PARACYCLIC COMPLEXES ARISING FROM α-DERIVATIONS

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1.Introduction

Let A be a k-algebra with 1, $\alpha: A \to A$ an automorphism of algebras. In [R-S] we described a construction of the graded α-differential algebra $\Omega_k^\alpha(A)$. Now we define Karoubi's operator κ for α-differential non-commutative forms, and study some of its properties.

This operator allows us to construct a parachain complex for $\Omega_k^\alpha(A)$. This may be the first step to get a mixed complex for $\Omega_k^\alpha(A)$, in order to define the α-cyclic homology of A.

2.Parachain complexes.

Definition 2.1. *A parachain complex is a graded k-module $\bigoplus_{i \in N} V_i$ with two operators $b: V_i \to V_{i-1}$, $B: V_i \to V_{i+1}$ such that*

(1) $b^2 = B^2 = 0$
(2) *the operator $T = 1 - (bB + Bb)$ is invertible.*

It may be easily checked that T commutes with b and B. When T is the identity, the two differentials b and B commute. Such a parachain complex is called a *mixed complex*.

Example. Let $(\overline{C}_*(A), b)$ be the normalized Hochschild complex given by $\overline{C}_n(A) = A^{\otimes(n+1)}/D_n$, where D_n is spanned by the elements $a_0 \otimes \cdots \otimes a_n$ such that $a_i = 1$ for some i with $1 \le i \le n$, and

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

Let $B: \overline{C}_n(A) \to \overline{C}_{n+1}(A)$ be the operator given by

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^i n 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}$$

Now $(\overline{C}_*(A), b, B)$ is a mixed complex, and the cyclic homology of A is

$$HC_*(A) = H_*(Tot(\overline{C}_*(A), b, B))$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition. *A bi-parachain complex is a N^2-graded k-module $\bigoplus_{(i,j)\in N^2} V_{i,j}$ with operators $b\colon V_{i,j} \to V_{i-1,j}$, $\overline{b}\colon V_{i,j} \to V_{i,j-1}$ $B\colon V_{i,j} \to V_{i+1,j}$, $\overline{B}\colon V_{i,j} \to V_{i,j+1}$ such that*

(1) $b^2 = \overline{b}^2 = B^2 = \overline{B}^2 = 0$
(2) *the operators* $T = 1-(bB+Bb)$ *and* $\overline{T} = 1-(\overline{bB}+\overline{Bb})$ *are invertible*
(3) b *and* B *commute in the graded sense with* $\overline{b}$ *and* $\overline{B}$.

Proposition. *There is a functor $V \to Tot(V)$ from bi-parachain complexes to parachain complexes, where $Tot(V)$ is:*

$$Tot_n(V) = \sum_{i+j=n} V_{i,j}$$

$$Tot(b) = b+\overline{b}, \qquad Tot(B) = \overline{B}+\overline{T}B \qquad \text{and} \qquad Tot(T) = T\overline{T}$$

So, when $Tot(T) = 1$, $Tot(V)$ is a mixed complex.

Proof. It follows immediately that $Tot(b)^2 = Tot(B)^2 = 0$. Now,

$$\begin{aligned} Tot(T) &= 1-(Tot(b)Tot(B)+Tot(B)Tot(b)) \\ &= 1-(b+\overline{b})(\overline{B}+\overline{T}B)-(\overline{B}+\overline{T}B)(b+\overline{b}) \\ &= 1-(bB+Bb)\overline{T}-(\overline{bB}+\overline{Bb}) \\ &= 1-(1-T)\overline{T}-(1-\overline{T}) \\ &= T\overline{T}. \end{aligned}$$

The definition and proposition above can be generalized, getting multi-parachain complexes, and a functor from multi-parachain complexes to parachain complexes.

So, as the above proposition shows, the construction of parachain complexes may be the first step to get mixed complexes.

Example. Let A be a k-algebra, and G a finite group acting on A by automorphisms.

