Morita equivalence for positive Hochschild homology and dihedral homology

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Abstract

In this paper we prove the invariance of positive Hochschild homology and dihedral homology with respect to hermitian Morita equivalence between involutive algebras. We also define the notion of hermitian k-congruence and prove some results on Morita invariance of HH_*^+ and HD_* in this context.

1 Introduction

Let us consider two rings A and B. They are Morita equivalent if and only if their module categories are equivalent. The Morita invariance of HH_* was proved by Dennis-Igusa [2].

If A and B are equipped with involutions ω_A and ω_B, we may consider their positive Hochschild homology HH_*^+ and their dihedral homology HD_* as well as the notion of hermitian Morita equivalence.

One of the main purposes of this paper is to show that HH_*^+ and HD_* are invariant under hermitian Morita equivalence, which is proved in Theorem 2.7.

If A and B are Morita equivalent and A is involutive, it is always possible to define an involution on B such that the equivalence is an hermitian one. Also, it may happen that $HH_*^+(A) \neq HH_*^+(B)$ even when they are Morita equivalent in the usual sense, so, for involutive algebras, HH_*^+ and HD_* are finer invariants than HH_* and HC_*.

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In the second part of this paper, we consider the following fact: for a given A, the functor $H_*(A, -)$ (resp. $H_*^+(A, -)$) is applied at the category of A-bimodules (resp. compatible A-bimodules). The proof of Morita invariance of Hochschild homology makes explicit use of the isomorphisms given by the Morita equivalence . So we study what happens under a weaker condition: the equivalence of the bimodule categories.

We use, after Schack [8], the notion of k-congruence, and we define the notion of hermitian k-congruence. We show, in Theorem 3.6, that HH_* and HH_*^+ are also invariant under k-congruences (resp. hermitian k-congruences). Finally, we show that if A is an involutive B- Azumaya algebra , where B is a commutative involutive k-algebra, then $HH_*^+(A/k) = HH_*^+(B/k)$.

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2 Hermitian Morita invariance

We first recall some definitions and fix notation. Let k be an arbitrary commutative ring with 1, A and B two unital k-algebras. We will denote with $_A mod$ the category of left A-modules, mod_A the category of right A-modules, $_A mod_B$ the category of bimodules (with left action of A and a right action of B) and $_A bimod$ the category of A-bimodules.

Definition 2.1 *A and B are **Morita equivalent** if and only if there exist two bimodules $_A P_B$ and $_B Q_A$ such that $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$ as bimodules, with the usual bimodule structure of A and B.*

This definition is equivalent to the following facts:

- The categories $_A mod$ and $_B mod$ are equivalent.

- The categories mod_A and mod_B are equivalent.

The equivalences are given by:

$$Q \otimes_A - : _A mod \to _B mod$$

$$- \otimes_A P : mod_A \to mod_B$$

As a consequence, $_A bimod$ and $_B bimod$ are equivalent, by the functor:

$$P \otimes_B - \otimes_B Q : _B bimod \to _A bimod$$

Remark(on Def. 2.1): If $_A mod$ and $_B mod$ are equivalent by a functor F, then take $Q = F(A)$ and $P = F^{-1}(B)$.

It is well known that if A and B are Morita equivalent, then $H_*(A, M) \cong H_*(B, Q \otimes_A M \otimes_B P)$ and $HC_*(A) \cong HC_*(B)$ (idem for cohomology). Suppose now that A and B are equipped with involutions of algebras ω_A and ω_B, and let M be an A-bimodule which has an involution ω_M compatible with the involution of A, namely, $\omega_M : M \to M$ is an additive map such that $\omega_M^2 = id_M$ and $\omega_M(ama') = \omega_A(a')\omega_M(m)\omega_A(a)$, for $m \in M$, $a, a' \in A$.

Example: $M = A \otimes A^{op}$ is a compatible A-bimodule, where $\omega_M : A \otimes A^{op} \to A \otimes A^{op}$ is defined by $\omega_M(a \otimes a') = \omega_A(a') \otimes \omega_A(a)$.

The A-bimodule M is equivalent to a left $A^e = A \otimes_k A^{op}$-module, via $(a \otimes a')m = ama'$. The category of compatible A-bimodules is actually

a category of modules over a certain ring R defined as follows: R is the twisted group algebra $R = \mathbb{Z}/2\mathbb{Z} \ltimes A^e$, where the generator σ of $\mathbb{Z}/2\mathbb{Z}$ acts by $\sigma(a \otimes b) = \omega_A b \otimes \omega_A a$. Precisely, $R = k[\mathbb{Z}/2\mathbb{Z}] \otimes A^e = A^e \oplus A^e \sigma$ equipped with the following multiplication: $(e_1 + f_1 \sigma)(e_2 + f_2 \sigma) = (e_1 e_2 + f_1 \sigma(f_2)) + (e_1 f_2 + f_1 \sigma(e_2))\sigma$ The fact that $_R mod = ((compatible A - bimodules))$ follows from the observation that R is the endomorphism ring of the projective generator $A^e \oplus A^e$ with involution $(e, f) \rightarrow (\sigma(f), \sigma(e))$. Explicitly, any compatible A-bimodule M is an R-module by $(e + f\sigma)m = em + f\omega_M(m)$.

