

COHERENT SEQUENCES OF POLYNOMIAL IDEALS ON BANACH SPACES

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ABSTRACT. This article deals with the relationship between an operator ideal and its natural polynomial extensions. We define the concept of coherent sequence of polynomial ideals and also the notion of compatibility between polynomial and operator ideals. We study the stability of these properties for maximal and minimal hulls, adjoint and composition ideals. We also relate these concepts with conditions on the underlying tensor norms.

INTRODUCTION

The concept of ideal of multilinear functionals on normed spaces was introduced by Pietsch [17] in 1983 and since then it has been studied by several authors. The related concept of ideal of n -homogeneous polynomials experienced a similar development, both as a tool to study holomorphic functions and by its own interest. Many of the well known polynomial ideals can be considered as the n -homogeneous analogous to some operator ideal. This is the case, for example, of the ideals of nuclear, integral, multiple r -summing or r -dominated polynomials. However, the extension of a linear operator ideal to higher degrees is not always obvious. For example, many extensions of the ideal of absolutely r -summing operators have been developed, among them, the absolutely, the multiple and the strongly r -summing polynomials and the r -dominated polynomials. Also, it is important to note that the multilinear and polynomial theories are far from being translations of the linear one.

One of the aims of this work is to clarify the relationship between an operator ideal \mathfrak{A} and a polynomial ideal defined in the spirit of \mathfrak{A} . For example, the ideal of nuclear n -homogeneous polynomials is the natural extension of the ideal of nuclear operators to the polynomial setting, so that it deserves to bare the name. In order to shed some light on what makes this extension natural, we introduce the concept of *compatibility* between a polynomial ideal and an operator ideal. This property is related with the natural operations of fixing variables and multiplying by linear functionals. Moreover, the polynomial extension of an operator ideal \mathfrak{A} usually gives rise to a sequence of polynomial ideals $\{\mathfrak{A}_k\}_k$, each \mathfrak{A}_k an ideal of k -homogeneous polynomials. Therefore, it is interesting to study not only the compatibility of each polynomial ideal \mathfrak{A}_k with \mathfrak{A} , but also the relationship between different \mathfrak{A}_k 's. To this end, the concept of *coherence* of a sequence of polynomial ideals is also introduced.

Along the article it is shown that there are always several polynomial ideals compatible with a given operator ideal. We describe the largest and the smallest among these polynomial ideals.

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Moreover, a characterization of all compatible polynomial ideals is obtained in terms of the largest and smallest ones. On the other hand, we show that for a given polynomial ideal there exists at most one operator ideal compatible with it.

In the first section, we define the properties of compatibility and coherence and prove some general results. We present some typical polynomial ideals and analyze if they satisfy these properties. In particular, we show that not all the usual polynomial extensions of the absolutely r -summing operator ideal are compatible with it. The second section is devoted to the study of the smallest and the largest polynomial ideals that are compatible with a fixed operator ideal. We show that the sequence of smallest (or largest) compatible polynomial ideals turns out to be coherent. The third section deals with composition ideals: in some situation, coherence and compatibility are preserved by the ideal composition. In the fourth section the presented concepts are related with tensor norms properties. Under certain circumstances, the coherence of a sequence of polynomial ideals can be formulated in terms of explicit relations between the underlying tensor norms. As an application, we exhibit examples of mixed tensor norms that are not equivalent to any (α, β) -norm, the existence of which was predicted in [14]. In the last section, we show that coherence and compatibility are preserved by the operations of taking maximal and minimal hulls and the adjoint of the polynomial ideals.

Even though the notions that we study and some of the results that we obtain can be stated for general polynomial and operator ideals, we prefer to restrict ourselves to the context of normed ideals.

We refer to [9, 16] for notation and results regarding polynomials in general, to [11, 12, 13, 14] for polynomial ideals and to [7, 10, 18] for tensor products of Banach spaces.

1. DEFINITIONS AND GENERAL RESULTS

Throughout this paper E , F and G will be complex Banach spaces. We denote by $\mathcal{P}^n(E, F)$ the Banach space of all continuous n -homogeneous polynomials from E to F . If $P \in \mathcal{P}^n(E, F)$, there exists a unique symmetric n -linear mapping $\overset{\vee}{P}: \underbrace{E \times \cdots \times E}_n \rightarrow F$ such that

$$P(x) = \overset{\vee}{P}(x, \dots, x).$$

We define $P_{a^k} \in \mathcal{P}^{n-k}(E, F)$ by

$$P_{a^k}(x) = \overset{\vee}{P}(a^k, x^{n-k}).$$

For $k = 1$, we write P_a instead of P_{a^1} .

We also denote $T_P : \bigotimes^{n,s} E \rightarrow F$ the linearization of P :

$$T_P\left(\sum_j x_j \otimes \cdots \otimes x_j\right) = \sum_j P(x_j).$$

Let us recall the definition of polynomial ideals [11, 12, 13, 14]. A **normed ideal of continuous n -homogeneous polynomials** is a pair $(\mathfrak{A}_n, \|\cdot\|_{\mathfrak{A}_n})$ such that:

- (i) $\mathfrak{A}_n(E, F) = \mathfrak{A}_n \cap \mathcal{P}^n(E, F)$ is a linear subspace of $\mathcal{P}^n(E, F)$ and $\|\cdot\|_{\mathfrak{A}_n(E, F)}$ is a norm on it.
- (ii) If $T \in \mathcal{L}(E_1, E)$, $P \in \mathfrak{A}_n(E, F)$ and $S \in \mathcal{L}(F, F_1)$, then $S \circ P \circ T \in \mathfrak{A}_n(E_1, F_1)$ and

$$\|S \circ P \circ T\|_{\mathfrak{A}_n(E_1, F_1)} \leq \|S\| \|P\|_{\mathfrak{A}_n(E, F)} \|T\|^n$$
- (iii) $z \mapsto z^n$ belongs to $\mathfrak{A}_n(\mathbb{C}, \mathbb{C})$ and has norm 1.

Many examples of polynomial ideals appear as generalizations of ideals of operators. We intend to clarify the relationship between an ideal of operators and its possible generalization to higher degrees. In particular, we are interested in properties that are shared by the operator and polynomial ideals.

Next lemma shows that any polynomial ideal is closed by the combined operation of fixing variables followed by multiplication by a power of a linear functional.

Lemma 1.1. *Let \mathfrak{A}_n be an ideal of n -homogeneous polynomials and $P \in \mathfrak{A}_n(E, F)$. If $T_1, \dots, T_n \in \mathcal{L}(G, E)$, then the n -homogeneous polynomial given by $Q(\cdot) = \check{P}(T_1(\cdot), \dots, T_n(\cdot))$ belongs to $\mathfrak{A}_n(G, F)$.*

Moreover, if $0 < j < n$, $\gamma \in E'$, and $a \in E$, then

(a) $\gamma^j P_{a^j}$ belongs to $\mathfrak{A}_n(E, F)$.

(b) $(\gamma^j P)_{a^j}$ belongs to $\mathfrak{A}_n(E, F)$.

Proof. The first assertion follows from the polarization formula. (a) follows from this fact and the equality

$$\gamma(x)^j P_{a^j}(x) = \check{P}(\gamma(x)a, \dots, \gamma(x)a, x, \dots, x).$$

To prove (b), we expand $(\gamma^j P)_{a^j}$ as

$$\begin{aligned} (\gamma^j P)_{a^j} &= \frac{1}{\binom{n+j}{j}} \sum_{i=0}^j \binom{j}{i} \binom{n}{j-i} \gamma(a)^i \gamma(x)^{j-i} \check{P}(a^{j-i}, x^{n-j+i}) \\ &= \frac{1}{\binom{n+j}{j}} \sum_{i=0}^j \binom{j}{i} \binom{n}{j-i} \gamma(a)^i (\gamma^{j-i} P_{a^{j-i}})(x), \end{aligned}$$

and use (a). □

This result suggests that the operations of fixing a variable or multiplying by a linear functional are inherent to the structure of polynomial ideals. This motivates the following definitions, that should be compared with [2, 3] and the concept of holomorphy type.

Definition 1.2. *Let \mathfrak{A} be a normed ideal of linear operators. We say that the normed ideal of n -homogeneous polynomials \mathfrak{A}_n is **compatible with** \mathfrak{A} if there exist positive constants A and B such that for every Banach spaces E and F , the following conditions hold:*

- (i) *For each $P \in \mathfrak{A}_n(E, F)$ and $a \in E$, $P_{a^{n-1}}$ belongs to $\mathfrak{A}(E; F)$ and*

$$\|P_{a^{n-1}}\|_{\mathfrak{A}(E, F)} \leq A \|P\|_{\mathfrak{A}_n(E, F)} \|a\|^{n-1}$$

(ii) For each $T \in \mathfrak{A}(E, F)$ and $\gamma \in E'$, $\gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E, F)$ and

$$\|\gamma^{n-1}T\|_{\mathfrak{A}_n(E, F)} \leq B\|\gamma\|^{n-1}\|T\|_{\mathfrak{A}(E, F)}$$

We are interested in relating not only ideals of polynomials and operators, but also ideals of homogeneous polynomials of different degrees:

Definition 1.3. Consider the sequence $\{\mathfrak{A}_k\}_{k=1}^N$, where for each k , \mathfrak{A}_k is an ideal of k -homogeneous polynomials and N is eventually infinite. We say that $\{\mathfrak{A}_k\}_k$ is a **coherent sequence of polynomial ideals** if there exist positive constants C and D such that for every Banach spaces E and F , the following conditions hold for $k = 1, \dots, N-1$:

(i) For each $P \in \mathfrak{A}_{k+1}(E, F)$ and $a \in E$, P_a belongs to $\mathfrak{A}_k(E; F)$ and

$$\|P_a\|_{\mathfrak{A}_k(E, F)} \leq C\|P\|_{\mathfrak{A}_{k+1}(E, F)}\|a\|$$

(ii) For each $P \in \mathfrak{A}_k(E, F)$ and $\gamma \in E'$, γP belongs to $\mathfrak{A}_{k+1}(E, F)$ and

$$\|\gamma P\|_{\mathfrak{A}_{k+1}(E, F)} \leq D\|\gamma\|\|P\|_{\mathfrak{A}_k(E, F)}$$

Let \mathfrak{A} be a linear operator ideal. We say that the sequence of k -homogeneous polynomial ideals $\{\mathfrak{A}_k\}_k$ is a **coherent sequence associated to \mathfrak{A}** if $\{\mathfrak{A}_k\}_k$ is a coherent sequence and $\mathfrak{A}_1 = \mathfrak{A}$.

Note that if $\{\mathfrak{A}_k\}_{k=1}^N$ is a coherent sequence, then for each $k = 1, \dots, N$, the polynomial ideal \mathfrak{A}_k is compatible with $\mathfrak{A} = \mathfrak{A}_1$ with constants $A \leq C^k$ and $B \leq D^k$. Nevertheless, in most of the natural examples one obtains a better estimates.

Also, coherent sequences are always holomorphy types.

The following lemmas show a kind of convers to conditions (i) and (ii) of Definitions 1.2 and 1.3.

Lemma 1.4. Let \mathfrak{A}_n be a normed ideal of n -homogeneous polynomials compatible with \mathfrak{A} and $T \in \mathcal{L}(E, F)$. Then:

a) $T \in \mathfrak{A}(E, F)$ if and only if $\gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E, F)$ for all $\gamma \in E'$ (or for some nonzero $\gamma \in E'$).

b) $T \in \mathfrak{A}(E, F)$ if and only if there exist $P \in \mathfrak{A}_n(E, F)$ and $a \in E$ such that $T = P_{a^{n-1}}$.

Proof. In both cases, only one of the implications needs to be proved since the other one follows from the definition of compatibility.

a) Suppose that $Q = \gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E, F)$ and choose $a \in E$ such that $\gamma(a) \neq 0$. Then $Q_{a^{n-1}} \in \mathfrak{A}(E, F)$, since \mathfrak{A}_n is compatible with \mathfrak{A} . Now,

$$Q_{a^{n-1}} = \frac{1}{n}\gamma^{n-1}(a)T + \frac{n-1}{n}\gamma^{n-2}(a)T(a)\gamma.$$

So, we can express T as

$$(1) \quad T = \frac{n}{\gamma^{n-1}(a)}Q_{a^{n-1}} - \frac{n-1}{\gamma(a)}T(a)\gamma.$$

Since T is a linear combination of a finite type operator and $Q_{a^{n-1}}$, we conclude that it belongs to $\mathfrak{A}(E, F)$.

b) Take $T \in \mathfrak{A}(E, F)$ and choose $a \in E$ and $\gamma \in E'$ such that $\gamma(a) = 1$. By equation (1), T can be written as

$$T = \left(n(\gamma^{n-1}T) - (n-1)T(a)\gamma^n \right)_{a^{n-1}}$$

and the polynomial $P = n(\gamma^{n-1}T) - (n-1)T(a)\gamma^n$ belongs to $\mathfrak{A}_n(E, F)$. \square

Lemma 1.5. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence of normed ideals of homogeneous polynomials and $P \in \mathcal{P}^k(E, F)$. Then:*

a) $P \in \mathfrak{A}_k(E, F)$ if and only if γP belongs to $\mathfrak{A}_{k+1}(E, F)$ for all $\gamma \in E'$ (or for some nonzero $\gamma \in E'$).

b) $P \in \mathfrak{A}_k(E, F)$ if and only if there exists $Q \in \mathfrak{A}_{k+1}(E, F)$ and $a \in E$ such that $P = Q_a$.

