taking infimum over all possible extension we conclude that $\|P\|_{e_F} = \|P^*\|_{\eta_F,\lambda_F}$. ■

Remark 2.5 *The proof of Proposition 2.4 shows that if X is a dual space, the infimum in (2) is actually a minimum.*

Corollary 2.6 *If X is a dual space, every polynomial $P : E \to X$ extends to F if and only if the e_F-norm is equivalent to the uniform norm on $\mathcal{P}_{e_F}(^kE, X)$.*

Proof: If the e_F-norm is equivalent to the uniform norm on $\mathcal{P}_{e_F}({}^kE, X)$, it follows that the η_F, λ_F-norm on $\left(\otimes_s^k E\right) \otimes Y$ is equivalent to the π, π-norm . Taking dual with both norms, it follows that every polynomial extend to F. The converse was commented above. ∎

3 Extendible polynomials

Following [10], we will say that a polynomial $P : E \to X$ is extendible if for all Banach spaces F containing E there exists $\tilde{P} \in \mathcal{P}({}^kF, X)$ an extension of P and we will denote the space of all such polynomials by $\mathcal{P}_e({}^kE, X)$.

For any Banach space E we have the natural (isometric) inclusions:

$$I_E : E \hookrightarrow C(B_{E'}, w*)$$

$$J_E : E \hookrightarrow l_\infty(B_{E'})$$

given by

$$I_E(x)(x') = x'(x) \text{ for } x' \in B_{E'}$$

$$J_E(x) = (x'(x))_{x' \in B_{E'}}$$

The following theorems shows the role played by these particular inclusions. Recall that a Banach space Y is said to have the metric extension property if for every Banach space E, every linear operator $T : E \to Y$ and every F containing E there exists a linear operator $\tilde{T} : F \to Y$ extending T with the same norm. $l_\infty(I)$ and $C(K)''$ have the metric extension property for every set I and every compact Hausdorff space K [4].

Theorem 3.1 *If X is a $\mathcal{C}_l$-space then a polynomial $P : E \to X$ is extendible if and only if P extends to $C(B_{E'}, w*)$. In this case, if P_0 is such an extension, then for every F containing E there exists an extension $\tilde{P}$ on F with $\|\tilde{P}\| \leq l\ \|P_0\|$.*

Proof: Let P_0 be an extension of P to $C(B_{E'})$ and $AB_p(P_0)$ its Aron-Berner extension to $C(B_{E'})''$ composed with the projection p ($\|p\| \leq l$). If F is any Banach space containing E, since $C(B_{E'})''$ has the metric extension property, the inclusion map $E \hookrightarrow C(B_{E'}) \hookrightarrow C(B_{E'})''$ extends to a norm-one operator $j : F \to C(B_{E'})''$ making the following diagram commutative:

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & C(B_{E'})'' \arrow[r, hook, "AB_p(P_0)"] & X \\ F \arrow[ur, "j"] & & \end{tikzcd}$$

Consequently, if we define $\tilde{P} = AB_p(P_0) \circ j$ we obtain an extension of P to F satisfying $\left\|\tilde{P}\right\| \leq \|AB_p(P_0)\| \; \|j\| \leq \|p\| \; \|P_0\|$. ■

If X is not a $\mathcal{C}_l$-space, a polynomial $P : E \to X$ could extend to $C(B_{E'})$ without being necessarily extendible. Indeed, since $(B_{l_1}, w*)$ is a metrizable compact set, the space $C(B_{l_1}, w*)$ is separable. Then, by the Sobczyck theorem (see, for example, [11]), c_0 is complemented in $C(B_{l_1})$ and consequently any polynomial on c_0 can be extended to $C(B_{l_1})$. However, there are non-extendible polynomials on c_0 (for example, the identity operator). Therefore, for a general Banach space X, to assure the extendibility of a polynomial P it will be necessary to extend it to a larger space than $C(B_{E'})$:

Theorem 3.2 *A polynomial $P : E \to X$ is extendible if and only if P extends to $l_\infty(B_{E'})$. In this case, if P_0 is such an extension, then for every F containing E there exists an extension $\tilde{P}$ on F with $\|\tilde{P}\| \leq \|P_0\|$.*

Proof: Let P_0 be an extension of P to $l_\infty(B_{E'})$. For any Banach space F containing E, since $l_\infty(B_{E'})$ has the metric extension property, the inclusion $J_E : E \hookrightarrow l_\infty(B_{E'})$ extends to a norm-one operator $\tilde{J}_E : F \hookrightarrow l_\infty(B_{E'})$. Therefore, $\tilde{P} = P_0 \circ \tilde{J}_E$ is an extension of P and satisfies $\left\|\tilde{P}\right\| \leq \|P_0\| \; \left\|\tilde{J}_E\right\| = \|P_0\|$. ■