At this step there are at least three questions to answer:

- Is $H_*^+(A, M)$ isomorphic to $H_*^+(B, Q \otimes_A M \otimes_A P)$?

- Is $HD_*(A)$ isomorphic to $HD_*(B)$?

- If M is a compatible A-bimodule, can we define an involution on $Q \otimes_A M \otimes_A P$ so that we obtain a compatible B-bimodule?

Remark: It is necessary to talk about bimodules because if not, the compatibility condition with the involution of the ring makes no sense.

An hermitian Morita theory for algebras with involution was introduced in [3] and generalized by Hahn in [5].

According to the definitions of Hahn, and fixing notation, we have:

Definition 2.2 A and B are **hermitian Morita equivalent** , *if they are Morita equivalent (in the usual sense) and (with the notations of definition 2.1):*

1. *P and Q are k-symmetric (i.e. $\lambda p = p\lambda \; \forall \lambda \in k, p \in P$, idem for Q)*

2. *There exist $\mu : P \otimes_B Q \rightarrow A$ and $\tau : Q \otimes_A P \rightarrow B$ isomorphisms of A-bimodules and B-bimodules respectively such that:*

   (a) $\mu(p \otimes q)p' = p\tau(q \otimes p')$ and
   (b) $\tau(q \otimes p)q' = q\mu(p \otimes q) \; \forall p, p' \in P, \; q, q' \in Q$

3. *There exists an additive bijection $\Theta : P \rightarrow Q$ such that*

   (a) $\mu(p \otimes \Theta(p')) = \omega_A(\mu(p' \otimes \Theta(p)))$
   (b) $\tau(\Theta(p) \otimes p') = \omega_B(\tau(\Theta(p') \otimes p))$ and
   (c) $\Theta(apb) = \omega_B(b)\Theta(p)\omega_A(a) \; \forall a \in A, b \in B, p, p' \in P, q, q' \in Q$

A pair of bimodules $_APB$ and $_BQ_A$ satisfying these properties will be called **invertible bimodules**.

Hahn shows that under these hypothesis the categories ((hermitian $_Amod$)) and ((hermitian $_Bmod$)) are equivalent, where ((hermitian $_Amod$)) is the category whose objects are A-modules equipped with a ω_A-sesquilinear form.

Another interpretation of this definition will be given by proposition 2.3. We will now fix some notation.

Let

$$FA :_A mod \to_{A^{op}} mod = mod_A$$

$$FB :_B mod \to_{B^{op}} mod = mod_B$$

be the functors induced by the involutions ω_A and ω_B. Namely, if M is a left A-module then $FA(M) = M$ as abelian group, and the right A-action is given by:

$$m.a = \omega_A(a).m \;\forall m \in M,\, a \in A$$

If $f :_A M \to_A N$ then $FA(f) = f$.

Now, if A and B are Morita equivalent we then have two possible functors from $_Amod$ to $_Bmod$. One is the usual $Q \otimes_A -$, and the other one is the composition:

$$_Amod \xrightarrow{FA} mod_A \xrightarrow{-\otimes_A P} mod_B \xrightarrow{FB} {}_Bmod$$

Proposition 2.3 *The following facts are equivalent:*

1. *$\exists \Theta : P \to Q$ $\mathbb{Z}$-isomorphism such that*

$$\Theta(a.p.b) = \omega_B(b)\Theta(p)\omega_A(a) \;\forall a \in A,\, p \in P,\, b \in B$$

2. *The functors $Q \otimes_A -$ and $FB(- \otimes_A P)FA$ are naturally equivalent.*

Proof:
1. $\Rightarrow$ 2.

We observe that $\Theta(apb) = \omega_B(b)\Theta(p)\omega_A(a) \;\forall a \in A,\, b \in B$ and $p \in P$ if and only if $\Theta^{-1}(bqa) = \omega_A(a)\Theta^{-1}(q)\omega_B(b) \;\forall a \in A,\, b \in B$ and $q \in Q$.

Let $\eta_M : Q \otimes_A M \to FB(FA(M) \otimes_A P)$ defined by

$$\eta_M(q \otimes_A m) = m \otimes_A \Theta^{-1}(q)$$

η_M is a well-defined B-module morphism because of the properties of Θ with respect to scalars.

η_M is an isomorphism with inverse $m \otimes_A p \mapsto \Theta(p) \otimes_A m$.

The naturality of η_M is easily checked.

2. $\Rightarrow$ 1.

We remark that $\omega_A : FA(A) \to A$ is an isomorphism of A-bimodules:

$$\omega_A(a'.a.a'') = \omega_A(\omega_A(a'').a.\omega_A(a')) = a'.\omega_A(a).a''$$

where the action on the first term is the action on $FA(A)$, and the usual one on the others.

