IDEALS OF MULTILINEAR FORMS - A LIMIT ORDER APPROACH

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Introduction

In 1983, A. Pietsch [20] presented his "designs of a theory" for ideals of multilinear forms. This work provided a general framework from which different lines of investigation developed. Some ideals of multilinear forms appeared as the multilinear natural extension of ideals of linear operators (e.g., nuclear and integral multilinear forms). However, it is not always clear what the multilinear analogous of a linear operators ideal should be. For example, the ideal of absolutely r-summing operators lead to the development of many ideals of multilinear forms: absolutely r-summing, r-dominated, multiple r-summing, etc. Also, some ideals of multilinear forms do not share this linear origin, but appeared by their own interest (sometimes, in relation with ideals of polynomials and holomorphy types). In any case, the theory of ideals of multilinear forms allows to deal with all the different situations in a unified way.

In the linear theory, the calculus of limit orders proved a useful tool, specially to compare different ideals of linear operators. In [5], we introduced the concept of limit order for multilinear forms. As an application, we could show that some properties of bilinear ideals were no longer valid for the n-linear case ($n \geq 3$).

In this article, we aim to develop a bit further the general theory of limit orders for multilinear forms. Even though some of our proofs are simple adaptations of the linear analogous results, we chose to present them here. Our motivation was to give a self-contained treatment of the subject, since the linear versions of these results come from different sources and with different notations.

In the first section, we recall the definitions of limit order and present some general properties. The second section deals with maximal and adjoint ideals of multilinear forms and their corresponding limit orders.

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In section 3, the ideal of r-dominated n-linear forms is shown to be dual to a tensor norm. This allows us to describe its adjoint ideal. In section 4 we estimate the limit order of the ideal of multiple 1-summing forms. Finally, in section 5 we study the diagonal of a multilinear form. We show that the limit order of a multilinear ideal gives estimates of the diagonal of any multilinear form in the ideal.

Given X, Y Banach spaces, we denote by $\mathcal{L}(X;Y)$ the space of continuous linear mappings $T : X \to Y$. If $X_1, \dots, X_n$ and Y are Banach spaces, $\mathcal{L}(X_1, \dots, X_n; Y)$ denotes the space of continuous n-linear mappings $T : X_1 \times \cdots \times X_n \to Y$. Whenever $X_1 = \cdots = X_n = X$ and $Y = \mathbb{C}$, the space of continuous n-linear mappings is simply denoted by $\mathcal{L}(^n X)$. We are going to deal with mappings $T \in \mathcal{L}(^n \ell_p)$. We denote by $x_1, \dots, x_n$ the elements in ℓ_p. If x is a sequence we write $x = (x(k))_{k=1}^{\infty}$, with $x(k) \in \mathbb{C}$.

Let us recall that $T \in \mathcal{L}(^n X)$ is **nuclear** if there are sequences $(x'_{1,k})_k, \dots, (x'_{n,k})_k$ in X' with $\|x'_{i,k}\| \leq 1$ for all k and $i = 1, \dots, n$ and there is $(\lambda(k))_k \in \ell_1$ so that for every $x_1, \dots, x_n \in X$

$$T(x_1, \dots, x_n) = \sum_k \lambda(k) \cdot x'_{1,k}(x_1) \cdots x'_{n,k}(x_n).$$

We denote by $\mathcal{N}(^n X)$ the space of nuclear n-linear forms on X.

A mapping $T \in \mathcal{L}(^n X)$ is called **integral** if there exists a positive Borel-Radon measure μ on $B_{X'} \times \cdots \times B_{X'}$ (with the weak*-topologies) such that

$$T(x_1, \dots, x_n) = \int_{B_{X'} \times \cdots \times B_{X'}} x'_1(x_1) \cdots x'_n(x_n) \; d\mu(x'_1, \dots, x'_n)$$

for all $x_1, \dots, x_n \in X$ (see [6, 4.5] and [1]). The space of integral n-linear forms on X is denoted by $\mathcal{I}(^n X)$.

A sequence $(x_n)_n$ in a Banach space X is **strongly p-summable** if $(\|x_n\|)_n \in \ell_p$. The space of strongly p-summable sequences is a Banach space with the norm

$$\|(x_n)_n\|_p = \left(\sum_n \|x_n\|^p \right)^{1/p}.$$

A sequence in a Banach space is **weakly p-summable** if $(x'(x_n))_n \in \ell_p$ for all $x' \in X'$. The space of weakly p-summable sequences endowed with the norm

$$w_p((x_n)_n) = \sup_{x' \in B_{X'}} \left(\sum_n |x'(x_n)|^p \right)^{1/p}$$

is a Banach space. These concepts can also be considered for finite sequences $(x_1, \dots, x_n)$ by means of the natural identification with $(x_1, \dots, x_n, 0, 0, \dots)$.

An operator $T \in \mathcal{L}(X;Y)$ is **absolutely r-summing** if there exists $C > 0$ such that for any finite choice of elements $x_1, \dots, x_n \in X$ we have

$$\|(T(x_i))_{i=1}^n\|_r \leq C \cdot \ w_r((x_i)_{i=1}^n).$$

We denote by $\Pi_r(X;Y)$ the space of absolutely r-summing operators between X and Y.

A map $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is said to be **absolutely $(s; r_1, \dots, r_n)$-summing** (where $\frac{1}{s} \leq \frac{1}{r_1} + \cdots + \frac{1}{r_n}$) [2, 15] if there exists $C > 0$ such that for any finite choice of elements $x_j^i \in X_j$, $j = 1, \dots, n$, $i = 1, \dots, m$ we have

$$\left(\sum_{i=1}^m \|T(x_1^i, \dots, x_n^i)\|^s \right)^{1/s} \leq C \cdot w_{r_1}(x_1^i) \cdots w_{r_n}(x_n^i).$$

A map $T \in \mathcal{L}((X_1, \dots, X_n; Y)$ is said to be **r-dominated** [21, 17] if it is absolutely $(r/n; r, \dots, r)$-summing; that is, there exists $C > 0$ such that for every $x_j^i \in X_j$, $j = 1, \dots, n$, $i = 1, \dots, m$,

$$\left(\sum_{i=1}^m \|T(x_1^i, \dots, x_n^i)\|^{r/n} \right)^{n/r} \leq C \cdot w_r(x_1^i) \cdots w_r(x_n^i).$$

We denote by $\mathcal{D}_r(^nX)$ the space of r-dominated n-linear forms on X.

Although all the results in the article are proved for complex Banach spaces, standard modifications can be made to obtain the real version of most of them.

1. Limit order of ideals of multilinear forms

In [5], the limit order for ideals of multilinear forms is defined. We recall the definitions and some basic facts.