Proof. a) Let $R = \gamma P \in \mathfrak{A}_{k+1}(E, F)$ and choose $a \in E$ such that $\gamma(a) = 1$. Proceeding as in the proof of [1, Proposition 5.3], we can write P as

$$(2) \quad P = \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} \gamma^{j-1} R_{a^j}.$$

By the coherence of $\{\mathfrak{A}_k\}_k$, $\gamma^{j-1} R_{a^j}$ belongs to $\mathfrak{A}_k(E, F)$ for each j and we conclude that so does P .

b) Let $P \in \mathfrak{A}_k(E, F)$ and fix $a \in E$ and $\gamma \in E'$ such that $\gamma(a) = 1$. It is not difficult to check that, for each $j = 1, \dots, n+1$, we have $(\gamma^j P_{a^{j-1}})_a = \gamma^{j-1} (\gamma P)_{a^j}$. Then, by equation (2) we obtain

$$P = \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} (\gamma^j P_{a^{j-1}})_a = \left(\sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} \gamma^j P_{a^{j-1}} \right)_a.$$

Clearly, $Q = \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} \gamma^j P_{a^{j-1}}$ belongs to $\mathfrak{A}_{k+1}(E, F)$ by the coherence of the sequence $\{\mathfrak{A}_k\}$. \square

The previous lemmas allows to infer relationships between operator ideals from compatible polynomial ideals. An analogous result hold for coherent sequences of polynomial ideals (see also [3, 2]).

Proposition 1.6. a) *Let \mathfrak{A}_n and \mathfrak{B}_n be normed ideals of n -homogeneous polynomials compatible with \mathfrak{A} and \mathfrak{B} respectively. If for some E and F , $\mathfrak{A}_n(E, F) \subset \mathfrak{B}_n(E, F)$, then $\mathfrak{A}(E, F) \subset \mathfrak{B}(E, F)$.*

b) *Let $\{\mathfrak{A}_k\}_k$ and $\{\mathfrak{B}_k\}_k$ be coherent sequences. If for some E and F and some k_0 , $\mathfrak{A}_{k_0}(E, F) \subset \mathfrak{B}_{k_0}(E, F)$, then $\mathfrak{A}_k(E, F) \subset \mathfrak{B}_k(E, F)$ for all $k \leq k_0$.*

Proof. We prove only a), since b) is similar. For $u \in \mathfrak{A}(E, F)$ and a nonzero $\gamma \in E'$, we have $\gamma^{n-1}u \in \mathfrak{A}_n(E, F)$. Thus, $\gamma^{n-1}u \in \mathfrak{B}_n(E, F)$, and by Lemma 1.4, $u \in \mathfrak{B}(E, F)$. \square

Note that we only need that \mathfrak{A} satisfies (ii) and \mathfrak{B} satisfies (i) in the definitions to obtain the conclusions of the proposition (in fact, none of the norm inequalities in (i) and (ii) are necessary).

The previous proposition asserts that an ideal of n -homogeneous polynomials \mathfrak{A}_n can be compatible with at most one operator ideal \mathfrak{A} . Moreover, there can be at most one coherent sequence $\{\mathfrak{B}_k\}_{k=1}^n$ with $\mathfrak{B}_n = \mathfrak{A}_n$.

Before presenting examples, we need some technical results that will be frequently used throughout this work.

Let $\sigma: \bigotimes^n E \rightarrow \bigotimes^{n,s} E$ be the symmetrization operator

$$\sigma(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\eta \in S_n} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)},$$

where S_n denotes the set of all permutations of $\{1, \dots, n\}$.

The following result can be derived from [15, Corollary 3]. However, we provide a simple proof for the sake of completeness.

Lemma 1.7. *Let $\sigma: \bigotimes^n E \rightarrow \bigotimes^{n,s} E$ be the symmetrization operator. Then, for any symmetric n -tensor norm α and all $a, b \in E$ we have*

$$\alpha\left(\sigma(a \otimes b \otimes \cdots \otimes b); \bigotimes^{n,s} E\right) \leq e \|a\| \|b\|^{n-1}.$$

Proof. Let $r \in \mathbb{C}$ be a primary root of the unit: $r^n = 1$ and $r^j \neq 1$ for every $1 \leq j < n$. Let us see that for all $t > 0$

$$\sigma(a \otimes b \otimes \cdots \otimes b) = \frac{1}{n^2} \sum_{j=0}^{n-1} r^j t^{n-1} \left(\frac{r^j}{t} b + a\right)^n.$$

Indeed, if $P \in \mathcal{P}^n(E)$, $\langle P, \sigma(a \otimes b \otimes \cdots \otimes b) \rangle = \bigvee P(a, b, \dots, b)$, and

$$\begin{aligned} \left\langle P, \frac{1}{n^2} \sum_{j=0}^{n-1} r^j t^{n-1} \left(a + \frac{r^j}{t} b\right)^n \right\rangle &= \frac{1}{n^2} \sum_{j=0}^{n-1} r^j t^{n-1} P\left(a + \frac{r^j}{t} b\right) \\ &= \frac{1}{n^2} \sum_{j=0}^{n-1} r^j t^{n-1} \sum_{i=0}^n \binom{n}{i} \bigvee P\left(a^{n-i}, \left(\frac{r^j}{t} b\right)^i\right) \\ &= \frac{t^{n-1}}{n^2} \sum_{i=0}^n \binom{n}{i} \frac{1}{t^i} \bigvee P(a^{n-i}, b^i) \sum_{j=0}^{n-1} r^{j(i+1)} \\ &= \frac{t^{n-1}}{n^2} \binom{n}{n-1} \frac{1}{t^{n-1}} \bigvee P(a, b^{n-1}) n \\ &= \bigvee P(a, b, \dots, b). \end{aligned}$$

Suppose that $\|a\| = \|b\| = 1$. Then for all $t > 0$ we have that

$$\begin{aligned} \alpha\left(\sigma(a \otimes b \otimes \cdots \otimes b); \bigotimes^{n,s} E\right) &\leq \frac{1}{n^2} \sum_{j=0}^{n-1} t^{n-1} \left\| \frac{r^j}{t} b + a \right\|^n \\ &\leq \frac{1}{n^2} \sum_{j=0}^{n-1} t^{n-1} \left(\frac{1}{t} + 1\right)^n = \frac{1}{n} t^{n-1} \left(\frac{1}{t} + 1\right)^n. \end{aligned}$$

Choosing $t = \frac{1}{n-1}$ we obtain

$$\alpha\left(\sigma(a \otimes b \otimes \cdots \otimes b); \bigotimes^{n,s} E\right) \leq \left(\frac{n}{n-1}\right)^{n-1} \leq e.$$

Thus for all $a, b \in E$,

$$\begin{aligned} \alpha\left(\sigma(a \otimes b \otimes \cdots \otimes b)\right) &= \|a\| \|b\|^{n-1} \alpha\left(\sigma\left(\frac{a}{\|a\|} \otimes \frac{b}{\|b\|} \otimes \cdots \otimes \frac{b}{\|b\|}\right)\right) \\ &\leq e \|a\| \|b\|^{n-1}. \quad \square \end{aligned}$$

From the previous proof we obtain the useful expression

$$(3) \quad \sigma(a \otimes b \otimes \cdots \otimes b) = \frac{1}{n^2} \frac{1}{(n-1)^{n-1}} \sum_{j=0}^{n-1} r^j ((n-1)r^j b + a)^n.$$

Corollary 1.8. *a) For any normed ideal \mathfrak{A}_n of n -homogeneous polynomials, $\gamma, \phi \in E'$ and $y \in F$, we have*

$$\|\gamma \phi^{n-1} y\|_{\mathfrak{A}_n(E,F)} \leq e \|\gamma\| \|\phi\|^{n-1} \|y\|.$$

b) Let $P \in \mathcal{P}^n(E, F)$, and $a, b \in E$. Then

$$\check{P}(a, b^{n-1}) = \frac{1}{n^2} \frac{1}{(n-1)^{n-1}} \sum_{j=0}^{n-1} r^j P((n-1)r^j b + a)$$

and

$$\|\check{P}(a, b^{n-1})\| \leq e \|P\| \|a\| \|b\|^{n-1}.$$

Proof. a) Define $T \in \mathcal{L}(\mathbb{C}, F)$ as $T(c) = cy$. Then

$$\|\gamma \phi^{n-1} y\|_{\mathfrak{A}_n(E,F)} \leq \|\gamma \phi^{n-1}\|_{\mathfrak{A}_n(E,\mathbb{C})} \|T\|_{\mathcal{L}(\mathbb{C},F)} = \|\gamma \phi^{n-1}\|_{\mathfrak{A}_n(E,\mathbb{C})} \|y\|_{\mathcal{L}(\mathbb{C},F)}$$

Since we always have the norm one inclusion $\bigotimes_{\pi_s}^{n,s} E' \hookrightarrow \mathfrak{A}_n(E, \mathbb{C})$, we obtain $\|\gamma \phi^{n-1}\|_{\mathfrak{A}_n(E,\mathbb{C})} \leq \pi_s(\sigma(\gamma \otimes \phi \otimes \cdots \otimes \phi); \bigotimes^{n,s} E') \leq e \|\gamma\| \|\phi\|^{n-1}$, which ends the proof.

b) This follows from expression (3) and the previous lemma. \square

Now we show some classical ideals of polynomials that satisfy the definitions.

Example 1.9. *Continuous homogeneous polynomials: \mathcal{P} .*

For each n , \mathcal{P}^n is compatible with the ideal \mathcal{L} of all continuous linear operators with constants $A = e$ and $B = 1$. Also, the sequence $\{\mathcal{P}^k\}_{k=1}^\infty$ is a coherent sequence with constants $C = e$ and $D = 1$. Both results follow from the previous corollary.

For sequences of polynomial ideals whose norms are the usual polynomial norm, it is only necessary to check that the sequence is closed under the operations of fixing variables and multiplying by linear functionals. Consequently, it is easy to prove that the sequences of approximable, compact, weakly compact, weakly sequentially continuous and weakly continuous on bounded sets polynomials are coherent and compatible with the corresponding operator ideals.

Example 1.10. *Nuclear polynomials: \mathcal{P}_N*

A polynomial $P \in \mathcal{P}^k(E; F)$ is said to be **nuclear** if it can be written as $P(x) = \sum_{i=1}^{\infty} \gamma_i(x)^k y_i$, where $\gamma_i \in E'$, $y_i \in F$ for all i and $\sum_{i=1}^{\infty} \|\gamma_i\|^k \|y_i\| < \infty$. The space of nuclear k -homogeneous polynomials from E into F will be denoted by $\mathcal{P}_N^k(E; F)$. It is a Banach space when we consider the norm

$$\|P\|_{\mathcal{P}_N^k(E; F)} = \inf \left\{ \sum_{i=1}^{\infty} \|\gamma_i\|^k \|y_i\| \right\}$$

where the infimum is taken over all representations of P as above.

Let $P \in \mathcal{P}_N^k(E, F)$. For $a \in E$, it is immediate that P_a is nuclear and $\|P_a\|_{\mathcal{P}_N^{k-1}(E; F)} \leq \|a\| \|P\|_{\mathcal{P}_N^k(E; F)}$. Also, by Corollary 1.8, we have $\|\gamma P\|_{\mathcal{P}_N^{k+1}(E; F)} \leq e \|\gamma\| \|P\|_{\mathcal{P}_N^k(E; F)}$ for any $\gamma \in E'$. Therefore, the sequence of nuclear polynomials is coherent with the ideal of nuclear operators with constants $C = 1$ and $D = e$. The compatibility with the same constants follows similarly.

Example 1.11. *Integral polynomials: \mathcal{P}_{PI} and \mathcal{P}_{GI}*

A polynomial $P \in \mathcal{P}^k(E, F)$ is **Pietsch-integral** if there exists a regular F -valued Borel measure μ , of bounded variation on $(B_{E'}, w^*)$ such that

$$P(x) = \int_{B_{E'}} \gamma(x)^n d\mu(\gamma)$$

for all $x \in E$. The space of k -homogeneous Pietsch-integral polynomials is denoted by $\mathcal{P}_{PI}^k(E, F)$ and the integral norm of a polynomial $P \in \mathcal{P}_{PI}^k(E, F)$ is defined as

$$\|P\|_{\mathcal{P}_{PI}^k(E, F)} = \inf \{ |\mu|(B_{E'}) \},$$

where the infimum is taken over all measures μ representing P .

The definition of **Grothendieck-integral** polynomials is analogous, but taking the measure μ to be F'' -valued. The space of Grothendieck-integral polynomials is denoted by $\mathcal{P}_{GI}^k(E, F)$.

The integral polynomials are compatible and coherent with the ideal of integral operators, with constants $A = C = 1$ and $B = D = e$, as we will see in Sections 4 and 5.