Take $V_{p,q} = k[G^{p+1} \otimes A^{\otimes p+1}$. Define the operators:

$$d_i : V_{p,q} \to V_{p-1,q}$$

$$\overline{(d)}_i : V_{p,q} \to V_{p,-1q}$$

$$t : V_{p,q} \to V_{p,q}$$

$$\overline{(t)} : V_{p,q} \to V_{p,q}$$

respectively by:

$$d_i(g_0,\ldots,g_p;a_0,\ldots,a_q) = (g_0,\ldots,g_p;a_0,\ldots,a_i.a_{i+1},\ldots,a_q)(0 \le i \le q-1)$$

$$d_q(g_0,\ldots,g_p;a_0,\ldots,a_q) = (g_0,\ldots,g_p;((g_0.g_1.\ldots.g_p)^{-1}a_q)a_0,\ldots,a_{q-1})$$

$$\overline{(d)}_i(g_0,\dots,g_p;a_0,\dots,a_q) = (g_0,\dots,g_i.g_{i+1},\dots,g_p;a_0,\dots,a_q)(0\le i\le p-1)$$

$$\overline{(d)}_q(g_0,\dots,g_p;a_0,\dots,a_q) = (g_p.g_0,\dots,g_{q-1};(g_p(a_0)),\dots,(g_p(a_q))$$

$$t(g_0,\dots,g_p;a_0,\dots,a_q) = (g_0,\dots,g_q;(g_0.g_1.\dots.g_p)^{-1}a_q),a_0,\dots,a_{q-1})$$

$$\overline{(t)}(g_0,\dots,g_p;a_0,\dots,a_q) = (g_p,g_0,\dots,g_{p-1};g_p(a_0),\dots,g_p(a_q))$$

Take $b=\sum_{i=0}^q d_i$, $\overline{(b)}=\sum_{i=0}^q \overline{(d)}_i$, $B=(1-t)sN$, $\overline{(B)}=(1-\overline{(t)})\overline{(s)}\overline{(N)}$, $T=t^(p+1)$ and $\overline{(T)}=\overline{(t)}^{q+1}$.

Then $(V,b,B,\overline{b},\overline{B})$ is a bi-parachain complex.

AS $T.\overline{(T)}=1$, in this case we obtain, by taking Tot, a mixed complex.

3.α-differential forms and the Karoubi operator κ.

Let A be an associative k-algebra with 1 and $\alpha\colon A\to A$ an automorphism of algebras. An α-derivation of A into an A-bimodule M is a k-linear map, $d_\alpha\colon A\to M$, such that

$$d_\alpha(ab)=\alpha(a)d_\alpha(b)+d_\alpha(a)b \qquad \text{for } a,b\in A.$$

In [R-S] we described the construction of $\Omega_k^\alpha(A)$. A and $A\otimes A^{op}$ are considered as A-bimodules with the structures defined respectiveley by $a\circ x\circ b=ax\alpha(b)$, and $a\circ(x\otimes y)\circ b=ax\otimes yb$. Now $\Omega_k^\alpha(A)=I_\alpha=\mathrm{Ker}(A\otimes A^{op}\xrightarrow{\mu_\alpha}A)$, where $\mu_\alpha(a\otimes b)=a\alpha(b)$, and $d_\alpha\colon A\to\Omega_k^\alpha(A)$ is defined by $d_\alpha(a)=1\otimes a-\alpha(a)\otimes 1$.

$\Omega_k^\alpha(A)$ is an A-bimodule, as μ_α is a morphism of A-bimodules.

The pair $(d_\alpha,\Omega_k^\alpha(A))$ is characterized by the following universal property:

Let δ be an α-derivation of A with values in an A-bimodule M, then there exists a unique homomorphism of bimodules $i_\delta\colon\Omega_k^\alpha(A)\to M$ such that $\delta=i_\delta\circ d_\alpha$.