If $T \in \mathcal{L}(^n\ell_p)$, we call it **diagonal** if there exists a sequence $\alpha = (\alpha(k))_k$ such that for all $x_1, \dots, x_n \in \ell_p$ we can write

$$T(x_1, \dots, x_n) = \sum_k \alpha(k) x_1(k) \cdots x_n(k).$$

We denote by T_α the diagonal multilinear mapping given by the sequence α. On the other hand, the diagonal linear operator from ℓ_p to ℓ_q associated to a sequence σ is defined by $D_\sigma(x) = (\sigma(k)x(k))_k$.

Given a diagonal multilinear form $T_\alpha \in \mathcal{L}(^n\ell_p)$, we consider a sequence σ such that $\sigma(k)^n = \alpha(k)$ for all k. We take the diagonal operator $D_\sigma : \ell_p \to \ell_n$ associated to σ and define a mapping $\Phi : \ell_n \times \cdots \times \ell_n \to \mathbb{C}$ by $\Phi(x_1, \dots, x_n) = \sum_k x_1(k) \cdots x_n(k)$. The fact that T is well defined on ℓ_p guarantees that $D_\sigma(\ell_p) \subset \ell_n$. Now, the diagonal n-linear mapping T can be rewritten as

$$T_\alpha(x_1, \dots, x_n) = \Phi(D_\sigma(x_1), \dots, D_\sigma(x_n)). \tag{1}$$

We use this decomposition several times.

Given $N \in \mathbb{N}$, we define the n-linear form Φ_N on $\mathbb{C}^N$ by:

$$\Phi_N(x_1, \dots, x_n) = \sum_{k=1}^{N} x_1(k) \cdots x_n(k).$$

We recall the notion of limit order for operators ideals (see [19, Section 14.4]). Given an operator ideal $\mathfrak{A}$, the limit order $\lambda(\mathfrak{A}; p, q)$ is the infimum over all $\lambda \geq 0$ such that every diagonal operator $D_\sigma : \ell_p \to \ell_q$ with $\sigma \in \ell_{1/\lambda}$ belongs to $\mathfrak{A}(\ell_p, \ell_q)$.

Ideals of multilinear forms were introduced in [20]. Let us recall the definition. An **ideal of multilinear forms** $\mathfrak{A}$ is a subclass of $\mathcal{L}$, the class of all multilinear forms such that, for any Banach spaces $X_1, \dots, X_n$ the set

$$\mathfrak{A}(X_1, \dots, X_n) = \mathfrak{A} \cap \mathcal{L}(X_1, \dots, X_n)$$

satisfies

(1) For any $\gamma_1 \in X_1', \dots, \gamma_n \in X_n'$, the mapping $(x_1, \dots, x_n) \mapsto \gamma_1(x_1) \cdots \gamma_n(x_n)$ belongs to $\mathfrak{A}(X_1, \dots, X_n)$.
(2) If $S, T \in \mathfrak{A}(X_1, \dots, X_n)$, then $S + T \in \mathfrak{A}(X_1, \dots, X_n)$.
(3) If $T \in \mathfrak{A}(X_1, \dots, X_n)$ and $S_i \in \mathcal{L}(Y_i, X_i)$ for $i = 1, \dots, n$, then $T \circ (S_1, \dots S_n) \in \mathfrak{A}(Y_1, \dots, Y_n)$.

In [5], the concept of limit order was defined for ideals of multilinear forms:

Definition 1.1. *Let $\mathfrak{A}$ be an ideal of multilinear forms. For $1 \leq p \leq \infty$, the limit order $\lambda_n(\mathfrak{A}; p)$ is given by:*

$$\lambda_n(\mathfrak{A}; p) = \inf\{\lambda : \textit{ for each } \alpha \in \ell_{1/\lambda},\ T_\alpha \textit{ belongs to } \mathfrak{A}(^n\ell_p)\}$$

With almost the same proof as in [19, Section 14.4], we obtain alternative expressions for $\lambda_n(\mathfrak{A};p)$. First, we have:

$$\lambda_n(\mathfrak{A};p) = \inf\{\lambda : \text{ if } \alpha = (k^{-\lambda})_k, \text{ then } T_\alpha \text{ belongs to } \mathfrak{A}(^n\ell_p)\}.$$

Also, if $\mathfrak{A}$ is quasi-normed and complete, then $\lambda_n(\mathfrak{A};p)$ is the infimum of all $\lambda \geq 0$ such that

$$\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \leq CN^\lambda \tag{2}$$

for all $N \in \mathbb{N}$, where $C > 0$ is a constant.

Clearly, the definition of limit order can be generalized to n-linear forms on $\ell_{p_1} \times \cdots \times \ell_{p_n}$. All the results in this article can be extended to this situation. However, for the reader's convenience, we prefer to state them in the simple case $p_1 = \cdots = p_n = p$.

It is also possible to define the limit order of ideals of n-homogeneous polynomials. But this will not lead us to new horizons. Indeed, in [8] the authors showed that, given a λ-normed ideal of n-homogeneous polynomials $\mathcal{Q}$, there exists a λ-normed ideal of n-linear forms $\mathcal{Q}^\vee$ with the following property:

> A polynomial P is in $\mathcal{Q}$ if and only if its associated symmetric n-linear form $\check{P}$ is in $\mathcal{Q}^\vee$.

Moreover, $P \leftrightarrow \check{P}$ is a one-to-one correspondence between diagonal n-homogeneous polynomials and diagonal symmetric n-linear forms. Then, the limit orders of ideals of homogeneous polynomials can be seen as limit orders of ideals of multilinear forms.

Examples: In [5] the limit orders of some ideals of multilinear forms are computed:

$$\lambda_n(\mathcal{L};p) = \begin{cases} 0 & \text{if } p \leq n \\ 1 - \frac{n}{p} & \text{if } p > n \end{cases}$$

$$\lambda_n(\mathcal{N};p) = \lambda_n(\mathcal{I};p) = \begin{cases} \frac{n}{p'} & \text{if } 1 \leq p < n' \\ 1 & \text{if } n' \leq p \end{cases}$$

$$\lambda_n(\mathcal{D}_r;p) = n\ \lambda(\Pi_r;p,n)$$

where $\lambda(\Pi_r;p,n)$ is the limit order of the ideal of absolutely r-summing linear operators (see [19, Section 22.4]). The relation of the limit order of r-dominated n-linear forms and absolutely r-summing operators follows from the following result:

Proposition 1.2. [5] *Let $T_\alpha \in \mathcal{L}(^n\ell_p)$ be diagonal and D_σ its associated diagonal operator. Then T_α is r-dominated if and only if D_σ is absolutely r-summing.*

Now we state some general properties of limit orders. As in the linear case, the following property holds for composition ideals:

Proposition 1.3. *Let $\mathfrak{A}$ be an ideal of n-linear forms and $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ linear operator ideals. Then:*

$$\lambda_n(\mathfrak{A} \circ (\mathfrak{B}_1, \dots, \mathfrak{B}_n); p) \leq \lambda(\mathfrak{B}_1; p, s) + \cdots + \lambda(\mathfrak{B}_n; p, s) + \lambda_n(\mathfrak{A}; s)$$

As a consequence, since $\mathfrak{A} = \mathfrak{A} \circ (\mathcal{L}, \dots, \mathcal{L})$, we have

$$|\lambda_n(\mathfrak{A}; p) - \lambda_n(\mathfrak{A}; p_0)| \leq n \left| \frac{1}{p} - \frac{1}{p_0} \right|.$$

Therefore, $\lambda_n(\mathfrak{A}; p)$ is a continuous function of $\frac{1}{p}$.

