Syzygies and sumsets, commutative algebra vs additive combinatorics

Philippe Gimenez (`pgimenez@uva.es`)
IMUVa - Universidad de Valladolid (Spain)

Abstract. Given a finite nonempty subset $\mathcal{A}$ in $\mathbb{N}^d$, for all $s \geq 0$, the set formed by all sums of s non necessarily different elements in $\mathcal{A}$, $s\mathcal{A} = \{a_1 + \cdots + a_s,\ a_i \in \mathcal{A}\}$, is called the s-fold iterated sumset of $\mathcal{A}$. Additive combinatorics studies sumsets of $\mathcal{A}$ and their cardinality. On the other hand, if one takes an algebraically closed field $\mathbb{K}$ and $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a_n}\} \subset \mathbb{N}^d$, one can associate to each $\mathbf{a_i} = (a_{i1}, \ldots, a_{id})$, the monomial $\mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i1}} \times \cdots \times t_d^{a_{id}} \in \mathbb{K}[t_1, \ldots, t_d]$, and define the ring homomorphism $\varphi_{\mathcal{A}}$: $\mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[t_1, \ldots, t_d]$, $x_i \mapsto \mathbf{t}^{\mathbf{a}_i}$. This parametrically defines a toric variety and provides a toric ideal $I_{\mathcal{A}} = \ker \varphi_{\mathcal{A}}$. Based on recent results in [1], [2], [3], [4], [5], and some work in progress, we will show in some specific cases (monomial curves, simplicial varieties with at most one singular point) how the sumsets structure of $\mathcal{A}$ is related to the syzygies and, in particular, to the Castelnuovo-Mumford regularity, of the toric ideal $I_{\mathcal{A}}$. This illustrates the interplay between additive combinatorics and commutative algebra, exhibiting how each area can help to solve problems in the other one.

This is based on joint work with Mario González-Sánchez (U. Valladolid) and Ignacio García-Marco (U. La Laguna)

References

[1] L. Colarte-Gómez, J. Elias and R.M. Miró-Roig (2023). Sumsets and Veronese varieties. *Collect. Math.*, 74, 353–374.

[2] S. Eliahou, E. Mazumdar (2022). Iterated sumsets and Hilbert functions. *J. Algebra*, 593, 274–294.

[3] J. Elias (2022). Sumsets and Projective Curves. *Mediterr. J. Math.*, 19:177, 11 pp.

[4] P. Gimenez and M. González Sánchez (2023). Castelnuovo-Mumford regularity of projective monomial curves via sumsets, *Mediterr. J. Math.*, 20:287, 24 pp.

[5] M. González-Sánchez (2025). *Syzygies, regularity, and their interplay with additive combinatorics.* PhD Thesis, University of Valladolid.

Synchronization and Non-Synchronization in Random Geometric Graphs

Pablo Groisman (`pgroisma@dm.uba.ar`)
Universidad de Buenos Aires and IMAS-CONICET

Abstract. Understanding the geometry of the energy function in physical systems with a large number of components is as important as elusive. A similar situation occurs with the loss functions of deep neural networks and other learning procedures. In both cases, any information about the local minima and other critical points is essential, as they govern the long-time behavior of gradient descent mechanisms.

The Kuramoto model is a gradient system of ordinary differential equations whose dynamics reproduce the behavior of an ensemble of coupled oscillators. The coupling is determined by a given graph. The goal is to understand which features of the graph determine the geometry of the energy.

I learnt from Alicia that by means of a change of variables, understanding the geometry of Kuramoto's energy can be stated as an algebraic geometry problem. In fact, I learnt from Alicia everything I know about algebraic geometry (thanks Alicia!). Unfortunately, it is not too much (sorry Alicia!).

In view of this, in this talk we will deal with the geometry of Kuramoto's energy with other tools. The goal is to study this energy in random geometric graphs on a given Riemannian manifold and to understand which properties of the underlying manifold govern the number and nature of the energy's local minima.

Higher order tangent spaces – the toric case

Ragni Piene (ragnip@math.uio.no)
University of Oslo, Norway

Abstract. Projective algebraic varieties can be described and classified by way of their tangent spaces, leading in particular to polar varieties and dual varieties. A finer study involves the higher order tangent spaces, which give rise to higher order polar and dual varieties, as well as higher order reciprocal polar varieties and distance degrees. For toric varieties this study is of particular interest due to the connection with convex geometry and combinatorics. A particular feature of toric varieties is that their higher order tangent spaces can be interpolated to give a toric variety that osculates the given one. The talk will survey results that are part of, or relevant to, joint work with Alicia.