When $M = A$, the natural transformation gives us a morphism:

$$Q \overset{\psi}{\to} Q \otimes_A A \overset{\eta_A}{\to} FB(FA(A) \otimes_A P) \overset{\underset{= \omega_A \otimes_A id_P}{FB(\omega_A \otimes_A id_P)}}{\longrightarrow} FB(A \otimes_A P) \overset{FB(\phi) = \phi}{\to} FB(P)$$

We define Θ^{-1} as the composition:

$$\Theta^{-1} = \phi(\omega_A \otimes_A id_P)\eta_A\psi : Q \to FB(P)$$

Θ^{-1} is composition of isomorphisms of B-modules, then a $\mathbb{Z}$-isomorphism with the property $\Theta^{-1}(bq) = b\Theta^{-1}(q) = \Theta^{-1}(q)\omega_B(b)$ (the action is on $FB(P)$).

If $a \in A$ we have the following:

$$\Theta^{-1}(qa) = \phi(\omega_A \otimes_A id_P)\eta_A\psi(qa) = \phi(\omega_A \otimes_A id_P)\eta_A(q \otimes_A a) =$$

$$= \phi(\omega_A \otimes_A id_P)\eta_A(id_Q \otimes_A \times a)(q \otimes_A 1)$$

where $\times a$ is the right multiplication by a. By the naturality of η_A, the following diagram is commutative:

$$
\begin{array}{ccc}
Q \otimes_A A & \overset{id_Q \otimes_A \times a}{\longrightarrow} & Q \otimes_A A \\
\downarrow{\eta_A} & & \downarrow{\eta_A} \\
FB(FA(A) \otimes_A P) & \underset{\times a \otimes_A id_P}{\longrightarrow} & FB(FA(A) \otimes_A P)
\end{array}
$$

Then:

$$\phi(\omega_A \otimes_A id_P)\eta_A(id_Q \otimes_A \times a)(q \otimes_A 1) = \phi(\omega_A \otimes_A id_P)(\times a \otimes_A id_P)\eta_A(q \otimes_A 1) =$$

$$= \phi(\omega_A \otimes_A id_P)(\times a \otimes_A id_P)(\omega_A \otimes_A id_P)^{-1}(\omega_A \otimes_A id_P)\eta_A(q \otimes_A 1) =$$

$$= \phi(\omega_A \times a\omega_A^{-1} \otimes_A id_P)(\omega_A \otimes_A id_P)\eta_A(q \otimes_A 1)$$

but $\omega_A(\times a)\omega_A^{-1}(a') = \omega_A(\omega_A(a').a) = \omega_A(a).a'$ then $\omega_A(\times a)\omega_A^{-1} = \omega_A(a)\times$ where $\omega_A(a)\times : A \to A$, $(a' \mapsto \omega_A(a).a')$.

So:

$$\Theta^{-1}(qa) = \phi(\omega_A(a) \times \otimes_A id_P)(\omega_A \otimes_A id_P)\eta_A(q \otimes_A 1) =$$

$$= \phi(\omega_A(a)\times\otimes_A id_P)\phi^{-1}\phi(\omega_A\otimes_A id_P)\eta_A(q\otimes_A 1) = \phi(\omega_A(a)\times\otimes_A id_P)\phi^{-1}\Theta^{-1}(q) =$$

$$= \phi(\omega_A(a) \times \otimes_A id_P)(1 \otimes_A \Theta^{-1}(q)) = \phi(\omega_A(a) \otimes_A \Theta^{-1}(q)) = \omega_A(a)\Theta^{-1}(q)$$

We obtain $\Theta(apb) = \omega_B(b)\Theta(p)\omega_A(a)$ $\forall a \in A$, $b \in B$ and $p \in P$.

Lemma 2.4 *If M is an A-bimodule which has an involution ω_M compatible with the involution ω_A of A, then $Q \otimes_A M \otimes_A P$ has, in a natural way, an involution compatible with the involution ω_B of B.*

Proof:

Let $\omega : Q \otimes_A M \otimes_A P \to Q \otimes_A M \otimes_A P$ be defined by:

$$\omega(q \otimes m \otimes p) = \Theta(p) \otimes \omega_M(m) \otimes \Theta^{-1}(q)$$

It is easy to verify that ω satisfies the required properties.

