Linear Recurrent Sequences

This text is the result of [1], which has been written for a very nice subject called "Seminario Elemental de Álgebra" (something like Elementary course of Algebra), and it was written after the study of [1] and during the specification and implementation in *Smalltalk* (an objects ambient,living together in a computer) of the linear recurrent sequences.

1. Some definitions

Some of the following definitions were taken from [1].

Definition. *A sequence* $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ *is said to be* **linear recurrent of order less or equal than** k *if there exists* k *complex numbers* $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ *such that*

$$x_{n+k} = \alpha_0.x_n + \alpha_1.x_{n+1} + \cdots + a_{k-1}.x_{n+k-1} \quad \forall n \in \mathbb{N} \tag{1}$$

Definition. *Given a linear recurrent sequence* $\{x_n\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$*, we define the* **order** *of* $\{x_n\}_{n\in\mathbb{N}}$ *(and noted* $ord(\{x_n\}_{n\in\mathbb{N}})$*) to the positive integer:*

i) *1 if* $\{x_n\}_{n\in\mathbb{N}}$ *is a linear recurrent sequence of order less or equal than 1, (e.g. if* $\{x_n\}_{n\in\mathbb{N}}$ *is a geometric progression).*

ii) k if $\{x_n\}_{n\in\mathbb{N}}$ is a linear recurrent sequence of order less or equal than k but is **not** a liner recurrent sequence of order less or equal than $(k-1)$.

Definition. *If* $p = \sum_{i=0}^{k} \alpha_i x^i \in \mathbb{C}[X] - \{0\}$ *is a polynomial of degree* k *and* $\{x_n\}_{n\in\mathbb{N}}$ *a linear recurrent sequence. We say that* $\{x_n\}_{n\in\mathbb{N}}$ **satisfies** p *if*

$$\alpha_0.x_n + \alpha_1.x_{n+1} + \cdots + a_{k-1}.x_{n+k-1} = 0 \quad \forall n \in \mathbb{N}$$

Definition. *We denote* $f = \mathcal{X}_{a_n}, f \in \mathbb{C}[X]$*, and we say that* f *is the* **characteristic polynomial** *of* $\{a_n\}_{n\in\mathbb{N}}$*, if* f *is monic,* $ord(\{a_n\}_{n\in\mathbb{N}}) = \deg(f)$ *and* $f(\{a_n\}) = 0$.

There exists a matrix representation of (1) and the characteristic polynomial of that matrix is the same as the minimal one, and the same as of the last definition. (for further details guess... yes! [1]).

2. Structure of the linear recurrent sequences

In [1] it's proved that

$$\mathcal{S}_p = \big\{\{x_n\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} / \{x_n\}_{n\in\mathbb{N}} \text{ is a linear recurrent sequence that satisfies } p\big\}$$

is a $\mathbb{C}$ vector subspace of $\mathbb{C}^{\mathbb{N}}$ of dimention $\deg(p)$. And that the set $\mathcal{S} \subset \mathbb{C}^{\mathbb{N}}$ of all linear recurrent sequences is a $\mathbb{C}$ vector subspace. In the proof they use a lemma which says that if $\{x_n\}_{n\in\mathbb{N}}$ satisfies p, then for all polynomial q, $\{x_n\}_{n\in\mathbb{N}}$ also satisfies $q \cdot p$.

2.1 Properties of evaluating the sequences at polynomials.

Let $f = \sum_{i=0}^{k} \alpha_i x^i$ and $g = \sum_{i=0}^{l} \beta_i x^i$ be two polynomials, where $f = \mathcal{X}_{a_n}, g = \mathcal{X}_{b_n}$, $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are not null sequences.

1. If $f = p \cdot q$ and $p, q \notin \mathbb{C}$, then $p(\{a_n\}) \neq 0$ and $q(\{b_n\}) \neq 0$ since f is the minimal that is zero at a_n.

2. $(f+g)\{c_n\} = 0 \not\Rightarrow f(\{c_n\}) = 0$ and $g(\{c_n\}) = 0$. A counterexample is $\overline{f} = f - \alpha_0$, where it's clear that $\alpha_o(\{a_n\}) \neq 0$ and $\overline{f}(\{a_n\}) \neq 0$ because f is minimal.

3. If $(f+g)(\{c_n\}) = 0$ and $f(\{c_n\}) = 0$, then $g(\{c_n\}) = 0$, filling with zeros if it's needed we have

$$\begin{aligned}
(f+g) &= \sum_{i=0}^{k} (\alpha_i + \beta_i) x^i \\
(f+g)(\{c_n\}) &= \sum_{i=0}^{k} (\alpha_i + \beta_i) c_{n+i} \\
&= \sum_{i=0}^{k} \alpha_i c_{n+i} + \sum_{i=0}^{k} \beta_i c_{n+i} \\
&= \sum_{i=0}^{k} \beta_i c_{n+i} \\
&= g(\{c_n\})
\end{aligned}$$

4. The set $\mathrm{I}_{a_n} = \{p \in \mathbb{C}[X] / p(\{a_n\}) = 0\}$ is the same as $f\mathbb{C}[X]$ for if $p(\{a_n\}) = 0$ then

$$\begin{aligned} p &= f \cdot q + r \\ p(\{a_n\}) &= (f \cdot q + r)(\{a_n\}) \\ &= r(\{a_n\}) \end{aligned}$$

hence $r = 0$ or $\deg(r) < \deg(f)$, so we have $r = 0$ because of the definition of f.

Corollary. $[f; g](\{a_n + b_n\}) = 0$.

This is evident hence $[f; g]$ is a multiple of f and of g so evaluated at a_n and b_n it's zero, but being $\mathcal{S}_{[f;g]}$ a linear subspace, we conclude that $[f; g](\{a_n + b_n\}) = 0$.

Corollary. *Let* $m = \mathcal{X}_{a_n + b_n}$ *and*

$$\overline{f} = \frac{f}{(f; g)} \quad y \quad \overline{g} = \frac{g}{(f; g)}$$

then

$$\text{máx}\{\deg(\overline{f} \cdot \overline{g}), 1\} \leq \deg(m) \leq \deg([f; g])$$

Hence $m(\{a_n + b_n\}) = 0$ and $f(\{a_n\}) = 0$ we have that $(m \cdot f)(\{a_n + b_n\}) = 0$ and $(m \cdot f)(\{a_n\}) = 0$, because $\mathcal{S}_{m \cdot f}$ is a linear subspace, $\{b_n\} \in \mathcal{S}_{m \cdot f}$, so $g / f \cdot m$ since g generates I_{b_n}, cancelling f and g common roots, we have that $\overline{g} / \ \overline{f} \cdot m$, so $\overline{g} / m$. Same way we have $\overline{f} / m$. Being $\overline{f} \perp \overline{g}$ we have the proof.

Bibliography.

[1] Juan Sabia, Susana Tesauri, Sucesiones Recursivas Lineales, *Notas de Matemática*, **5** (U.B.A 1997).