Setting $\Omega^{0,\alpha}(A)=A$, $\Omega^{1,\alpha}(A)=\Omega_k^\alpha(A)$, and $\Omega^{n,\alpha}(A)=\Omega^{1,\alpha}(A)\otimes_A\cdots\otimes_A\Omega^{1,\alpha}(A)$, $\Omega^\alpha(A)=\bigoplus\Omega^{n,\alpha}(A)$ is naturally a graded algebra, and there is a unique α-differential d_α on $\Omega^\alpha(A)$ extending the derivation $d_\alpha\colon\Omega^{0,\alpha}(A)\to\Omega^{1,\alpha}(A)$. The graded α-differential algebra $\Omega^\alpha(A)$ is characterized by the following universal property: *Let $\phi\colon A\to\Omega'$ be an homomorphism of algebras with units where Ω' is a graded α-differential algebra, then there is a unique homomorphism of graded α-differential algebras $\widehat{\phi}\colon\Omega^\alpha(A)\to\Omega'$ which extends ϕ.*

Lemma 3.1. *The map $x\otimes\overline{y}\to x\otimes y-x\alpha(y)\otimes 1$ is an isomorphism of left A-modules*

$$A\otimes\overline{A}\xrightarrow{\cong}\Omega_k^\alpha(A)$$

Proof. One first remarks that $x\otimes y-x\alpha(y)\otimes 1$ depends only on the class of y in $\overline{A}$, so the map is well defined, and its image is in I_α. The quotient of $A\otimes A^{op}$ by the relations $x\otimes y-x\alpha(y)\otimes 1$ maps isomorphically to A (with inverse map given by $x\to$ class of $x\otimes 1$). Therefore the kernel of this factor map is isomorphic to the kernel of μ_α.

Let us introduce the following usual notation: $x \otimes \overline{y}$ (or equivalently $x \otimes y - x\alpha(y) \otimes 1$) is written $xd_\alpha(y)$.

I_α is an A-bimodule because it is a sub-A-bimodule of $A \otimes A^{op}$. So, by the isomorphism shown in the previous Lemma, $A \otimes \overline{A}$ becomes an A-bimodule. The left module structure is simply

$$a(xd_\alpha(y)) = axd_\alpha(y)$$

The right module structure is

$$(xd_\alpha(y))b = xd_\alpha(yb) - x\alpha(y)d_\alpha(b)$$

because, in I_α,

$$\begin{aligned}(x \otimes y - x\alpha(y) \otimes 1)b &= x \otimes yb - x\alpha(y) \otimes b \\ &= (x \otimes yb - x\alpha(yb) \otimes 1) - (x\alpha(y) \otimes b - x\alpha(yb) \otimes 1)\end{aligned}$$

So we have the classical formula

$$d_\alpha(yb) = \alpha(y)d_\alpha(b) + d_\alpha(y)b$$

Now,

$$\Omega^{n,\alpha}(A) = I_\alpha \otimes_A \cdots \otimes_A I_\alpha = A \otimes \overline{A}^n$$

with the identification

$$a_0 d_\alpha(a_1) \ldots d_\alpha(a_n) = a_0 \otimes \overline(a)_1 \otimes \cdots \otimes ovrline(a)_n$$

and the operator $d_\alpha \colon \Omega^{n,\alpha}(A) \to \Omega^{n+1,\alpha}(A)$ is given by

$$d_\alpha(a_0 \otimes \cdots \otimes a_n) = 1 \otimes \overline(a)_0 \otimes \cdots \otimes \overline(a)_n$$

At this point it is interesting to remark a difference with the case A commutative and $\alpha = id$, where the cohomology of the complex $(\Omega^{*,\alpha}(A), d)$ is trivial (by Poincaré's Lemma). When $\alpha \neq id$ this fact not necessarily holds, take for example $A = k[t]$ where k is a field, $\mathrm{car}(k) = 0$, q an n-th root of 1.