Definition 2.5 *In the conditions of definition 2.2, we shall say that*

$(A, \Theta :_A P_B \to_B Q_A, B)$ *is an* **hermitian set of equivalence data**

Lemma 2.6 *If Θ, μ, τ satisfy conditions 1. 2. and 3. of definition 2.2, then μ and τ are compatible with the involutions,*
i.e.: $\mu(\omega(p \otimes q)) = \omega_A(\mu(p \otimes q))$ and $\tau(\omega'(q \otimes p)) = \omega_B(\tau(q \otimes p))$.

   where:

   $\omega : P \otimes_A Q \to P \otimes_A Q \;\; \omega(p \otimes q) = \Theta^{-1}(q) \otimes \Theta(p)$
   $\omega' : Q \otimes_B P \to Q \otimes_A P \;\; \omega'(q \otimes p) = \Theta(p) \otimes \Theta^{-1}(q)$

Proof: Immediate.
Examples:

1. Let $B = M_n(A)$ with $(\omega_B(m))_{ij} = \omega_A(m_{ji})$ if $m \in M_n(A)$, $P = A^{1 \times n}$ (row vectors), $Q = A^{n \times 1}$ (column vectors) and $\Theta : P \to Q$ given by $\Theta(a_1, ..., a_n) = (\omega_A(a_1), ..., \omega_A(a_n))^t$.

   The conditions of definition 2.2 are easily verified.

2. Let A be a k-algebra with involution, $B = A$, $\omega_B(a) = \gamma \omega_A(a) \gamma$ where $\gamma \in A$ is such that $\gamma^2 = 1$, $\Theta : A \to A$, $\Theta(x) = \gamma \omega_A(x)$.

3. Let (A, ω_A) be an involutive k-algebra, P a finitely generated projective A-module, $B = End_A(P)$. Then $P \in_A mod_B$ and if B is equipped with the involution induced by the involution of $M_n(A)$ we have an hermitian set of equivalence data.

4. Let M be a C^∞ locally compact manifold, and G a discrete group, acting on M without fixed points.

Let $A = C^\infty(M/G)$ be the algebra of complex-valued C^∞-functions that vanish at the infinity. $B = C^\infty(M) \rtimes G$.

   We consider the involution ω_A in A given by complex conjugation, and the involution ω_B in B given by:

$$\omega_B(\phi, g) = (g^{-1}\phi^*, g^{-1})$$

   Where $\phi \in C^\infty(M)$, $g \in G$, and if $m \in M$: $\phi^*(m) = \phi(m)^*$ (the complex conjugation). The action of G in $C^\infty(M)$ is given by $(h\phi)(m) = \phi(h^{-1}m)$ $\forall m \in M$, $h \in G$.

   We take $P = Q = C^\infty(M)$, equipped with the following structure:

1. Left A-module by: $(f\phi)(m) = f([m])\phi(m)$

2. Right B-module by: $(\phi(\phi_1, g))(m) = \phi(gm)\phi_1(gm)$

8

3. Right A-module as in 1.

4. Left B-module by: $((\phi_1, g)\phi)(m) = \phi_1(m)\phi_{(g^{-1}m)}$

where $f \in C^\infty(M/G)$; $\phi_1, \phi \in C^\infty(M)$; $m \in M$, $g \in G$ and $[m]$ denote the class of m in M/G.

We consider the additive morphism $\Theta : P \to Q = P$ given by complex conjugation. We check that $\Theta(apb) = \omega_B(b)\Theta(p)\omega_A(a)$ $\forall a \in A$, $b \in B$, $p \in P$.

Finally we define the bimodule isomorphisms $\mu : P \otimes_B Q \to A$ and $\tau : Q \otimes_A P \to B$ by:

$$\tau(\phi, \psi) = \sum_{x \in G}(F_x, x) \qquad \text{where } F_x \in C^\infty(M), F_x(m) = \phi(m)\psi(x^{-1}m)$$

and

$$\mu(\phi, \psi)([m]) = \sum_{x \in G} \phi(x^{-1}m)\psi(x^{-1}m) \qquad (\phi, \psi \in C^\infty(M))$$

1. $\tau(\Theta(p) \otimes p') = \omega_B(\tau(\Theta(p') \otimes p))$

2. $\mu(p \otimes q)p' = p\tau(q \otimes p')$ and

3. $\tau(q \otimes p)q' = q\mu(p \otimes q')$ $\forall p, p' \in P$, $q, q' \in Q$

Then it will follow from next Theorem and 2.8 that $HH_*^+(A) \cong HH_*^+(B)$ and $HD_*(A) \cong HD_*(B)$.

Remarks:

1. These last equalities show the usefulness of computations of HH_*^+ and HD_* of crossed products.

2. The last example is a particular case of the notion of "Strong Morita equivalence" for C^*-algebras defined by Rieffel [7].

Theorem 2.7 *In the conditions of Def.2.2 and Lemma 2.6, if $1/2 \in k$:*

$$H_*^+(A, M) \cong H_*^+(B, Q \otimes_A M \otimes_A P)$$

Proof: During this proof, we will write $\otimes$ instead of $\otimes_A$ or $\otimes_B$ when the meaning of $\otimes$ is clear by the context.