Agebraic winding numbers

Marie-Françoise Roy (marie-françoise.roy@univ-rennes.fr)
Université de Rennes, France

Abstract. In this paper, we propose a new algebraic winding number and prove that it computes the number of complex roots of a polynomial in a rectangle, including roots on edges or vertices with appropriate counting. The definition makes sense for the algebraic closure C = R[i] of a real closed field R, and the root counting result also holds in this case. We study in detail the properties of the algebraic winding number defined in [1] with respect to complex root counting in rectangles. We extend both winding numbers to rational functions, obtaining then algebraic versions of the argument principle for rectangles.
Joint work with Daniel Perrucci, University of Buenos Aires
[1] M. Eisermann, The fundamental theorem of algebra made effective: an elementary real-algebraic proof via Sturm chains. Amer. Math. Monthly 119 (2012), no. 9, 715–752.

The scattering correspondence: particle physics meets algebraic statistics

Hal Schenck (hks0015@auburn.edu)
Auburn University

Abstract. An arrangement of hypersurfaces in projective space is strict normal crossing (SNC) if and only if its Euler discriminant is nonzero. We study the critical loci of arbitrary Laurent monomials in the equations of the smooth hypersurfaces. The family of these loci forms an irreducible variety in the product of two projective spaces, known in algebraic statistics as the likelihood correspondence and in particle physics as the scattering correspondence. We establish an explicit determinantal

representation for the minimal generators of the bihomogeneous prime ideal that defines this variety. Joint work with Thomas Kahle, Bernd Sturmfels, and Max Wiesmann.

Maximal Mumford Curves from Planar Graphs

Bernd Sturmfels (bernd@mis.mpg.de)
MPI Leipzig

Abstract. We discuss algebraic curves from the perspectives of real and tropical geometry. A curve of genus g is maximal Mumford (MM) if it has g+1 ovals and g tropical cycles. We construct families of MM curves that are full-dimensional in their moduli space. Our curves deformations of line arrangements, to be illustrated with colorful pictures.

Absolute Concentration Robustness of Zero-One Networks with Conservation Laws

Xiaoxian Tang (xiaoxian@buaa.edu.cn)
Beihang University

Abstract. Absolute concentration robustness (ACR) means the concentration of certain species stays the same in all the steady states. In this talk, we study how conservation laws might effect ACR in zero-one reaction networks. We wonder whether ACR can be preserved or precluded by adding species that depend on the existing species. For simplifying the problem, we focus on the non-redundant networks, which means in each reaction, no species appear in the reactant and the product simultaneously. We prove the following results. (i)For the one-dimensional case, a two-species network has no ACR for a generic choice of rate constants. (ii)For the two-dimensional case, if there are at least four different rows in the stoichiometric matrix, then the network has no ACR for a generic choice of rate constants. These results implies that many conservation laws prevent ACR in zero-one reaction networks.

Algebraic tools for recovering measures from moments

Mauricio Velasco (mvelasco@cmat.edu.uy)
Universidad de la República (Uruguay)

Abstract. The moments of a measure μ in a compact set $X \subseteq \mathbb{R}^n$ are the averages of monomials according to μ. A problem with applications ranging from statistical inference to stochastic control is the (possibly approximate) recovery of a measure's density from a collection of moments. In this talk I will discuss several generalizations of the classical Christoffel-Darboux kernel method for carrying out such recovery procedures and prove several quantitative guarantees and showcase some recent applications. The talk will discuss joint work (some ongoing) with several coauthors (L. Bentancur, R.Chhaibi, F. Gamboa, C. Meroni and D. Henrion).

Polynomial Methods for Handwriting Recognition

Stephen M. Watt (smwatt@uwaterloo.ca)
Cheriton School of Computer Science
University of Waterloo, Canada

Abstract. Modern devices such as tablets and telephones capture digital ink strokes as sequences of (x, y, t) points where t is an explicit time coordinate or implicit from a sampling frequency. Most handwriting work has been based on analyzing these point sequences. We take a different approach and treat digital ink strokes as segments of plane curves. This gives a representation for the digital ink that does not depend device resolution or on writing speed. Approximating the curves as polynomials in an orthogonal basis leads to highly efficient recognition explainable in geometric terms.

Framework We consider an ink trace to be a segment of a plane curve $\big(x(s), y(s)\big), s \in [0, L]$. Since we are concerned with handwriting recognition, or other curve classification problems, we want curves that look similar to have similar representations. To achieve this, we choose a geometric parameterization and project the curves onto a low-dimensional function space. For the parameterization, we have found arc length, given by $ds^2 = dx^2 + dy^2$, to be an effective choice. For the low-dimensional function space, choosing an orthogonal polynomial basis allows projection that may approximate the functions $x(s)$ and $y(s)$ as closely as desired. We choose a basis $\{B_i\}$ of $d + 1$ polynomials that are orthogonal with respect to an inner product $\langle \cdot, \cdot \rangle_I$ and obtain the approximation $\big(\hat{x}(s), \hat{y}(s)\big)$ where $\hat{x}(s) = \sum_{i=0}^{d} \hat{x}_i B_i(s) \approx x(s)$ and $\hat{y}(s) = \sum_{i=0}^{d} \hat{y}_i B_i(s) \approx y(s)$. The curves are thus be represented as points in $\mathbb{R}^{2(d+1)}$. The components $\hat{x}_i$ and $\hat{y}_i$ may be obtained by numerical integration, *e.g.* $\hat{x}_i = \langle x, B_i \rangle_I / h_i$ where $h_i = \langle B_i, B_i \rangle_I$ [4].