Therefore, we have

$$\mathcal{P}_e({}^kE, X) = \mathcal{P}_{e_{C(B_{E'})}}({}^kE, X)$$

for a $\mathcal{C}_l$-space X and

$$\mathcal{P}_e({}^kE, X) = \mathcal{P}_{e_{l_\infty(B_{E'})}}({}^kE, X)$$

for the general case. It also follows from Theorems 3.1 and 3.2 that the extendible norm defined in [10]:

$$\|P\|_e = \inf\{c > 0 : \text{ for all } F \text{ there is an extension of } P \text{ to } F \text{ with norm} \leq c\} \qquad (3)$$

is well defined and coincides with $\|P\|_{e_{l_\infty(B_{E'})}}$ (and with $\|P\|_{e_{C(B_{E'})}}$ if X is a $\mathcal{C}_1$-space, which occurs for example when X is a dual space). Consequently, $\left(\mathcal{P}_e({}^kE, X), \| \; \|_e\right)$ is a Banach space and if every polynomial $P : E \to X$ is extendible, $\| \; \|$ and $\| \; \|_e$ are equivalent on $\mathcal{P}({}^kE, X)$.

Remark 3.3 *Since all $C(K)$ have the Dunford-Pettis property [7], it follows [14] that extendible scalar-valued polynomials are weakly sequentially continuous. The converse is not true: in l_2 extendible polynomials are nuclear ([10]), while there are approximable (and therefore weakly sequentially continuous) polynomials that are not nuclear.*

There are many properties of polynomials that are preserved by Aron-Berner extensions and composition with linear operators, such as being of finite type, nuclear, compact, etc. As an application of the previous theorems we show that the extendibility is one of this properties.

Theorem 3.4 *If $P : E \to X$ is extendible and $T : G \to E$ is a continuous linear operator, then $P \circ T : G \to X$ is extendible and $\|P \circ T\|_e \leq \|P\|_e \; \|T\|^k$.*

Proof: Thanks to Theorem 3.2, we need only to extend $P \circ T$ to $l_\infty(B_{G'})$. Let $T' : E' \to G'$ be the transpose of T and $T_1' = \frac{T'}{\|T\|}$. Since $T_1'(B_{E'}) \subseteq B_{G'}$, we can define $T_0 : l_\infty(B_{G'}) \to l_\infty(B_{E'})$ by

$$T_0\,(a) = \|T\| \;\; (a_{T_1'(x')})_{x' \in B_{E'}}$$

for $a = (a_{y'})_{y' \in B_{G'}} \in l_\infty(B_{G'})$, giving the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{T} & E \\ J_G \downarrow & & \downarrow J_E \\ l_\infty(B_{G'}) & \xrightarrow{T_0} & l_\infty(B_{E'}) \end{array}$$

If $P_0 : l_\infty(B_{E'}) \longrightarrow X$ is an extension of P, then $P_0 \circ T_0$ is an extension of $P \circ T$ to $l_\infty(B_{G'})$. By the theorem, $P \circ T$ is extendible and we have $\|P \circ T\|_e \leq \|P_0\| \; \|T_0\|^k = \|P_0\| \; \|T\|^k$ for any extension P_0 of P. Hence, $\|P \circ T\|_e \leq \|P\|_e \; \|T\|^k$. ■

Observe that the operator T need not be extendible for the composition to be extendible. The statement in Theorem 3.4 is not true if we replace T by a non-extendible polynomial: the polynomial $P(z) = z$ on $\mathbf{R}$ or $\mathbf{C}$ is obviously extendible but for any non-extendible polynomial Q the composition $P \circ Q = Q$ is not extendible.

Corollary 3.5 *Any restriction of an extendible polynomial is also extendible, with not larger extendible norm.*

Proof: The result follows from the theorem, taking T as the inclusion. ■

Theorem 3.6 *If $P \in \mathcal{P}_e({}^kE, X)$, then $AB(P) \in \mathcal{P}_e({}^kE'', X'')$ and $\|AB(P)\|_e \leq \|P\|_e$.*

Proof: Assume that P is an extendible polynomial and take P_0 an extension of P to $l_\infty(B_{E'})$. Since $P = P_0 \circ J_E$, the Aron-Berner extension of P is $AB(P_0) \circ J_E''$ (see section 1). On the other hand, since $l_\infty(B_{E'})''$ has the metric extension property, the operator

$$J_E'' : E'' \longrightarrow l_\infty(B_{E'})''$$

extends to $l_\infty(B_{E'''}) \supset E''$, with the same norm. If we call this extension j, we have the commutative diagram:

$$\begin{tikzcd} E \arrow[r, "J_E"] \arrow[d] & l_\infty(B_{E'}) \arrow[r, "P_0"] \arrow[d] & X \arrow[d] \\ E'' \arrow[r, "J''_E"] \arrow[d, "J_{E''}"'] & l_\infty(B_{E'})'' \arrow[r, "AB(P_0)"] & X'' \\ l_\infty(B_{E'''}) \arrow[ur, "j"] & & \end{tikzcd}$$

where the unlabeled vertical arrows are the canonical inclusions in the biduals. This shows that $AB(P_0)\circ j$ is an extension of $AB(P_0)\circ J''_E = AB(P)$. Hence, $AB(P)$ is extendible and $\|AB(P)\|_e \leq \|AB(P_0)\circ j\| \leq \|P_0\|$. Since P_0 is an arbitrary extension of P to $l_\infty(B_{E'})$, the result follows. ∎