2. Maximal and adjoint ideals of multilinear mappings

Let $\mathfrak{A}$ be a quasi-normed ideal of n-linear forms. The maximal hull $\mathfrak{A}^{max}$ of $\mathfrak{A}$ is defined as the class of all n-linear forms T such that

$$\|T\|_{\mathfrak{A}^{max}(E_1,\dots,E_n)} := \sup\{\|T|_{M_1\times\cdots\times M_n}\|_{\mathfrak{A}(M_1,\dots,M_n)} : M_i \subset E_i,\ dim M_i < \infty\}$$

is finite.

$\mathfrak{A}^{max}$ is always complete and it is the largest ideal whose quasi-norm coincides with $\|\cdot\|_{\mathfrak{A}}$ in finite dimensional spaces. Hence, if $\mathfrak{A}$ is complete, equation (2) says that

$$\lambda_n(\mathfrak{A}; p) = \lambda_n(\mathfrak{A}^{max}; p)$$

for all p.

A quasi-normed ideal $\mathfrak{A}$ is called maximal if $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}}) = (\mathfrak{A}^{max}, \|\cdot\|_{\mathfrak{A}^{max}})$.

Following [10, 13] we define now the adjoint of an ideal of multilinear mappings.

Let $\mathfrak{A}$ be a normed ideal of n-linear mappings. The equality

$$\left(\bigotimes_{i=1}^n M_i; \alpha \right) \overset{1}{=} \mathfrak{A}(M_1', \dots, M_n')$$

defines a tensor norm in FIN (the class of all finite dimensional normed spaces). This norm can be extended to a finitely generated tensor norm on the class of normed spaces by

$$\|s\|_{(\bigotimes_{i=1}^n E_i;\alpha)} := \inf \left\{ \|s\|_{(\bigotimes_{i=1}^n M_i;\alpha)} : M_i \in \text{FIN}(E_i),\ s \in \bigotimes_{i=1}^n M_i \right\}$$

In this case, the tensor norm α and the ideal $\mathfrak{A}$ are said to be associated.

Given a normed ideal $\mathfrak{A}$ associated to the finitely generated tensor norm α, its adjoint ideal $\mathfrak{A}^*$ is defined by

$$\mathfrak{A}^*(E_1, \ldots, E_n) := \left(\bigotimes_{i=1}^{n} E_i; \alpha \right)'.$$

The adjoint ideal is called dual ideal in [10].

The tensor norm associated to $\mathfrak{A}^*$ is denoted by α^*. We also have the representation theorem [13, Section 3.2] (see also [10, Section 4]):

$$\mathfrak{A}^{max}(E_1, \ldots, E_n) = \left(\bigotimes_{i=1}^{n} E_i; \alpha^* \right)'.$$

In particular, this shows that the adjoint ideal $\mathfrak{A}^*$ is maximal.

The limit orders of an ideal and its adjoint ideal are related, as in the linear case, by the following equality

Lemma 2.1. $\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \|\Phi_N\|_{\mathfrak{A}^*(^n\ell_{p'}^N)} = N$

Proof. First, we note that

$$\begin{aligned} N &= \sum_{k=1}^{N} \Phi_N(e_k, \ldots, e_k) = \Phi_N \left(\sum_{k=1}^{N} e_k \otimes \cdots \otimes e_k \right) \\ &\leq \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \left\| \sum_{k=1}^{N} e_k \otimes \cdots \otimes e_k \right\|_{\left(\bigotimes^n \ell_p^N; \alpha^*\right)} = \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \cdot \|\Phi_N\|_{\mathfrak{A}^*(^n\ell_{p'}^N)} \end{aligned}$$

For the reverse inequality, let us choose a norm one $s \in \left(\bigotimes^n \ell_p^N; \alpha^*\right)$ such that $\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} = |\Phi_N(s)|$. Then

$$\|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \cdot \|\Phi_N\|_{\mathfrak{A}^*(^n\ell_{p'}^N)} = \|\Phi_N(s) \cdot \Phi_N\|_{\mathfrak{A}^*(^n\ell_{p'}^N)}.$$

We write $s = \sum_{i_1,\ldots,i_n=1}^{N} \alpha_{i_1,\ldots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}$ and define $s_0 := \sum_{i=1}^{N} \alpha_{i,\ldots,i} e_i \otimes \cdots \otimes e_i$ the diagonal of s. Clearly, $\Phi_N(s) = \Phi_N(s_0)$.

Now we define $S : \ell_{p'}^N \to \ell_{p'}^N$ by $S(x_1, x_2, \ldots, x_N) = (x_2 \ldots, x_N, x_1)$. It is easy to see that for each $1 \leq j \leq n$, $\sum_{k=1}^{N} (e_j \otimes \cdots \otimes e_j) \circ$

$(S^k, \cdots, S^k) = \Phi_N \in \mathfrak{A}^*({}^n\ell_{p'}^N)$. Therefore,

$$\begin{aligned}\|\Phi_N(s)\cdot\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} &= \|\Phi_N(s_0)\cdot\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)}\\ &= \left\|\sum_{i=1}^N \alpha_{i,\dots,i}\cdot\sum_{k=1}^N (e_i\otimes\cdots\otimes e_i)\circ(S^k,\cdots,S^k)\right\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)}\\ &= \left\|\sum_{k=1}^N s_0\circ(S^k,\dots,S^k)\right\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} \le N\cdot\|s_0\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)}.\end{aligned}$$

Here we identify tensors with multilinear forms. Proceeding as in Proposition 5.1 and noting that in this situation the operator Λ_t has norm one, we can see that $\|s_0\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} \le \|s\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} = \|s\|_{\left(\bigotimes^n \ell_p^N;\alpha^*\right)} = 1$. This completes the proof. □

Immediatelly from this and (2) we have

Corollary 2.2. *Let $\mathfrak{A}$ be a normed ideal of multilinear forms. Then $\lambda_n(\mathfrak{A}, p) + \lambda_n(\mathfrak{A}^*; p') \ge 1$.*

The following definition and results are multilinear versions of those introduced in [14] for linear operators. As in [14], we denote $\ell_n(\mathfrak{A}, p) := \{\alpha \in \ell_\infty : T_\alpha \in \mathfrak{A}({}^n\ell_p)\}$. The sequence space $\ell_n(\mathfrak{A}, p)$ is a Banach space if we consider the norm

$$\|\alpha\|_{\ell_n(\mathfrak{A},p)} = \|T_\alpha\|_{\mathfrak{A}({}^n\ell_p)}$$

Definition 2.3. *Let $\mathfrak{A}$ be an ideal of n-linear forms and $1 \le p \le \infty$. We define the **defect** by*

$$d_n(\mathfrak{A}, p) = \inf\left\{\frac{1}{r} - \frac{1}{s} : \ell_r \subset \ell_n(\mathfrak{A}, p) \subset \ell_s\right\}.$$

Proposition 2.4. *Let $\mathfrak{A}$ be a Banach ideal of n-linear forms. Then $d_n(\mathfrak{A}, p)$ is the infimum of $\lambda - \mu$ where $\lambda, \mu \ge 0$ are such that*

$$CN^\mu \le \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \le DN^\lambda \tag{3}$$

for all $N \in \mathbb{N}$, for some constants $C, D > 0$.