Example 1.12. *Extendible polynomials: \mathcal{P}_e*

A polynomial $P : E \rightarrow F$ is **extendible** if for any Banach space G containing E there exists $\tilde{P} \in \mathcal{P}^k(G, F)$ an extension of P . We will denote the space of all such polynomials by $\mathcal{P}_e^k(E, F)$. For $P \in \mathcal{P}_e^k(E, F)$, its extendible norm is given by

$$\|P\|_{\mathcal{P}_e^k(E, F)} = \inf \{ c > 0 : \text{ for all } G \supset E \text{ there is an extension of } P \text{ to } G \text{ with norm } \leq c \}.$$

To see the coherence of the sequence $\{\mathcal{P}_e^k\}_k$, take $P \in \mathcal{P}_e^k(E, F)$.

(i) Let $a \in E$ and suppose $E \subset G$. If $\tilde{P} \in \mathcal{P}^k(G, F)$ is any extension of P to G , by the polarization formula, $(\tilde{P})_a$ is an extension of P_a to G with norm $\|(\tilde{P})_a\| \leq e \|\tilde{P}\|$ (Corollary 1.8). This implies that P_a is extendible and $\|P_a\|_{\mathcal{P}_e^{k-1}} \leq e \|P\|_{\mathcal{P}_e^k}$.

(ii) Let $\gamma \in E'$. If $\tilde{\gamma}$ is an extension of γ and \tilde{P} an extension of P to $G \supset E$, then $\tilde{\gamma}\tilde{P}$ is an extension of γP to G with norm at most $\|\gamma\|\|P\|$. Thus $\gamma P \in \mathcal{P}_e^{k+1}(E, F)$ and $\|\gamma P\|_{\mathcal{P}_e^{k+1}} \leq \|\gamma\|\|P\|_{\mathcal{P}_e^k}$.

Therefore, the sequence of extendible polynomials is coherent with the ideal of extendible operators, with constants $C = e$ and $D = 1$.

Analogously, we see that they are compatible with the same constants.

Before the next examples, we need to recall the definition of the weak r -norm of a sequence: for $x^1, \dots, x^m \in E$, we define

$$w_r((x^i)_{i=1}^m) = \sup_{\gamma \in B_{E'}} \left(\sum_i |\gamma(x^i)|^r \right)^{1/r}.$$

Example 1.13. *Multiple r -summing polynomials: \mathcal{M}_r*

A k -homogeneous polynomial P from E to F is **multiple r -summing** if there exists $C > 0$ such that for every choice of finite sequences $(x_j^{i_j})_{i_j=1}^{m_j} \subset E$ the following holds

$$\left(\sum_{i_1, \dots, i_k=1}^{m_1, \dots, m_k} \|P(x_1^{i_1}, \dots, x_k^{i_k})\|^r \right)^{\frac{1}{r}} \leq C \cdot w_r((x_1^{i_1})_{i_1=1}^{m_1}) \cdots w_r((x_k^{i_k})_{i_k=1}^{m_k}),$$

The least of such constants C is called the **multiple r -summing norm** and denoted $\|P\|_{\Pi_r^k(E, F)}$.

Let $P \in \Pi_r^k(E, F)$. For $a \in E$, it is immediate that P_a is multiple r -summing and $\|P_a\|_{\Pi_r^{k-1}} \leq \|a\|\|P\|_{\Pi_r^k}$. Also for any $\gamma \in E'$, we have

$$(\gamma P)^\vee(x_1, \dots, x_{k+1}) = \frac{1}{k+1} \sum_{j=1}^{k+1} \gamma(x_j) \check{P}(x_1, \dots, \tilde{x}_j, \dots, x_{k+1})$$

where \tilde{x}_j means that this coordinate is omitted. Then, by the triangle inequality,

$$\begin{aligned} & \left(\sum_{i_1, \dots, i_{k+1}=1}^{m_1, \dots, m_{k+1}} \|(\gamma P)^\vee(x_1^{i_1}, \dots, x_{k+1}^{i_{k+1}})\|^r \right)^{\frac{1}{r}} \leq \\ & \leq \frac{1}{k+1} \sum_{j=1}^{k+1} \left(\sum_{i_1, \dots, i_{k+1}=1}^{m_1, \dots, m_{k+1}} |\gamma(x_j^{i_j})|^r \left\| \check{P}(x_1^{i_1}, \dots, \tilde{x}_j^{i_j}, \dots, x_{k+1}^{i_{k+1}}) \right\|^r \right)^{1/r} \\ & \leq \frac{1}{k+1} \sum_{j=1}^{k+1} \left(\sum_{i_j=1}^{m_j} |\gamma(x_j^{i_j})|^r \|P\|_{\Pi_r^k}^r \prod_{l=1, l \neq j}^{k+1} w_r((x_l^{i_l})_{i_l=1}^{m_l})^r \right)^{1/r} \\ & \leq \|\gamma\|\|P\|_{\Pi_r^k} w_r((x_1^{i_1})_{i_1=1}^{m_1}) \cdots w_r((x_{k+1}^{i_{k+1}})_{i_{k+1}=1}^{m_{k+1}}). \end{aligned}$$

Hence, γP is multiple r -summing with $\|\gamma P\|_{\Pi_r^{k+1}} \leq \|\gamma\|\|P\|_{\Pi_r^k}$.

Therefore, $\{\Pi_r^k\}_k$ is a coherent sequence associated with the ideal of absolutely r -summing operators with constants $C = D = 1$. Consequently, compatibility constants are also $A = B = 1$.

Example 1.14. *r -dominated polynomials: \mathcal{D}_r*

A k -homogeneous polynomial P from E to F is **r -dominated** if there exists $C > 0$ such that for every finite sequence $(x^i)_{i=1}^m \subset E$ the following holds

$$\left(\sum_{i=1}^m \|P(x^i)\|^{\frac{r}{k}} \right)^{\frac{k}{r}} \leq C \cdot w_r((x^i)_{i=1}^m)^k.$$

The least of such constants C is called the **r -dominated norm** and denoted $\|P\|_{\mathcal{D}_r^k(E,F)}$.

The ideals of r -dominated polynomials are compatible and coherent with the ideal of absolutely r -summing operators. This is a particular case of the composition ideals considered in Section 3.

Now we see that not all the usual polynomial extensions of an operator ideal are compatible.

Example 1.15. *Absolutely 1-summing polynomials.*

A k -homogeneous polynomial P from E to F is **absolutely 1-summing** if there exists $C > 0$ such that for every finite sequence $(x^i)_{i=1}^m \subset E$ the following holds

$$\left(\sum_{i=1}^m \|P(x^i)\| \right) \leq C \cdot w_1((x^i)_{i=1}^m)^k.$$

The least of such constants C is called the **absolutely 1-summing norm**.

To see that the ideal of absolutely 1-summing polynomials is not a compatible extension of the ideal of absolutely 1-summing operators, we exhibit a 2-homogeneous absolutely 1-summing polynomial that does not verify condition (i). This example also shows that the sequence of absolutely 1-summing polynomial ideals is not coherent.

Let $P: \ell_2 \rightarrow \ell_2 \otimes_{\pi} \ell_2$ be the polynomial given by $P(x) = x \otimes x$. Since weakly 1-summing sequences in ℓ_2 are always strongly 2-summing, it is immediate to check that P is absolutely 1-summing. For any nonzero $a \in \ell_2$, the linear operator P_a is not completely continuous and therefore it is not absolutely 1-summing.

The same example shows that for real Banach spaces, the ideal of **strongly r -summing** polynomials is not compatible with the ideal of absolutely r -summing operators for any r [8, Example 3.4].

2. THE SMALLEST AND THE LARGEST COMPATIBLE IDEALS.

Consider a normed ideal \mathfrak{A} of linear operators. In this section we define normed ideals of n -homogeneous polynomials, $\mathcal{M}_n^{\mathfrak{A}}$ and $\mathcal{F}_n^{\mathfrak{A}}$, compatible with \mathfrak{A} with the following property: if \mathfrak{A}_n is another ideal compatible with \mathfrak{A} then for each E, F ,

$$\mathcal{F}_n^{\mathfrak{A}}(E, F) \subset \mathfrak{A}_n(E, F) \subset \mathcal{M}_n^{\mathfrak{A}}(E, F).$$

In other words, $\mathcal{M}_n^{\mathfrak{A}}$ y $\mathcal{F}_n^{\mathfrak{A}}$ are, respectively, the largest and the smallest ideal of n -homogeneous polynomials compatible with \mathfrak{A} .

Define, for Banach spaces E and F ,

$$(4) \quad \mathcal{M}_n^{\mathfrak{A}}(E, F) = \{P \in \mathcal{P}^n(E, F) / P_{a^{n-1}} \in \mathfrak{A}(E, F), \forall a \in E\}$$

with norm

$$\|P\|_{\mathcal{M}_n^{\mathfrak{A}}(E,F)} := \sup_{\|a\|=1} \|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)}.$$

Also, we define

$$(5) \quad \mathcal{F}_n^{\mathfrak{A}}(E, F) = \left\{ P \in \mathcal{P}^n(E, F) \mid P = \sum_{i=1}^m \gamma_i^{n-1} T_i, \ T_i \in \mathfrak{A}(E, F), \ \gamma_i \in E' \right\}$$

with norm

$$\|P\|_{\mathcal{F}_n^{\mathfrak{A}}(E,F)} := \inf \left\{ \sum_{i=1}^m \|\gamma_i\|^{n-1} \|T_i\|_{\mathfrak{A}(E,F)} \right\},$$

where the infimum is taken over all possible representations of P as in equation (5).

In the case of \mathfrak{A} being complete, we define also

$$(6) \quad \mathcal{N}_n^{\mathfrak{A}}(E, F) = \left\{ P \in \mathcal{P}^n(E, F) \mid P = \sum_{i=1}^{\infty} \gamma_i^{n-1} T_i \right\}$$

where $T_i \in \mathfrak{A}(E, F)$, $\gamma_i \in E'$ and $\sum_{i=1}^{\infty} \|\gamma_i\|^{n-1} \|T_i\|_{\mathfrak{A}(E,F)} < \infty$, with norm

$$\|P\|_{\mathcal{N}_n^{\mathfrak{A}}(E,F)} := \inf \left\{ \sum_{i=1}^{\infty} \|\gamma_i\|^{n-1} \|T_i\|_{\mathfrak{A}(E,F)} \right\},$$

where the infimum is taken over all possible representations of P as in equation (6).

Remark 2.1. It is easy to prove the following isometric identifications for the previously defined ideals:

$$\mathcal{M}_n^{\mathfrak{A}}(E, F) \stackrel{1}{=} \mathcal{P}^{n-1}(E, \mathfrak{A}(E, F)),$$

$$\mathcal{F}_n^{\mathfrak{A}}(E, F) \stackrel{1}{=} \mathcal{P}_f^{n-1}(E, \mathfrak{A}(E, F)),$$

$$\mathcal{N}_n^{\mathfrak{A}}(E, F) \stackrel{1}{=} \mathcal{P}_N^{n-1}(E, \mathfrak{A}(E, F)),$$

where \mathcal{P}_f denotes the ideal of finite type polynomials.

Now we show that these polynomial ideals are the extreme cases among those compatible with \mathfrak{A} :

Proposition 2.2. *Let \mathfrak{A} be a normed ideal of linear operators. Then:*

- a) $\mathcal{M}_n^{\mathfrak{A}}$ is the largest normed ideal of n -homogeneous polynomials compatible with \mathfrak{A} .
- b) $\mathcal{F}_n^{\mathfrak{A}}$ is the smallest normed ideal of n -homogeneous polynomials compatible with \mathfrak{A} .
- c) If \mathfrak{A} is complete, then $\mathcal{N}_n^{\mathfrak{A}}$ is the smallest Banach ideal of n -homogeneous polynomials compatible with \mathfrak{A} .

Moreover, in all the cases compatibility constants are $A = B = 1$.

Proof. a) It is clear that $\mathcal{M}_n^{\mathfrak{A}}$ is a normed ideal of n -homogeneous polynomials. Moreover, if \mathfrak{A}_n is compatible with \mathfrak{A} and $P \in \mathfrak{A}_n(E, F)$, then $P_{a^{n-1}} \in \mathfrak{A}(E, F)$. Therefore $P \in \mathcal{M}_n^{\mathfrak{A}}(E, F)$ and hence $\mathfrak{A}_n \subset \mathcal{M}_n^{\mathfrak{A}}$.

It remains to verify that $\mathcal{M}_n^{\mathfrak{A}}$ is compatible with \mathfrak{A} . Condition (i) is clearly satisfied with constant $A = 1$.

To see that $\mathcal{M}_n^{\mathfrak{A}}$ satisfies (ii), let $T \in \mathfrak{A}(E, F)$, $\gamma \in E'$ and $a \in E$. We have

$$(\gamma^{n-1}T)_{a^{n-1}} = \frac{1}{n}\gamma(a)^{n-1}T + \frac{n-1}{n}\gamma(a)^{n-2}T(a)\gamma.$$

Then, $(\gamma^{n-1}T)_{a^{n-1}} \in \mathfrak{A}(E, F)$ and therefore, $\gamma^{n-1}T \in \mathcal{M}_n^{\mathfrak{A}}(E, F)$. Moreover, by the triangle inequality,

$$\|(\gamma^{n-1}T)_{a^{n-1}}\|_{\mathfrak{A}(E, F)} \leq \|\gamma\|^{n-1}\|T\|_{\mathfrak{A}(E, F)}\|a\|^{n-1}.$$

Thus (ii) is satisfied with constant $B = 1$.