Corollary 3.7 *A polynomial $P \in \mathcal{P}(^kE)$ is extendible if and only if its Aron-Berner extension $AB(P)$ is extendible. In this case, $\|P\|_e = \|AB(P)\|_e$.*

Proof: If $AB(P)$ is extendible, being P its restriction to E , P is extendible by corollary 3.5 and $\|P\|_e \leq \|AB(P)\|_e$. The converse and the reverse inequality follow from Theorem 3.6. ∎

In general, the converse of theorem 3.6 is not true: The identity operator on $Id_{c_0} : c_0 \to c_0$ is not extendible (it cannot be extended to l_∞). However, its Aron-Berner extension $AB(Id_{c_0}) = Id''_{c_0} = Id_{l_\infty}$, being l_∞ an injective space, is extendible. In the case that X is a $\mathcal{C}_l$-space, we combine theorems 3.4 and 3.6 and obtain:

Corollary 3.8 *If X is a $\mathcal{C}_l$-space, a polynomial $P : E \to X$ is extendible if and only if its Aron-Berner extension is extendible. In this case, $\|P\|_e \leq \|AB(P)\|_e \leq l\ \|P\|_e$.*

For scalar valued polynomials something more can be said. We define on $\otimes_s^k E$ the following norm:

$$\|s\|_\eta := \|s\|_{\eta_{C(B_{E'})}} = \| \otimes_s^k I_E(s)\|_{\pi_{C(B_{E'})}}$$

The η-norm coincides with the one defined in [10]. The following corollary was also proved in [10].

Corollary 3.9 $\left(\otimes_s^k E, \|\ \|_\eta\right)' = \left(\mathcal{P}_e(^kE), \|\ \|_e\right)$ *isometrically.*

Proof: $\left(\otimes_s^k E, \|\ \|_\eta\right)' = \left(\otimes_s^k E, \|\ \|_{\eta_{C(B_{E'})}}\right)' = \left(\mathcal{P}_{e_{C(B_{E'})}}(^kE), \|\ \|_e\right) = \left(\mathcal{P}_e(^kE), \|\ \|_e\right)$ ∎

Hence, a scalar-valued polynomial is extendible if and only if it is η-continuous. As was the case for fixed F, this is not true for vector-valued polynomials, since for degree 1 $\|\ \|_\eta$ is just the norm on E. In fact, $\eta-$continuity is related to weak extendibility:

Definition 3.10 *A polynomial $P : E \to X$ is said to be weakly extendible if for every linear functional γ on X, the scalar-valued polynomial $\gamma \circ P$ is extendible.*

Using the correspondence between extendibility and extendibility to $C(B_{E'})$, we can reformulate the results of the previous section:

Proposition 3.11 *A polynomial $P \in \mathcal{P}(^kE, X)$ is weakly extendible if and only if its associated operator $T_P : \otimes_s^k E \to X$ is $\eta-$continuous.*

Corollary 3.12 *For a Banach space X the following are equivalent:*
i) X is injective.
ii) For all Banach spaces E, a polynomial (of any degree) from E to X is extendible if and only if it is weakly extendible.

Proof: i) $\Rightarrow$ ii) If $P : E \to X$ is weakly extendible, the previous proposition says that its associated operator $T_P : \otimes_s^k E \to X$ is $\eta-$continuous. Observe that, since both theorems 3.1 and 3.2 apply for scalar-valued polynomials, the $\eta-$norm coincides with the $\eta_{l_\infty(B_{E'})}-$norm. Therefore, $\otimes_{s,\eta}^k E$ is an isometric subspace of $\otimes_{s,\pi}^k l_\infty(B_{E'})$ and since X is injective, T_P extends to a continuous operator from $\otimes_{s,\pi}^k l_\infty(B_{E'})$ to X. Hence, P extends to $l_\infty(B_{E'})$ and by theorem 3.2 is extendible.
ii) $\Rightarrow$ i) Since every linear operator to X is weakly extendible, they are all extendible. This means that X is injective. ∎

It is also simple now to find a predual of the space of extendible polynomials when $X = Y'$ is a dual space. In this case X is a $\mathcal{C}_1$-space, so if we put $\lambda = \lambda_{C(B_{E'})}$, we have:

Proposition 3.13 *a) If every weakly extendible polynomial from E to Y' is extendible, then $\big((\otimes_{s,\eta}^k E) \otimes_\pi Y\big)' = \mathcal{P}_e(^kE, Y')$ isomorphically.*
b)$\big((\otimes_{s,\eta}^k E) \otimes_\lambda Y\big)' = \mathcal{P}_e(^kE, Y')$ isometrically.

Corollary 3.14 *If X is a dual space, the infimum in (3) is actually a minimum.*

Corollary 3.15 *If X is a dual space, every polynomial $P : E \to X$ is extendible if and only if the e$-$norm is equivalent to the uniform norm on $\mathcal{P}_e(^kE, X)$.*

This last corollary extends a result given in [10] for scalar valued polynomials.

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