Proof. Take r, s such that $\ell_r \subset \ell_n(\mathfrak{A}, p) \subset \ell_s$. From the closed graph theorem, the inclusions $\ell_r \hookrightarrow \ell_n(\mathfrak{A}, p)$ and $\ell_n(\mathfrak{A}, p) \hookrightarrow \ell_s$ are continuous. Thus, there exist constants C and D such that, for all $N \in \mathbb{N}$,

$$CN^{\frac{1}{s}} \le \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} = \|(1,\dots,1,0,\dots)\|_{\ell_n(\mathfrak{A},p)} \le DN^{\frac{1}{r}}.$$

Conversely, suppose $CN^{\frac{1}{s}} \le \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \le DN^{\frac{1}{r}}$ for all $N \in \mathbb{N}$. It is enough to show that, for all $\varepsilon > 0$, we have $\ell_{r-\varepsilon} \subset \ell_n(\mathfrak{A}, p) \subset \ell_{s+\varepsilon}$.

The first inclusion follows from the equivalence between the definitions of limit order. For the second inclusion, let $\sigma \in \ell_n(\mathfrak{A}, p)$. First consider the case in which $(\sigma_n)_n$ converges to 0. We can assume that $(|\sigma_n|)_n$ is not increasing. We can factor Φ_N as

$$\begin{array}{ccccccc} \ell_p^N & \times & \cdots & \times & \ell_p^N & \xrightarrow{\Phi_N} & \mathbb{C} \\ D_\alpha \downarrow & & & & D_\alpha \downarrow & \nearrow_{T_\sigma} & \\ \ell_p^N & \times & \cdots & \times & \ell_p^N & & \end{array}$$

where $\alpha = \sigma^{-1/n}$. Therefore,

$$\begin{aligned} CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} &\leq \|T_\sigma\|_{\mathfrak{A}(^n\ell_p^N)} \cdot \|D_\alpha\|^n \\ &\leq \|T_\sigma\|_{\mathfrak{A}(^n\ell_p)} \cdot |\sigma_N|^{-1} \end{aligned}$$

Consequently, $|\sigma_N| \leq \frac{\|T_\sigma\|_{\mathfrak{A}(^n\ell_p)}}{DN^{1/s}}$ for all $N \in \mathbb{N}$, which implies that $\sigma \in \ell_{s+\varepsilon}$.

Now we assume that $(\sigma_n)_n$ does not converge to 0. This means that there are constants a and b and a subsequence of σ_n such that $a \leq \sigma_{n_k} \leq b$ for all k. We will see that in this case, both $\ell_n(\mathfrak{A}, p)$ and ℓ_s coincide with ℓ_∞.

Let $\beta \in \ell_\infty$. We define $S : \ell_p \to \ell_p$ by

$$S(e_k) = \left(\frac{\beta_k}{\sigma_{n_k}}\right)^{\frac{1}{n}} e_{n_k}$$

Since $T_\beta = T_\sigma \circ (S, \dots, S)$, we have that $T_\beta \in \mathfrak{A}(^n\ell_p)$ and then $\beta \in \ell_n(\mathfrak{A}, p)$.

Now, since $\ell_n(\mathfrak{A}, p) = \ell_\infty$, we have $\Phi = T_{(1,1,\dots)} \in \mathfrak{A}(^n\ell_p)$. From the inequality $CN^{\frac{1}{s}} \leq \|\Phi_N\|_{\mathfrak{A}(^n\ell_p^N)} \leq \|\Phi\|_{\mathfrak{A}(^n\ell_p)}$ for all N, we obtain $s = \infty$. □

Proposition 2.5. *Let $\mathfrak{A}$ be a Banach ideal of n-linear forms and $\mathfrak{A}^*$ its adjoint ideal. Then*

$$\lambda_n(\mathfrak{A}; p) + \lambda_n(\mathfrak{A}^*; p') = 1 + d_n(\mathfrak{A}; p).$$

Proof.

$$\begin{aligned}
\lambda_n(\mathfrak{A};p)+\lambda_n(\mathfrak{A}^*;p') =& \inf\{\lambda>0:\ \exists D>0 \text{ with } \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \le DN^{\lambda},\ \forall N\}\\
&+\inf\{\nu>0:\ \exists C>0 \text{ with } \|\Phi_N\|_{\mathfrak{A}^*({}^n\ell_{p'}^N)} \le CN^{\nu},\ \forall N\}\\
=& \inf\{\lambda>0:\ \exists D>0 \text{ with } \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \le DN^{\lambda},\ \forall N\}\\
&+\inf\{\nu>0:\ \exists \widetilde{C}>0 \text{ with } \widetilde{C}N^{1-\nu} \le \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)},\ \forall N\}\\
=& 1+\inf\{\lambda-\mu:\ \widetilde{C}N^{\mu} \le \|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)} \le DN^{\lambda}\}\\
=& 1+d_n(\mathfrak{A};p).
\end{aligned}$$

□

Note that Corollary 2.2 can also be obtained as a consequence of Proposition 2.5. We also get the following analogous to [14, Corollary 1].

Corollary 2.6. *Let $\mathfrak{A}$ be a Banach ideal. The following are equivalent:*
(a) $\lambda_n(\mathfrak{A};p)+\lambda_n(\mathfrak{A}^*;p')=1$.
(b) There exists $r>0$ such that for all $\varepsilon>0$, $\ell_{r-\varepsilon}\subset\ell_n(\mathfrak{A},p)\subset\ell_{r+\varepsilon}$.
(c) There exists $\lambda\ge 0$ such that for all $\varepsilon>0$ and all $N\in\mathbb{N}$, $CN^{\lambda-\varepsilon}\le\|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)}\le DN^{\lambda+\varepsilon}$ for some constants $C,D>0$.
(d) $\left(\frac{\log\|\Phi_N\|_{\mathfrak{A}({}^n\ell_p^N)}}{\log N}\right)_{N\in\mathbb{N}}$ *converges as $N\to\infty$.*

Moreover, if these equivalences hold, $\frac{1}{r}=\lambda=\lambda_n(\mathfrak{A},p)$

Remark 2.7. The definition of limit order implies that, for all $\varepsilon>0$,

$$\ell_{1/\lambda_n(\mathfrak{A},p)-\varepsilon}\subset\ell_n(\mathfrak{A},p).$$

Therefore, the equality $\lambda_n(\mathfrak{A};p)+\lambda_n(\mathfrak{A}^*;p')=1$ is equivalent to the inclusion $\ell_n(\mathfrak{A},p)\subset\ell_{1/\lambda_n(\mathfrak{A},p)+\varepsilon}$ for all $\varepsilon>0$.