At the beginning, the proof is similar to the proof of Morita invariance of Hochschild homology. Namely, we know that there exist

$\{p_1, ... p_s\} \subset P$, $\{p'_1, ..., p'_m\} \subset P$, $\{q_1, ..., q_s\} \subset Q$, $\{q'_1, ..., q'_m\} \subset Q$ such that:

$$\mu(\sum_{i=1}^{s} p_i \otimes q_i) = 1_A \qquad \tau(\sum_{j=1}^{m} q'_j \otimes p'_j) = 1_B$$

Then we define a morphism of complexes:

$$\psi : M \otimes A^{\otimes *} \to (Q \otimes_A M \otimes_A P) \otimes B^{\otimes *}$$

$$\phi : (Q \otimes_A M \otimes_A P) \otimes B^{\otimes *} \to M \otimes A^{\otimes *}$$

as in ([6],p.19).

For ψ (the other is similar), one has:

$$\psi_n(m \otimes a_1 \otimes ... \otimes a_n) = \sum_{1 \le j_i \le s} (q_{j_0} \otimes m \otimes p_{j_1} \tau(q_{j_1} \otimes a_1 \otimes p_{j_2}) \otimes ... \otimes \tau(q_{j_n} \otimes a_n \otimes p_{j_0}))$$

$$\omega(\psi_n(m \otimes a_1 \otimes ... \otimes a_n)) =$$

$$= (-1)^{n(n+1)/2} \sum_{1 \le j_i \le s} \omega(q_{j_0} \otimes m \otimes p_{j_1}) \otimes \omega_B(\tau(q_{j_n} \otimes a_n \otimes p_{j_0})) \otimes ... \otimes \omega_B(\tau(q_{j_1} \otimes a_1 \otimes p_{j_2}))$$

$$= (-1)^{n(n+1)/2} \sum_{1 \le j_i \le s} \Theta(p_{j_1}) \otimes \omega_M(m) \otimes \Theta^{-1}(q_{j_0}) \otimes \tau\omega(q_{j_n} \otimes a_n \otimes p_{j_0}) \otimes ... \otimes \tau\omega(q_{j_1} \otimes a_1 \otimes p_{j_2})$$

$$= (-1)^{n(n+1)/2} \sum_{1 \le j_i \le s} \Theta(p_{j_1}) \otimes \omega_M(m) \otimes \Theta^{-1}(q_{j_0}) \otimes \tau(\Theta(a_n p_{j_0}) \otimes \Theta^{-1}(q_{j_n})) \otimes ... \otimes \tau(\Theta(a_1 p_{j_2}) \otimes \Theta^{-1}(q_{j_1}))$$

$$= (-1)^{n(n+1)/2} \sum_{1 \le j_i \le s} \Theta(p_{j_1}) \otimes \omega_M(m) \otimes \Theta^{-1}(q_{j_0}) \otimes \tau(\Theta(p_{j_0})\omega_A(a_n) \otimes \Theta^{-1}(q_{j_n})) \otimes ... \otimes \tau(\Theta(p_{j_2})\omega_A(a_1) \otimes \Theta^{-1}(q_{j_1}))$$

If $(m \otimes a_1 \otimes ... \otimes a_n) \in (M \otimes A^{\otimes *})^+$, then

$$(m \otimes a_1 \otimes ... \otimes a_n) = (-1)^{n(n+1)/2}\omega_M(m) \otimes \omega_A(a_n) \otimes ... \otimes \omega_A(a_1)$$

and then:

$$\psi_n(\omega(m \otimes a_1 \otimes ... \otimes a_n)) = \psi_n((-1)^{n(n+1)/2}\omega_M(m) \otimes \omega_A(a_n) \otimes ... \otimes \omega_A(a_1))$$

So, $\sum_{1 \le j_i \le s}(q_{j_0} \otimes m \otimes p_{j_1} \otimes \tau(q_{j_1} \otimes a_1 p_{j_2}) \otimes ... \otimes \tau(q_{j_n} \otimes a_n p_{j_0})) =$

$$= (-1)^{n(n+1)/2} \sum_{1 \le j_i \le s} (q_{j_0} \otimes \omega_M(m) \otimes p_{j_1}) \otimes \tau(q_{j_n} \otimes \omega_A(a_n)p_{j_0}) \otimes ... \otimes \tau(q_{j_1} \otimes \omega_A(a_1)p_{j_2})$$

As $\tau(\sum_{i=1}^{s} q_{j_i} \otimes p_{j_i}) = 1_B$, then $1_B = \tau(\sum_{i=1}^{s} \Theta(p_{j_i}) \otimes \Theta^{-1}(q_{j_i}))$

And as τ is an isomorphism, $\sum_{i=1}^{s} q_{j_i} \otimes p_{j_i} = \sum_{i=1}^{s} \Theta(p_{j_i}) \otimes \Theta^{-1}(q_{j_i})$.