The product in the α-differential graded algebra $(\Omega^\alpha(A), d_\alpha)$ is performed by using the rules of d_α, for instance,

$$\begin{aligned}(1 \otimes x)(y \otimes z) &= d_\alpha(x)(yd_\alpha(z)) = (d_\alpha(x)y)d_\alpha(z) \\ &= (d_\alpha(xy) - \alpha(x)d_\alpha(y))d_\alpha(z) \\ &= 1 \otimes xy \otimes z - \alpha(x) \otimes y \otimes z\end{aligned}$$

Now we will use the identification $\Omega^{n,\alpha}(A) = A \otimes \overline{A}^n$, so the α-Hochschild homology $HH_{\alpha,*}(A) = HH_{\alpha,*}(A, A)$ is the homology of the complex

$$\ldots \to \Omega^{2,\alpha}(A) \xrightarrow{b} \Omega^{1,\alpha}(A) \xrightarrow{b} A \to 0$$

with

$$
\begin{aligned}
&b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\
&= (a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i (a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\
&+ (-1)^n (\alpha(a_n) a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})
\end{aligned}
$$

Let the Karoubi operator $\kappa \colon \Omega^\alpha(A) \to \Omega^\alpha(A)$ be the degree zero operator given by

$$
\begin{aligned}
&\kappa(\omega d_\alpha(a_n)) = \kappa(a_0 d_\alpha(a_1) \dots d_\alpha(a_n)) \\
&= (-1)^{|\omega|} \left(d_\alpha(a_n)\omega + d_\alpha \left((\alpha - id)(a_n) a_0\right) d_\alpha(a_1) \dots d_\alpha(a_{n-1})\right) \\
&= (-1)^{n+1} \left(d_\alpha(\alpha(a_n) a_0) d_\alpha(a_1) \dots d_\alpha(a_{n-1}) - \alpha(a_n) d_\alpha(a_0) d_\alpha(a_1) \dots d_\alpha(a_{n-1})\right)
\end{aligned}
$$

Lemma 3.2.

(1) $bd_\alpha + d_\alpha b = 1 - \kappa$
(2) $b\kappa = \kappa b$ *and* $d_\alpha \kappa = \kappa d_\alpha$

Proof. (1) follows easily by direct computation, and (2) follows immediately by (1).

Let us define $\alpha \colon \Omega^{n,\alpha}(A) \to \Omega^{n,\alpha}(A)$ by

$$\alpha(a_0 d_\alpha(a_1) \dots d_\alpha(a_n)) = \alpha(a_0) d_\alpha(\alpha(a_1)) \dots d_\alpha(\alpha(a_n))$$

Lemma 3.3. *On* $\Omega^{n,\alpha}(A)$*, we have the identities:*

(1) $\kappa^{n+1} d_\alpha = d_\alpha \ \alpha$
(2) $\kappa^n = \alpha + b\kappa^n d_\alpha$
(3) $\kappa^{n+1} = \alpha - d_\alpha b \alpha$
(4) κ *is invertible*

Proof.

(1) Using the identification $\Omega^{n,\alpha}(A) = A \otimes \overline{A}^n$, we have

$$
\begin{aligned}
\kappa(a_0 \otimes \cdots \otimes \overline{(}a)_{n+1}) &= (-1)^{n+1} (\alpha(a_{n+1}) \otimes \overline{(}a)_0 \otimes \cdots \otimes \overline{(}a)_n) \\
&+ (-1)^n (1 \otimes \overline{(}\alpha(a_{n+1}) a_0) \otimes \overline{(}a)_1 \otimes \cdots \otimes \overline{(}a)_n)
\end{aligned}
$$

Now,

$$\kappa(1 \otimes \overline{(}a)_0 \otimes \cdots \otimes \overline{(}a)_n) = (-1)^n (1 \otimes \overline{(}\alpha(a_n)) \otimes \overline{(}a)_0 \otimes \cdots \otimes \overline{(}a)_{n-1})$$

showing that $\kappa^{n+1} d_\alpha = d_\alpha \ \alpha$ on $\Omega^{n,\alpha}(A)$.