The proof of b) is a simpler version of the proof of c).

c) It is easy to see that $\mathcal{N}_n^{\mathfrak{A}}$ is a normed ideal of n -homogeneous polynomials. Completeness follows from Remark 2.1.

We now prove that if \mathfrak{A}_n is a Banach ideal of n -homogeneous polynomials compatible with \mathfrak{A} (with constants \tilde{A} and \tilde{B}) then $\mathcal{N}_n^{\mathfrak{A}}(E, F) \subset \mathfrak{A}_n(E, F)$. Consider $P \in \mathcal{N}_n^{\mathfrak{A}}(E, F)$ with representation $P = \sum_{i=1}^{\infty} \gamma_i^{n-1}T_i$, where $T_i \in \mathfrak{A}(E, F)$, $\gamma_i \in E'$. For every $k \in \mathbb{N}$, by the compatibility of \mathfrak{A}_n with \mathfrak{A} , we have that $\sum_{i=1}^k \gamma_i^{n-1}T_i \in \mathfrak{A}_n(E, F)$. Moreover, the series is convergent in $\mathfrak{A}_n(E, F)$ since

$$\sum_{i=1}^{\infty} \|\gamma_i^{n-1}T_i\|_{\mathfrak{A}_n(E, F)} \leq \tilde{B} \sum_{i=1}^{\infty} \|\gamma_i\|^{n-1}\|T_i\|_{\mathfrak{A}(E, F)}.$$

Hence $P \in \mathfrak{A}_n(E, F)$ and $\|P\|_{\mathfrak{A}_n(E, F)} \leq \tilde{B}\|P\|_{\mathcal{N}_n^{\mathfrak{A}}(E, F)}$.

Finally, we prove that $\mathcal{N}_n^{\mathfrak{A}}$ is compatible with \mathfrak{A} . It is immediate that (ii) is satisfied with constant $B = 1$. To prove (i), consider $a \in E$, and $P \in \mathcal{N}_n^{\mathfrak{A}}(E, F)$ and choose a representation $P = \sum_{i=1}^{\infty} \gamma_i^{n-1}T_i$. Then

$$P_{a^{n-1}} = \sum_{i=1}^{\infty} \left[\frac{1}{n}\gamma_i(a)^{n-1}T_i + \frac{n-1}{n}\gamma_i(a)^{n-2}T_i(a)\gamma_i \right].$$

For each $k \in \mathbb{N}$, $\sum_{i=1}^k \left[\frac{1}{n}\gamma_i(a)^{n-1}T_i + \frac{n-1}{n}\gamma_i(a)^{n-2}T_i(a)\gamma_i \right] \in \mathfrak{A}(E, F)$, and

$$\sum_{i=1}^{\infty} \left\| \frac{1}{n}\gamma_i(a)^{n-1}T_i + \frac{n-1}{n}\gamma_i(a)^{n-2}T_i(a)\gamma_i \right\|_{\mathfrak{A}} \leq \left(\sum_{i=1}^{\infty} \|\gamma_i\|^{n-1}\|T_i\|_{\mathfrak{A}} \right) \|a\|^{n-1},$$

for every representation of P . Since \mathfrak{A} is a complete ideal, we obtain that $P_{a^{n-1}} \in \mathfrak{A}(E, F)$ and

$$\|P_{a^{n-1}}\|_{\mathfrak{A}} \leq \|a\|^{n-1}\|P\|_{\mathcal{N}_n^{\mathfrak{A}}},$$

for every $a \in E$. Thus (i) is satisfied with constant $A = 1$. \square

The following proposition is an immediate consequence of the definition of $\mathcal{M}_n^{\mathfrak{A}}$, $\mathcal{F}_n^{\mathfrak{A}}$ and $\mathcal{N}_n^{\mathfrak{A}}$. It provides a kind of converse of Proposition 1.6 for the special cases of largest and smallest compatible and coherent ideals:

Proposition 2.3. *Let \mathfrak{A} and \mathfrak{B} be normed operator ideals. If for some E and F , $\mathfrak{A}(E, F) \subset \mathfrak{B}(E, F)$, then for all $n \geq 1$, $\mathcal{M}_n^{\mathfrak{A}}(E, F) \subset \mathcal{M}_n^{\mathfrak{B}}(E, F)$ and $\mathcal{F}_n^{\mathfrak{A}}(E, F) \subset \mathcal{F}_n^{\mathfrak{B}}(E, F)$. If \mathfrak{A} and \mathfrak{B} are complete, then we have also that $\mathcal{N}_n^{\mathfrak{A}}(E, F) \subset \mathcal{N}_n^{\mathfrak{B}}(E, F)$.*

Proposition 2.2 allows us to obtain the following characterization of all the polynomial ideals which are compatible with a given operator ideal.

Proposition 2.4. *Let \mathfrak{A} be a normed operator ideal and \mathfrak{A}_n a normed ideal of n -homogeneous polynomials. Then \mathfrak{A}_n is compatible with \mathfrak{A} with constants A and B if and only if the following two conditions hold for every Banach spaces E and F :*

- (i) $\mathcal{F}_n^{\mathfrak{A}}(E, F) \subset \mathfrak{A}_n(E, F)$ and the inclusion has norm less than or equal to B ,
- (ii) $\mathfrak{A}_n(E, F) \subset \mathcal{M}_n^{\mathfrak{A}}(E, F)$ and the inclusion has norm less than or equal to A .

Let \mathfrak{A} be a linear operator ideal. If $\{\mathfrak{A}_k\}_k$ is any coherent sequence of normed ideals of homogeneous polynomials with $\mathfrak{A}_1 = \mathfrak{A}$, then for each $k \in \mathbb{N}$, \mathfrak{A}_k is compatible with \mathfrak{A} . Thus, by Proposition 2.2 we have

$$\mathcal{F}_k^{\mathfrak{A}}(E, F) \subset \mathfrak{A}_k(E, F) \subset \mathcal{M}_k^{\mathfrak{A}}(E, F),$$

for every Banach spaces E and F . Therefore, if we show that $\{\mathcal{M}_k^{\mathfrak{A}}\}_k$ and $\{\mathcal{F}_k^{\mathfrak{A}}\}_k$ are coherent sequences (note that $\mathcal{M}_1^{\mathfrak{A}} = \mathcal{F}_1^{\mathfrak{A}} = \mathfrak{A}$), we can derive that they are, respectively, the largest and the smallest coherent sequence associated to \mathfrak{A} . Analogously, if \mathfrak{A} is complete, we obtain that $\{\mathcal{N}_k^{\mathfrak{A}}\}_k$ is the smallest coherent sequence of Banach ideals associated to \mathfrak{A} . However, in the three cases the obtained coherence constants are larger than the compatibility constants of Proposition 2.2.

Proposition 2.5. *Let \mathfrak{A} be a linear operator ideal. Then*

- a) *The sequence $\{\mathcal{M}_k^{\mathfrak{A}}\}_k$ is coherent with constants $C = e$ and $D = 1$. Thus, $\{\mathcal{M}_k^{\mathfrak{A}}\}_k$ is the largest coherent sequence associated to \mathfrak{A} .*
- b) *The sequence $\{\mathcal{F}_k^{\mathfrak{A}}\}_k$ is coherent with constants $C = 1$ and $D = e$. Thus, $\{\mathcal{F}_k^{\mathfrak{A}}\}_k$ is the smallest coherent sequence associated to \mathfrak{A} .*
- c) *If \mathfrak{A} is complete, the sequence $\{\mathcal{N}_k^{\mathfrak{A}}\}_k$ is coherent with constants $C = 1$ and $D = e$. Thus, $\{\mathcal{N}_k^{\mathfrak{A}}\}_k$ is the smallest coherent sequence of Banach ideals associated to \mathfrak{A} .*

Proof. We prove conditions (i) and (ii) of Definition 1.3 for the sequences $\{\mathcal{M}_k^{\mathfrak{A}}\}_k$ and $\{\mathcal{N}_k^{\mathfrak{A}}\}_k$. The case b) follows similarly.

a) (i) Let $P \in \mathcal{M}_k^{\mathfrak{A}}(E, F)$ and $a \in E$. We have to show that $P_a \in \mathcal{M}_{k-1}^{\mathfrak{A}}(E, F)$ with $\|P_a\|_{\mathcal{M}_{k-1}^{\mathfrak{A}}(E, F)} \leq e\|a\|\|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E, F)}$. For this, we need to prove that $(P_a)_{b^{k-2}} \in \mathfrak{A}(E, F)$ for all $b \in B_E$ and $\|(P_a)_{b^{k-2}}\|_{\mathfrak{A}(E, F)} \leq e\|a\|\|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E, F)}$.

As in Lemma 1.7 we take $r \in \mathbb{C}$ a primitive $(k-1)$ -root of 1. Then, for each $x \in E$ and $t > 0$, we have

$$\begin{aligned} (P_a)_{b^{k-2}}(x) &= (P_x)^\vee(a, b^{k-2}) \\ &= \frac{1}{(k-1)^2} \sum_{j=0}^{k-2} r^j t^{k-2} P_x \left(\frac{r^j}{t} b + a \right) \\ &= \frac{1}{(k-1)^2} \sum_{j=0}^{k-2} r^j t^{k-2} P_{\left(\frac{r^j}{t} b + a\right)^{k-1}}(x). \end{aligned}$$

Since $P \in \mathcal{M}_k^{\mathfrak{A}}(E, F)$, $(P_a)_{b^{k-2}}$ belongs to $\mathfrak{A}(E, F)$.

Choosing $t = \frac{1}{k-2}$ we obtain, for $\|a\| = \|b\| = 1$, that

$$\begin{aligned} \|(P_a)_{b^{k-2}}\|_{\mathfrak{A}(E,F)} &\leq \frac{1}{(k-1)^2} \sum_{j=0}^{k-2} t^{k-2} \left\| P_{\left(\frac{r^j}{t}b+a\right)^{k-1}} \right\|_{\mathfrak{A}(E,F)} \\ &\leq \frac{1}{(k-1)^2} \sum_{j=0}^{k-2} \left(\frac{1}{k-2}\right)^{k-2} (k-1)^{k-1} \|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)} \\ &\leq e \|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)}. \end{aligned}$$

Therefore, for each $a \in E$,

$$\|P_a\|_{\mathcal{M}_{k-1}^{\mathfrak{A}}(E,F)} = \sup_{\|b\|=1} \|(P_a)_{b^{k-2}}\|_{\mathfrak{A}(E,F)} \leq e \|a\| \|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)}.$$

(ii) Let $P \in \mathcal{M}_k^{\mathfrak{A}}(E, F)$, $\gamma \in E'$ and $a \in E$. Then

$$(\gamma P)_{a^k} = \frac{1}{k+1} \left(P(a)\gamma + k\gamma(a)P_{a^{k-1}} \right).$$

This implies that $(\gamma P)_{a^k} \in \mathfrak{A}(E, F)$ and thus $\gamma P \in \mathcal{M}_{k+1}^{\mathfrak{A}}(E, F)$. Moreover,

$$\begin{aligned} \|\gamma P\|_{\mathcal{M}_{k+1}^{\mathfrak{A}}(E,F)} &= \sup_{\|a\|=1} \|(\gamma P)_{a^k}\|_{\mathfrak{A}(E,F)} \\ &\leq \frac{1}{k+1} \sup_{\|a\|=1} \left(\|P\|_{\mathcal{P}^k(E,F)} \|\gamma\| + k \|\gamma\| \|P_{a^{k-1}}\|_{\mathfrak{A}(E,F)} \right) \\ &\leq \|\gamma\| \|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)}. \end{aligned}$$

c) (i) Let $P \in \mathcal{N}_{k+1}^{\mathfrak{A}}(E, F)$, fix a representation $P = \sum_{i=1}^{\infty} \gamma_i^k T_i$ and let $a \in E$. Then

$$P_a = \frac{1}{k+1} \sum_{i=1}^{\infty} \left(T_i(a) \gamma_i^k + k \gamma_i(a) (\gamma_i^{k-1} T_i) \right).$$

The partial sums of the above series belong to $\mathcal{N}_k^{\mathfrak{A}}(E, F)$. Furthermore,

$$\begin{aligned} \frac{1}{k+1} \sum_{i=1}^{\infty} \left\| \left(T_i(a) \gamma_i^k + k \gamma_i(a) (\gamma_i^{k-1} T_i) \right) \right\|_{\mathcal{N}_k^{\mathfrak{A}}} \\ \leq \sum_{i=1}^{\infty} \frac{\|a\| \|T_i\|_{\mathcal{L}} \|\gamma_i\|^k + k \|a\| \|\gamma_i\|^k \|T_i\|_{\mathfrak{A}}}{k+1} \\ \leq \|a\| \sum_{i=1}^{\infty} \|\gamma_i\|^k \|T_i\|_{\mathfrak{A}}. \end{aligned}$$

Then, P_a belongs to $\mathcal{N}_k^{\mathfrak{A}}(E, F)$ and, since the above inequality is valid for every representation of P , we obtain that

$$\|P_a\|_{\mathcal{N}_k^{\mathfrak{A}}(E,F)} \leq \|a\| \|P\|_{\mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)}.$$