If we define ψ_n^{Θ} by the same formula but replacing the sets $\{p_1, ..., p_s\}$ and $\{q_1, ..., q_s\}$ by $\{\Theta^{-1}(q_1), ..., \Theta^{-1}(q_s)\}$ and $\{\Theta(p_1), ..., \Theta(p_s)\}$ respectively, we have that:

$$\omega \psi_n = \psi_n^{\Theta} \omega$$

As a consequence we have the following commutative diagrams:

$$
\begin{array}{ccc}
C_n(A, M) & \xrightarrow{\psi_n} & C_n(B, Q \otimes_A M \otimes_A P) \\
\downarrow{\omega} & & \downarrow{\omega} \qquad \Rightarrow \\
C_n(A, M) & \xrightarrow{\psi_n^{\Theta}} & C_n(B, Q \otimes_A M \otimes_A P)
\end{array}
$$

$$
\begin{array}{ccc}
H_n(A, M) & \xrightarrow{\psi_*} & H_n(B, Q \otimes_A M \otimes_A P) \\
\downarrow{\omega_*} & & \downarrow{\omega_*} \\
H_n(A, M) & \xrightarrow{\psi_*^{\Theta}} & H_n(B, Q \otimes_A M \otimes_A P)
\end{array}
$$

Loday shows that $\psi\phi$ is homotopic to the identity by defining an explicit homotopy h_i (and the same for $\phi\psi$), then $\psi_* = \phi_*^{-1}$.

If one considers h_i^{Θ} defined by the same formula as h_i but replacing p_i' by $\Theta^{-1}(q_i)$ and q_i' by $\Theta^{-1}(p_i)$, then $\psi^{\Theta}\phi$ is homotopic to the identity, so $\psi_*^{\Theta} = \phi_*^{-1}$. As a consequence $\psi_*^{\Theta} = \psi_*$.

This proves that

$$H_n^+(A, M) \cong^{\psi_*} H_n^+(B, Q \otimes_A M \otimes_A P)$$

Remark: Sometimes we may have $\Theta(q_i) = p_i$, for all i (for example in the case $B = M_n(A)$, , see [6]). If this is the case, the hypothesis that $1/2 \in k$ is not necessary, but the adequate version of positive Hochschild homology in this case, denoted H^{inv} is the homology of the double complex-called homotopy colimit- having alll columns equal to the Hochschild complex with 2-periodic row boundary map $1 \pm \omega$. As now the chain map giving Morita invariance commutes with the operator ω, it follows that Morita invariance also holds for this homology.

Proposition 2.8 *Under the same hypothesis of Theorem 2.7, $HD_*(A) \cong HD_*(B)$*

Proof: We have explicit maps $HD_*(A) \to HD_*(B)$, $HH^+_*(A) \to HH^-_*(B)$ and $HH^-_*(A) \to HH^+_*(B)$ that induce commutative diagrams between the Connes's long exact sequence in dihedral homology for A and B. From Theorem 2.7 and a similar version for HH^-_*, we deduce that $HD_*(A) \to HD_*(B)$ is an isomorphism, using a five-Lemma argument.

3 Equivalences between bimodule categories

The proof of Morita invariance of HH_* and HC_* makes use of the existence of an equivalence between the categories of A-bimodules and B-bimodules, and the existence of invertible bimodules $_BP_A$ and $_AQ_B$ allows us to build explicitly the bimodule equivalence.

The equivalence between the categories $_Abimod$ and $_Bbimod$ is in general weaker than the equivalence between $_Amod$ and $_Bmod$. In fact, $_Amod$ and $_Bmod$ are equivalent if and only if mod_A and mod_B are equivalent. Both imply that $_Abimod$ is equivalent to $_Bbimod$, but if for example A is a (non commutative) ring such that $\otimes_A$ is not symmetric, then the converse does not hold.

Example: Let $\mathbb{H}$ denote the quaternionic numbers, i.e. $\mathbb{H}$ is the real algebra generated by i, j, k with relations $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$; $jk = i = -kj$; $ki = j = -ik$. Since the set of real numbers is a field, every ring Morita equivalent to it is a matrix ring with real coefficients. $\mathbb{H}$ has dimension 4 (as a real vector space), if it is Morita equivalent to the real numbers then it must be isomorphic to $M_2(\mathbb{R})$, but that is not possible because $\mathbb{H}$ is a division ring. Nevertheless $\mathbb{H} \cong \mathbb{H}^{op}$ and $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$ (isomorphism of real algebras) and $M_4(\mathbb{R})$ is Morita equivalent to the real numbers. That shows that the category of $\mathbb{H}$-bimodules=$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{op}$-modules is equivalent to the category of real vector spaces, but the category of $\mathbb{H}$-modules is not.

This is a a particular example of the following situation: Consider an algebra A over a commutative ring R. A is called an **Azumaya algebra** over R if the category of A-bimodules is Morita equivalent to the category of R-modules. Examples of Azumaya algebras are Clifford algebras of even ranked quadratic modules, in particular the quaternion algebra $\mathbb{H}$. A result due to Schack [8] and also proved in [1] states that if R is a commutative algebra over a field k and A is an Azumaya algebra over R, then $HH_*(A/k) =$

$HH_*(R/k)$. We shall study this situation for involutive k-algebras at the end of this Section.