3. Dominated multilinear forms

First we show that r-dominated n-linear forms are dual to a tensor norm whenever $r\ge n$. Next, this duality will be used to study the adjoint ideal $\mathcal{D}_r^*$.

For $r\ge n$, we define in $\bigotimes_{i=1}^n X_i$,

$$\alpha_{r'}^n(s)=\inf\left\{\ell_u(\lambda(i))\cdot w_r(x_i^1)\cdots w_r(x_i^n):\ s=\sum_{i=1}^N\lambda(i)\cdot x_i^1\otimes\cdots\otimes x_i^n\right\}$$

where $\frac{1}{u}+\frac{n}{r}=1$.

A straightforward application of [9, §1.2] gives

Proposition 3.1. *$\alpha^n_{r'}$ is a finitely generated tensor norm of order n.*

Now we can show the desired duality:

Proposition 3.2. *If $r \geq n$, $\mathcal{D}_r(^nX) = (\bigotimes^n X; \alpha^n_{r'})'$.*

Proof. Let $T \in \mathcal{D}_r(^nX)$. For $s = \sum_{i=1}^N \lambda(i) \cdot x_i^1 \otimes \cdots \otimes x_i^n \in \bigotimes^n X$ we have

$$\begin{aligned} |T(s)| = \left| \sum_{i=1}^N \lambda(i) \cdot T(x_i^1, \ldots, x_i^n) \right| &\leq \ell_{\frac{r}{n}} \left(\left| T(x_i^1, \ldots, x_i^n) \right| \right) \cdot \ell_u(\lambda(i)) \\ &\leq \|T\|_{\mathcal{D}_r(^nX)} \cdot w_r(x_i^1) \cdots w_r(x_i^n) \cdot \ell_u(\lambda(i)). \end{aligned}$$

Since this is valid for any representation of s, we obtain that $T \in (\bigotimes^n X; \alpha^n_{r'})'$ and $\|T\|_{(\bigotimes^n X; \alpha^n_{r'})'} \leq \|T\|_{\mathcal{D}_r(^nX)}$.

Conversely, let $T \in (\bigotimes^n X; \alpha^n_{r'})'$. For any sequences $(x_i^1)_{i=1}^N, \ldots, (x_i^n)_{i=1}^N$ in X, there exist scalars $\lambda_1, \ldots, \lambda_N$ with $\ell_u(\lambda_i) = 1$ such that

$$\begin{aligned} \ell_{\frac{r}{n}} \left(\left| T(x_i^1, \ldots, x_i^n) \right| \right) &= \sum_{i=1}^N \lambda_i \cdot T(x_i^1, \ldots, x_i^n) \\ &= T \left(\sum_{i=1}^N \lambda_i \cdot x_i^1 \otimes \cdots \otimes x_i^n \right) \\ &\leq \|T\|_{(\bigotimes^n X; \alpha^n_{r'})'} \cdot \alpha^n_{r'} \left(\sum_{i=1}^N \lambda_i \cdot x_i^1 \otimes \cdots \otimes x_i^n \right) \\ &\leq \|T\|_{(\bigotimes^n X; \alpha^n_{r'})'} \cdot w_r(x_i^1) \cdots w_r(x_i^n) \cdot \ell_u(\lambda_i). \end{aligned}$$

Thus, $T \in \mathcal{D}_r(^nX)$ and $\|T\|_{\mathcal{D}_r(^nX)} \leq \|T\|_{(\bigotimes^n X; \alpha^n_{r'})'}$. □

Inspired by [6, Chapters 17 and 18], we study $\mathcal{D}_r^*$, the adjoint ideal to the ideal of r-dominated multilinear mappings.

Note that, since $\mathcal{D}_r$ is a maximal ideal, we have $\mathcal{D}_r = (\mathcal{D}_r^*)^*$. Therefore, for $M_1, \ldots, M_n \in$ FIN,

$$\mathcal{D}_r^*(M_1, \ldots, M_n) \overset{1}{=} \left(\bigotimes_{i=1}^n M_i; \alpha^n_{r'} \right).$$

Let $T \in \mathcal{D}_r^*(M_1, \ldots, M_n)$ and fix $\varepsilon > 0$. T admits a representation

$$T(x_1, \ldots, x_n) = \sum_{k=1}^N \lambda(k) \cdot \gamma_k^1(x_1) \cdots \gamma_k^n(x_n), \tag{4}$$

where $(\lambda(k))_k \subset \mathbb{C}$, $(\gamma_k^i)_k \subset M_i'$ satisfy

$$\ell_u(\lambda(k)) \cdot w_r(\gamma_k^1) \cdots w_r(\gamma_k^n) = \|T\|_{\mathcal{D}_r^*(M_1,\dots,M_n)} \cdot (1+\varepsilon)$$

with $\frac{1}{u} + \frac{n}{r} = 1$.

Then we can factor T as:

$$\begin{array}{ccccccc} M_1 & \times & \cdots & \times & M_n & \xrightarrow{T} & \mathbb{C} \\ R^1 \downarrow & & & & R^n \downarrow & \nearrow_{T_\lambda} & \\ \ell_r^N & \times & \cdots & \times & \ell_r^N & & \end{array} \tag{5}$$

where $R^i(x) = (\gamma_k^i(x))_{k=1}^N$. Since $\|T_\lambda\| = \ell_u(\lambda(k))$ and $\|R^i\| = w_r(\gamma_k^i)$, we have

$$\|T_\lambda\| \cdot \|R^1\| \cdots \|R^n\| = \|T\|_{\mathcal{D}_r^*({}^n M)} \cdot (1+\varepsilon).$$

Following these steps backwards, we obtain for each factorization of T as in (5), a representation of T as in equation (4). Therefore, we have

$$\|T\|_{\mathcal{D}_r^*({}^n M)} = \inf \left\{ \|T_\lambda\| \cdot \|R^1\| \cdots \|R^n\| : T \text{ factors as in (5)} \right\}.$$