(ii) Let $P \in \mathcal{N}_k^{\mathfrak{A}}(E, F)$. Suppose first that $P = \gamma^{k-1} T$, with $\gamma \in E'$ and $T \in \mathcal{L}(E, F)$. Let $\phi \in E'$. Then, proceeding as in Lemma 1.7 and Corollary 1.8 (applied to $x \mapsto (\phi \gamma^{k-1})(x)$) we obtain the expression

$$(\phi P)(x) = (\phi \gamma^{k-1} T)(x) = \frac{1}{k^2} \sum_{j=0}^{k-1} t^{k-1} r^j \left(\frac{r^j}{t} \gamma(x) + \phi(x) \right)^k T(x),$$

where $t > 0$ and $r \in \mathbb{C}$ is a primary k -root of the unit. This means that $\phi P \in \mathcal{N}_{k+1}^{\mathfrak{A}}(E, F)$ and

$$\|\phi P\|_{\mathcal{N}_{k+1}^{\mathfrak{A}}(E, F)} \leq e\|\phi\|\|\gamma\|^{k-1}\|T\|_{\mathfrak{A}(E, F)}.$$

Consider now $P \in \mathcal{N}_k^{\mathfrak{A}}(E, F)$ and a representation $P = \sum_{i=1}^{\infty} \gamma_i^{k-1} T_i$ with $\gamma_i \in E'$ and $T_i \in \mathcal{L}(E, F)$. Then the finite sums $\sum_{i=1}^m \phi \gamma_i^{k-1} T_i$ belong to $\mathcal{N}_{k+1}^{\mathfrak{A}}(E, F)$ and the series converges since

$$\sum_{i=1}^{\infty} \|\phi \gamma_i^{k-1} T_i\|_{\mathcal{N}_{k+1}^{\mathfrak{A}}(E, F)} \leq e\|\phi\| \sum_{i=1}^{\infty} \|\gamma_i\|^{k-1} \|T_i\|_{\mathfrak{A}(E, F)}.$$

Thus $P \in \mathcal{N}_{k+1}^{\mathfrak{A}}(E, F)$. The above inequality is valid for every representation of P , hence

$$\|\phi P\|_{\mathcal{N}_{k+1}^{\mathfrak{A}}(E, F)} \leq e\|\phi\|\|P\|_{\mathcal{N}_k^{\mathfrak{A}}(E, F)}. \quad \square$$

Remark 2.6. Note that if \mathfrak{A} is any operator ideal, the polynomial ideals $\mathcal{M}_n^{\mathfrak{A}}$, $\mathcal{F}_n^{\mathfrak{A}}$ and $\mathcal{N}_n^{\mathfrak{A}}$ are always different. Indeed, in the scalar valued case we have:

$$\mathcal{M}_n^{\mathfrak{A}}(E) = \mathcal{P}^n(E), \quad \mathcal{F}_n^{\mathfrak{A}}(E) = \mathcal{P}_f^n(E) \quad \text{and} \quad \mathcal{N}_n^{\mathfrak{A}}(E) = \mathcal{P}_N^n(E)$$

with equivalent norms. This means, in particular, that for any operator ideal, there are several different coherent sequences $\{\mathfrak{A}_k\}_k$ with $\mathfrak{A}_1 = \mathfrak{A}$.

3. COMPOSITION IDEALS

Let \mathfrak{A}_n be an ideal of n -homogeneous polynomials and \mathfrak{B} and \mathfrak{C} operator ideals. Following [11], we say that a polynomial P is in the **composition ideal** $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ if it admits a factorization $P = S \circ Q \circ T$, with $S \in \mathfrak{C}$, $Q \in \mathfrak{A}_n$ and $T \in \mathfrak{B}$. The ideals being normed, we define the composition quasi-norm

$$(7) \quad \|P\|_{\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}} = \inf \{ \|S\|_{\mathfrak{C}} \|Q\|_{\mathfrak{A}_n} \|T\|_{\mathfrak{B}}^n : \text{all factorizations of } P \}.$$

This quasi-norm is actually a λ -norm for some $0 < \lambda \leq 1$ [11]. We say that the composition ideal $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is normed whenever the composition quasi-norm (7) is a norm.

A normed ideal of linear operators (polynomials) is closed if the norm considered is the usual linear operator (polynomial) norm. If \mathfrak{B} and \mathfrak{C} are closed operator ideals and \mathfrak{A}_n is normed, then $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is normed.

This section is devoted to show that the composition of a coherent sequence of polynomial ideals with two fixed operator ideals is still coherent in the following situations: if either both linear operator ideals or the polynomial ideals are closed. We also address the question of compatibility for both types of composition. For the first case we have the following:

Proposition 3.1. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence of normed polynomial ideals with constants C and D , and let \mathfrak{C} and \mathfrak{B} be closed ideals of linear operators. Then $\{\mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}\}_k$ is a coherent sequence with constants C and D .*

Proof.

We check condition (i) of Definition 1.2: Let $P \in \mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}(E, F)$. Then $P = SQT$ with $S \in \mathfrak{C}(\tilde{F}, F)$, $T \in \mathfrak{B}(E, \tilde{E})$ y $Q \in \mathfrak{A}_k(\tilde{E}, \tilde{F})$.

Also, $P_a(\cdot) = S\check{Q}(T(a), T(\cdot), \dots, T(\cdot)) = SQ_{T(a)}T$, and $Q_{T(a)} \in \mathfrak{A}_{k-1}(\tilde{E}, \tilde{F})$ since $\{\mathfrak{A}_k\}$ is coherent. Therefore $P_a \in \mathfrak{C} \circ \mathfrak{A}_{k-1} \circ \mathfrak{B}(E, F)$.

Moreover,

$$\begin{aligned} \|P_a\|_{\mathfrak{C} \circ \mathfrak{A}_{k-1} \circ \mathfrak{B}(E, F)} &\leq C\|T(a)\|\|S\|_{\mathfrak{C}(\tilde{F}, F)}\|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})}\|T\|_{\mathfrak{B}(E, \tilde{E})}^{k-1} \\ &\leq C\|a\|\|S\|_{\mathfrak{C}(\tilde{F}, F)}\|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})}\|T\|_{\mathfrak{B}(E, \tilde{E})}^k, \end{aligned}$$

and this holds for every factorization $P = SQT$. Hence

$$\|P_a\|_{\mathfrak{C} \circ \mathfrak{A}_{k-1} \circ \mathfrak{B}(E, F)} \leq C\|a\|\|P\|_{\mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}(E, F)}.$$

Condition (ii): Again, we take $P = SQT$ with $S \in \mathfrak{C}(\tilde{F}, F)$, $T \in \mathfrak{B}(E, \tilde{E})$ and $Q \in \mathfrak{A}_k(\tilde{E}, \tilde{F})$.

Consider $\gamma \in E'$ and define the operators

$$\begin{aligned} \tilde{T} &\in \mathfrak{B}(E, \tilde{E} \times \mathbb{C}), & \tilde{T}(x) &= (T(x), \gamma(x)) = (i_1 \circ T)(x) + (i_2 \circ \gamma)(x) \\ R &\in \mathfrak{A}_{k+1}(\tilde{E} \times \mathbb{C}, \tilde{F}), & R(y, \lambda) &= Q(y)\lambda = (Q \circ \pi_1)(y, \lambda) \cdot \pi_2(y, \lambda), \end{aligned}$$

where i_1, i_2 are the inclusions

$$\tilde{E} \xrightarrow{i_1} \tilde{E} \times \mathbb{C}, \quad \mathbb{C} \xrightarrow{i_2} \tilde{E} \times \mathbb{C},$$

and π_1, π_2 are the projections to the first and second coordinate.

We have that SRT belongs to $\mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)$ and

$$SRT(x) = SR(T(x), \gamma(x)) = S(\gamma(x)Q(T(x))) = \gamma(x)P(x).$$

As a consequence, $\gamma P \in \mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)$.

For the inequality of norms, we may assume that $\|T\|_{\mathfrak{B}(E, \tilde{E})} = \|T\|_{\mathfrak{L}(E, \tilde{E})} = 1$. Let us consider in $\tilde{E} \times \mathbb{C}$ the norm $\|(y, \lambda)\| = \max\{\|y\|, |\lambda|\}$ and suppose $\|\gamma\| = 1$. Then, we have

$$\|R\|_{\mathfrak{A}_{k+1}(\tilde{E} \times \mathbb{C}, \tilde{F})} \leq D\|Q \circ \pi_1\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})}\|\pi_2\| \leq D\|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})},$$

and

$$\|\tilde{T}\|_{\mathfrak{B}(E, \tilde{E} \times \mathbb{C})} = \|\tilde{T}\|_{\mathfrak{L}(E, \tilde{E} \times \mathbb{C})} = \max\{\|T\|_{\mathfrak{L}(E, \tilde{E})}, \|\gamma\|\} = 1.$$

Thus we obtain

$$\begin{aligned} \|\gamma P\|_{\mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)} &\leq \|S\|_{\mathfrak{C}(\tilde{F}, F)}\|R\|_{\mathfrak{A}_{k+1}(\tilde{E} \times \mathbb{C}, \tilde{F})}\|\tilde{T}\|_{\mathfrak{B}(E, \tilde{E} \times \mathbb{C})}^{k+1} \\ &\leq D\|S\|_{\mathfrak{C}(\tilde{F}, F)}\|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})} \end{aligned}$$

and this is true for every factorization $P = SQT$ with $\|T\|_{\mathfrak{B}(E, \tilde{E})} = 1$. Hence, for a general $\gamma \in E'$,

$$\|\gamma P\|_{\mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)} \leq D\|\gamma\|\|P\|_{\mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}(E, F)}. \quad \square$$

Similarly one can prove the following

Proposition 3.2. *Let \mathfrak{A} be a normed ideal of linear operators and \mathfrak{A}_n a normed ideal of n -homogeneous polynomials compatible with \mathfrak{A} with constants A and B . If \mathfrak{C} and \mathfrak{B} are closed operator ideals, then $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is compatible with $\mathfrak{C} \circ \mathfrak{A} \circ \mathfrak{B}$ with constants A and B .*

Now we turn our attention to the composition of a closed polynomial ideal with arbitrary operator ideals. As a particular case, if \mathfrak{B} is an operator ideal, the composition ideal $\mathcal{P}^n \circ \mathfrak{B}$ is analogous to the Pietsch Factorization method of multilinear mappings. In [2] the authors norm the polynomial ideal obtained by this method considering in equation (7) all the factorizations of the multilinear operator \check{P} rather than the factorizations of P . That is, the norm for the polynomial ideal keeps the multilinear essence of Pietsch method. Although for a fixed n the norm given by (7) and the norm in [2] are equivalent, the equivalence constants depend on n . Therefore, compatibility, coherence and the corresponding constants for $\mathcal{P}^n \circ \mathfrak{B}$ do not follow from the analogous properties for the norm defined in [2].

As we mentioned before, composition ideals are generally not normed but λ -normed. Since our work is devoted to normed polynomial ideals we choose to restrict ourselves to this case.

Proposition 3.3. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence of closed polynomial ideals. If \mathfrak{B} and \mathfrak{C} are normed operator ideals such that the composition $\mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}$ is normed for all k , then $\{\mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}\}_k$ is a coherent sequence with constants $C = e$ and $D = 1$.*

Proof. Condition (i): Let $P = SQT$, with $T \in \mathfrak{B}(E, \tilde{E})$, $Q \in \mathfrak{A}_k(\tilde{E}, \tilde{F})$ and $S \in \mathfrak{C}(\tilde{F}, F)$. Then $P_a = (SQT)_a = SQ_{T(a)}T \in \mathfrak{C} \circ \mathfrak{A}_{k-1} \circ \mathfrak{B}(E, F)$.

Moreover, by Example 1.9,

$$\begin{aligned} \|P_a\|_{\mathfrak{C} \circ \mathfrak{A}_{k-1} \circ \mathfrak{B}(E, F)} &\leq \|S\|_{\mathfrak{C}(\tilde{F}, F)} \|Q_{T(a)}\|_{\mathfrak{A}_{k-1}(\tilde{E}, \tilde{F})} \|T\|_{\mathfrak{B}(E, \tilde{E})}^{k-1} \\ &\leq e \|a\| \|S\|_{\mathfrak{C}(\tilde{F}, F)} \|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})} \|T\|_{\mathfrak{B}(E, \tilde{E})}^k. \end{aligned}$$

Since this holds for every factorization $P = SQT$, we obtain $\|P_a\|_{\mathfrak{C} \circ \mathfrak{A}_{k-1} \circ \mathfrak{B}(E, F)} \leq e \|a\| \|P\|_{\mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}(E, F)}$.

Condition (ii): Let $P = SQT$, $T \in \mathfrak{B}(E, \tilde{E})$, $Q \in \mathfrak{A}_k(\tilde{E}, \tilde{F})$ and $S \in \mathfrak{C}(\tilde{F}, F)$. Define as in Proposition 3.1 the operators

$$\begin{aligned} \tilde{T} &\in \mathfrak{B}(E, \tilde{E} \times \mathbb{C}), & \tilde{T}(x) &= (T(x), \gamma(x)) = (i_1 \circ T)(x) + (i_2 \circ \gamma)(x) \\ R &\in \mathfrak{A}_{k+1}(\tilde{E} \times \mathbb{C}, \tilde{F}), & R(y, \lambda) &= Q(y)\lambda = Q \circ \pi_1(y, \lambda) \cdot \pi_2(y, \lambda). \end{aligned}$$

Then, $SR\tilde{T} \in \mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)$ and

$$SR\tilde{T}(x) = S(R(T(x), \gamma(x))) = S(Q(T(x)))\gamma(x) = \gamma(x)P(x).$$

Thus $\gamma P \in \mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)$.