In [8] Schack defines the notion of k-congruence (weak and strong) between two unital k-algebras A and B as follows:

Definition 3.1 *An equivalence of categories $T :_A bimod \to_B bimod$ is called a k-congruence if and only if:*

1. *(weak) $T(A) \cong^\gamma B$ (where γ denotes the isomorphism)*

2. *(strong) $T(M \otimes_A N) \cong T(M) \otimes_B T(N)$*

Remark: Strong k-congruence implies weak k-congruence.

Schack shows that every Morita equivalence induces a strong k-congruence, and claims that if A and B are such that there exists a weak k-congruence between them, then $H^*(A, M) \cong H^*(B, T(M))$ (where M is any A-bimodule).

Let now (A, ω_A) and (B, ω_B) be two unital involutive k-algebras. We are interested in the bimodule equivalences T which preserve involutions.

We know that every equivalence between $_A bimod =_{A^e} mod$ and $_{B^e} mod$ is obtained by tensoring with a module $R \in {}_{B^e} mod_{A^e}$, where $R = T(A^e)$. Let's suppose that R is equipped with an involution ω_R compatible with A and B (i.e. $\omega_R(ara') = \omega_A(a')\omega_R(r)\omega_A(a)$ and $\omega_R(brb') = \omega_B(b')\omega_R(r)\omega_A(b)$ $\forall a, a' \in A$, $b, b' \in B$ and $r \in R$); then if M is a compatible A^e-module, $T(M) = R \otimes_{A^e} M$ has a well-defined involution $\omega_{R \otimes_{A^e} M} = \omega_R \otimes_{A^e} \omega_M$ that makes $T(M)$ into a compatible B^e-module.

In this context we make the following:

Definition 3.2 *T is an **involutive k-congruence** if and only if it is a (weak) k-congruence and $\omega_B(b) = \gamma(\omega_A(a) \otimes_{A^e} \omega_R(r))$ $\forall b \in B$, where $\gamma(a \otimes_{A^e} r) = b$. We write then $T((A, \omega_A)) = (B, \omega_B)$, ($\gamma$ is the isomorphism of the definition 3.1).*

Proposition 3.3 *Every hermitian equivalence $\overline{T} :_A mod \to_B mod$ induces an involutive k-congruence*
$T : ((compatible\ _A bimod)) \to ((compatible\ _B bimod))$. i.e. a k-congruence
$T :_{Z/2Z \rtimes A^e} -mod \to_{Z/2Z \rtimes B^e} -mod$

Proof: An hermitian Morita equivalence is given by a pair of invertible bimodules $_A P_B$ and $_B Q_A$ and an additive map $\Theta : P \to Q$ satisfying the

conditions of 2.2. Then we can take $R = Q \otimes_k P$ with the involution
$\omega_R : Q \otimes_k P \to Q \otimes_k P$, $\omega_R(q \otimes p) = \Theta(p) \otimes \Theta^{-1}(q)$. The induced equivalence
is then $T(M) = R \otimes_{A^e} M = (Q \otimes_k P) \otimes_{A^e} M \cong Q \otimes_A M \otimes_A P$.

$T(A) \cong Q \otimes_A A \otimes_A P \cong Q \otimes_A P \cong B$, and we can easily check that
$T(A, \omega_A) = (B, \omega_B)$.

As Schack has already remarked, every k-linear functor
$T :_A bimod \to_B bimod$ defines a morphism between the Yoneda extensions
$Ext^*_{A^e}(-, -)$ and $Ext^*_{B^e}(T(-), T(-))$, applying T to every bimodule in each
class of extension, which is clearly an isomorphism when T is an equivalence.
In particular, when A and B are k-projective and $T(A) \cong B$ we have
$H^*(A, M) \cong H^*(B, T(M))$ for every A-bimodule M. In the case of invo-
lutive k-algebras we want to show that $H^{*+}(A, M) \cong H^{*+}(B, T(M))$ for
every compatible A-bimodule M, and analogously for homology.

We will suppose, until the end of this section, that A and B are both
k-projective, and if T is an equivalence between the categories of $_{A^e}mod$
and $_{B^e}mod$, let $_{B^e}R_{A^e}$ and $_{A^e}S_{B^e}$ be bimodules such that $R \otimes_{A^e} S \cong B^e$,
$S \otimes_{B^e} R \cong A^e$ (bimodule isomorphisms) and $T = R \otimes_{A^e} (-)$. The condition
that T is a k-congruence implies that $R \otimes_{A^e} A \cong B$ as B^e-module. Now we
require also that $A \otimes_{A^e} S \cong B$.

The condition of projectivity of A and B over k allows us to compute
$H_*(A, M)$ as $Tor^{A^e}_*(A, M)$, using A^e-projective resolutions of A.