In [11], the ideal of r-integral polynomials is defined. We define in an analogous way the ideal $\mathcal{I}_r$ of r-integral multilinear forms. If $r \geq n$, we say that $T \in \mathcal{L}(X_1, \dots, X_n)$ is r-**integral** if there exist a finite measure space (Ω, μ) and operators $S_i : X_i \to L_r(\mu)$ such that $T = Q_{\mu,r}^n \circ (S_1, \dots, S_n)$, where $Q_{\mu,r}^n \in \mathcal{L}({}^n L_r(\mu))$ is the integrating n-linear form $Q_{\mu,r}^n(f_1, \dots, f_n) = \int_\Omega f_1 \cdots f_n \, d\mu$:

$$\begin{array}{ccccccc} X_1 & \times & \cdots & \times & X_n & \xrightarrow{T} & \mathbb{C} \\ S_1 \downarrow & & & & S_n \downarrow & \nearrow_{Q_{\mu,r}^n} & \\ L_r(\mu) & \times & \cdots & \times & L_r(\mu) & & \end{array}$$

$\mathcal{I}_r$ is a Banach ideal with the r-integral norm:

$$\|T\|_{\mathcal{I}_r(X_1,\dots,X_n)} = \inf \left\{ \|S_1\| \cdots \|S_n\| \cdot \|Q_{\mu,r}^n\| : T = Q_{\mu,r}^n \circ (S_1, \dots, S_n) \right\}.$$

Lemma 3.3. *The n-linear form $Q_{\mu,r}^n$ belongs to $\mathcal{D}_r^*$ and $\|Q_{\mu,r}^n\|_{\mathcal{D}_r^*({}^n L_r(\mu))} = \|Q_{\mu,r}^n\| = \mu(\Omega)^{1/u}$, where $\frac{1}{u} + \frac{n}{r} = 1$.*

Proof. We have to show that $Q_{\mu,r}^n$ is a continuous linear form on $(\bigotimes^n L_r(\mu); (\alpha_{r'}^n)^*)$ with norm $\mu(\Omega)^{1/u}$. This is shown in [6, Proposition 18.2] for bilinear forms. Their proof is also valid for $n \geq 3$. □

Corollary 3.4. *If $r \geq n$, $\mathcal{I}_r \subset \mathcal{D}_r^*$ and $\|T\|_{\mathcal{D}_r^*} \leq \|T\|_{\mathcal{I}_r}$ for each r-integral n-linear form T.*

Proof. If T is r-integral, it can be written as $T = Q^n_{\mu,r} \circ (S_1, \dots, S_n)$. Lemma 3.3 implies that $T \in \mathcal{D}_r^*$ and

$$\|T\|_{\mathcal{D}_r^*} \le \|Q^n_{\mu,r}\|_{\mathcal{D}_r^*} \cdot \|S_1\| \cdots \|S_n\| = \|Q^n_{\mu,r}\| \cdot \|S_1\| \cdots \|S_n\|.$$

Taking the infimum over all representations of T we obtain the desired inequality. □

Theorem 3.5. *For $r \ge n$, we have $\mathcal{D}_r^* \stackrel{1}{=} \mathcal{I}_r^{max}$.*

Proof. It is enough to show, for $M_1, \dots, M_n \in$ FIN, that $\|T\|_{\mathcal{I}_r(M_1,\dots,M_n)} = \|T\|_{\mathcal{D}_r^*(M_1,\dots,M_n)}$. One inequality is given in Corollary 3.4. For the other one, we factor $T \in \mathcal{D}_r^*(M_1, \dots, M_n)$ as

$$\begin{array}{ccccccc} M_1 & \times & \cdots & \times & M_n & \xrightarrow{T} & \mathbb{C} \\ R^1 \downarrow & & & & R^n \downarrow & \nearrow_{T_\lambda} & \\ \ell_r^N & \times & \cdots & \times & \ell_r^N & & \end{array}$$

Let us show now that T_λ can be factored as

$$\begin{array}{ccccccc} \ell_r^N & \times & \cdots & \times & \ell_r^N & \xrightarrow{T_\lambda} & \mathbb{C} \\ J \downarrow & & & & J \downarrow & \nearrow_{Q^n_{\mu,r}} & \\ L_r(\mu) & \times & \cdots & \times & L_r(\mu) & & \end{array}$$

with $\|Q^n_{\mu,r}\| \cdot \|J\|^n \le \|T_\lambda\|$. Since T_λ factors through $T_{|\lambda|}$, we can assume that $\lambda(k) \ge 0$ for each k. Let (Ω, μ) a measure space that can be split Ω as a disjoint union of subsets $A_1, \dots, A_n$ with $\mu(A_k) = \lambda(k)^u$. Let $J : \ell_r^N \to L_r(\mu)$ be defined as

$$J(x) = \sum_{k=1}^N x(k) \cdot \lambda(k)^{\frac{1-u}{n}} \cdot \chi_{A_k}.$$

Simple computations show that $\|J\| = 1$ and $Q^n_{\mu,r} \circ (J, \dots, J) = T_\lambda$. Since $\|Q^n_{\mu,r}\| = \mu(\Omega)^{1/u} = \ell_u(\lambda(k))$ and $\|T_\lambda\| = \ell_u(\lambda(k))$, we are done. □

In [19, Section 22.4], it is shown that absolutely r-summing operators satisfy $\lambda(\Pi_r, p, q) + \lambda(\Pi_r^*, q, p) = 1$. By [14, Corollary 1] (which is analogous to Corollary 2.6), this means that $\ell(\Pi_r, p, q) \subset \ell_{1/\lambda(\Pi_r,p,q)+\varepsilon}$ for all $\varepsilon > 0$. Now, by Proposition 1.2, $\ell_n(\mathcal{D}_r, p) \subset \ell_{1/\lambda_n(\mathcal{D}_r,p)+\varepsilon}$ for all $\varepsilon > 0$. Consequently, by Corollary 2.6 and the remark following it, we have that $\lambda_n(\mathcal{D}_r; p) + \lambda_n(\mathcal{D}_r^*; p') = 1$. So we have, for $r \ge n$,

$$\lambda_n(\mathcal{I}_r; p) = \lambda_n(\mathcal{D}_r^*; p) = 1 - \lambda_n(\mathcal{D}_r; p'). \tag{6}$$

We use the results of this sections to obtain some properties of the ideal of r-integral n-linear forms. For $n = 2$, the ideal of r-integral bilinear forms is maximal [11, 4.4]. Then, by Theorem 3.5, it is the adjoint of the ideal of r-dominated bilinear forms. By [6] (see also [5]), r-dominated and 2-dominated bilinear forms coincide for all $r \geq 2$. Thus, the same holds for r-integral bilinear forms: $\mathcal{I}_r(^2X) = \mathcal{I}_2(^2X)$ for all Banach space X and all $r \geq 2$. This result is not longer true for $n \geq 3$. In fact, from [5, Proposition 2.6] and equality 6 we have:

Corollary 3.6. *Let $n \geq 3$. Given $r \geq n$, there exists p such that, for any $s > r$, there are diagonal s-integral n-linear forms on ℓ_p which are not r-integral.*

4. Multiple 1-summing forms

Multiple summing operators have been introduced independently by M. Matos [16] and F. Bombal, D. Pérez-García and I. Villanueva [4]. A multilinear operator $T \in \mathcal{L}(X_1, \dots, X_n; Y)$ is **multiple r-summing** ($T \in \Pi_r(X_1, \dots, X_n; Y)$) if there exists $C > 0$ such that for every choice of finite sequences $(x_j^{i_j}) \subseteq X_j$ the following holds

$$\left(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \|T(x_1^{i_1}, \dots, x_n^{i_n})\|^r \right)^{\frac{1}{r}} \leq C \cdot w_r((x_1^{i_1})_{i_1=1}^{m_1}) \cdots w_r((x_n^{i_n})_{i_n=1}^{m_n}).$$

The least of such constants C is called the **multiple p-summing norm** and denoted $\|T\|_{\Pi_r(X_1,\dots,X_n;Y)}$.