To prove the inequality of norms we now consider $\tilde{E} \times \mathbb{C}$ with the norm $\|(y, \lambda)\| = \|y\| + |\lambda|$. Since, for every k , the norm in \mathfrak{A}_k is the usual polynomial norm,

$$\begin{aligned} \|R\|_{\mathfrak{A}_{k+1}(\tilde{E} \times \mathbb{C}, \tilde{F})} &= \sup_{\|y\| + |\lambda| \leq 1} \|\lambda Q(y)\| \leq \|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})} \sup_{\|y\| + |\lambda| \leq 1} |\lambda| \|y\|^k \\ &\leq \frac{k^k}{(k+1)^{k+1}} \|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})}. \end{aligned}$$

Also,

$$\begin{aligned}\|\tilde{T}\|_{\mathfrak{B}(E, \tilde{E} \times \mathbb{C})} &= \|(i_1 \circ T)(\cdot) + (i_2 \circ \gamma)(\cdot)\|_{\mathfrak{B}} \leq \|i_1\| \|T\|_{\mathfrak{B}(E, \tilde{E})} + \|i_2\| \|\gamma\| \\ &= \|T\|_{\mathfrak{B}(E, \tilde{E})} + \|\gamma\|.\end{aligned}$$

Then,

$$\begin{aligned}\|\gamma P\|_{\mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)} &\leq \|S\|_{\mathfrak{C}(\tilde{F}, F)} \|R\|_{\mathfrak{A}_{k+1}(\tilde{E} \times \mathbb{C}, \tilde{F})} \|\tilde{T}\|_{\mathfrak{B}(E, \tilde{E} \times \mathbb{C})}^{k+1} \\ &\leq \|S\|_{\mathfrak{C}(\tilde{F}, F)} \frac{k^k}{(k+1)^{k+1}} \|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})} \left(\|T\|_{\mathfrak{B}(E, \tilde{E})} + \|\gamma\| \right)^{k+1}.\end{aligned}$$

If we consider $T_t = tT$ and $Q_t = t^{-k}Q$, we obtain a new factorization of P and thus of γP . The previous inequality applied to this factorization gives

$$\|\gamma P\|_{\mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)} \leq \|S\|_{\mathfrak{C}(\tilde{F}, F)} \frac{k^k}{(k+1)^{k+1}} \frac{\|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})}}{t^k} \left(t \|T\|_{\mathfrak{B}(E, \tilde{E})} + \|\gamma\| \right)^{k+1}.$$

This expression is minimum when $t = \frac{k\|\gamma\|}{\|T\|_{\mathfrak{B}(E, \tilde{E})}}$. In this case,

$$\begin{aligned}\|\gamma P\|_{\mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)} &\leq \|S\|_{\mathfrak{C}(\tilde{F}, F)} \frac{k^k}{(k+1)^{k+1}} \left(\frac{\|T\|_{\mathfrak{B}(E, \tilde{E})}}{k\|\gamma\|} \right)^k \|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})} ((k+1)\|\gamma\|)^{k+1} \\ &= \|\gamma\| \|S\|_{\mathfrak{C}(\tilde{F}, F)} \|Q\|_{\mathfrak{A}_k(\tilde{E}, \tilde{F})} \|T\|_{\mathfrak{B}(E, \tilde{E})}^k.\end{aligned}$$

This is true for each factorization $P = SQT$ and therefore,

$$(8) \quad \|\gamma P\|_{\mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)} \leq \|\gamma\| \|P\|_{\mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}(E, F)},$$

which completes the proof. \square

Regarding compatibility, a similar proof gives the following:

Proposition 3.4. *Let \mathfrak{A}_n be a closed ideal of n -homogeneous polynomials compatible with a closed operator ideal \mathfrak{A} . If \mathfrak{B} and \mathfrak{C} are normed ideals of linear operators such that $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ and $\mathfrak{C} \circ \mathfrak{A} \circ \mathfrak{B}$ are normed, then $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is compatible with $\mathfrak{C} \circ \mathfrak{A} \circ \mathfrak{B}$ with constants $A = e$ and $B = 1$.*

The ideal of r -dominated k -homogeneous polynomials \mathcal{D}_r^k is the composition ideal $\mathcal{P}^k \circ \Pi_r$ [19], where Π_r is the ideal of absolutely r -summing operators. For $r \geq k$, \mathcal{D}_r^k is a normed polynomial ideal and the r -dominated norm coincides with the composition norm. The two previous propositions show that the sequence of r -dominated homogeneous polynomials is coherent and compatible with the ideal of absolutely r -summing operators, as we promised in Section 1.

Corollary 3.5. *Let $1 < r < \infty$ and N be the largest integer not greater than r . The sequence $\{\mathcal{D}_r^k\}_{k=1}^N$ is coherent with constants $C = e$ and $D = 1$. Moreover, for each $n \leq N$, \mathcal{D}_r^n is compatible with Π_r with the same constants.*

4. RELATION WITH TENSOR NORMS

The aim of this section is to relate the concepts of compatibility and coherence with tensor norm properties. Given an operator ideal \mathfrak{A} , a symmetric n -tensor norm α and Banach spaces E and F , we can define $\mathfrak{A}_n(E, F) \stackrel{1}{=} \mathfrak{A}(\bigotimes_{\alpha}^{n,s} E, F)$, where we identify P with the linear operator T_P (i.e. $P \in \mathfrak{A}_n(E, F)$ if and only if $T_P \in \mathfrak{A}(\bigotimes_{\alpha}^{n,s} E, F)$ and $\|P\|_{\mathfrak{A}_n(E, F)} = \|T_P\|_{\mathfrak{A}(\bigotimes_{\alpha}^{n,s} E, F)}$). Since α is a tensor norm and \mathfrak{A} an operator ideal, it is easy to see that \mathfrak{A}_n is a normed ideal of n -homogeneous polynomials. We have also

Proposition 4.1. *Let α be a symmetric n -tensor norm and \mathfrak{A} an operator ideal. Then, $\mathfrak{A}_n(E, F) = \mathfrak{A}(\bigotimes_{\alpha}^{n,s} E, F)$ is compatible with \mathfrak{A} , with constants $A = e$ and $B = e$.*

Proof. Let us check condition (i) of Definition 1.2. Take $P \in \mathfrak{A}_n(E, F)$. For $a \in E$, we define $\Phi_{a^{n-1}} : E \rightarrow \bigotimes_{\alpha}^{n,s} E$ by

$$\Phi_{a^{n-1}}(x) = \sigma(a \otimes \cdots \otimes a \otimes x).$$

By Lemma 1.7, $\Phi_{a^{n-1}}$ is continuous and

$$\alpha(\Phi_{a^{n-1}}(x); \bigotimes_{\alpha}^{n,s} E) \leq e\|a\|^{n-1}\|x\|.$$

Moreover, $P_{a^{n-1}}(x) = (T_P \circ \Phi_{a^{n-1}})(x)$. Then, $P_{a^{n-1}}$ belongs to $\mathfrak{A}(E, F)$ and

$$\|P_{a^{n-1}}\|_{\mathfrak{A}} \leq \|T_P\|_{\mathfrak{A}} \|\Phi_{a^{n-1}}\|_{\mathcal{L}(E, \bigotimes_{\alpha}^{n,s} E)} \leq e\|a\|^{n-1}\|P\|_{\mathfrak{A}_n},$$

which gives condition (i) with $A = e$.

Now we prove condition (ii). For $\gamma \in E'$, define $\Psi_{\gamma^{n-1}} : \bigotimes_{\alpha}^{n,s} E \rightarrow E$, as $\Psi_{\gamma^{n-1}}(x^n) = \gamma(x)^{n-1}x$.

To see that $\Psi_{\gamma^{n-1}}$ is continuous, it is enough to consider the case $\alpha = \varepsilon_s$. If $z = \sum_{i=1}^m x_i^n$, we have

$$\begin{aligned} \|\Psi_{\gamma^{n-1}}(z)\|_E &= \left\| \sum_{i=1}^m \gamma(x_i)^{n-1} x_i \right\|_E = \sup_{\varphi \in B_{E'}} \left| \sum_{i=1}^m \gamma(x_i)^{n-1} \varphi(x_i) \right| \\ &= \sup_{\varphi \in B_{E'}} |\langle \gamma^{n-1} \varphi, z \rangle| \leq \sup_{\varphi \in B_{E'}} \|\gamma^{n-1} \varphi\|_{\mathcal{P}_I^n(E)} \varepsilon_s(z) \end{aligned}$$

By Corollary 1.8, $\|\gamma^{n-1} \varphi\|_{\mathcal{P}_I^n(E)} \leq e\|\gamma\|^{n-1}\|\varphi\|$, so $\Psi_{\gamma^{n-1}}$ is continuous and

$$\|\Psi_{\gamma^{n-1}}\| \leq e\|\gamma\|^{n-1}.$$

Take now $u \in \mathfrak{A}(E, F)$. Since $T_{\gamma^{n-1}u} = u \circ \Psi_{\gamma^{n-1}}$, we have $T_{\gamma^{n-1}u} \in \mathfrak{A}(\bigotimes_{\alpha}^{n,s} E, F)$. Therefore, $\gamma^{n-1}u \in \mathfrak{A}_n(E, F)$ and

$$\|\gamma^{n-1}u\|_{\mathfrak{A}_n} \leq \|u\|_{\mathfrak{A}} \|\Psi_{\gamma^{n-1}}\| \leq e\|\gamma\|^{n-1}\|u\|_{\mathfrak{A}},$$

from which (ii) follows, with constant $B = e$. \square

The previous proposition gives a simple way to obtain a great variety of compatible polynomial ideals from a fixed operator ideal. Recall that, on the other hand, given a polynomial ideal, there exists at most one operator ideal compatible with it.

Suppose now that for each k we have a symmetric k -tensor norm α_k and set $\mathfrak{A}_k(E, F) = \mathfrak{A}(\bigotimes_{\alpha_k}^{k,s} E, F)$. Each \mathfrak{A}_k is compatible with \mathfrak{A} but, in order to obtain a coherent sequence,

some coherence properties for the sequence of tensor norms $\{\alpha_k\}_k$ are needed. Let us establish this coherence. For $a \in E$, $\gamma \in E'$ we define the following mappings for each k (we omit the dependence on k in the notation for the sake of simplicity):

$$\begin{aligned} \Phi_a : \bigotimes^{k-1,s} E &\rightarrow \bigotimes^{k,s} E & \Psi_\gamma : \bigotimes^{k+1,s} E &\rightarrow \bigotimes^{k,s} E \\ x^{k-1} &\mapsto \sigma(a \otimes x^{k-1}) & x^{k+1} &\mapsto \gamma(x)x^k. \end{aligned}$$

For $P \in \mathcal{P}^k(E)$, we have $T_{P_a} = T_P \circ \Phi_a$ and $T_{\gamma P} = T_P \circ \Psi_\gamma$. From this, it is not difficult to obtain the following:

Proposition 4.2. *Let \mathfrak{A} be an operator ideal. Suppose we have, for each k , a symmetric k -tensor norm α_k and define $\mathfrak{A}_k(E, F) = \mathfrak{A}(\bigotimes_{\alpha_k}^{k,s} E, F)$. Then, $\{\mathfrak{A}_k\}_k$ is a coherent sequence of polynomial ideals (with constants C and D) if and only if for every Banach space E the mappings*

$$\Phi_a : \left(\bigotimes^{k-1,s} E, \alpha_{k-1} \right) \longrightarrow \left(\bigotimes^{k,s} E, \alpha_k \right)$$

and

$$\Psi_\gamma : \left(\bigotimes^{k+1,s} E, \alpha_{k+1} \right) \longrightarrow \left(\bigotimes^{k,s} E, \alpha_k \right)$$

are continuous for every k (with $\|\Phi_a\| \leq C\|a\|$ and $\|\Psi_\gamma\| \leq D\|\gamma\|$).

We use the results of this section to prove that the sequences $\{\mathcal{P}_{PI}^k\}_k$ and $\{\mathcal{P}_{GI}^k\}_k$ of Pietsch and Grothendieck integral polynomials are coherent and compatible with the ideals of Pietsch and Grothendieck integral operators.