If $(P_n, d_n)_{n \geq 0}$ is an A^e-projective resolution of A, the fact that $- \otimes_{A^e} S$
is also an equivalence implies that $(P_n \otimes_{A^e} S, d_n \otimes id_S)_{n \geq 0}$ is a B^e-projective
resolution of $A \otimes_{A^e} S \cong B$. Then $H_*(B, T(M)) = Tor^{B^e}_*(B, R \otimes_{A^e} M)$ is
the homology of the complex
$(P_n \otimes_{A^e} S \otimes_{B^e} R \otimes_{A^e} M, d_n \otimes id_S \otimes id_R \otimes id_M)_{n \geq 0} \cong$
$\cong (P_n \otimes_{A^e} A^e \otimes_{A^e} M, d_n \otimes id_{A^e} \otimes id_M)_{n \geq 0} \cong (P_n \otimes_{A^e} M, d_n \otimes id_M)_{n \geq 0}$. The
homology of this last complex computes $Tor^{A^e}_*(A, M)$. So we have proved
the following:

Proposition 3.4 *With the notations as above, if A and B are k-projective
and $A \otimes_{A^e} S \cong B$ as a B-bimodule, then $H_*(A, M) \cong H_*(B, T(M))$*

Let us now return to the case where A and B are involutive algebras and
T is an involutive k-congruence.

Suppose we have an A^e-projective resolution of A $(P_n, d_n)_{n \geq 0}$ such that
every P_n is a compatible A^e-module, equipped with an involution such that
$\omega_P.d = d.\omega_P$. In this situation we will say that we have an **involutive**

projective resolution, which is the same thing that a projective resolution in the category of $\mathbb{Z}/2\mathbb{Z} \rtimes A^e$-modules.

Lemma 3.5 *If M is a compatible A-bimodule, then:*

$$H_*^{inv}(A, M) = Tor_*^{Z/2Z \rtimes A^e}(A, M)$$

Proof: Let $C'(A)$ be the Hochschild resolution of A as an A^e-module. Define $y' : C'_n(A) \to C'_n(A)$ as $y'(a \otimes a_1 \otimes \ldots \otimes a_n \otimes a') = (-1)^{n(n+1)/2}(\omega_A a' \otimes \omega_A a_n \otimes \ldots \otimes \omega_A a_1 \otimes \omega_A a)$. Let $\mathbb{Z}/2\mathbb{Z}$ act on $C'_*(A) \oplus C'_*(A)$ by $(x_1, x_2) \to (y'(x_2), y'(x_1))$, and form the corresponding homotopy colimit as in Section 2. The result is a double complex whose total complex is a projective $\mathbb{Z}/2\mathbb{Z} \rtimes A^e$-module in each degree. Moreover, tensoring the latter complex over $\mathbb{Z}/2\mathbb{Z} \rtimes A^e$ with M is exactly the complex that computes $H_*^{inv}(A, M)$ as defined before. This concludes the proof.

Theorem 3.6 *Under the same notations and hypothesis as in 3.5:*

$$H_*^{inv}(A, M) \cong H_*^{inv}(B, T(M))$$

Proof: It is enough to remark the following facts:

- The standard Hochschild resolution of A, equipped with the involution y' defined in the previous lemma is a $\mathbb{Z}/2\mathbb{Z} \rtimes A^e$- projective resolution.

- $H_*^{inv}(A, M) = Tor_*^{Z/2Z \rtimes A^e}(A, M)$.

- With the same argument as in proposition 3.4, one obtains an involutive B^e-projective resolution of B.

So, the theorem is a corollary of the above discussion.

Remarks

1. An analogous proof shows that the same holds for cohomology.

2. If A is a commutative k-projective algebra, we can define an explicit morphism $f_* : HH_*(A) \to HH_*(B)$ such that f_0 is an isomorphism and f_* makes the long exact sequences commutative as follows.

Let $r_0 \in R$ be such that $\gamma(r_0 \otimes_{A^e} 1_A) = 1_B$, and let $f_n : HH_n(A) \to HH_n(B)$ be defined by
$$f_n([a_0 \otimes ... \otimes a_n]) = [\gamma(r_0 \otimes a_0) \otimes ... \otimes \gamma(r_0 \otimes a_n)]$$
f_n is well-defined because $\gamma(r_0 \otimes a)\gamma(r_0 \otimes a') - \gamma(r_0 \otimes aa')$ is in general a linear combination of terms of the type $\gamma(r_i \otimes a_i'')$ with $a_i'' \in [A, A]$, but A is commutative.

Then, since in the above situation HH_* is an universal functor, we know that $HH_*(A) \cong HH_*(B)$, and the commutativity of f_* with the involutions shows that $HH_*^+(A) \cong^{f_*} HH_*^+(B)$.

Finally, looking at the case of involutive Azumaya algebras, we remark that, as a corollary of 3.6 and Schack's Theorem 3 [8], we have:

Proposition 3.7 *Let B be an involutive and commutative k-algebra, and let A be an Azumaya B-algebra Then any hermitian B-congruence $T :_B$ bimod $\to_A$ bimod induces an isomorphism*

$$H_*^{inv}(B, -) \to H_*^{inv}(A, T(-))$$

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