A. Defant and D. Pérez-García used multiple 1-summing in [7] to show that its associated tensor norm preserves unconditionality. Some other properties of the norm were also proved and applied in [7, Section 6] to compute the limit order for bilinear multiple 1-summing operators. Their result can be written with our notation as

$$\lambda_2(\Pi_1; p) = \begin{cases} \dfrac{1}{p'} & \text{if } 2 \leq p \\ \dfrac{3}{2} - \dfrac{2}{p} & \text{if } \frac{4}{3} \leq p < 2 \\ 0 & \text{if } 1 \leq p < \frac{4}{3} \end{cases}$$

Our aim is now to compute the limit order of Π_1 for higher n. In fact, what we do is to compute the Π_1-norm of $\Phi_N : \ell_p^N \times \cdots \times \ell_p^N \to \mathbb{C}$.

Let us begin by considering the case $p \le 2$. We follow the steps of [7, Section 6]. First of all, if $T \in \mathcal{L}({}^n\ell_2^N)$ then

$$\|T\|_{\Pi_1({}^n\ell_2^N)} \asymp \left(\sum_{i_1,\dots,i_n=1}^{N} |T(e_{i_1},\dots,e_{i_n})|^2 \right)^{1/2} \tag{7}$$

(see [18, Theorem 4.2], also [7, Theorem 5.1]). On the other hand, by [7, Theorem 5.2], if X has 1-unconditional basis, has cotype 2 and $\dim X = N$, then for $S \in \mathcal{L}({}^nX)$ we have

$$\|S\|_{\Pi_1({}^nX)} \asymp \sup_{\sigma_j} \|S \circ (D_{\sigma_1},\dots,D_{\sigma_n})\|_{\Pi_1({}^n\ell_2^N)}, \tag{8}$$

where $D_{\sigma_j} : \ell_2^N \to X$ are norm-one diagonal operators.

Applying (7) and (8) to Φ_N we obtain

$$\begin{aligned} \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} &\asymp \sup_{\sigma_j} \left(\sum_{k_1,\dots,k_n=1}^{N} |\sigma_1(k_1)\cdots\sigma_n(k_n)\Phi_N(e_{k_1},\dots,e_{k_n})|^2 \right)^{1/2} \\ &= \sup_{\sigma_j} \left(\sum_{k=1}^{N} |\sigma_1(k)\cdots\sigma_n(k)|^2 \right)^{1/2} \end{aligned}$$

where the supremum is taken over all σ_j such that $D_{\sigma_j} : \ell_2^N \to \ell_p^N$, $j = 1,\dots,n$ are norm-one operators. Note that $\|D_{\sigma_j}\| = \|\sigma_j\|_{\ell_r^N}$, where $\frac{1}{r} = \frac{1}{2} - \frac{1}{p'}$. If $r \ge n$, we have

$$\begin{aligned} \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} &\asymp \sup \left\{ \left(\sum_{k=1}^{N} |\sigma_1(k)\cdots\sigma_n(k)|^2 \right)^{1/2} : \ \sigma_j \in B_{\ell_r^N} \right\} \\ &= \sup \left\{ \left(\sum_{k=1}^{N} |\sigma(k)|^2 \right)^{1/2} : \ \sigma \in B_{\ell_{r/n}^N} \right\} \\ &= \|id : \ell_{r/n}^N \to \ell_2^N\| = \begin{cases} 1 & \text{if } 1 \le \frac{r}{n} \le 2 \\ N^{\frac{1}{2}-\frac{n}{r}} & \text{if } 2 < \frac{r}{n} \end{cases} \end{aligned}$$

If $r < n$, with the same procedure we obtain $\|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} \preceq 1$. Since the reverse inequality is always true, we also have $\|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} \asymp 1$ for this case. This gives:

$$\|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} \asymp \begin{cases} 1 & \text{if } 1 \le p \le \frac{2n}{n+1} \\ N^{\frac{n+1}{2}-\frac{n}{p}} & \text{if } \frac{2n}{n+1} \le p \le 2. \end{cases}$$

We now consider $p > 2$. Let us see that in this case

$$N^{1-\frac{n-1}{p}}(\log N)^{1/p'} \prec \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} \prec N^{\frac{n+1}{2}-\frac{n}{p}}.$$

First we show the lower bound by induction on n.

By [7, Lemma 3.4] we have the isometry $\Pi_1({}^n\ell_p^N;\mathbb{C}) \overset{1}{=} \Pi_1({}^{n-1}\ell_p^N;\ell_{p'}^N)$, denoted by $T \leftrightarrow \widetilde{T}$. If $n = 2$, then $\widetilde{\Phi}_N = id : \ell_p^N \to \ell_{p'}^N$. Therefore, $\|\Phi_N\|_{\Pi_1({}^2\ell_p^N)} = \|id\|_{\Pi_1(\ell_p^N;\ell_{p'}^N)} \asymp (N\log N)^{1/p'}$, by [19, 22.4.11].

Let us now consider $\Sigma_N : \ell_{p'}^N \to \mathbb{C}$ given by $z \rightsquigarrow \sum_{k=1}^N z(k)$ and $\Psi_N = \Sigma_N \circ \widetilde{\Phi}_N : \overbrace{\ell_p^N \times \cdots \times \ell_p^N}^{(n-1) \text{ times}} \to \ell_{p'}^N \to \mathbb{C}$. By the induction hypothesis

$$N^{1-\frac{n-2}{p}}(\log N)^{1/p'} \prec \|\Psi_N\|_{\Pi_1({}^{n-1}\ell_p^N)} \leq \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)}\|\Sigma_N\|.$$

Now, since $\|\Sigma_N\| = N^{\frac{1}{p}}$, we have the desired lower bound.