First, recall that $\mathcal{P}_{PI}^k(E, F) = \mathcal{L}_{PI}(\bigotimes_{\varepsilon_s}^{k,s} E, F)$ and $\mathcal{P}_{GI}^k(E, F) = \mathcal{L}_{GI}(\bigotimes_{\varepsilon_s}^{k,s} E, F)$ isometrically [6, 20]. An application of Lemma 1.7 shows that the sequence of symmetric injective tensor norms verifies the conditions of the previous proposition with constants $C = 1$ and $D = e$. Also, we obtain the same constants for compatibility. Therefore, we have

Corollary 4.3. *The sequences $\{\mathcal{P}_{PI}^k\}_k$ and $\{\mathcal{P}_{GI}^k\}_k$ are coherent with constants $C = 1$ and $D = e$. Moreover, they are compatible with the ideals of Pietsch and Grothendieck integral operators respectively, with constants $A = 1$ and $B = e$.*

If \mathfrak{A} is a maximal operator ideal, by the representation theorem [7, Section 17] there exists a finitely generated (2-fold) tensor norm β such that:

$$\begin{aligned} \mathfrak{A}(E, F) &\stackrel{1}{=} (E \otimes_\beta F')' \cap \mathcal{L}(E, F) \\ \mathfrak{A}(E, F') &\stackrel{1}{=} (E \otimes_\beta F)' \end{aligned}$$

In this case, we write $\mathfrak{A} = \mathcal{L}_\beta$ and say that \mathfrak{A} is dual to the tensor norm β . Floret [11] extends these concepts to the polynomial setting with the introduction of mixed tensor norms. We recall their definition: a mixed tensor norm δ of order $n + 1$ assigns to each pair (E, F) a norm on $\bigotimes^{n,s} E \otimes F$ such that

- (a) $\delta(\bigotimes^{n,s} 1 \otimes 1, \bigotimes^{n,s} \mathbb{C} \otimes \mathbb{C}) = 1$
- (b) δ satisfies the metric mapping property.

A polynomial ideal \mathfrak{A}_n is maximal if and only if it is dual to a finitely generated mixed tensor norm δ in the following sense (see [11, 7.8]):

$$\begin{aligned}\mathfrak{A}_n(E, F) &\stackrel{1}{=} (\bigotimes^{n,s} E \otimes F', \delta)' \cap \mathcal{P}^n(E, F) \\ \mathfrak{A}_n(E, F') &\stackrel{1}{=} (\bigotimes^{n,s} E \otimes F, \delta)'\end{aligned}$$

In this case we say that \mathfrak{A}_n is dual to the tensor norm δ and write $\mathfrak{A}_n = \mathcal{P}_\delta^n$.

Following [14], if α is a symmetric n -tensor norm and β is a 2-fold tensor norm, we denote (α, β) the mixed tensor norm on $\bigotimes^{n,s} E \otimes F$ given by

$$\left(\bigotimes^{n,s} E \otimes F, (\alpha, \beta) \right) \stackrel{1}{=} \bigotimes_{\alpha}^{n,s} E \otimes_{\beta} F.$$

If $\mathfrak{A} = \mathcal{L}_\beta$ and \mathfrak{A}_n is the polynomial ideal given by $\mathfrak{A}_n(E, F) = \mathfrak{A}(\bigotimes_{\alpha}^{n,s} E, F)$, it follows that \mathfrak{A}_n is dual to (α, β) . In particular, if α and β are finitely generated, \mathfrak{A}_n is maximal.

In [14], it is conjectured that not every maximal ideal is dual to a mixed tensor norm of the form (α, β) . We now show that this conjecture is true, presenting a maximal polynomial ideal that is not dual to any (α, β) norm. First we need the following result, which is of independent interest:

Proposition 4.4. *Let \mathfrak{A} be an operator ideal and \mathfrak{A}_n an ideal of n -homogeneous polynomials compatible with \mathfrak{A} . If $\mathfrak{A}_n = \mathcal{P}_{(\alpha, \beta)}^n$ for some 2-fold tensor norm β and some symmetric n -tensor norm α , then:*

- a) $\mathfrak{A} = \mathcal{L}_\beta$;
- b) $\mathfrak{A}_n(E, F) = \mathfrak{A}(\bigotimes_{\alpha}^{n,s} E, F)$.

Proof. We have $\mathfrak{A}_n(E, F) = (\bigotimes_{\alpha}^{n,s} E \otimes_{\beta} F')' \cap \mathcal{P}^n(E, F) = \mathcal{L}_\beta(\bigotimes_{\alpha}^{n,s} E, F)$. Then, by Proposition 4.1, \mathfrak{A}_n is compatible with \mathcal{L}_β . By uniqueness of the compatible operator ideal (Proposition 1.6), $\mathfrak{A} = \mathcal{L}_\beta$. \square

In [5] it is shown that the ideal of r -dominated n -linear forms is maximal and a finitely generated n -tensor norm is presented to which it is dual. Using the same ideas we prove an analogous statement for vector-valued polynomials.

For $z \in \bigotimes^{n,s} E \otimes F$ we define

$$\delta_r^n(z, \bigotimes^{n,s} E \otimes F) = \inf \left\{ w_r((x_i)_{i=1}^m)^n \ell_u((y_i)_{i=1}^m) : z = \sum_{i=1}^m x_i^n \otimes y_i \right\},$$

where $\frac{1}{u} + \frac{n}{r} = 1$ and $\ell_u((y_i)_{i=1}^m) = (\sum_{i=1}^m \|y_i\|^u)^{\frac{1}{u}}$.

Proceeding as in [7, 12.5] it can be seen that δ_r^n is a finitely generated mixed tensor norm. Also, we have:

Lemma 4.5. *\mathcal{D}_r^n is dual to the mixed tensor norm δ_r^n . In particular, \mathcal{D}_r^n is a maximal polynomial ideal.*

Proof. We show that $\mathcal{D}_r^n(E, F') = (\bigotimes^{n,s} E \otimes F, \delta_r^n)'$. In a similar way it can be proved that $\mathcal{D}_r^n(E, F) = (\bigotimes^{n,s} E \otimes F', \delta_r^n)' \cap \mathcal{P}^n(E, F)$.

Let $P \in \mathcal{D}_r^n(E, F')$ and $z \in (\bigotimes^{n,s} E \otimes F, \delta_r^n)$. For any representation $z = \sum_{i=1}^m x_i^n \otimes y_i$, we have

$$\begin{aligned} |\langle P, z \rangle| &= \left| \sum_{i=1}^m P(x_i)(y_i) \right| \leq \ell_u((y_i)_{i=1}^m) \ell_{\frac{r}{n}}((P(x_i))_{i=1}^m) \\ &\leq \|P\|_{D_r^n(E, F')} w_r((x_i)_{i=1}^m)^n \ell_u((y_i)_{i=1}^m). \end{aligned}$$

This is true for any representation of z , thus $P \in (\bigotimes^{n,s} E \otimes F, \delta_r^n)'$ and $\|P\|_{(\bigotimes^{n,s} E \otimes F, \delta_r^n)'} \leq \|P\|_{D_r^n(E, F')}$.

Conversely, let $P \in (\bigotimes^{n,s} E \otimes F, \delta_r^n)'$, choose $\varepsilon > 0$ and a sequence $(x_i)_{i=1}^m \subset E$. Then for each $i = 1, \dots, m$ there exists an element $y_i \in B_F$, such that $\|P(x_i)\| \leq P(x_i)(y_i) + \frac{\varepsilon}{m}$. Also, we can find a sequence $(\lambda_i)_{i=1}^m$ of positive numbers, with $\ell_u((\lambda_i)_{i=1}^m) = 1$, such that $\sum_{i=1}^m \lambda_i (P(x_i)(y_i) + \frac{\varepsilon}{m}) = \ell_{\frac{r}{n}}\left((P(x_i)(y_i) + \frac{\varepsilon}{m})_{i=1}^m\right)$. Then

$$\begin{aligned} \left(\sum_{i=1}^m \|P(x_i)\|^{\frac{r}{n}} \right)^{\frac{n}{r}} &\leq \left(\sum_{i=1}^m \left(P(x_i)(y_i) + \frac{\varepsilon}{m} \right)^{\frac{r}{n}} \right)^{\frac{n}{r}} \\ &= \sum_{i=1}^m \lambda_i \left(P(x_i)(y_i) + \frac{\varepsilon}{m} \right) \\ &\leq \langle P, \sum_{i=1}^m x_i^n \otimes \lambda_i y_i \rangle + \varepsilon \\ &\leq \|P\|_{(\bigotimes^{n,s} E \otimes F, \delta_r^n)'} w_r((x_i)_{i=1}^m)^n \ell_u((\lambda_i y_i)_{i=1}^m) + \varepsilon \end{aligned}$$

Since $\ell_u((\lambda_i y_i)_{i=1}^m) \leq 1$ and this is valid for any $\varepsilon > 0$, we have that $P \in \mathcal{D}_r^n(E, F')$ and $\|P\|_{D_r^n(E, F')} \leq \|P\|_{(\bigotimes^{n,s} E \otimes F, \delta_r^n)'}$. \square

Corollary 4.6. *For any $n \geq 2$, the ideal of r -dominated n -homogeneous polynomials \mathcal{D}_r^n is maximal but is not dual to any mixed tensor norm of the form (α, β) .*

Proof. We have seen that \mathcal{D}_r^n is maximal in the previous lemma. Suppose that there exists n such that $\mathcal{D}_r^n = \mathcal{P}_{(\alpha, \beta)}^n$, for some 2-fold tensor norm β and some symmetric n -tensor norm α .

Since the ideal of r -dominated polynomials is compatible with the ideal of absolutely r -summing operators, the previous proposition would assure that $\mathcal{D}_r^n(E, F) = \Pi_r(\bigotimes_{\alpha}^{n,s} E, F)$. Now consider $E = F = \ell_1$ and $P \in \mathcal{P}^n(\ell_1, \ell_1)$ given by

$$P(x) = \sum_{j=1}^{\infty} x_j^n.$$

Since P factors through the absolutely 1-summing inclusion $\ell_1 \hookrightarrow \ell_2$, P is r -dominated for all $r \geq 1$ (in particular, for $r \geq n$). However, we have that $T_P(e_j^n) = e_j$, and therefore $T_P \in \mathcal{L}(\bigotimes_{\alpha}^{n,s} \ell_1, \ell_1)$ cannot be weakly compact (independently of the choice of α). Consequently, T_P is not absolutely r -summing, which leads to a contradiction. Therefore, there is no α and β such that $\mathcal{D}_r^n = \mathcal{P}_{(\alpha, \beta)}^n$. \square

Note that the previous corollary also gives the following:

Corollary 4.7. *There are mixed tensor norms that are not equivalent to any (α, β) -norm.*

Since there are mixed tensor norms that are not of the form (α, β) , it is now desirable to point out conditions on a mixed tensor norm (or a sequence of mixed tensor norms) that ensure compatibility (or coherence) with an operator ideal.

For $a \in E$ and $\gamma \in E'$ we define the following mappings:

$$\begin{aligned} \Phi_a^F : \bigotimes^{k-1, s} E \otimes F &\rightarrow \bigotimes^{k, s} E \otimes F & \Psi_\gamma^F : \bigotimes^{k+1, s} E \otimes F &\rightarrow \bigotimes^{k, s} E \otimes F \\ x^{k-1} \otimes y &\mapsto \sigma(a \otimes x^{k-1}) \otimes y & x^{k+1} \otimes y &\mapsto \gamma(x)x^k \otimes y \\ \\ \Phi_{a^{k-1}}^F : E \otimes F &\rightarrow \bigotimes^{k, s} E \otimes F & \Psi_{\gamma^k}^F : \bigotimes^{k+1, s} E \otimes F &\rightarrow E \otimes F \\ x \otimes y &\mapsto \sigma(a \otimes x^{k-1}) \otimes y & x^{k+1} \otimes y &\mapsto \gamma(x)^k x \otimes y \end{aligned}$$

Note each of the conditions (i) and (ii) of Definitions 1.3 and 1.2 can be seen as dual to continuity properties of one of the four mappings defined above when we consider F' instead of F . This allows us to state the following results, whose proofs we omit.

Proposition 4.8. *Let β be a 2-fold tensor norm and δ a mixed tensor norm of order $n+1$. Then, \mathcal{P}_δ^n is compatible with \mathcal{L}_β (with constants A and B) if and only if the mappings $\Phi_{a^{n-1}}^{F'}$ and $\Psi_{\gamma^{n-1}}^{F'}$ are δ -to- β and β -to- δ continuous for every E and F , (with $\|\Phi_{a^{n-1}}^{F'}\| \leq A\|a\|^{n-1}$ and $\|\Psi_{\gamma^{n-1}}^{F'}\| \leq B\|\gamma\|^{n-1}$).*

Proposition 4.9. *Let \mathfrak{A} be an operator ideal. Suppose we have, for each k , a mixed tensor norm δ_k of order $k+1$. Then, $\mathcal{P}_{\delta_k}^k$ is a coherent sequence of polynomial ideals (with constants C and D) if and only if the mappings $\Phi_a^{F'}$ and $\Psi_\gamma^{F'}$ are δ_{k-1} -to- δ_k and δ_{k+1} -to- δ_k continuous for every k , E and F (with $\|\Phi_a^{F'}\| \leq C\|a\|$ and $\|\Psi_\gamma^{F'}\| \leq D\|\gamma\|$).*

5. MAXIMAL, MINIMAL AND ADJOINT IDEALS

In this section we show that other natural ideal operations preserve coherence and compatibility. First we consider the adjoint of a polynomial ideal.