(2) A direct computation shows that

$$
\begin{aligned}
\kappa^n d_\alpha(a_0 \otimes \cdots \otimes \overline{(}a)_n) &= \kappa^n (1 \otimes \overline{(}a)_0 \otimes \cdots \otimes \overline{(}a)_n) \\
&= (-1)^n (1 \otimes \overline{(}\alpha(a_1)) \otimes \cdots \otimes \overline{(}\alpha(a_n)) \otimes \overline{(}a)_0)
\end{aligned}
$$

and

$$\begin{aligned}\kappa^n(a_0 \otimes \cdots \otimes \bar{(}a)_n) = (-1)^n[&(\alpha(a_1) \otimes \cdots \otimes \bar{(}\alpha(a_n)) \otimes \bar{(}a)_0) \\ &+ \sum_{i=1}^{n-1}(-1)^i(1 \otimes \bar{(}\alpha(a_1)) \otimes \cdots \otimes \bar{(}\alpha(a_i a_{i+1})) \otimes \cdots \otimes \bar{(}\alpha(a_n)) \otimes \bar{(}a)_0) \\ &+ (-1)^n(1 \otimes \alpha(a_1) \otimes \cdots \otimes \alpha(a_n)a_0)]\end{aligned}$$

Now, it is immediate that $\kappa^n = \alpha + b\kappa^n d_\alpha$.

(3) By (1) and (2), we have that

$$\begin{aligned}\kappa^{n+1} &= \kappa\ \kappa^n = \kappa(\alpha + b\kappa^n d_\alpha) \\ &= \kappa\ \alpha + b\kappa^{n+1}d_\alpha = (\kappa + bd_\alpha)\ \alpha \\ &= (1 - d_\alpha b)\ \alpha\end{aligned}$$

(4) The polynomial $(\kappa^n - \alpha)(\kappa^{n+1} - \alpha)$ has constant term α^2, which is invertible, and

$$(\kappa^n - \alpha)(\kappa^{n+1} - \alpha) = (b\kappa^n d_\alpha)(-d_\alpha b\ \alpha) = 0$$

So κ is invertible.

We define the Connes operator B on $\Omega^{n,\alpha}(A)$ by

$$B = \sum_{j=0}^{n} \kappa^j d_\alpha$$

Proposition 3.4. *$(\Omega^\alpha(A), B, b)$ is a "parachain complex" (see* [G-J]*).*

Proof. We can compute $\kappa^{n(n+1)}$ in two ways. First, using (2) and (1), we have

$$\begin{aligned}\kappa^{n(n+1)} - \alpha^{n+1} &= \sum_{j=0}^{n} \alpha^{n-j}\kappa^{nj}(\kappa^n - \alpha) = \sum_{j=0}^{n} \alpha^{n-j} b\kappa^{nj+n} d_\alpha \\ &= \alpha^n bB\end{aligned}$$

On the other hand, using (3) and (1), we have

$$\begin{aligned}\kappa^{n(n+1)} - \alpha^n &= \sum_{j=0}^{n-1} \alpha^{n-1-j}\kappa^{(n+1)j}(\kappa^{n+1} - \alpha) = -\sum_{j=0}^{n-1} \alpha^{n-1-j}\kappa^{(n+1)j} d_\alpha b\ \alpha \\ &= -\alpha^n Bb\end{aligned}$$

Thus we obtain

$$\kappa^{n(n+1)} = \alpha^{n+1} + \alpha^n bB = \alpha^n - \alpha^n Bb$$

So

$$bB + Bb = 1 - \alpha$$

The above proposition is a technical result. However, in some cases, (for example if α is given by the action of a group on A) we'll have the possibility of constructing a bi-parachain complex V such that $(Tot(V), Totb, TotB)$ is a mixed complex.

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