Curve Similarity The previous approach to determining curve similarity was that of "dynamic time warping", originally proposed in [6]. This is a sequence alignment method where sample points of two curves are matched seeking a sample point correspondence that minimizes the sum of squared distances. Discovering the optimal alignment can be time consuming when the samples are spaced differently along the two curves.

Instead, we take the approach to compute a variational integral between the curves to be compared. If the two curves are $\big(x(s), y(s)\big)$ and $\big(\bar{x}(s), \bar{y}(s)\big)$, then we let $\xi(s) = x(s) - \bar{x}(s)$ and $\eta(s) = y(s) - \bar{y}(s)$ and use the distance measure

$$\|\xi\|_I^2 + \|\eta\|_I^2 = \langle \xi, \xi \rangle_I + \langle \eta, \eta \rangle_I = \Big\langle \sum_{i=0}^{d} \xi_i B_i, \sum_{i=0}^{d} \xi_i B_i \Big\rangle_I + \Big\langle \sum_{i=0}^{d} \eta_i B_i, \sum_{i=0}^{d} \eta_i B_i \Big\rangle_I = \sum_{i=0}^{d} (\xi_i^2 + \eta_i^2) h_i. \quad (1)$$

This reduces the similarity computation from thousands of machine instructions to dozens.

Choice of Basis We may choose any orthogonal basis to obtain the fast comparison of (1), and we have explored the use of Legendre, Chebyshev, Legendre-Sobolev and Chebyshev-Sobolev basis polynomials [1]. The Sobolev variants use inner products of the form $\int_a^b f(s)g(s)w(s)ds + \mu \int_a^b f'(s)g'(s)w(s)ds$ where $w(s)$ is the weight function for the corresponding orthogonal polynomial family. This helps match the the changes of direction of the curves. The Legendre variants have a weight function 1 so the inner products can be integrated in real time, as the curve is being written, and then rescaled in small constant time when the pen is lifted. Since conversion of basis is numerically ill-conditioned, we wish to perform various operations such as differentiation, root finding and gcd in the orthogonal basis representation, so we have developed algorithms to do this for bases of interest.

Consequences of Linear Separability When characters are labelled with their type, we find that their coefficient vectors form clusters, linearly separable by type, with a few only very poorly written outliers. This linear separability has a number of useful consequences: very few training samples are needed; all points inside the convex hull of the samples correspond to recognizable characters of the same class; thus linear homotopies between points of a class remain in the class. We have used these properties to develop confidence measures based on distances to SVM separating planes and to the convex hull of k nearest neighbours [3]. We have also used homotopies to track special points on characters, allowing baseline determination form characters (as is useful in mathematical handwriting recognition), as opposed to *vice versa* [5].

Additional Directions For recognition of rotated characters or characters subjected to shear, we may represent integral invariants, rather than the (x, y) coordinates, in an orthogonal polynomial basis [2]. Because samples can be represented compactly, and only a few are needed for each character class, it is possible to rapidly adapt to a user's handwriting style as feedback is obtained from correct and incorrect recognitions. Moreover, it is straightforward to find an average example for each character in a user's handwriting style and perform automatic handwriting neatening.

Conclusion We have found that it is quite practical to recognize similar plane curves represented using orthogonal polynomial bases, and that the necessary operations can be performed without leaving that representation.

References

[1] R. Corless, D. Singh Kalhan, and S.M. Watt. Well-conditioned polynomial representations for mathematical handwriting recognition. In *Proc. SYNASC 2025*. IEEE Computer Society, 2025.

[2] O. Golubitsky, V. Mazalov, and S.M. Watt. Orientation-independent recognition of handwritten characters with integral invariants. In *Proc. ASCM and MCIS 2009, COE Lecture Notes Vol 22*. Kyushu University, 2009.

[3] O. Golubitsky and S.M. Watt. Confidence measures in recognizing handwritten mathematical symbols. In *CICM 2009*. Springer Verlag LNCS 5625, 2009.

[4] O. Golubitsky and S.M. Watt. Distance-based classification of handwritten symbols. *International Journal on Document Analysis and Recognition (IJDAR)*, 13(2):133–146, 2010.

[5] Rui Hu and S.M. Watt. Identifying features via homotopy on handwritten mathematical symbols. In *Proc. SYNASC 2013*. IEEE COmputer Socieity, 2013.

[6] J. Kurskal and M. Liberman. The symmetric time-warping problem: from continuous to discrete. In D. Sankoff and J. Kruskal, editors, *Time Warps, String Edits, and Macromolecules: The Theory and Practice of Sequence Comparisons*, Reading, Massachusetts USA, 1983. Addison-Wesley.