Let \mathfrak{A}_n be a normed ideal of n -homogeneous polynomials. For each pair of Banach spaces E and F , we define the adjoint ideal $\mathfrak{A}_n^*(E, F)$ as a vector-valued version of [11, 4.3]: for $M \in FIN(E)$, $N \in FIN(F)$ we define λ the mixed tensor norm of order $n+1$ given by

$$\left(\bigotimes^{n, s} M \otimes N, \lambda \right) \stackrel{1}{=} \mathfrak{A}_n(M', N).$$

That is, $z = \sum_i x_i^n \otimes y_i \in \bigotimes^{n, s} M \otimes N$ is associated to $P^z \in \mathfrak{A}_n(M', N)$, where $P^z(x') = \sum_i x'(x_i)^n y_i$; and $\lambda(z; \bigotimes^{n, s} M \otimes N) := \|P^z\|_{\mathfrak{A}_n(M', N)}$.

For $z \in \bigotimes^{n, s} E \otimes F$, we define

$$\lambda(z; \bigotimes^{n, s} E \otimes F) := \inf \left\{ \lambda(z; \bigotimes^{n, s} M \otimes N) \right\},$$

where the infimum is taken over all $M \in FIN(E)$, $N \in FIN(F)$ such that $z \in \bigotimes^{n, s} M \otimes N$. Finally, the adjoint ideal \mathfrak{A}_n^* is

$$\mathfrak{A}_n^*(E, F) := \left(\bigotimes^{n, s} E \otimes F', \lambda \right)' \cap \mathcal{P}^n(E, F).$$

Proposition 5.1. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence with constants C and D . Then $\{\mathfrak{A}_k^*\}_k$ is a coherent sequence with constants D and C .*

Proof. Condition (i). By Proposition 4.9 it suffices to verify that

$$\begin{aligned} \Phi_a^{F'} : \left(\bigotimes^{k-1,s} E \otimes F', \lambda \right) &\rightarrow \left(\bigotimes^{k,s} E \otimes F', \lambda \right) \\ x^{k-1} \otimes y &\mapsto \sigma(a \otimes x^{k-1}) \otimes y \end{aligned}$$

is continuous and that $\|\Phi_a^{F'}\| \leq D\|a\|$, for all $a \in E$.

Consider $M \in \text{FIN}(E)$; $N \in \text{FIN}(F')$ and $z = \sum_i x_i^{k-1} \otimes y_i \in \bigotimes^{k-1,s} M \otimes N$. Then $P^{\Phi_a^{F'}(z)} \in \mathfrak{A}_k(M'_a, N)$, where $M_a = M \oplus [a]$. If $x' \in M'_a$

$$P^{\Phi_a^{F'}(z)}(x') = \sum_i \sigma(a \otimes x_i^{k-1})(x') y_i = \sum_i x'(a) x'(x_i)^{k-1} y_i = (aP^z)(x').$$

Thus,

$$\begin{aligned} \lambda\left(\Phi_a^{F'}(z); \bigotimes^{k,s} E \otimes F'\right) &\leq \lambda\left(\Phi_a^{F'}(z); \bigotimes^{k,s} M_a \otimes N\right) \\ &= \|P^{\Phi_a^{F'}(z)}\|_{\mathfrak{A}_k(M'_a, N)} = \|aP^z\|_{\mathfrak{A}_k(M'_a, N)} \\ &\leq D\|a\| \|P^z\|_{\mathfrak{A}_{k-1}(M'_a, N)} \\ &\leq D\|a\| \lambda(z; \bigotimes^{k-1,s} M \otimes N). \end{aligned}$$

And hence, $\lambda\left(\Phi_a^{F'}(z); \bigotimes^{k,s} E \otimes F'\right) \leq D\|a\| \lambda(z; \bigotimes^{k-1,s} E \otimes F')$.

Condition (ii). We must verify that the mapping

$$\begin{aligned} \Psi_\gamma^{F'} : \left(\bigotimes^{k+1,s} E \otimes F', \lambda \right) &\rightarrow \left(\bigotimes^{k,s} E \otimes F', \lambda \right) \\ x^{k+1} \otimes y &\mapsto \gamma(x) x^k \otimes y \end{aligned}$$

is continuous and that $\|\Psi_\gamma^{F'}\| \leq C\|\gamma\|$, for all $\gamma \in E'$.

Consider $\gamma \in E'$ and let $M \in \text{FIN}(E)$, $N \in \text{FIN}(F')$ and $z_i = \sum x_i^{k+1} \otimes y_i \in \bigotimes^{k+1,s} M \otimes N$. Then $P^{\Psi_\gamma^{F'}(z)} \in \mathfrak{A}_k(M', N)$ and, if $x' \in M'$,

$$P^{\Psi_\gamma^{F'}(z)}(x') = \sum_i \gamma(x_i) x_i(x')^k y_i = \sum_i x_i(\gamma) x_i(x')^k y_i = (P^z)_\gamma(x').$$

Therefore,

$$\begin{aligned} \lambda\left(\Psi_\gamma^{F'}(z); \bigotimes^{k,s} E \otimes F'\right) &\leq \lambda\left(\Psi_\gamma^{F'}(z); \bigotimes^{k,s} M \otimes N\right) \\ &= \|P^{\Psi_\gamma^{F'}(z)}\|_{\mathfrak{A}_k(M', N)} = \|(P^z)_\gamma\|_{\mathfrak{A}_k(M', N)} \\ &\leq C\|\gamma\| \|P^z\|_{\mathfrak{A}_{k+1}(M', N)} \\ &= C\|\gamma\| \lambda(z; \bigotimes^{k+1,s} M \otimes N). \end{aligned}$$

Thus, $\lambda\left(\Psi_\gamma^{F'}(z); \bigotimes^{k,s} E \otimes F'\right) \leq C\|\gamma\| \lambda(z; \bigotimes^{k+1,s} E \otimes F')$. □

Loosely speaking, the maximal hull \mathfrak{A}_k^{max} of a normed polynomial ideal \mathfrak{A}_k is the largest normed ideal that coincides isometrically with \mathfrak{A} in finite dimensional spaces [11, 14]. The

sequence of maximal hulls of the polynomial ideals \mathfrak{A}_k also preserves the coherence of the original sequence. Indeed, since \mathfrak{A}_k^{\max} coincides with \mathfrak{A}_k^{**} , we have the following:

Corollary 5.2. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence with constants C y D . Then $\{\mathfrak{A}_k^{\max}\}_k$ is a coherent sequence with constants C y D .*

Similarly, using Proposition 4.8 we can deal with the compatibility of the adjoint or the maximal hull of a polynomial ideal:

Proposition 5.3. *Let \mathfrak{A} be a normed operator ideal. If \mathfrak{A}_n is a normed ideal of n -homogeneous polynomials compatible with \mathfrak{A} with constants A and B , then \mathfrak{A}_n^* is compatible with \mathfrak{A}^* with constants B and A .*

Corollary 5.4. *If \mathfrak{A}_n is compatible with \mathfrak{A} with constants A and B , then \mathfrak{A}_n^{\max} is compatible with \mathfrak{A}^{\max} with constants A and B .*

The results of this section allows us to obtain a different proof of the compatibility and coherence of Grothendieck integral polynomials. Since $\mathcal{P}_{GI}^n = (\mathcal{P}_N^n)^{\max}$, it is an immediate consequence of Corollaries 5.2 and 5.4, and Example 1.10 that the Grothendieck integral polynomials are compatible and coherent with the ideal of Grothendieck integral operators with constants $A = C = 1$ and $B = D = e$. The same conclusion follows using that $\mathcal{P}_{GI}^n = (\mathcal{P}^n)^*$ and Propositions 5.1 and 5.3 and Example 1.9.

Remark 5.5. The reciprocal of Proposition 5.1 and Corollary 5.2 is not true. A counterexample for both is the sequence $\{\mathfrak{A}_k\}_k$, where

$$\mathfrak{A}_k(E, F) = \begin{cases} \mathcal{P}_N^k(E, F) & \text{if } k \text{ is even} \\ \mathcal{P}_{GI}^k(E, F) & \text{if } k \text{ is odd.} \end{cases}$$

Then $\{\mathfrak{A}_k^{\max}\}_k = \{\mathcal{P}_{GI}^k\}$ and $\{\mathfrak{A}_k^*\}_k = \{\mathcal{P}^k\}_k$ are coherent sequences, but $\{\mathfrak{A}_k\}_k$ is not.

Similarly, the reciprocal of Proposition 5.3 and Corollary 5.4 is false in general too.

We end this section with considering the minimal hull of a polynomial ideal. Recall that given a Banach ideal of n -homogeneous polynomials \mathfrak{A}_n , the minimal ideal \mathfrak{A}_n^{\min} is defined as

$$\mathfrak{A}_n^{\min} = \overline{\mathcal{F}} \circ \mathfrak{A}_n \circ \overline{\mathcal{F}},$$

where $\overline{\mathcal{F}}$ is the ideal approximable operators; and

$$\|P\|_{\mathfrak{A}_n^{\min}} = \inf \|S\|_{\overline{\mathcal{F}}} \|Q\|_{\mathfrak{A}_n} \|T\|_{\overline{\mathcal{F}}}^n,$$

where the infimum is taken over all factorizations $P = SQT$ with $S, T \in \overline{\mathcal{F}}$, $Q \in \mathfrak{A}_n$.

Since $\overline{\mathcal{F}}$ is a closed operator ideal, the following two corollaries are just particular cases of Propositions 3.1 and 3.2 respectively.

Corollary 5.6. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence of Banach polynomial ideals with constants C y D . Then $\{\mathfrak{A}_k^{\min}\}_k$ is a coherent sequence with constants C y D .*

Corollary 5.7. *If \mathfrak{A}_n is a Banach polynomial ideal compatible with a Banach operator ideal \mathfrak{A} , with constants A and B , then \mathfrak{A}_n^{\min} is compatible with \mathfrak{A}^{\min} with constants A and B .*

Finally, we relate the results of this section with the largest and smallest compatible polynomial ideals defined in Section 2.

Proposition 5.8. *Let \mathfrak{A} be a normed operator ideal. Then:*

- (i) $(\mathcal{M}_n^{\mathfrak{A}})^{\max} \stackrel{1}{=} (\mathcal{M}_n^{\mathfrak{A}^{\max}})$;
- (ii) $(\mathcal{N}_n^{\mathfrak{A}})^{\min} \stackrel{1}{=} (\mathcal{N}_n^{\mathfrak{A}^{\min}})$.

Proof. We only prove (i) (the proof of (ii) is similar).

Since $\mathcal{M}_n^{\mathfrak{A}}$ is compatible with \mathfrak{A} with constants $A = B = 1$, its maximal hull $(\mathcal{M}_n^{\mathfrak{A}})^{\max}$ is also compatible with \mathfrak{A}^{\max} with constants $A = B = 1$ by Corollary 5.4. Therefore $(\mathcal{M}_n^{\mathfrak{A}})^{\max} \subset \mathcal{M}_n^{\mathfrak{A}^{\max}}$, and by Proposition 2.4 the inclusion has norm 1.

Conversely, for $M, N \in FIN$ we have $\mathfrak{A}(M, N) \stackrel{1}{=} \mathfrak{A}^{\max}(M, N)$. As a consequence, $\mathcal{M}_n^{\mathfrak{A}}(M, N) = \mathcal{M}_n^{\mathfrak{A}^{\max}}(M, N)$ isometrically, which implies that $\mathcal{M}_n^{\mathfrak{A}^{\max}} \subset (\mathcal{M}_n^{\mathfrak{A}})^{\max}$ with norm one inclusion. \square

Recall that a normed polynomial ideal \mathfrak{A}_n is maximal if $\mathfrak{A}_n^{\max} = \mathfrak{A}_n$. Analogously, a Banach polynomial ideal is minimal if $\mathfrak{A}_n^{\min} = \mathfrak{A}_n$. Thus, last proposition states in particular that if \mathfrak{A} is a maximal (respectively minimal) normed operator ideal then $\mathcal{M}_n^{\mathfrak{A}}$ (respectively $\mathcal{N}_n^{\mathfrak{A}}$) is maximal (respectively minimal).

We also have the following result that establishes a duality relationship between the smallest and largest Banach polynomial ideals compatible with an operator ideal.

Proposition 5.9. *Let \mathfrak{A} be a normed operator ideal. Then $(\mathcal{N}_n^{\mathfrak{A}})^* = \mathcal{M}_n^{\mathfrak{A}^*}$.*

Proof. Let $M, N \in FIN$. Then

$$\begin{aligned} (\mathcal{N}_n^{\mathfrak{A}})^*(M, N) &= \mathcal{N}_n^{\mathfrak{A}}(M', N')' \\ &= \mathcal{P}_N^{n-1}(M', \mathfrak{A}(M', N'))' \\ &= \mathcal{P}^{n-1}(M, \mathfrak{A}(M', N'))' \\ &= \mathcal{P}^{n-1}(M, \mathfrak{A}^*(M, N)) \\ &= \mathcal{M}_n^{\mathfrak{A}^*}(M, N), \end{aligned}$$

where second and last identities follow from Remark 2.1 and all equalities are isometric.

Since every adjoint ideal is maximal, we have $(\mathcal{N}_n^{\mathfrak{A}})^* = (\mathcal{M}_n^{\mathfrak{A}^*})^{\max}$. Therefore, $(\mathcal{M}_n^{\mathfrak{A}^*})^{\max} = \mathcal{M}_n^{\mathfrak{A}^*}$, by Proposition 5.8 (see the above comments). \square

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