To get the upper bound, let us factor Φ_N in the following way

$$\begin{array}{ccccccc} \ell_p^N & \times & \cdots & \times & \ell_p^N & \xrightarrow{\Phi_N} & \mathbb{C} \\ \downarrow & & & & \downarrow & \nearrow & \\ \ell_2^N & \times & \cdots & \times & \ell_2^N & & \end{array}$$

With this we get

$$\begin{aligned} \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} &\leq \|id : \ell_p^N \to \ell_2^N\|^n \|\Phi_N\|_{\Pi_1({}^n\ell_2^N)} \\ &\prec (N^{\frac{1}{2}-\frac{1}{p}})^n\sqrt{N} = N^{\frac{n+1}{2}-\frac{n}{p}}. \end{aligned}$$

This altoghether gives the following situation

$$\begin{array}{rlll} & \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} & \asymp 1 & \text{if } 1 \leq p \leq \frac{2n}{n+1} \\ & \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} & \asymp N^{\frac{n+1}{2}-\frac{n}{p}} & \text{if } \frac{2n}{n+1} \leq p \leq 2 \\ N^{1-\frac{n-1}{p}}(\log N)^{1/p'} \prec & \|\Phi_N\|_{\Pi_1({}^n\ell_p^N)} & \prec N^{\frac{n+1}{2}-\frac{n}{p}} & \text{if } 2 \leq p. \end{array}$$

We reformulate this results in terms of limit orders and defects:
For $p \leq 2$:

$$\lambda_n(\Pi_1;p) = \begin{cases} 0 & \text{if } 2 \leq 1 \leq p \leq \frac{2n}{n+1} \\ \dfrac{n+1}{2} - \dfrac{n}{p} & \text{if } \frac{2n}{n+1} \leq p \leq 2 \end{cases}$$

and $d_n(\Pi_1,p) = 0$.
For $p > 2$:

$$1 - \frac{n-1}{p} \leq \lambda_n(\Pi_1;p) \leq \frac{n+1}{2} - \frac{n}{p}$$

and $d_n(\Pi_1,p) \leq \frac{n-1}{2} - \frac{1}{p}$.

5. The diagonal of a multilinear form

Throughout this section, X_i will be a Banach space with unconditional basis $\{e_j^i\}_j$ $(i = 1, \dots, n)$. We define the application $D : \mathcal{L}(X_1, \dots, X_n) \to \mathcal{L}(X_1, \dots, X_n)$ given by

$$D(T)(x_1, \dots, x_n) = \sum_{j=1}^{\infty} T(e_j^1, \dots, e_j^n) x_1(j) \cdots x_n(j).$$

Note that $D(T)$ is the diagonal n-linear form given by the diagonal of T. The linear mapping D is well defined and continuous [8, Proposition 1.3]. Now we show that it preserves some ideals of multilinear forms.

Proposition 5.1. *Let β be a tensor norm of order n. If $T \in \mathcal{L}(X_1, \dots, X_n)$ is β-continuous (i.e., $T \in (\bigotimes_{i=1}^n X_i; \beta)'$), then $D(T)$ is also β-continuous.*

Proof. For $0 \leq t \leq 1$ and $i = 1, \dots, n$, we define $\Lambda_t^i : X_i \to X_i$ by

$$\Lambda_t^i(x) = \sum_{j=1}^{\infty} x(j) r_j(t) e_j^i,$$

where $\{r_j\}_j$ are the generalized n-Rademacher functions [3, Section 1]. By the unconditionality of the basis, Λ_t^i is continuous and $\|\Lambda_t^i\| \leq 2K_i$ (being K_i the unconditionality constant of the basis).

Let $s \in \bigotimes_{i=1}^n X_i$, $s = \sum_{k=1}^M x_k^1 \otimes \cdots \otimes x_k^n$. We have

$$\begin{aligned}
|D(T)(s)| &= \left| \sum_{k=1}^{M} D(T)(x_k^1, \dots, x_k^n) \right| \\
&= \left| \int_0^1 \sum_{k=1}^{M} T\left(\Lambda_t^1(x_k^1), \dots, \Lambda_t^n(x_k^n)\right) dt \right| \\
&= \left| \int_0^1 T\left((\Lambda_t^1 \otimes \cdots \otimes \Lambda_t^n)(s)\right) dt \right| \\
&\leq \|T\|_{\left(\bigotimes_{i=1}^n X_i; \beta\right)'} \cdot \|\Lambda_t^1\| \cdots \|\Lambda_t^n\| \cdot \|s\|_\beta \\
&\leq \|T\|_{\left(\bigotimes_{i=1}^n X_i; \beta\right)'} \cdot 2^n K_1 \cdots K_n \cdot \|s\|_\beta
\end{aligned}$$

□

Corollary 5.2. *If $\mathfrak{A}$ is a maximal ideal of n-linear forms, then $D : \mathfrak{A}(X_1, \dots, X_n) \to \mathfrak{A}(X_1, \dots, X_n)$ is well defined and continuous.*

In particular, the ideals of integral, extendible, r-dominated ($r \geq n$) and multiple r-summing n-linear forms are preserved by D. An example of an ideal which is not maximal but is preserved by D is given in the following:

Proposition 5.3. *If $T \in \mathcal{L}(X_1, \dots, X_n)$ is weakly sequentially continuous, then so is $D(T)$*

Proof. Let $(x_k^i)_k \subseteq X_i$ be weakly convergent to $x^i \in X_i$. With the notation of the proof of Proposition 5.1, we have

$$D(T)(x_k^1, \dots, x_k^n) = \int_0^1 T\left(\Lambda_t^1(x_k^1), \dots, \Lambda_t^n(x_k^n)\right) dt.$$

Since each Λ_t^i is linear and T is weakly sequentially continuous, $T\left(\Lambda_t^1(x_k^1), \dots, \Lambda_t^n(x_k^n)\right)$ converges to $T\left(\Lambda_t^1(x^1), \dots, \Lambda_t^n(x^n)\right)$ for every $t \in [0,1]$. Now the result follows from the dominated convergence theorem. ☐

However, not every ideal is preserved by D:

Example 5.4. *The ideals of nuclear, approximable and weakly continuous on bounded sets multilinear forms are not preserved by D:*

Let $T \in \mathcal{L}(^n\ell_1)$ be given by

$$T(x^1, \dots, x^n) = \left(\sum_{j=1}^{\infty} x^1(j)\right) \cdots \left(\sum_{j=1}^{\infty} x^n(j)\right).$$

Clearly, T is a finite type n-linear form. Now,

$$D(T)(x^1, \dots, x^n) = \sum_{j=1}^{\infty} x^1(j) \cdots x^n(j)$$

which is not weakly continuous on bounded sets (hence neither nuclear, nor approximable).

From the above results about the operator D and the definitions of limit order and defect, we can obtain some information about the diagonal of any n-linear form belonging to certain ideal.

Proposition 5.5. *If $\mathfrak{A}$ is a maximal ideal of n-linear forms (or any ideal of n-linear forms preserved by D) and $T \in \mathfrak{A}(^n\ell_p)$, then, for every $\varepsilon > 0$,*

$$\left(T(e_j, \dots, e_j)\right)_j \in \ell_{r+\varepsilon}$$

where $r = \frac{1}{\lambda_n(\mathfrak{A};p) - d_n(\mathfrak{A};p)}$ and $(e_j)_j$ is the canonical basis of ℓ_p.

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